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# Stability analysis of non-linear sampled data systems with time varying sample period and delay in the feedback loop

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**Abstract:** In almost all practical applications of control, technological and economical considerations impose limits on communication speed, frequency of communication, and frequency of actuator adjustment. Such limits turned the analysis of sampled data systems into a flourishing field. Water systems pose a particular challenge: the systems are networks of canals and reservoirs spread over large areas, and the actuators are relatively large and exposed to the elements. In this study, a theorem on the local exponential stability of sampled data systems with variable control time step and variable delay in the communication between the non-linear continuous time process and the non-linear discrete time controller is presented. To illustrate the application of the theorem, it is applied to a simple water system.



*Keywords:* sampled data system, communication delays, stability, open channel flow, water management
*1991 MSC:* [2008]93C57

## 1. INTRODUCTION

There is a long history of water level control in water management. In fact, even a fixed weir can act as a water level regulator for a lake with a varying inflow (van Nooijen and Kolechkina, 2020b). Hand operated moveable weirs or gates may even go back to prehistory (van Nooijen et al., 2021). In modern water management, digital computers are used to calculate the control actions and implement those actions by adjusting moveable weirs or gates, or adjust pump speeds (Mareels et al., 2005; Hadid et al., 2019; van Nooijen and Kolechkina, 2018). Networks of open channels form an important category of environmental systems. They are used not only to transport irrigation and drainage water, but also as highways for barges transporting raw materials and goods. Automatic control of these systems poses specific problems (van Nooijen and Kolechkina, 2020a). Remotely operated gates and weirs are adjusted by electric motors. Using these has an associated cost in terms of wear and tear. It therefore makes sense to limit the number of adjustments to actuator settings. This also helps save communications bandwidth. Allowing some time to implement a control action may also help with the larger actuators, where rapid adjustment is costly or undesirable. Evidently, the time step depends on the time scale on which the system operates. The control of the water level of a large lake has another rhythm than the management of a small irrigation canal. The question now arises how non-trivial communication delays and control time step variations affect these systems.

More general communications problems, such as dropped packets or out of order arrival, will not be treated here. For such problems, see Zhang et al. (2013) and references therein. The increased generality achieved there entails either restrictions on plant or controller form, or assumptions on the existence of functions or functionals that may take considerable effort to find. Examples are the treatment of global stability in van de Wouw et al. (2012), which achieves generality by assuming existence of a family of approximate discrete time plant models that meet specific criteria and a family of auxiliary functions, and Tolić (2020), which assumes existence of a suitable functionals and involves solving several matrix inequalities. Dealing with sampled data systems with aperiodic sampling has also received considerable attention, see for example Hetel et al. (2017) and references therein, but there additional delay in the feedback loop is not explicitly considered. Again, for many of the methods discussed, specific functionals need to be found.

The emphasis in this paper is on a simple method to check local asymptotic stability of a physical network of waterways where the actuators are controlled by a discrete controller. The approach of Hu and Michel (2000) for sampled data systems is extended to sampled data systems with variable delays in the feedback loop by borrowing an idea used in the treatment of linear sampled systems by Åström and Wittenmark (1997). The most attractive aspect of this approach is that local asymptotic stability is linked to a specific property of a matrix that can be constructed automatically once the derivatives of the time


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evolution functions and output functions of the plant and controller have been determined. Those derivatives themselves can in principle be determined automatically either through symbolic differentiation in suitable software or through automatic differentiation (Bischof et al., 2008; Naumann, 2012). As an illustration, the resulting theorem is applied to a simple process consisting of a lake fed by a stream and discharging into a river over an adjustable weir. Such a process could be a component in a larger network. A PI controller is used to keep the exposition as clear as possible. For this simple system the relation between controller coefficients, stability, and system response is examined for different controller time steps and delays.

## 2. A GENERAL DESCRIPTION OF THE TYPE OF SYSTEM UNDER CONSIDERATION

Consider a continuous process with an $n_{\mathrm{p}}$-dimensional state vector $x_{\mathrm{p}}$

$$\dot{x}_{\mathrm{p}}(t) = f_{\mathrm{p},1}(x_{\mathrm{p}}(t), u_{\mathrm{p}}(t)) \tag{1}$$

$$y_{\mathrm{p}}(t) = f_{\mathrm{p},2}(x_{\mathrm{p}}(t)) \tag{2}$$

$$x_{\mathrm{p}}(0) = x_{\mathrm{p}}^{(0)} \tag{3}$$

that is linked to a discrete time controller with an $n_{\mathrm{c}}$-dimensional state vector $x_{\mathrm{c}}$

$$x_{\mathrm{c}}(k+1) = f_{\mathrm{c},1}(x_{\mathrm{c}}(k), u_{\mathrm{c}}(k)) \tag{4}$$

$$y_{\mathrm{c}}(k) = f_{\mathrm{c},2}(x_{\mathrm{c}}(k), u_{\mathrm{c}}(k)) \tag{5}$$

$$x_{\mathrm{c}}(0) = x_{\mathrm{c}}^{(0)} \tag{6}$$

by a sampler that also models the variable feedback loop delay $\rho_k$

$$u_{\mathrm{c}}(k) = \begin{cases} 0 & k = 0 \\ y_{\mathrm{p}}(\tau_k - \rho_k) & k > 0 \end{cases} \tag{7}$$

and a zero order hold that links the controller output to the process input

$$u_{\mathrm{p}}(t) = y_{\mathrm{c}}(k), \tau_k \le t < \tau_{k+1} \tag{8}$$

where the $\tau_k$ are the points in time when the input to the process changes. To keep the notation compact, from this point onward the convention $x(\tau_0 - \rho_0) \triangleq 0$ will be used. As in Hu and Michel (2000), the system can be simplified by inserting (8) into (1) and (7) into (4) and (5). Next (2) and the modified version of (5) are used to eliminate $y_{\mathrm{p}}$ and $y_{\mathrm{c}}$. With new variables

$$x(t) = x_{\mathrm{p}}(t)\ ;\ u(k) = x_{\mathrm{c}}(k) \tag{9}$$

and new time evolution functions

$$f(x, \upsilon, u) = f_{\mathrm{p},1}(x, f_{\mathrm{c},2}(u, f_{\mathrm{p},2}(\upsilon))) \tag{10}$$

$$g(\upsilon, u) = f_{\mathrm{c},1}(u, f_{\mathrm{p},2}(\upsilon)) \tag{11}$$

the evolution in time can be written as

$$\dot{x}(t) = f(x(t), x(\tau_k - \rho_k), u(k)), \tau_k \le t < \tau_{k+1} \tag{12}$$

$$u(k+1) = g(u(k), x(\tau_k - \rho_k)) \tag{13}$$

$$x(\tau_0) = x_{\mathrm{p}}^{(0)} \tag{14}$$

$$u(\tau_0) = x_{\mathrm{c}}^{(0)} \tag{15}$$

where it is assumed that the controller starts up with $u_{\mathrm{c}}(0) = 0$. It will be assumed that

$$f \in C^1(\mathbb{R}^{n_{\mathrm{p}}} \times \mathbb{R}^{n_{\mathrm{p}}} \times \mathbb{R}^{n_{\mathrm{c}}}, \mathbb{R}^{n_{\mathrm{p}}}), f(0,0,0) = 0 \tag{16}$$

$$g \in C^1(\mathbb{R}^{n_{\mathrm{c}}} \times \mathbb{R}^{n_{\mathrm{p}}}, \mathbb{R}^{n_{\mathrm{c}}}), g(0,0) = 0 \tag{17}$$

and that there exists a $\mu$ such that the $\tau_k$ satisfy

$$\lim_{k\to\infty} \tau_k = \infty,\ \sup_{k\in\mathbb{N}} \{\tau_{k+1} - \tau_k\} = \mu < \infty \tag{18}$$

and the $\rho_{k+1}$ satisfy

$$\tau_k < \tau_{k+1} - \rho_{k+1} < \tau_{k+1} \tag{19}$$

To prepare for the formulation of the theorem, some auxiliary definitions are needed, namely, the matrices that describe the linearisation of the system

$$A = \left.\frac{\partial f(x,\upsilon,u)}{\partial x}\right|_{(0,0,0)},\ A_0 = \left.\frac{\partial f(x,\upsilon,u)}{\partial \upsilon}\right|_{(0,0,0)},$$
$$B = \left.\frac{\partial f(x,\upsilon,u)}{\partial u}\right|_{(0,0,0)} \tag{20}$$

and two functions $F \in C(\mathbb{R}^{n_{\mathrm{p}}} \times \mathbb{R}^{n_{\mathrm{p}}} \times \mathbb{R}^{n_{\mathrm{c}}}, \mathbb{R}^{n_{\mathrm{p}}})$ and $G \in C(\mathbb{R}^{n_{\mathrm{c}}} \times \mathbb{R}^{n_{\mathrm{p}}}, \mathbb{R}^{n_{\mathrm{c}}})$ that represent the remainder terms after linearisation of the continuous and discrete system respectively

$$\lim_{(x,\upsilon,u)\to(0,0,0)} \frac{F(x,\upsilon,u)}{\sqrt{\|x\|^2 + \|\upsilon\|^2 + \|u\|^2}} = 0 \tag{21}$$

$$\lim_{(u,\upsilon)\to(0,0)} \frac{G(u,\upsilon)}{\sqrt{\|\upsilon\|^2 + \|u\|^2}} = 0 \tag{22}$$

Given these definitions, the time evolution of the system follows from

$$\dot{x}(t) = Ax(t) + A_0 x(\tau_k - \rho_k) + Bu(\tau_k) \tag{23}$$
$$+F(x(t), u(\tau_k - \rho_k), u(\tau_k)), \tau_k \le t < \tau_{k+1}$$

$$u(\tau_{k+1}) = Cu(\tau_k) + Dx(\tau_k - \rho_k) \tag{24}$$
$$+G(x(\tau_k - \rho_k), u(\tau_k))$$

From (23) it follows that

$$x(t) = x(\tau_k) + (t-\tau_k) A_0 x(\tau_k - \rho_k) + (t - \tau_k) Bu(\tau_k)$$
$$+ \int_{\tau=\tau_k}^{t} Ax(\tau) + F(x(\tau), x(\tau_k - \rho_k), u(\tau_k))\, d\tau \tag{25}$$

or

$$x(t) = \mathrm{e}^{(t-\tau_k)A} x(\tau_k)$$
$$+ \int_{\tau=\tau_k}^{t} \mathrm{e}^{(t-\tau)A} d\tau A_0 x(\tau_k - \rho_k) + \int_{\tau=\tau_k}^{t} \mathrm{e}^{(t-\tau)A} d\tau Bu(\tau_k)$$
$$+ \int_{\tau=\tau_k}^{t} \mathrm{e}^{(t-\tau)A} F(x(\tau), x(\tau_k - \rho_k), u(\tau_k))\, d\tau \tag{26}$$

For a proof of (26) see, for instance, Sideris (2013, Corollary 4.1). Next, two vectors and a matrix are defined that link the continuous problem to a discrete problem.

$$\omega(k) = \begin{bmatrix} x(\tau_k - \rho_k) \\ x(\tau_k) \\ u(\tau_k) \end{bmatrix} \tag{27}$$

$$\Omega(k) = \tag{28}$$

$$\begin{bmatrix} \displaystyle\int_{\tau=\tau_k}^{\tau_{k+1}-\rho_{k+1}} \mathrm{e}^{(\tau_{k+1}-\rho_{k+1}-\tau)A} F(x(\tau), x(\tau_k - \rho_k), u(\tau_k))\, d\tau \\ \displaystyle\int_{\tau=\tau_k}^{\tau_{k+1}} \mathrm{e}^{(\tau_{k+1}-\tau)A} F(x(\tau), x(\tau_k - \rho_k), u(\tau_k))\, d\tau \\ G(x(\tau_k - \rho_k), u(\tau_k)) \end{bmatrix}$$

$$H_k = \begin{bmatrix} J(k,\rho_{k+1}) A_0 & e^{(\tau_{k+1}-\rho_{k+1}-\tau_k)A} & J(k,\rho_{k+1}) B \\ J(k,0) A_0 & e^{(\tau_{k+1}-\tau_k)A} & J(k,0)\, d\tau B \\ D & 0 & C \end{bmatrix} \quad (29)$$

where

$$J(k,\rho) = \int_{\tau=\tau_k}^{\tau_{k+1}-\rho} e^{(\tau_{k+1}-\rho-\tau)A} d\tau \quad (30)$$

This makes it possible to write

$$\omega(k+1) = H_k \omega(k) + \Omega(k) \quad (31)$$

## 3. THEORETICAL RESULTS

For matrices $\|\cdot\|$ is the norm induced by the Euclidean vector norm.

*Lemma 1.* Let $A$, $A_0$, $B$, $F$, $\mu$, $\tau_k$ and $\rho_k$ be as defined earlier. Let the time evolution of $x$ be given by (12). There is a $\delta_3 > 0$ such that for $\|\omega(k)\| \le \delta_3$ and $t \in [\tau_k, \tau_{k+1})$

$$\|x(t)\| \le (1 + \mu(\|A_0\| + \|B\| + 2)) e^{(1+\|A\|)(t-\tau_k)} \|\omega(k)\| \quad (32)$$

The proof of the bound (32) was extracted from the proof of Lemma 2.2 in Hu and Michel (2000).

**Proof.** From (21), continuity of $F$, and $F(0,0,0) = 0$, it follows that there is a $\delta_2 > 0$ such that

$$\|F(x,\upsilon,u)\| \le \|x\| + \|\upsilon\| + \|u\| \quad (33)$$

whenever $\max(\|x\|, \|\upsilon\|, \|u\|) \le \delta_2$. Now suppose that for all $\delta_3 > 0$ there is $\omega(k)$ with $\|\omega(k)\| < \delta_3$ and a $t_1 > \tau_k$ such that $\|x(t_1)\| > \delta_2$. Now let

$$c_0 = (1 + (\tau_{k+1} - \tau_k)(\|A_0\| + \|B\| + 2)) e^{\mu(\|A\|+1)} \quad (34)$$

and take $\delta_3 = \delta_2/c_0$. From $\|x(\tau_k)\| < \delta_2$, $\|x(t_1)\| > \delta_2$, and continuity, it follows that there is a $t_0 \in [\tau_k, t_1)$ such that $\|x(t)\| < \delta_2$ for $\tau_k \le t < t_0$ and $\|x(t_0)\| = \delta_2$. From (25), it follows that for $t \in [\tau_k, \tau_{k+1})$

$$\begin{aligned} \|x(t)\| &\le \|x(\tau_k)\| \\ &+ (t-\tau_k)(\|A_0\| \|x(\tau_k - \rho_k)\| + \|B\| \|u(\tau_k)\|) \quad (35) \\ &+ \int_{\tau=\tau_k}^{t} \|A\| \|x(\tau)\| + \|F(x(\tau), x(\tau_k-\rho_k), u(\tau_k))\| d\tau \end{aligned}$$

and from (33), it now follows that

$$\begin{aligned} \|x(t)\| &\le \|x(\tau_k)\| \\ &+ (t-\tau_k)((\|A_0\|+1)\|x(\tau_k-\rho_k)\| + (\|B\|+1)\|u(\tau_k)\|) \\ &+ \int_{\tau=\tau_k}^{t} (\|A\|+1)\|x(\tau)\| d\tau \\ &\le (1 + \mu(\|A_0\| + \|B\| + 2)) \|\omega(k)\| \\ &+ \int_{\tau=\tau_k}^{t} (\|A\|+1)\|x(\tau)\| d\tau \quad (36) \end{aligned}$$

By the Grönwall inequality this implies that

$$\begin{aligned} &\|x(t_0)\| \le \\ &(1 + \mu(\|A_0\| + \|B\| + 2)) \|\omega(k)\| e^{(t_0-\tau_k)(\|A\|+1)} \\ &\le \frac{(1 + \mu(\|A_0\| + \|B\| + 2)) e^{(t_0-\tau_k)(\|A\|+1)}}{(1 + (\tau_{k+1}-\tau_k)(\|A_0\| + \|B\| + 2)) e^{\lambda(\|A\|+1)}} \delta_2 \\ &< \delta_2 \quad (37) \end{aligned}$$

which contradicts our assumption. Therefore, there must be a $0 < \delta_3 < \delta_2/c_0$ such that for $\|\omega(k)\| \le \delta_3$, it follows that $\|x(t)\| \le \delta_2$ for all $t \in [\tau_k, \tau_{k+1}]$. For $\|\omega(k)\| \le \delta_3$, it follows that

$$\|x(t)\| \le (1 + \mu(\|A_0\| + \|B\| + 2)) \|\omega(k)\| e^{(t-\tau_k)(\|A\|+1)}$$

*Lemma 2.* Let $A$, $A_0$, $B$, $C$, $D$, $F$, $G$, $\Omega$, $\omega$, $\mu$, $\tau_k$ and $\rho_k$ be as defined earlier. For any given $\nu > 0$ there exists a $\delta_1(\nu) > 0$ such that

$$\|x(t)\| \le c_0 \|\omega(k)\| \quad (38)$$

$$\|\Omega(k)\| \le \nu \|\omega(k)\| \quad (39)$$

whenever $\|\omega(k)\| \le \delta_1$ for $k \in \mathbb{N}$ and $\tau_k \le t < \tau_{k+1}$ where $c_0$ as in (34) and with $\mu$ as in (18). This is a variation on Lemma 2.2 in Hu and Michel (2000).

**Proof.** From (32) it follows that there is a $\delta_3 > 0$ such that for $\|\omega(k)\| \le \delta_3$

$$\|x(t)\| \le (1 + \mu(\|A_0\| + \|B\| + 2)) e^{(1+\|A\|)(t-\tau_k)} \|\omega(k)\|$$

for all $t \in [\tau_k, \tau_{k+1})$. This proves (38). Next, suppose that $\nu$ is given. Take $\varepsilon_1 > 0$ such that

$$\nu = \varepsilon_1 \left(1 + 2\mu \exp(\mu \|A\|) \sqrt{c_0^2 + 1}\right) \quad (40)$$

By (21), (22), and continuity, there is a $\delta_4 > 0$ such that

$$\|F(x,\upsilon,u)\| \le \varepsilon_1 \sqrt{\|x\|^2 + \|u\|^2 + \|\upsilon\|^2} \quad (41)$$

$$\|G(u,\upsilon)\| \le \varepsilon_1 \sqrt{\|\upsilon\|^2 + \|u\|^2} \quad (42)$$

whenever $\sqrt{\|x\|^2 + \|u\|^2 + \|\upsilon\|^2} \le \delta_4$. Next take $\delta_1 = \min\left(\delta_3, \delta_4/\left(c_0\sqrt{3}\right)\right)$. Now, $\|\omega(k)\| \le \delta_1 \le \delta_3$, so for $t \in [\tau_k, \tau_{k+1}]$

$$\|x(t)\| \le c_0 \|\omega(k)\| \le \delta_4/\sqrt{3},\ \|x(\tau_k - \rho_k)\| \le \delta_4/\sqrt{3}, \\ \|u(\tau_k)\| \le \delta_4/\sqrt{3}$$

and therefore $\sqrt{\|x(t)\|^2 + \|x(\tau_k-\rho_k)\|^2 + \|u(\tau_k)\|^2} \le \delta_4$ for $t \in [\tau_k, \tau_{k+1}]$. Now,

$$\begin{aligned} &\|\Omega(k)\| \le \\ &\int_{\tau=\tau_k}^{\tau_{k+1}-\rho_{k+1}} e^{(\tau_{k+1}-\rho_{k+1}-\tau)\|A\|} \|F(x(\tau), x(\tau_k-\rho_k), u(\tau_k))\| d\tau \\ &+ \int_{\tau=\tau_k}^{\tau_{k+1}} e^{(\tau_{k+1}-\tau)\|A\|} \|F(x(\tau), x(\tau_k-\rho_k), u(\tau_k))\| d\tau \\ &+ \|G(x(\tau_k-\rho_k), u(\tau_k))\| \end{aligned}$$

By using (41) and (42), we find

$$\begin{aligned}
&\|\Omega(k)\| \leq 2\exp(\mu\|A\|)\\
&\times \int_{\tau=\tau_k}^{\tau_{k+1}} \varepsilon_1 \sqrt{\|x(\tau)\|^2 + \|x(\tau_k-\rho_k)\|^2 + \|u(\tau_k)\|^2} d\tau\\
&+\varepsilon_1 \sqrt{\|x(\tau_k-\rho_k)\|^2 + \|u(\tau_k)\|^2}\\
&\leq 2\exp(\mu\|A\|) \int_{\tau=\tau_k}^{\tau_{k+1}} \varepsilon_1 \sqrt{c_0^2\|\omega(k)\|^2 + \|\omega(k)\|^2} d\tau\\
&+\varepsilon_1\|\omega(k)\|\\
&\leq \varepsilon_1\left(1+2\mu\exp(\mu\|A\|)\sqrt{c_0^2+1}\right)\|\omega(k)\|\\
&= \nu\|\omega(k)\| \qquad (43)
\end{aligned}$$

which proves (39).

With Lemma 2, the approach from Hu and Michel (2000) can be used to prove the following theorem.

*Theorem 3.* Assume that $\tau_k$, $\rho_k$, $f$, and $g$ satisfy (18), (19), (16), and (17) respectively, and $H_k$ is as defined in (29). If

$$\limsup_{k\to\infty} \max_{\lambda\in\sigma(H_k)} |\lambda| < 1 \qquad (44)$$

where $\sigma(H_k)$ is the spectrum of $H_k$, and each subsequence of $\{H_k\}_{k=1}^{\infty}$ contains a subsequence which converges to a Schur stable matrix, and the solutions $P_k$ of $H_k^\top P_k H_k - P_k = -I$ satisfy

$$\limsup_{k\to\infty} \|P_{k+1} - P_k\| < 1 \qquad (45)$$

then the trivial solution $(x,u) = (0,0)$ of (12-13) is exponentially stable. (For the details of the proof see Appendix A).

## 4. A SIMPLE EXAMPLE OF A WATER SYSTEM

The system to be used as example is a small lake with an area of $a = 1\,\mathrm{hm}^2$ receiving an unregulated inflow from a stream and with an outflow to a river that is regulated by a moveable weir (Fig. 1). It receives an inflow of $q_{\mathrm{in}}$. Losses due to evaporation and seepage from the lake to the groundwater are neglected. The desired water level or set-point is $h^*$ which is given relative to a height datum, for instance, the Amsterdam Ordnance Datum (NAP). The aim of the control system will be to keep the mean water level $h$ within a given margin $\Delta h = 0.2\,\mathrm{m}$ of the set-point $h^*$. It is assumed that the inflow is a stationary process with a long term average of $q^* = 0.3\,\mathrm{m}^3/\mathrm{s}$. As disturbance, an additional inflow of $0.2\,\mathrm{m}^3/\mathrm{s}$ over a period of 3600 s is used (Fig. 2). An automated measurement station with several sensors does some data preprocessing to determine the average water level and removes all of the measurement noise. It is also assumed that for level fluctuations within the margins, the area of the lake may be taken to be constant. The lake will be modelled by

$$\dot{h}(t) = \frac{q_{\mathrm{in}}(t) - q_{\mathrm{w}}(h(t), h_{\mathrm{cr}}(t))}{a} \qquad (46)$$

$$h(0) = h^* \qquad (47)$$

where

$$q_{\mathrm{w}}(h, h_{\mathrm{cr}}) = bc_{\mathrm{w}}\left(\frac{2}{3}\right)^{3/2}\sqrt{\mathfrak{g}}\max(0, h-h_{\mathrm{cr}})^{3/2} \qquad (48)$$

models the flow over the weir, $h$ is the water level in the lake upstream of the weir, $h_{\mathrm{cr}}$ is the crest level, $b = 1.75\,\mathrm{m}$

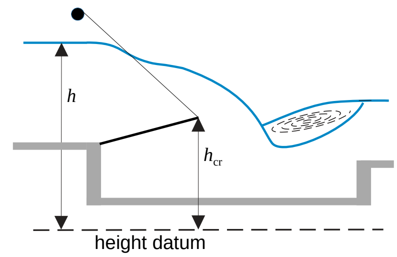



Fig. 1. A typical moveable weir

is the width of the weir, $c_{\mathrm{w}} = 1.0$ is a constant depending on the weir design, and $\mathfrak{g} = 9.81\,\mathrm{m/s}^2$ is the gravitational acceleration at the location of the weir. Let level $h_{\mathrm{cr}}^* < h^*$ be the crest level such that $q_{\mathrm{w}}(h^*, h_{\mathrm{cr}}^*) = q^*$ which for the above values follows from

$$q^* = 1.75\left(\frac{2}{3}\right)^{3/2}\sqrt{\mathfrak{g}}(h^* - h_{\mathrm{cr}}^*)^{3/2} = 0.3\,\mathrm{m}^3\mathrm{s} \qquad (49)$$

so

$$h^* - h_{\mathrm{cr}}^* = \frac{3}{2}\left(\frac{q^*}{bc_{\mathrm{w}}\sqrt{\mathfrak{g}}}\right)^{2/3} \simeq 0.216\mathrm{m} \qquad (50)$$

A Taylor series expansion around the setpoint gives

$$\begin{aligned} q_{\mathrm{w}}(h, h_{\mathrm{cr}}) = q^* + c_{\mathrm{L}}((h-h^*) - (h_{\mathrm{cr}} - h_{\mathrm{cr}}^*)) \\ + F(h-h^*, h_{\mathrm{cr}} - h_{\mathrm{cr}}^*) \end{aligned} \qquad (51)$$

where

$$F(x,z) = q_{\mathrm{w}}(h^*+x, h_{\mathrm{cr}}^*+z) - q^* - c_{\mathrm{L}}(x-z) \qquad (52)$$

is the remainder term, and the linear terms in the expansion are $c_{\mathrm{L}}$ and $-c_{\mathrm{L}}$ with

$$c_{\mathrm{L}} = \frac{3}{2}\frac{q^*}{h^* - h_{\mathrm{cr}}^*} \simeq 2.081\,\mathrm{m}^2/\mathrm{s} \qquad (53)$$

The controller will act at times $\tau_k = k\Delta\tau$ for $k = 0, 1, \ldots$, and the communication delay will be $\rho$ with $0 < \rho < \Delta\tau$. As system state $x$, we take $x(t) = h(t) - h^*$. In this example, a discrete PI controller with coefficients $c_{\mathrm{P}}$ and $c_{\mathrm{I}}$ is used, which is modelled by

$$u(k+1) = u(k) + x(\tau_k - \rho_k) \qquad (54)$$

$$h_{\mathrm{cr}}(\tau_k) = h_{\mathrm{cr}}^* - c_{\mathrm{P}}x(\tau_k - \rho_k) - c_{\mathrm{I}}u(k) \qquad (55)$$

This results in

$$\dot{x}(t) = \frac{q_{\mathrm{in}}(t) - q_{\mathrm{w}}(h^* + x(t), h_{\mathrm{cr}}^* - c_{\mathrm{P}}x(\tau_k - \rho_k) - c_{\mathrm{I}}u(k))}{a} \qquad (56)$$

$$u(k+1) = u(k) + x(\tau_k - \rho_k) \qquad (57)$$

If $q_{\mathrm{in}}(t) = q^*$ and $q_{\mathrm{w}}$ is replaced by (51), then (56) can be written as

$$\dot{x}(t) = \frac{-c_{\mathrm{L}}(x - c_{\mathrm{P}}x(\tau_k - \rho_k) - c_{\mathrm{I}}u(k))}{a} - \frac{F(x, -c_{\mathrm{P}}x(\tau_k - \rho_k) - c_{\mathrm{I}}u(k))}{a} \qquad (58)$$

so in this case, $H_k$ defined in (29) is constructed using

$$A = -\frac{c_{\mathrm{L}}}{a}, A_0 = \frac{c_{\mathrm{L}}c_{\mathrm{P}}}{a}, B = \frac{c_{\mathrm{L}}c_{\mathrm{I}}}{a}, C = 1, D = 1 \qquad (59)$$

## 5. NUMERICAL EXPERIMENTS

Numerical experiments were conducted for $\tau_k = k\Delta\tau$ and $\rho_k = \rho$ with various combinations of $\Delta\tau$ and $\rho$. A disturbance of an amplitude $0.2\,\mathrm{m}^3/\mathrm{s}$ and a duration of

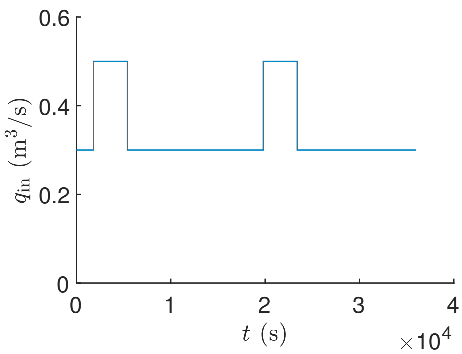



Fig. 2. Inflow for all tests

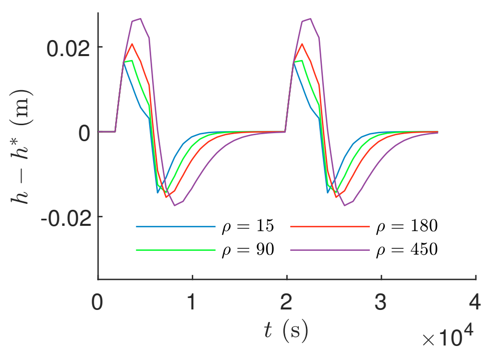



Fig. 3. Level as a function of time for optimal $c_{\mathrm{P}}$ and $c_{\mathrm{I}}$ with $\Delta\tau = 900\,\mathrm{s}$ and different $\rho$

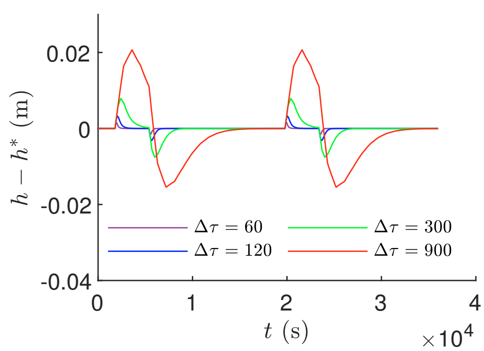
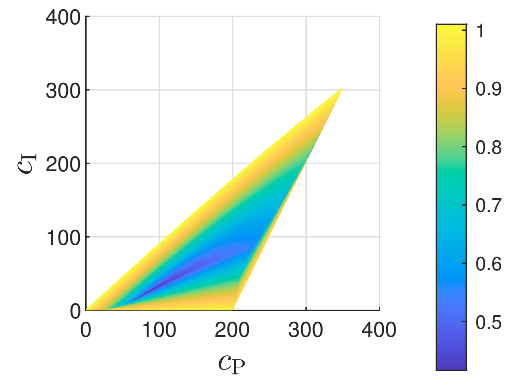
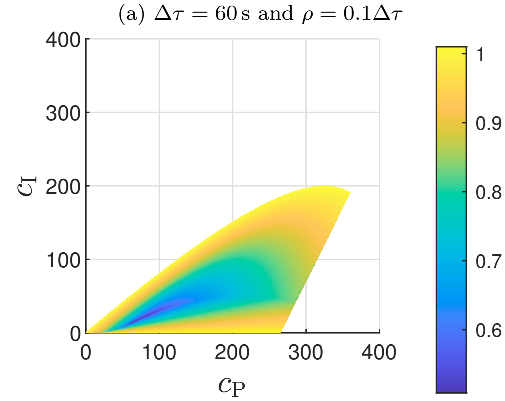


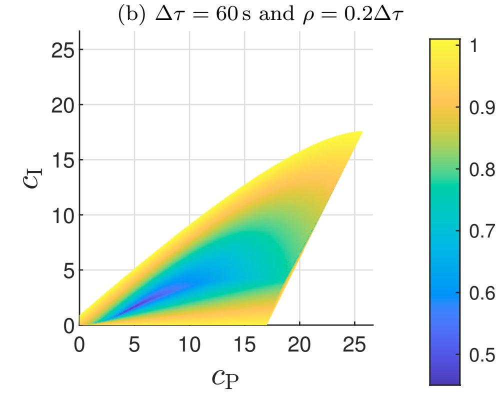

Fig. 4. Level as a function of time for optimal $c_{\mathrm{P}}$ and $c_{\mathrm{I}}$ for different $\Delta\tau$ with $\rho = 0.2\Delta\tau$




(a) $\Delta\tau = 60\,\mathrm{s}$ and $\rho = 0.1\Delta\tau$




(b) $\Delta\tau = 60\,\mathrm{s}$ and $\rho = 0.2\Delta\tau$




(c) $\Delta\tau = 900\,\mathrm{s}$ and $\rho = 0.2\Delta\tau$

Fig. 5. Plots of the region where $\max_{\lambda\in\sigma(H)} |\lambda| \le 1$ for the lake with weir and PI controller for several values of $\Delta\tau$ and $\rho$ (colour indicates the value of $|\lambda|$)

3600 s (Fig. 2) will be used to test system behaviour. Runs for different $\Delta\tau$ and $\rho$ with $c_{\mathrm{P}}$ and $c_{\mathrm{I}}$ chosen to minimize the modulus of the largest eigenvalue were performed. For fixed $\Delta\tau$ and increasing $\rho$ the maximum deviation from set-point and the time needed to get back to the set-point increase with $\rho$. (Fig. 3). A similar pattern is seen when $\Delta\tau$ is varied, but the ratio $\rho/\Delta\tau$ is held constant (Fig. 4). For constant $\Delta\tau$, the range of allowed $c_{\mathrm{I}}$ decreases with increasing $\rho$ (Fig. 5a, 5b). The size of the region of values of $c_{\mathrm{P}}$ and $c_{\mathrm{I}}$ for which the system is stable shrinks with increasing $\Delta\tau$ (Fig. 5b, 5c). Numerical experiments showed that for points outside the stability region the systems did indeed become unstable.

## 6. CONCLUSIONS

A theorem was presented that provides sufficient conditions for stability for a non-linear continuous time system with a non-linear discrete time controller where the control time step may vary and where the feedback loop contains a variable delay factor. A strong point of the method is that, once time evolution functions and output functions of the plant and controller have been determined, the derivatives of these functions needed for the theorem can be determined by symbolic or automatic differentiation. This removes the need for manual construction of Lyapunov functionals. Extension to delays in the feedback loop longer than one time step is expected to be a matter of extending the state vector of the controller and perhaps some technical modifications to the theorem. A bigger challenge will be adapting the theorem to allow for time delays within the continuous process while keeping verification of stability relatively simple.

Application of the results to a simple water system provided insight into the effect of control time step size and communication delay on controller performance. For the example system, the range of allowed values for $c_\mathrm{P}$ and $c_\mathrm{I}$ decreases with increasing time step size.

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## Appendix A. PROOF OF THEOREM 3

Some lemmas are used to split the proof up into manageable parts. These will be stated and proved first.

*Lemma 4.* Suppose $\{H_k\}_{k=1}^{\infty}$ is a sequence of square matrices such that

$$\limsup_{k\to\infty} \max_{\lambda\in\sigma(H_k)} |\lambda| < 1 \tag{A.1}$$

where $\sigma(H_k)$ is the spectrum of $H_k$ and each subsequence of $\{H_k\}_{k=1}^{\infty}$ contains a subsequence which converges to a Schur stable matrix. In that case there is a $k_0$ such that for $k \geq k_0$ for each $H_k$ there is a positive definite symmetric solution $P_k$ of

$$H_k^\top P_k H_k - P_k = -I \tag{A.2}$$

and there is a $m \in \mathbb{R}$ such that

$$\sup_{k\geq k_0} \|P_k\| = m \tag{A.3}$$

$$I \leq P_k \leq mI \tag{A.4}$$

$$\|H_k\| \leq \sqrt{m-1} \tag{A.5}$$

**Proof.** According to (A.1), there exists a $\delta > 0$ such that

$$\limsup_{k\to\infty} \max_{\lambda\in\sigma(H_k)} |\lambda| = 1-\delta$$

therefore there is a $k_0$ such that

$$\sup_{k\geq k_0} \max_{\lambda\in\sigma(H_k)} |\lambda| \leq 1-\frac{\delta}{2}$$

so for all $k \geq k_0$, $H_k$ is Schur stable, and there is a positive definite symmetric solution $P_k$ of $H_k^\top P_k H_k - P_k = -I$. Now suppose that $\|P_k\|$ for $k \geq k_0$ is not bounded. In that case for each $m \in \mathbb{N}$, $m > 0$ there must be a $k_m > k_0$ such that $\|P_{k_m}\| > m$. But the sequence $H_{k_m}$ must contain a subsequence converging to a Schur stable matrix $H$. Next suppose that $P$ is the solution of $H^\top PH - P = -I$. By continuity we must have $P_{k_m} \to P$ which contradicts the unboundedness of $\|P_{k_m}\|$.

From (A.2) follows that $P_k \geq I$. It then follows that $\|H_k\| = \sqrt{\max_{\lambda\in\sigma(H_k)}\left(H_k^\top H_k\right)} \leq \sqrt{m-1}$.

*Lemma 5.* If $0 < q < 1$ and $m \geq 1$ and

$$p(z) = q + 2zm\sqrt{m-1} + mz^2$$

then there is a $\nu$ with $0 < \nu < 1$ such that

$$0 < p(\nu) < 1$$

**Proof.** If $m = 1$ then $p(z) = q + z^2$ and any $0 < z < \sqrt{1-q}$ will do. If $m > 1$ then consider

$$q - 1 + 2zm\sqrt{m-1} + mz^2 = 0$$

This has roots

$$z_{1,2} = \frac{-2m\sqrt{m-1} \pm \sqrt{4m^2(m-1) + 4m(1-q)}}{2m} = -\sqrt{m-1} \pm \sqrt{(m-1) + \frac{1}{m}(1-q)}$$

Clearly $z_1 < 0 < z_2$, $p(z_1) = p(z_2) = 1$, and $p(0) = q$ so there must be a $0 < \nu < \min(z_2, 1)$ such that $p(\nu) < 1$.

*Lemma 6.* Let $A$, $A_0$, $B$, $C$, $D$, $F$, $G$, $\Omega$, $\omega$, $\mu$, $\tau_k$ and $\rho_k$ be as defined earlier. For any $k_1 \in \mathbb{N}$ and any $\varepsilon_1 > 0$ there is a $\delta(k_1, \varepsilon_1)$ such that $\|\omega(0)\| \leq \delta(k_1, \varepsilon_1)$ implies that $\|\omega(k_1)\| \leq \varepsilon_1$.

**Proof.** Pick $\nu \in (0,1)$ and let $\delta_1(v)$ be as in Lemma 2. The proof will use finite induction on $k$. Define

$$\delta(k_1, \varepsilon_1) = \frac{\min\{\varepsilon_1, \delta_1(v)\}}{\prod_{j=0}^{k_1-1}(\|H_j\|+1)} \tag{A.6}$$

Now according to (31) and (39)

$$\begin{aligned}\|\omega(1)\| &\leq \|H_0\|\,\|\omega(0)\| + \|\Omega(0)\| \\ &\leq \|H_0\|\,\|\omega(0)\| + \nu\,\|\omega(0)\| \\ &\leq (\|H_0\|+1)\,\|\omega(0)\|\end{aligned}$$

so

$$\begin{aligned}\|\omega(1)\| &\leq (\|H_0\|+1)\,\|\omega(0)\| \\ &\leq \frac{\min\{\varepsilon_1, \delta_1(v)\}}{\prod_{j=1}^{k_1-1}(\|H_j\|+1)} \leq \delta_1(v)\end{aligned}$$

Induction step: suppose that for $k' \leq k < k_1$ we have

$$\|\omega(k')\| \leq \frac{\min\{\varepsilon_1, \delta_1(v)\}}{\prod_{j=k}^{k_1-1}(\|H_j\|+1)}$$

then according to (31) and (39)

$$\begin{aligned}\|\omega(k'+1)\| &\leq (\|H_k\|+1)\,\|\omega(k')\| \\ &\leq \frac{\min\{\varepsilon_1, \delta_1(v)\}}{\prod_{j=k'+1}^{k_1-1}(\|H_j\|+1)}\end{aligned}$$

which implies that

$$\begin{aligned}\|\omega(k_1)\| &\leq (\|H_{k_1}\|+1)\,\|\omega(k_1-1)\| \\ &\leq (\|H_{k_1}\|+1)\,\frac{\min\{\varepsilon_1, \delta_1(v)\}}{\prod_{j=k_1-1}^{k_1-1}(\|H_j\|+1)} \\ &= \min\{\varepsilon_1, \delta_1(v)\} \leq \varepsilon_1\end{aligned}$$

Now we get to the proof of Theorem 3.

**Proof.** The proof of Theorem 3 is similar to the proof of the corresponding theorem in Hu and Michel (2000). According to Lemma 4, there is a $k_0$ such that (A.3) holds and for $k \geq k_0$ there is a symmetric positive definite $P_k$ such that $P_k = H_k^\top P_k H_k + I$. From (44) and (45) it follows that there is a $k_0$ and a $q$ such that $0 < q < 1$ and for $k \geq k_0$

$$\max_{\lambda \in \sigma(H_k)} |\lambda| < q, \|P_{k+1} - P_k\| < q \tag{A.7}$$

Next define

$$V(\omega(k)) = \omega(k)^\top P_{k-1}\omega(k) \tag{A.8}$$

and calculate

$$\begin{aligned}&V(\omega(k+1)) - V(\omega(k)) = \\ &-\omega(k)^\top \omega(k) + 2\omega(k)^\top H_k^\top P\Omega(k) + \Omega(k)^\top P_k\Omega(k) \\ &+\omega(k)^\top (P_k - P_{k-1})\,\omega(k)\end{aligned}$$

so

$$\begin{aligned}&V(\omega(k+1)) - V(\omega(k)) = \\ &\leq -(1-q)\,\|\omega(k)\|^2 \\ &+2m\sqrt{m-1}\,\|\Omega(k)\|\,\|\omega(k)\| + m\,\|\Omega(k)\|^2\end{aligned} \tag{A.9}$$

According to Lemma 4, for any value of $\tilde{\nu} > 0$ it is possible to find a $\delta_1(\tilde{\nu})$ such that $\|\Omega(k)\| \leq \tilde{\nu}\,\|\omega(k)\|$ provided that $\|\omega(k)\| \leq \delta_1(\tilde{\nu})$. This gives

$$\begin{aligned}&V(\omega(k+1)) - V(\omega(k)) \leq \\ &\left(-1 + q + 2m\sqrt{m-1}\tilde{\nu} + m\tilde{\nu}^2\right)\|\omega(k)\|^2\end{aligned} \tag{A.10}$$

According to Lemma 5, we can pick a $\nu$ $(0 < \nu < 1)$ such that

$$r = \sqrt{m - 1 + q + 2\nu m\sqrt{m-1} + m\nu^2} < 1 \tag{A.11}$$

Now define

$$\alpha = \min(1, -\ln r) \tag{A.12}$$

and for every $\varepsilon > 0$ define

$$\varepsilon_0 = \frac{\varepsilon}{c_0\sqrt{m}}\mathrm{e}^{-\alpha\mu} \tag{A.13}$$

Next use Lemma 6

$$\delta = \delta\left(k_0, \min\left(\frac{\delta_1(v)}{\sqrt{m}}, \varepsilon_0\mathrm{e}^{-k_0\alpha}\right)\right) \tag{A.14}$$

with $\delta_1(v)$ from Lemma 2 to get

$$\|\omega(k_0)\| \leq \min\left(\frac{\delta_1(v)}{\sqrt{m}}, \varepsilon_0\mathrm{e}^{-k_0\alpha}\right) \tag{A.15}$$

From (A.9) and $\|\omega(k_0)\| \leq \frac{\delta_1(v)}{\sqrt{m}}$ we get

$$\begin{aligned}&V(\omega(k_0+1)) - V(\omega(k_0)) \leq \\ &\left(q - 1 + 2vm\sqrt{m-1} + mv^2\right)\|\omega(k_0)\|^2\end{aligned} \tag{A.16}$$

so

$$V(\omega(k_0+1)) \leq \left(q + 2vm\sqrt{m-1} + mv^2\right)V(\omega(k_0))$$

where $\|\omega(k_0)\|^2 \leq V(\omega(k_0))$ follows from (A.8) and $k \geq k_0$ and therefore $P_{k_0} \geq I$. Now

$$V(\omega(k_0+1)) \leq r^2 V(\omega(k_0))$$

and

$$\begin{aligned}\|\omega(k_0+1)\|^2 &\leq V(\omega(k_0+1)) \leq r^2 V(\omega(k_0)) \\ &\leq r^2 m\,\|\omega(k_0)\|^2 < (\delta_1(v))^2\end{aligned}$$

Now suppose that for a given $k > k_0$ for all $k'$ such that for all $k_0 \leq k' \leq k-1$

$$\|\omega(k')\| \leq \delta_1(v)$$

and

$$V(\omega(k'+1)) \leq r^2 V(\omega(k'))$$

then

$$\begin{aligned}V(\omega(k+1)) &\leq \left(q + 2vm\sqrt{m-1} + mv^2\right)V(\omega(k)) \\ &\leq r^2 V(\omega(k)) \leq r^{2(k-k_0)}V(\omega(k_0))\end{aligned}$$

and therefore

$$\begin{aligned}\|\omega(k+1)\|^2 &\leq V(\omega(k+1)) \leq r^2 V(\omega(k)) \\ &\leq r^{2(k+1-k_0)} m\,\|\omega(k_0)\|^2 < (\delta_1(v))^2\end{aligned}$$

and

$$\|\omega(k+1)\| \leq r^{(k+1-k_0)}\mathrm{e}^{-k_0\alpha}\frac{\varepsilon\sqrt{m}}{c_0\sqrt{m}}\mathrm{e}^{-\alpha\mu}$$

Now for all $k \geq k_0$ and $t \in [\tau_k, \tau_{k+1})$ we have $\|\omega(k)\| \leq \delta_1(v)$ so

$$\begin{aligned}\|x(t)\| &\leq (1 + \mu(\|A_0\| + \|B\| + 2))\,\mathrm{e}^{(1+\|A\|)(t-\tau_k)}\,\|\omega(k)\| \\ &\leq \frac{(1 + \mu(\|A_0\| + \|B\| + 2))\,\mathrm{e}^{(1+\|A\|)(t-\tau_k)}}{(1 + \mu(\|A_0\| + \|B\| + 2))\,\mathrm{e}^{(1+\|A\|)(\tau_{k+1}-\tau_k)}}\,\|\omega(k)\| \\ &\leq r^{(k+1)}\frac{\varepsilon}{\sqrt{m}}\mathrm{e}^{-\alpha\mu} \leq \frac{\varepsilon}{\sqrt{m}}\mathrm{e}^{-(k+1-\alpha)\mu}\end{aligned}$$

and therefore $(x, u) = (0, 0)$ is exponentially stable.