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A general convergence result for viscosity solutions of Hamilton-Jacobi equations and non-linear semigroups

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ABSTRACT

We extend the Barles-Perthame procedure [4] (see also [22]) of semi-relaxed limits of viscosity solutions of Hamilton-Jacobi equations of the type $f - \lambda H f = h$ to the context of non-compact spaces.

The convergence result allows for equations on a 'converging sequence of spaces' as well as Hamilton-equations written in terms of two equations in terms of operators $H_\dagger$ and $H_\ddagger$ that serve as natural upper and lower bounds for the 'true' operator H.

In the process, we establish a strong relation between non-linear pseudo-resolvents and viscosity solutions of Hamilton-Jacobi equations. As a consequence we derive a convergence result for non-linear semigroups.

1. Introduction

In this paper, we will study the relation between three of the major objects in the field of semigroup theory: the semigroup, the resolvent and the generator.

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Consider the following problem posed by [6]. Find all maps $f : \mathbb{R}^+ \to \mathbb{C}$ satisfying

$$\begin{cases} \phi(t+s) = \phi(t)\phi(s) & \text{for all } s,t \geq 0, \\ \phi(0) = 1. \end{cases}$$

Assuming that ϕ is continuous (the conclusion holds under much weaker assumptions), it can be shown that all maps of this type are of the form $\phi_a(t) := e^{ta} = \lim_{k\to\infty} \left(1 - \frac{t}{k}a\right)^{-k}$ with $a \in \mathbb{C}$.

The factor a, which can be found by $a = \partial_t \phi_a(t)|_{t=0}$, captures all essential information of the semigroup ϕ_a. In addition, the dependence of ϕ_a on a is robust under convergence: for a sequence of $a_n \in \mathbb{C}$ with $a_n \to a$, it holds that $\phi_{a_n} \to \phi_a$ uniformly on compacts.

Semigroup theory generalizes these three concepts to the level of semigroups on Banach and locally convex spaces. We will focus here on non-linear semigroups on the space of bounded measurable functions $M_b(X)$ on a completely regular space X. The three objects of interest are

((Sa)) A generator $H \subseteq M_b(X) \times M_b(X)$;
((Sb)) A resolvent $R(\lambda) = (\mathbb{1} - \lambda H)^{-1}$, $\lambda > 0$;
((Sc)) A semigroup $V(t) = \lim_{k\to\infty} R\left(\frac{t}{k}\right)^k$, $t \geq 0$.

In addition, one wants to establish relations between $H_n \to H$, $R_n(\lambda) \to R(\lambda)$ and $V_n(t) \to V(t)$.

In the context of linear semigroups on some Banach space Y, these results are all well known, cf. [17,46]. Two main results in this context are the Hille-Yosida generation theorem [27] relating (Sa), (Sb) and (Sc), whereas the Trotter-Kato-Kurtz approximation theorem [45,28,49] establishes various implications between convergence of these three objects. For the more complex non-linear context, see [9,38].

Both the generation and approximation type results, however, assume that the Hamilton-Jacobi equation $f - \lambda H f = h$ can be solved in the classical sense. While in the linear context this is often possible, in the non-linear context this is difficult or even impossible. In the context of operators satisfying the maximum principle on $C_b(X)$, this issue can be resolved by working with viscosity solutions. These solutions were introduced by [10] and they showed that viscosity solutions can be used to define an extension $\widehat{H}$ of H that also satisfies the maximum principle and for which $f - \lambda \widehat{H} f = h$ can be solved classically. These properties single out viscosity solutions as the correct weak solution concept for the study of semigroup theory on $C_b(X)$.

A second key feature of viscosity solutions is the possibility to pass to limits in a straightforward fashion. The first stability result was already obtained in [10]. However, it was the procedure of semi-relaxed limits of Barles and Perthame [4] that allows to pass to the limit with viscosity sub- or super solutions without a-priori estimates. A comparison principle for the limiting equation then establishes local uniform convergence of viscosity solutions.

In non-locally compact, or infinite dimensional spaces, the situation is more difficult. A first result in the context of Hilbert spaces appeared in [36], but the result is not correct as the example of [41] or [21, Example 3.43] shows. Key problem is that the equation does not satisfy appropriate continuity properties. In the context of infinite dimensional spaces various generalizations of the method of semi-relaxed limits were developed that takes into account the difficulties arising from equations with bad continuity properties. See [34] and [21, Section 3.9] for a variant working with B-continuity, see also [11,12,42]. See [21, Section 3.4] for a discussion on more classical approximation results requiring appropriate uniform continuity estimates.

A third line of thinking was developed in a different direction in the context of large deviations of Markov processes by Feng and Kurtz [22,19, Chapters 6 and 7], who return to the point of view of using test-functions instead of semi-jets. Indeed, they find that in the infinite dimensional context, it pays of to work with two types of test functions, following earlier works for first-order equations by [13,43,44] that are suited for either sub- or supersolutions. To be precise [22] introduce three important improvements over the classical Barles-Perthame procedure:

(1) Instead of working on a single Polish space X (not necessarily finite dimensional nor Hilbertian), a sequence of spaces X_n that are mapped into X are considered. Conditions are given that imply the convergence of viscosity solutions of $f - \lambda H_n f = h_n$ on a space X_n to a viscosity solution of $f - \lambda H f = h$ on X. For this result, Feng and Kurtz work with a generalized notion of buc (bounded and uniform on compacts) convergence that applies to functions on different spaces.
(2) Instead of working with a limiting operator H, [22,19] follow initial papers for Hamilton-Jacobi equations on infinite dimensional spaces, see [13,43,44], allowing for the possibility for a relaxed upper bound $H_\dagger$ and a relaxed lower bound $H_\ddagger$. Thus, in the limit a subsolution to $f - \lambda H_\dagger f = h$ and a supersolution to $f - \lambda H_\ddagger f = h$ is obtained.
(3) The operators $H_\dagger$ and $H_\ddagger$ can take their images in the space of measurable functions $M(Y)$ on a space Y instead of X, where Y is a space containing more information than X, allowing for the use of the approximation theory in the setting of e.g. homogenization and multi-scale systems.

Various applications of these extended methods are given in Chapters 11, 12 and 13 of [22] and [19] and have recently been applied in large deviation theory in various settings [15,39,20,8,33].

The methods of [22], however, have two major drawbacks and lack one desirable property.

- The extension of [22] in their Chapter 7, which includes the generalizations (1), (2) and (3), is based on the property that the Hamiltonians H_n are given in terms of an exponential tilt of an operator A_n which is the generator of a Markov process. [22]

then approximate A_n by its bounded Yosida approximant $A_n^\varepsilon = A_n(\mathbb{1} - \varepsilon A_n)^{-1}$. This leads to a continuous operator H_n^ε that is easier to treat. In general, the replacement of H_n by a continuous approximation H_n^ε is not possible, therefore making it impossible to widely use the Feng-Kurtz stability result in a general setting, excluding e.g. an application in the context of Gamma convergence [26,47].

- A second major drawback arises from the realization that in general, and in particular in infinite dimensions cf. [43,44,13,19,23,1,24], it is advantageous to work with an upper and lower bound $H_\dagger$ and $H_\ddagger$ instead of a single Hamiltonian H. Therefore, instead of working with operators H_n to obtain a limiting upper and lower bounds $H_\dagger, H_\ddagger$, one would like to work with pairs $H_{n,\dagger}, H_{n,\ddagger}$ instead.
- Finally, a lacking desirable property is that the result in [22] is based on the assumption that X_n are mapped into X. This leads to problems for example in the setting of hydrodynamic limits, see e.g. [29]. In this context a Markov process is considered in which particles move around on a discrete lattice, e.g. $\mathbb{Z}^d$. A typical state-space would be $X_n := \{0,1\}^{\mathbb{Z}_d}$. After rescaling the lattice and speeding up time appropriately, the empirical measure associated to the particle locations converges to the solution of a diffusion equation, say in $X := L^1(\mathrm{d}x)$. The convergence of measures to a profile in $L^1(\mathbb{R}^d, \mathrm{d}x)$, however, is considered with respect to the vague topology on $\mathcal{X} := \mathcal{M}(\mathbb{R}^d)$.

  Thus, instead of considering spaces X_n that get mapped into X one wants to consider an auxiliary space $\mathcal{X}$, in which both X_n and X get mapped. The convergence of elements is then considered as elements in $\mathcal{X}$.

Our Theorem 5.7 extends the Feng-Kurtz version of the Barles-Perthame procedure to remedy these three issues. As a consequence, we show that the Kurtz [49] convergence result yields the convergence of semigroups, see Theorem 6.1.

To illustrate the use of the methods introduced above, we consider two examples in which we can establish convergence of semigroups, one in the context of the homogenisation of a one-dimensional Hamilton-Jacobi equation, where the corrector can be explicitly produced, see e.g. [37], and one example where we consider Hamilton-Jacobi equations arising in the theory of large deviations on discrete spaces. Applications to infinite dimensional settings where the results of this paper can replace the semigroup convergence results by [22] can be found in [22,19,18], further applications are work in progress.

Two more theoretical applications are considered in [32,31]. In [32] the extended procedure is used for a new proof of path-space large deviations for Markov processes. In [31] we give a framework to establish Gamma convergence of functionals on path-space.

We proceed with an overview of the paper. As all the generalizations are quite technical, we start out in Section 2 with stating (without giving definitions of the required notions) a basic version of the convergence of viscosity solutions and the convergence of semigroups. This allows to quickly grasp the kind of results possible by using the theory of the paper.

To set the stage for the more general results, we start in Section 3 with some preliminaries that include a treatment of basic properties of our notion of convergence taking place on spaces $X_n, X, \mathcal{X}$. All these results can be skipped on first reading assuming that $X_n = X = \mathcal{X}$ and coincide with the ones of [22] in the context that the X_n are mapped into X (in this case $X = \mathcal{X}$).

We proceed in Section 4 on a basic study of viscosity solutions for the Hamilton-Jacobi equation $f - \lambda Hf = h$, as well as a study of pseudo-resolvents. To some extent these results are known in the literature, but as the results and proofs will be used as input for our main results later on, we collect these results for completeness. First we show that pseudo-resolvents can be used to construct viscosity solutions. Second, given well-posedness of the Hamilton-Jacobi equation, viscosity solutions can be used to construct a pseudo-resolvent. Finally, in this context, the pseudo-resolvent can be used to define a new Hamiltonian that satisfies the conditions for the semi-group generation result by [9].

In Sections 5 and 6 we proceed with our convergence statements, containing the two main Theorems 5.7 and 6.1. In Section 7 we discuss how to use the comparison principle for Hamilton-Jacobi equations to establish density of the domain of an operator constructed out of viscosity solutions.

Finally, we end in Section 8 with two short examples, one involving the homogenization of a first-order Hamilton-Jacobi equation, and one where a discrete system approximates a continuous one, in which we show the applicability of our results.

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2. Two basic convergence results

To anticipate the general version of our two main results, we state in this section two simplified versions of these results. We will not give definitions of the required notions, as these will follow in more general context in Section 3. The notion of a pseudo-resolvent can be found as Definition 4.1.

We start with the convergence of viscosity solutions of Hamilton-Jacobi equations. A more general version is given as Theorem 5.7 below.

Theorem 2.1. *Let X be completely regular. Suppose there are contractive pseudo-resolvents $R_n(\lambda) : C_b(X) \to C_b(X)$, $\lambda > 0$ and operators $H, H_n \subseteq C_b(X) \times C_b(X)$, $n \geq 1$. Suppose in addition that*

(a) For each $n \geq 1$, $\lambda > 0$ and $h \in C_b(X)$ the function $R_n(\lambda)h$ is a viscosity solution to $f - \lambda H_n f = h$.

(b) We have local strict equi-continuity on bounded sets: for all compact sets $K \subseteq X$, $\delta > 0$ and $\lambda_0 > 0$, there is a compact set $\hat{K} = \hat{K}(K, \delta, \lambda_0)$ such that for all n and $h_1, h_2 \in C_b(X)$ and $0 < \lambda \leq \lambda_0$ we have

$$\sup_{y \in K} \{R_n(\lambda)h_1(y) - R_n(\lambda)h_2(y)\} \\ \leq \delta \sup_{x \in X} \{h_1(x) - h_2(x)\} + \sup_{y \in \hat{K}} \{h_1(y) - h_2(y)\}.$$

(c) For each $(f, g) \in H$ there are $(f_n, g_n) \in H_n$ such that $buc - \lim f_n = f$ and $buc - \lim g_n = g$.

(d) There is a buc-dense set $D \subseteq C_b(X)$ such that the comparison principle holds for the Hamilton-Jacobi equation $f - \lambda H f = h$ for all $h \in D$ and $\lambda > 0$.

Then there are operators $R(\lambda) : C_b(X) \to C_b(X)$ for $\lambda > 0$ that are locally strictly equi-continuous on bounded sets: for all compact sets $K \subseteq X$, $\delta > 0$ and $\lambda_0 > 0$, there is a compact set $\hat{K} = \hat{K}(K, \delta, \lambda_0)$ such that for all $h_1, h_2 \in C_b(X)$ and $0 < \lambda \leq \lambda_0$ we have

$$\sup_{y \in K} \{R(\lambda)h_1(y) - R(\lambda)h_2(y)\} \leq \delta \sup_{x \in X} \{h_1(x) - h_2(x)\} + \sup_{y \in \hat{K}} \{h_1(y) - h_2(y)\}.$$

For each $\lambda > 0$ and $h \in D$ the function $R(\lambda)h$ is the unique viscosity solution to $f - \lambda H f = h$. In addition if $h \in C_b(X)$ and $h_n \in C_b(X_n)$ are such that $buc - \lim h_n = h$ then $buc - \lim R_n(\lambda)h_n = R(\lambda)h$.

The next result uses the convergence of the pseudo-resolvents to obtain the convergence of semigroups. The key ingredient in this context are the semigroup generation and convergence results of [9,49]. A more general version follows from Theorems 6.1 and Proposition 7.1.

Theorem 2.2. *Suppose we are in the setting of Theorem 2.1. In addition suppose that there are operators $V_n(t) : C_b(X) \to C_b(X)$, $t \geq 0$ that form a semigroup. Suppose that*

(a) there is a buc-dense subset $\mathcal{D}_n$ such that for every $n \geq 1$, $f \in \mathcal{D}_n$ and $x \in X$:

$$\lim_{m \to \infty} R_n\left(\frac{t}{m}\right)^m f(x) = V_n(t)f(x).$$

(b) We have local strict equi-continuity on bounded sets for the semigroups: for all compact sets $K \subseteq X$, $\delta > 0$ and $t_0 > 0$, there is a compact set $\hat{K} = \hat{K}(K, \delta, \lambda_0)$ such that for all n and $h_1, h_2 \in C_b(X)$ and $0 \leq t \leq t_0$ that

$$\sup_{y \in K} \{V_n(t)h_1(y) - V_n(t)h_2(y)\}$$

$$\leq \delta \sup_{x \in X} \left\{h_1(x) - h_2(x)\right\} + \sup_{y \in \hat{K}} \left\{h_1(y) - h_2(y)\right\}.$$

(c) We have $buc - \lim_{\lambda \downarrow 0} R(\lambda)h = h$ *and for each* n *we have* $buc - \lim_{\lambda \downarrow 0} R_n(\lambda)h = h$.

Then there are operators $V(t) : C_b(X) \to C_b(X)$ *for* $t \geq 0$ *that are locally strictly equicontinuous on bounded sets: for all compact sets* $K \subseteq X$, $\delta > 0$ *and* $t_0 > 0$, *there is a compact set* $\hat{K} = \hat{K}(K, \delta, t_0)$ *such that for all* $h_1, h_2 \in C_b(X)$ *and* $0 \leq t \leq t_0$ *we have*

$$\sup_{y \in K} \left\{V(t)h_1(y) - V(t)h_2(y)\right\} \leq \delta \sup_{x \in X} \left\{h_1(x) - h_2(x)\right\} + \sup_{y \in \hat{K}} \left\{h_1(y) - h_2(y)\right\}.$$

In addition there is a buc-dense subset $\mathcal{D} \subseteq C_b(X)$ *such that for all* $x \in X$

$$\lim_{m \to \infty} R\left(\frac{t}{m}\right)^m f(x) = V(t)f(x)$$

Finally, if $f \in C_b(X)$ *and* $f_n \in C_b(X_n)$ *such that* $buc - \lim f_n = f$ *and* $t_n \to t$, *then* $buc - \lim V_n(t_n)f_n = V(t)f$.

Both results will follow as special cases of much more general result that we will prove in the following sections.

3. Preliminaries

3.1. Basic definitions

All spaces in this paper are assumed to be completely regular spaces. Let X be a space then we denote by $C_b(X)$ the set of continuous and bounded functions into $\mathbb{R}$. We denote by $Ba(X)$ the space of Baire measurable sets (the σ-algebra generated by $C_b(X)$). By $M(X)$, we denote by set of Baire measurable functions from X into $\overline{\mathbb{R}} := [-\infty, \infty]$. $M_b(X)$ denotes the set of bounded Baire measurable functions. Denote

$$USC_u(X) := \left\{ f \in M(X) \,\middle|\, f \text{ upper semi-continuous}, \sup_x f(x) < \infty \right\},$$
$$LSC_l(X) := \left\{ f \in M(X) \,\middle|\, f \text{ lower semi-continuous}, \inf_x f(x) > \infty \right\}.$$

For $g \in M(X)$ denote by $g^*, g_* \in M(X)$ the upper and lower semi-continuous regularizations of g.

3.2. Viscosity solutions

Let X and Y be two spaces. Let $\gamma : Y \to X$ be continuous and surjective.

We consider operators $A \subseteq M(X) \times C(Y)$. If A is single valued and $(f,g) \in A$, we write $Af := g$. We denote $\mathcal{D}(A)$ for the domain of A and $\mathcal{R}(A)$ for the range of A.

Definition 3.1. Let $A_\dagger \subseteq LSC_l(X) \times USC_u(Y)$ and $A_\ddagger \subseteq USC_u(X) \times LSC_l(Y)$. Fix $h_1, h_2 \in M(X)$. Consider the equations

$$f - A_\dagger f = h_1, \tag{3.1}$$

$$f - A_\ddagger f = h_2. \tag{3.2}$$

Classical solutions We say that u is a *classical subsolution* of equation (3.1) if there is a g such that $(u,g) \in A_\dagger$ and $u - g \leq h$. We say that v is a *classical supersolution* of equation (3.2) if there is g such that $(v,g) \in A_\ddagger$ and $v - g \geq h$. We say that u is a *classical solution* if it is both a sub- and a supersolution.

Viscosity subsolutions We say that $u : X \to \mathbb{R}$ is a *subsolution* of equation (3.1) if $u \in USC_u(X)$ and if, for all $(f,g) \in A_\dagger$ such that $\sup_x u(x) - f(x) < \infty$ there is a sequence $y_n \in Y$ such that

$$\lim_{n\to\infty} u(\gamma(y_n)) - f(\gamma(y_n)) = \sup_x u(x) - f(x), \tag{3.3}$$

and

$$\limsup_{n\to\infty} u(\gamma(y_n)) - g(y_n) - h_1(\gamma(y_n)) \leq 0. \tag{3.4}$$

Viscosity supersolution We say that $v : X \to \mathbb{R}$ is a *supersolution* of equation (3.2) if $v \in LSC_l(X)$ and if, for all $(f,g) \in A_\ddagger$ such that $\inf_x v(x) - f(x) > -\infty$ there is a sequence $y_n \in Y$ such that

$$\lim_{n\to\infty} v(\gamma(y_n)) - f(\gamma(y_n)) = \inf_x v(x) - f(x), \tag{3.5}$$

and

$$\liminf_{n\to\infty} v(\gamma(y_n)) - g(y_n) - h_2(\gamma(y_n)) \geq 0. \tag{3.6}$$

Viscosity solution We say that u is a *solution* of the pair of equations (3.1) and (3.2) if it is both a subsolution for $A_\dagger$ and a supersolution for $A_\ddagger$.

Comparison principle We say that (3.1) and (3.2) satisfy the *comparison principle* if for every subsolution u to (3.1) and supersolution v to (3.2), we have

$$\sup_x u(x) - v(x) \leq \sup_x h_1(x) - h_2(x). \tag{3.7}$$

If $H = A_\dagger = A_\ddagger$, we will say that the comparison principle holds for $f - \lambda Af = h$, if for any subsolution u for $f - \lambda Af = h_1$ and supersolution v of $f - \lambda Af = h_2$ the estimate in (3.7) holds.

Often, as in Section 4 below, $Y = X$ and $\gamma(x) = x$ simplifying the definitions above.

3.3. Operators

For an operator $A \subseteq M(X) \times M(Y)$ and $c \geq 0$ we write $cA \subseteq M(X) \times M(Y)$ for the operator

$$c \cdot A := \{(f, c \cdot g) \,|\, (f, g) \in A\} .$$

Here we write $c \cdot g$ for the function

$$c \cdot g(x) := \begin{cases} cg(x) & \text{if } g(x) \in \mathbb{R}, \\ \infty & \text{if } g(x) = \infty, \\ -\infty & \text{if } g(x) = -\infty. \end{cases}$$

The next set of properties is mainly relevant in the setting that $Y = X$.

Definition 3.2.

Contractivity We say that $T \subseteq M(X) \times M(X)$ is *contractive* if for all $f_1, f_2 \in \mathcal{D}(T)$:

$$\sup_x Tf_1(x) - Tf_2(x) \leq \sup_x f_1(x) - f_2(x),$$
$$\inf_x Tf_1(x) - Tf_2(x) \geq \inf_x f_1(x) - f_2(x).$$

If in addition $T0 = 0$, contractivity implies that $\sup_x Tf(x) \leq \sup_x f(x)$ and $\inf_x Tf(x) \geq \inf_x f(x)$.

Dissipativity We say $A \subseteq M(X) \times M(X)$ is *dissipative* if for all $(f_1, g_1), (f_2, g_2) \in A$ and $\lambda > 0$ we have

$$\|f_1 - \lambda g_1 - (f_2 - \lambda g_2)\| \geq \|f_1 - f_2\| ;$$

The range condition We say $A \subseteq M(X) \times M(X)$ satisfies *the range condition* if for all $\lambda > 0$ we have: the uniform closure of $\mathcal{D}(A)$ is a subset of $\mathcal{R}(\mathbb{1} - \lambda A)$.

The following theorem was proven for accretive operators but can be easily translated into dissipative operators by changing A by $-A$.

Theorem 3.3 *(Crandall-Liggett [9]). Let A be an operator on a Banach space E. Suppose that*

(a) A is dissipative,

(b) A satisfies the range condition.

Denote by $R(\lambda, A) = (\mathbb{1} - \lambda A)^{-1}$. *Then there is a strongly continuous (for the norm) contraction semigroup* $S(t)$ *defined on the uniform closure of* $\mathcal{D}(A)$ *and for all* $t \geq 0$ *and* f *in the uniform closure of* $\mathcal{D}(A)$

$$\lim_n \left\| R\left(\tfrac{t}{n}, A\right)^n f - S(t)f \right\| = 0.$$

3.4. Operators and the strict and buc topology

In addition to normed spaces, we consider bounded and uniform convergence on compacts (buc-convergence). This notion of convergence for functions on $C_b(X)$ is more natural from an applications point of view. This is due to the fact that it is the restriction of the locally convex strict topology restricted to sequences, see e.g. [5,40]. Indeed, it is the strict topology for which most well known results generalize (under appropriate conditions on the topology, e.g. X Polish): Stone-Weierstrass, Arzelà-Ascoli and the Riesz representation theorem. We define both notions.

Definition 3.4 *(buc convergence).* Let $f_n \in C_b(X)$ and $f \in C_b(X)$. We say that f_n converges *bounded and uniformly on compacts* (buc) if $\sup_n \|f_n\| < \infty$ and if for all compact $K \subseteq X$:

$$\lim_n \sup_{x \in K} |f_n(x) - f(x)| = 0. \tag{3.8}$$

Note (3.8) can be replaced by $f_n(x_n) \to f(x)$ for all sequences $x_n \in K$ that converge to $x \in K$.

Definition 3.5. The *(sub) strict topology* β on the space $C_b(X)$ for a completely regular space X is generated by the collection of semi-norms

$$p(f) := \sup_n a_n \sup_{x \in K_n} |f(x)|$$

where K_n are compact sets in X and where $a_n \geq 0$ and $a_n \to 0$.

Remark 3.6. The (sub)strict topology is the finest locally convex topology that coincides with the compact open topology on bounded sets. Thus, a sequence converges strictly if and only if it converges buc.

In the literature on locally convex spaces, the strict topology is usually referred to as the substrict topology, but on Polish spaces, among others, these topologies coincide, see [40].

Definition 3.7.

(a) Denote $B_r := \{f \in C_b(X) \mid \|f\| \leq r\}$. We say that a set D is *quasi-closed* if for all $r \geq 0$ the set $D \cap B_r$ is closed for the strict topology (or equivalently for the compact open or buc topologies).
(b) We say that $\widehat{D}$ is the *quasi-closure* of D if $\widehat{D} = \bigcup_{r>0} \widehat{D}_r$, where $\widehat{D}_r$ is the strict closure of $D \cap B_r$.
(c) We say that D_1 is *quasi-dense* in D_2 if $D_1 \cap B_r$ is strictly dense in $D_2 \cap B_r$ for all $r \geq 0$.

Next, we consider operators with respect to a hierarchy of statements regarding continuity involving the strict topology. The proof can be found in Appendix A.

Proposition 3.8. *Let $T : C_b(X) \to C_b(X)$. Consider*

(a) T is strictly continuous.
(b) For all $\delta > 0$, $r > 0$, and compact sets K there are $C_0(r)$, $C_1(\delta, r)$ and a compact set $\hat{K}(K, \delta, r)$ such that

$$\sup_{x \in K} |Tf(x) - Tg(x)| \leq \delta C_0(r) + C_1(\delta, r) \sup_{x \in \hat{K}(K,\delta,r)} |f(x) - g(x)|$$

for all $f, g \in C_b(X)$ such that $\|f\| \vee \|g\| \leq r$.
(c) T is strictly continuous on bounded sets.

Then (a) implies (b) and (b) implies (c).

Remark 3.9. There is not much room between properties (a) and (c). In the case that X is Polish space, and T is linear then (a) and (c) are equivalent, see e.g. [40, Corollary 3.2 and Theorem 9.1]. It is unclear to the author whether (b) and (c) are equivalent in general.

At various points in the paper, we will work with operators that are constructed by taking closures on dense sets. To do so, we need continuity properties. Even though working with (a) of 3.8 would be the desirable from a functional analytic point of view, (b) is much more explicit, and also suffices for our analysis.

The following result is proven in [22, Lemma A.11].

Lemma 3.10. *Suppose that an operator $T : D \subseteq C_b(X) \to C_b(X)$ satisfies (b) of Proposition 3.8. Then T has an extension to the quasi-closure $\widehat{D}$ of D that also satisfies property (b) of Proposition 3.8 (with the same choice of $\hat{K}$).*

3.5. A general setting of convergence of spaces

In previous section, we have studied buc convergence and the strict topology. This suffices for convergence problems in the context where all Hamilton-Jacobi equations $f - \lambda H_n f = h$ and $f - \lambda H f = h$ are based on the same space X. In practice, however, one runs into situations where this is not natural. In the context of simple slow-fast systems for example, one typically works with $X_n = E \times F$ and $X = E$. That is, we have a slow system on E that depends on a fast system taking values on F. Taking limits, we end up with a slow system on E with coefficients that are suitable averages over F. Thus, we need to connect X_n to X via a mapping η_n (e.g. a projection on the first coordinate) and extend the notion of buc convergence to allow for functions f_n on X_n to converge to X.

To extend the notion of buc convergence, we need to decide what 'uniform convergence' on compacts means. Following Definition 3.4, we saw that f_n converges to f buc if $\sup_n \|f_n\| < \infty$ and if for all compact sets K we have $f_n(x_n) \to f(x)$ for all sequences $x_n \in K$ converging to $x \in K$.

In the context of distinct X_n and X, there is no natural analogue of the compact set K. Instead, we will work a sequence of compact sets. Namely, we will choose compact sets $K_n \subseteq X_n$ that 'converge' to a compact set $K \subseteq X$. Then f_n converges to f if $\sup_n \|f_n\| < \infty$ and if for each of these sequences of compact sets and $x_n \in K_n$ converging to $x \in K$, we have $f_n(x_n) \to f(x)$.

We turn to the rigorous definition. We will slightly extend our discussion above by allowing spaces X_n and X such that the X_n are not naturally embedded in X. Instead, we will map all spaces to a common space $\mathcal{X}$ in which 'X_n converges to X'.

Assumption 3.11. Consider spaces X_n and X, some space $\mathcal{X}$, Baire measurable maps $\eta_n : X_n \to \mathcal{X}$ and a Baire measurable injective map $\eta : X \to \mathcal{X}$.

Definition 3.12 *(Kuratowski convergence).* Let $\{O_n\}_{n\geq 1}$ be a sequence of subsets in a space $\mathcal{X}$. We define the *limit superior* and *limit inferior* of the sequence as

$$\limsup_{n\to\infty} O_n := \{x \in \mathcal{X} \,|\, \forall\, U \in \mathcal{U}_x \,\forall\, N \geq 1 \,\exists\, n \geq N : \, O_n \cap U \neq \emptyset\},$$

$$\liminf_{n\to\infty} O_n := \{x \in \mathcal{X} \,|\, \forall\, U \in \mathcal{U}_x \,\exists\, N \geq 1 \,\forall\, n \geq N : \, O_n \cap U \neq \emptyset\},$$

where $\mathcal{U}_x$ is the collection of open neighborhoods of x in $\mathcal{X}$. If $O := \limsup_n O_n = \liminf_n O_n$, we write $O = \lim_n O_n$ and say that O is the Kuratowski limit of the sequence $\{O_n\}_{n\geq 1}$.

Assumption 3.13. There is a directed set $\mathcal{Q}$ (partially ordered set such that every two elements have an upper bound). For each $q \in \mathcal{Q}$, we have compact sets $K_n^q \subseteq X_n$ a compact set $K^q \subseteq X$ such that

(a) If $q_1 \leq q_2$, we have $K^{q_1} \subseteq K^{q_2}$ and for all n we have $K_n^{q_1} \subseteq K_n^{q_2}$.
(b) For all $q \in \mathcal{Q}$ and each sequence $x_n \in K_n^q$, every subsequence of x_n has a further subsequence that is converging to a limit $x \in K^q$ (that is: $\eta_n(x_n) \to \eta(x)$ in $\mathcal{X}$).
(c) For each compact set $K \subseteq X$, there is a $q \in \mathcal{Q}$ such that

$$\eta(K) \subseteq \liminf_n \eta_n(K_n^q).$$

Remark 3.14. Note that (b) implies that $\limsup_n \eta_n(K_n^q) \subseteq \eta(K^q)$. Note that (b) follows if $\bigcup_n \eta_n(K_n^q)$ is a subset of $\eta(K^q)$ and the topology on K^q is metrizable.

Conditions (b) should be interpreted in the sense that K^q is larger than the 'limit' of the sequence K_n, whereas (c) should be interpreted in the sense that each compact K in X is contained in a limit of that type.

We will say that a sequence $x_n \in X_n$ converges to $x \in X$ in the sense that $\eta_n(x_n) \to \eta(x)$ in $\mathcal{X}$. Dual to the notion of convergence in a topological space, there is the notion of convergence of functions.

Definition 3.15. Let Assumptions 3.11 and 3.13 be satisfied. For each n let $f_n \in M_b(X_n)$ and $f \in M_b(X)$. We say that $\mathrm{LIM}\, f_n = f$ if

- $\sup_n \|f_n\| < \infty$,
- if for all $q \in \mathcal{Q}$ and $x_n \in K_n^q$ converging to $x \in K^q$ we have

$$\lim_{n\to\infty} |f_n(x_n) - f(x)| = 0.$$

The notion of bounded and uniform on compacts (buc) is the prime example of a notion of LIM. For a second example see Example 2.7 in [22].

Example 3.16 *(buc convergence).* Consider some space X in which all compact sets are metrizable, and suppose that $X_n = X$ and that η_n is the identity map for all n. In this context, we can choose $\mathcal{X} = X$ and η the identity map. $\mathbb{Q}$ is the set of compact subsets. For $\mathcal{K} \in \mathcal{Q}$, we take $K_n^{\mathcal{K}} = K^{\mathcal{K}} = \mathcal{K}$.

Note that we need metrizable compact sets to extract converging subsequences for Assumption 3.13 (b).

We have $\mathrm{LIM}\, f_n = f$ if and only if $\sup_n \|f_n\| < \infty$ and if for all $\mathcal{K}$ and all sequences $x_n \in \mathcal{K}$ converging to $x \in \mathcal{K}$, we have $\lim_n f_n(x_n) = f(x)$.

Remark 3.17. In the setting that $\mathcal{X} = X$ whose topologies coincide, we can compare the notion of LIM we introduced to that which is used in [22]. Indeed, it is straightforward to show that both notions of $\mathrm{LIM}\, f_n = f$ for a sequence of functions coincide if the limiting function f is continuous.

Remark 3.18. The notion of LIM is subtle. It does not require $f_n(x_n) \to f(x)$ for all sequences x_n such that $\eta_n(x_n) \to \eta(x)$ in $\mathcal{X}$.

For example, let $X_n = \mathbb{R} \times \mathbb{R}$, $\mathcal{X} = X = \mathbb{R}$, $\eta_n(x,y) = x$ and $\eta(x) = x$. We could work with an index set $\mathcal{Q}$ consisting of all compact sets $[a,b] \times [c,d]$ in $\mathbb{R}^2$. Then $K_n^{[a,b]\times[c,d]} = [a,b] \times [c,d]$ and $K^{[a,b]\times[c,d]} = [a,b]$. Clearly the sequence $x_n := (x,n)$ satisfies $\eta_n(x,n) = x$ which converges to x. There is however, no compact set $[a,b] \times [c,d]$ such that (x,n) lies in this set for all n. Thus, we do not need to check convergence along this sequence in Definition 3.15.

Remark 3.19. Proceeding with last remark. Note that we could have chosen different compact sets with the same index set. E.g., we could have chosen $K_n^{[a,b]\times[c,d]} = [a,b] \times [nc,nd]$ and $K^{[a,b]\times[c,d]} = [a,b]$. This leads to a larger collection of sequences for which we have to verify convergence for LIM.

In Section 3.6 below, we will see that we can define a notion of equi-continuity of operators on the spaces X_n based on the set $\mathcal{Q}$ and compacts K_n^q.

Indeed, in Condition 5.5, key for our main results, we will assume that we have converge of Hamiltonians in the sense of LIM, and have equi-continuity for the resolvents in terms of K_n^q. This leads to a careful balance: choose small sets K_n^q, then verifying convergence with LIM is easy whereas verifying equi-continuity becomes hard and vice versa. Thus, the choice of K_n^q is context dependent and requires insight into the problem at hand.

The characterization of $f = \mathrm{LIM}\, f_n$ allows for generalization of the lim sup and lim inf as well.

Definition 3.20. Let Assumptions 3.11 and 3.13 be satisfied. Let $f_n \in M(X_n)$.

(a) Let $f \in USC_u(X)$. We say that $\mathrm{LIM\,SUP}\, f_n = f$ if

- $\sup_n \sup_{x \in X_n} f_n(x) < \infty$,
- if

$$f(x) = \sup_{q \in \mathcal{Q}} \sup \left\{ \limsup_{n\to\infty} f_n(x_n) \,\middle|\, x_n \in K_n^q, \, \eta_n(x_n) \to \eta(x) \right\}.$$

(b) Let $f \in LSC_l(X)$. We say that $\mathrm{LIM\,INF}_n\, f_n = f$ if

- $\inf_n \inf_{x \in X_n} f_n(x) > -\infty$,
- if

$$f(x) := \inf_{q \in \mathcal{Q}} \inf \left\{ \liminf_{n\to\infty} f_n(x_n) \,\middle|\, x_n \in K_n^q, \, \eta_n(x_n) \to \eta(x) \right\}.$$

The following is immediate.

Lemma 3.21. *Let Assumptions 3.11 and 3.13 be satisfied. Suppose that* $\mathrm{LIM\,SUP}_n f_n \leq f \leq \mathrm{LIM\,INF}\, f_n$*, then* $\mathrm{LIM}\, f_n = f$*.*

3.6. Joint equicontinuity of operators and LIM

It is a general fact from topology that if T_n are equi-continuous functions on some space and if $f_{1,n} \to f$ and $f_{2,n} \to f$ are elements in this space, then $T_n f_{1,n} - T f_{2,n} \to 0$.

We now show that equi-continuity in the sense of (b) of Proposition 3.8 combines with the notion of LIM in a similar way. Afterwards, we will show that we can use LIM and a collection of equi-continuous operators to define a limiting operator.

Definition 3.22. Let Assumptions 3.11 and 3.13 be satisfied. Let $T_n : B_n \subseteq M_b(X_n) \to B_n$ be operators.

We say that the collection $\{T_n\}_{n \geq 1}$ is *strictly equi-continuous on bounded sets* if the following holds. For all $q \in \mathcal{Q}$, $r > 0$ and $\delta > 0$, there is a $\hat{q} \in \mathcal{Q}$ and constants $C_0(r), C_1(\delta, r)$ such that for all n and $h_1, h_2 \in B_n$ with $\|h_1\| \vee \|h_2\| \leq r$ we have

$$\sup_{y \in K_n^q} \{T_n h_1(y) - T_n h_2(y)\} \leq \delta C_0(r) + C_1(\delta, r) \sup_{y \in K_n^{\hat{q}}} \{h_1(y) - h_2(y)\}.$$

Lemma 3.23. *Let Assumptions 3.11 and 3.13 be satisfied. Let* $T_n : B_n \subseteq M_b(X_n) \to B_n$ *be a collection of operators that is strictly equi-continuous on bounded sets.*

Suppose that $h_{1,n}, h_{2,n} \in B_n$ *and that* $\mathrm{LIM}\, h_{1,n} = \mathrm{LIM}\, h_{2,n}$*. Then it holds that* $\mathrm{LIM}\, T_n h_{1,n} - T_n h_{2,n} = 0$*. In particular, if* $\mathrm{LIM}\, T_n h_{1,n}$ *exists, then* $\mathrm{LIM}\, T_n h_{2,n}$ *exists also and is the same.*

Proof. Pick $h_{1,n}, h_{2,n} \in B_n$ and that $\mathrm{LIM}\, h_{1,n} = \mathrm{LIM}\, h_{2,n}$. We establish that $\mathrm{LIM\,SUP}\, T_n h_{1,n} - T_n h_{2,n} \leq 0$. By interchanging the roles of $h_{1,n}$ and $h_{2,n}$ this yields the statement for LIM INF which establishes the claim.

To do so, it suffices for any $q \in \mathcal{Q}$ and $x_n \in K_n^q$ and x such that $\eta_n(x_n) \to \eta(x)$ to establish that

$$\limsup_n T_n h_{1,n}(x_n) - T_n h_{2,n}(x_n) \leq 0.$$

As $\mathrm{LIM}\, h_{1,n}, \mathrm{LIM}\, h_{2,n}$ exists, there is some $r > 0$ such that $\sup_n \|h_{1,n}\| \vee \|h_{2,n}\| \leq r$. Thus, by joint strict local equi-continuity of the operators $\{T_n\}$ we can find for any $\delta > 0$ a $\hat{q}$ and constants $C_0(\delta), C_1(\delta, r)$ such that

$$\begin{aligned} &\limsup_n T_n h_{1,n}(x_n) - T_n h_{2,n}(x_n) \\ &\qquad \leq \delta C_0(r) + C_1(r, \delta) \limsup_n \sup_{y \in K_n^{\hat{q}}} h_{1,n}(y) - h_{2,n}(y). \end{aligned}$$

As $\operatorname{LIM} h_{1,n} = \operatorname{LIM} h_{2,n}$ the $\limsup_n$ on the right equals 0. Sending $\delta \to 0$, the first claim follows. The final claim is a direct consequence of the triangle inequality. $\square$

In next proposition, we show how to use the result of previous lemma to construct a limiting operator our of a sequence of operators that are strictly equi-continuous on bounded sets.

Proposition 3.24. *Let Assumptions 3.11 and 3.13 be satisfied. Let $T_n : B_n \subseteq M_b(X_n) \to B_n$ be a strictly equi-continuous on bounded sets. Suppose the spaces B_n are such that there is a $M > 0$ such that for all $h \in C_b(X)$ there are $h_n \in B_n$ such that $\operatorname{LIM} h_n = h$ and $\sup_n \|h_n\| \leq M \|h\|$.*

Set

$$\mathcal{D}(T) := \{h \in C_b(X) \,|\, \exists\, h_n \in B_n : h = \operatorname{LIM} h_n, \operatorname{LIM} T_n h_n \, \textit{exists and is continuous}\}$$

and $Th = \operatorname{LIM} T_n h_n$. Note that T is well defined because of Lemma 3.23

Then:

(a) T is strictly continuous on bounded sets in the sense of (b) of Proposition 3.8.
(b) The set $\mathcal{D}(T)$ is quasi-closed in $C_b(X)$.
(c) If $h \in \mathcal{D}(T)$ and $h_n \in B_n$ such that $\operatorname{LIM} h_n = h$, then $\operatorname{LIM} T_n h_n = Th$.

Remark 3.25. In Lemma 6.4 below, we will see that in the context that if the maps η_n are continuous and η is a homeomorphism onto its image, we can indeed always find h_n such that $\operatorname{LIM} h_n = h$ and $\sup_n \|h_n\| \leq \|h\|$.

The proof is inspired by Lemma 7.16 (b) and (c) in [22].

Proof of Proposition 3.24. We start by proving (a). Fix $r > 0$, a compact set $K \subseteq X$ and $\delta > 0$. We prove that there are constants $\hat{C}_0(r), \hat{C}_1(\delta, r)$ and a compact set $\widehat{K} = \widehat{K}(K, \delta, r)$ such that

$$\sup_{x \in K} |Tf(x) - Tg(x)| \leq \delta \hat{C}_0(r) + \hat{C}_1(\delta, r) \sup_{x \in \widehat{K}} |f(x) - g(x)|$$

for all $f, g \in \mathcal{D}(T)$ such that $\|f\| \vee \|g\| \leq r$.

Thus, fix $f, g \in \mathcal{D}(T)$ such that $\|f\| \vee \|g\| \leq r$ and let $x_0 \in K$ be such that

$$Tf(x_0) - Tg(x_0) = \sup_{y \in K} Tf(y) - Tg(y)$$

Let $f_n, g_n \in B_n$ such that $\operatorname{LIM} f_n = f$ and $\operatorname{LIM} g_n = g$ and $\sup_n \|f_n\| \leq M \|f\|$, $\sup_n \|g_n\| \leq M \|g\|$. By Assumption 3.13 (c), there is a q with $K \subseteq \liminf_n \eta_n(K_n^q)$.

Because of this, we can choose $x_n \in K_n^q$ such that $\lim_n \eta_n(x_n) = \eta(x_0)$ in $\mathcal{X}$. By definition of $\mathcal{D}(T)$ and T, we have $Tf(x_0) - Tg(x_0) = \lim_n T_n f_n(x_n) - T_n g_n(x_n)$. It follows by strict equi-continuity on bounded sets that there is a $\hat{q}$, $C_0(r)$ and $C_1(r,\delta)$ such that

$$\begin{aligned} \sup_{y \in K} Tf(y) - Tg(y) &= \lim_n T_n f_n(x_n) - T_n g_n(x_n) \\ &\leq \lim_n \left(\delta C_0(r) M + C_1(r,\delta) \sup_{y \in K_n^{\hat{q}}} \{f_n(y) - g_n(y)\} \right). \end{aligned}$$

Set $\hat{C}_0(r) = MC_0(r)$ and $\hat{C}_1(\delta, r) = C_1(\delta, r)$. Finally, let $y_n \in K_n^{\hat{q}}$ be such that $f_n(y_n) - g_n(y_n) + n^{-1} \geq \sup_{y \in K_n^{\hat{q}}} \{f_n(y) - g_n(y)\}$. By Assumption 3.13 (b), y_n has a converging subsequence with a limit $y_0 \in K^{\hat{q}}$. We obtain that

$$\begin{aligned} \sup_{y \in K} \{Tf(y) - Tg(y)\} &\leq \delta \hat{C}_0(r) + \hat{C}_1(\delta, r) \left(f(y_0) - g(y_0)\right) \\ &\leq \delta \hat{C}_0(r) + \hat{C}_1(\delta, r) \sup_{y \in K^{\hat{q}}} \{f(y) - g(y)\}. \end{aligned}$$

This establishes strict continuity on bounded sets for T.

We proceed with the proof of (b). First note that by strict continuity on bounded sets, and Lemma 3.10, we can extend T to the quasi-closure of $\mathcal{D}(T)$, on which T is also strictly continuous on bounded sets.

Next, we show that $\mathcal{D}(T)$ was in fact quasi-closed to begin with. Let h be in the strict closure of $\mathcal{D}(T) \cap B_R$ for some $R > 0$. We prove that $h \in \mathcal{D}(T) \cap B_R$. Using the extension of T to the quasi-closure of $\mathcal{D}(T)$, we can define $f := Th$. Let $h_n \in B_n$ such that $\mathrm{LIM}\, h_n = h$ and $\sup_n \|h_n\| \leq M \|h\|$. To establish (b), we need to prove that $\mathrm{LIM}\, T_n h_n = f$.

On bounded sets, the strict topology coincides with the compact-open topology. Thus, there are functions $h^{K,\varepsilon} \in \mathcal{D}(T)$ such that $\sup_{K,\varepsilon} \left\|h^{K,\varepsilon}\right\| \leq R$ and

$$\sup_{y \in K} \left|h(y) - h^{K,\varepsilon}(y)\right| \leq \varepsilon.$$

Define $f^{K,\varepsilon} := Th^{K,\varepsilon}$. Furthermore, find $f_n, f_n^{K,\varepsilon}, h_n^{K,\varepsilon}$ such that

$$\mathrm{LIM}\, f_n = f, \qquad \mathrm{LIM}\, f_n^{K,\varepsilon} = f^{K,\varepsilon}, \qquad \mathrm{LIM}\, h_n^{K,\varepsilon} = h^{K,\varepsilon},$$

and such that we have an upper bound r for the norms of all involved functions. To establish that $\mathrm{LIM}\, T_n h_n = f$, it suffices by Lemma 3.23 to prove that $\mathrm{LIM}\, T_n h_n - f_n = 0$.

Fix an arbitrary q and ε. Then it suffices to prove that

$$\lim_{n \to \infty} \sup_{y \in K_n^q} |T_n h_n(y) - f_n(y)| \leq 4\varepsilon. \tag{3.9}$$

To do so, fix $\delta > 0$ such that $C_0(r)\delta \leq \varepsilon$.

By strict equi-continuity on bounded sets of the operators T_n, starting with q, r and δ, we find a $\hat{q}$ and set $K := K^{\hat{q}}$. Note that we can use the same compact set K for the strict continuity estimate for the limiting operator T also.

Using this specific compact set K, it follows by the triangle inequality that

$$\begin{aligned}
&|T_n h_n(y) - f_n(y)| \\
&\quad \leq \left|T_n h_n(y) - T_n h_n^{\varepsilon,K}(y)\right| + \left|T_n h_n^{\varepsilon,K}(y) - f_n^{K,\varepsilon}(y)\right| + \left|f_n^{K,\varepsilon}(y) - f_n(y)\right| \\
&\quad \leq \left|T_n h_n(y) - T_n h_n^{\varepsilon,K}(y)\right| + \left|T_n h_n^{\varepsilon,K}(y) - f_n^{K,\varepsilon}(y)\right| + \left|f_n^{K,\varepsilon}(y) - f_n(y)\right|
\end{aligned}$$

After taking $\limsup_n$ and $\sup_{y \in K_n^q}$ over the three terms separately, we estimate as follows:

- By equi-continuity of the T_n and our choice of $\delta > 0$ and $\hat{q}$, we find that

$$\sup_{y \in K_n^q} \left|T_n h_n(y) - T_n h_n^{\varepsilon,K}(y)\right| \leq \varepsilon + \sup_{y \in K_n^{\hat{q}}} \left|h_n(y) - h_n^{\varepsilon,K}(y)\right|$$

  As $\mathrm{LIM}\, h_n = h$ and $\mathrm{LIM}\, h_n^{\varepsilon,K} = h^{\varepsilon,K}$, and the fact that $\sup_{y \in K} |h(y) - h^{K,\varepsilon}(y)| \leq \varepsilon$ implies that the $\limsup$ over the right-hand side is bounded above by 2ε.
- As $h^{K,\varepsilon}$ in $\mathcal{D}(T)$, we have $\mathrm{LIM}\, T_n h_n^{K,\varepsilon} = T h^{\varepsilon,K} = f^{\varepsilon,K}$. We also have $\mathrm{LIM}\, f_n^{K,\varepsilon} = f^{K,\varepsilon}$ so by Lemma 3.23 the middle term vanishes by choosing n large.
- To estimate $\limsup_n \sup_{y \in K_n^q} \left|f_n^{K,\varepsilon}(y) - f_n(y)\right|$ note that as $\mathrm{LIM}\, f_n^{K,\varepsilon} = f^{K,\varepsilon}$ and $\mathrm{LIM}\, f_n = f$, we find

$$\limsup_n \sup_{y \in K_n^q} \left|f_n^{K,\varepsilon}(y) - f_n(y)\right| \leq \sup_{y \in K^q} \left|f^{K,\varepsilon}(y) - f(y)\right| = \sup_{y \in K^q} \left|T h^{K,\varepsilon}(y) - T h(y)\right|$$

  Using the strict continuity on bounded sets of T, we find

$$\limsup_n \sup_{y \in K_n^q} \left|f_n^{K,\varepsilon}(y) - f_n(y)\right| \leq \varepsilon + \sup_{y \in K} \left|h^{K,\varepsilon}(y) - h(y)\right|.$$

  Thus the right-hand side is bounded by 2ε.

This establishes (3.9) which concludes the proof of (b). (c) follows by a direct application of Lemma 3.22. □

3.7. An extension of notions of convergence to a space containing additional information

In the context of problems that involve homogenisation or slow-fast systems, it often pays of to work with multi-valued Hamiltonians whose range naturally takes values in a space of functions with a domain that is larger. This larger domain takes into account a variable that we homogenize over or the ‘fast’ variable. We extend the setting of Section 3.5. We will not need the extension of all results therein, but restrict to the bare essentials.

Assumption 3.26. Consider spaces X_n and X, Y, and two spaces $\mathcal{X}, \mathcal{Y}$. We consider Baire measurable maps $\eta_n : X_n \to \mathcal{X}$, $\widehat{\eta}_n : X_n \to \mathcal{Y}$ and Baire measurable injective maps $\eta : X \to \mathcal{X}$, $\widehat{\eta} : Y \to \mathcal{Y}$. Finally, there are continuous surjective maps $\gamma : Y \to X$ and $\widehat{\gamma} : \mathcal{Y} \to \mathcal{X}$. The maps are such that the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{Y} \arrow[dd, "\widehat{\gamma}"] & Y \arrow[l, "\widehat{\eta}"'] \arrow[dd, "\gamma"] \\
X_n \arrow[ur, "\widehat{\eta}_n"] \arrow[dr, "\eta_n"'] & & \\
& \mathcal{X} & X \arrow[l, "\eta"']
\end{tikzcd}$$

Assumption 3.27. There is a directed set $\mathcal{Q}$ (partially ordered set such that every two elements have an upper bound). For each $q \in \mathcal{Q}$, we have compact sets $K_n^q \subseteq X_n$ a compact sets $K^q \subseteq X$ and $\widehat{K}^q \subseteq Y$ such that

(a) If $q_1 \leq q_2$, we have $K^{q_1} \subseteq K^{q_2}$, $\widehat{K}^{q_1} \subseteq \widehat{K}^{q_2}$ and for all n we have $K_n^{q_1} \subseteq K_n^{q_2}$.
(b) For all $q \in \mathcal{Q}$ and each sequence $x_n \in K_n^q$, every subsequence of x_n has a further subsequence $x_{n(k)}$ such that $\widehat{\eta}_{n(k)}(x_{n(k)}) \to \widehat{\eta}(y)$ in $\mathcal{Y}$ for some $y \in \widehat{K}^q$.
(c) For each compact set $K \subseteq X$, there is a $q \in \mathcal{Q}$ such that

$$\eta(K) \subseteq \liminf_n \eta_n(K_n^q).$$

(d) We have $\gamma(\widehat{K}^q) \subseteq K^q$.

Note the subtle difference with Assumption 3.13 in the sense that here (b) is written down in terms of convergence in $\mathcal{Y}$, whereas (c) is still written down in terms of convergence in $\mathcal{X}$.

Remark 3.28. Note that (b) and (d) imply that $\eta_{n(k)}(x_{n(k)}) \to \eta(\gamma(y))$ in $\mathcal{X}$ with $\gamma(y) \in K^q$. Thus, the Assumption 3.27 implies the conditions for $X, X_n, \mathcal{X}$ for Assumption 3.13.

Thus, in the context of Assumptions 3.26 and 3.27, we can use all notions of the previous sections, if we talk about functions or operators on $X, \mathcal{X}$ and X_n.

Example 3.29 *(Reduction of the dimension).* Consider two spaces X and Z and let $Y := X \times Z$, $X_n := X \times Z$ with maps $\eta_n(x, z) = x$, $\hat{\eta}_n(x, z) = (x, z)$ and $\gamma(x, z) = x$.

Assumption 3.27 is satisfied for example with $\mathcal{Q}$ the collection of pairs of compact sets in X and Z:

$$\{(K_1, K_2) \,|\, \forall K_1 \subseteq X, K_2 \subseteq Z \text{ compact}\},$$

and $K_n^{(K_1,K_2)} = K_1 \times K_2$, $K^{(K_1,K_2)} = K_1$ and $\widehat{K}^{(K_1,K_2)} = K_1 \times K_2$.

We have $\mathrm{LIM}\, f_n = f$ if and only if $\sup_n \|f_n\| < \infty$ and for all compact $K_1 \subseteq X$ and $K_2 \subseteq Z$ and sequences $(x_n, z_n) \in K_1 \times K_2$ and $x \in K_X$ such that $x_n \to x$, we have $f_n(x_n, z_n) \to f(x)$.

Note that the dependence of f_n on z_n should vanish in the limit.

4. Pseudo-resolvents and Hamilton-Jacobi equations

Having developed the topological and analytic machinery, we turn to the study of pseudo-resolvents $\{R(\lambda)\}_{\lambda>0}$ and viscosity solutions for Hamilton-Jacobi equations $f - \lambda H f = h$. In the linear context, the relation between pseudo-resolvents and 'classical' solutions to the Hamilton-Jacobi equation is well established. In particular, pseudo-resolvents have been used as tools in the approximation theory of linear semigroups and generators, see for example [17, Section 3.4]. We will show that the linearity of these the involved operators, however, is not essential.

In this section, we will not consider approximation problems yet, and restrict ourselves to the setting $Y = X$ and $\gamma(x) = x$.

Definition 4.1 *(Pseudo-resolvents).* Consider a space X and a subset B such that $C_b(X) \subseteq B \subseteq M(X)$ on which we have a family of operators $R(\lambda) : B \to B$, for $\lambda > 0$. We say that this family is a *pseudo-resolvent* if $R(\lambda)0 = 0$ for $\lambda > 0$ and if for all $\alpha < \beta$ we have

$$R(\beta) = R(\alpha)\left(R(\beta) - \alpha \frac{R(\beta) - \mathbb{1}}{\beta}\right).$$

We extend our notion of strict continuity on bounded sets to a collection of operators similar to what we did in Definition 3.22. Whereas in that Definition we worked with bounded functions only, we extend to unbounded functions in the next definition. In addition, we use that the constants that appear for pseudo-resolvents are typically 1.

Definition 4.2 *(Local strict equicontinuity on bounded sets).* Let B be a collection of functions $C_b(X) \subseteq B \subseteq M(X)$. We say that the pseudo-resolvent $R(\lambda) : B \to B$ is *locally strictly equicontinuous on bounded sets* if for each $\lambda_0 > 0$, each compact set $K \subseteq X$ and $r, \delta > 0$, there is a compact set $\widehat{K} = \widehat{K}(K, \delta, r, \lambda_0)$ such that

$$\sup_{x \in K} |R(\lambda)f(x) - R(\lambda)g(x)| \leq \delta \sup_{x \in X} |f(x) - g(x)| + \sup_{x \in \hat{K}(K,\delta,r,\lambda_0)} |f(x) - g(x)|$$

for all $0 < \lambda \leq \lambda_0$, $f, g \in B$ such that $\|f\| \vee \|g\| \leq r$.

Note that the present definition, restricted to a collection $B_0 \subseteq B \cap M_b(X)$, reduces to a definition like the one in Definition 3.22.

The three main results of this section are

- Proposition 4.4, which shows that a pseudo-resolvent R such that $R(\lambda)(f - \lambda Hf) = f$ yields viscosity solutions to the Hamilton-Jacobi equation for H. In other words, if the pseudo-resolvent is a left-inverse (classically) of $\mathbb{1} - \lambda H$, then it is also a right-inverse (in the viscosity sense). This implies that a pseudo-resolvent can be used to identify the resolvent of H.
- Proposition 4.8 establishes the converse, viscosity solutions to the Hamilton-Jacobi equation, if unique, can be used to construct a pseudo-resolvent.
- Proposition 4.10 shows that the pseudo-resolvent of the previous results can be used to define a new operator, which satisfies the conditions of the Crandall-Liggett theorem.

Even though, we will always formally think of the Hamilton-Jacobi equation $f - \lambda Hf = h$, we will work with two operators $H_\dagger, H_\ddagger$ instead. These operators should be interpreted as a upper and lower bound of the 'true' H.

Thus, in all sections below we will work with $H_\dagger \subseteq LSC_l(X) \times USC_u(X)$ and $H_\ddagger \subseteq USC_u(X) \times LSC_l(X)$ and study the Hamilton Jacobi equations

$$f - \lambda H_\dagger f = h, \qquad f - \lambda H_\ddagger f = h,$$

with $\lambda > 0$ and $h \in M(X)$.

Before proceeding with the announced results, we note that at various points, it is of interest to know whether the domain of definition of the resolvents and operators B includes $C_b(X)$. At least for resolvents, we can under some assumptions include $C_b(X)$ in the domain by continuous extension. We will therefore henceforth work with pseudo-resolvents that have $C_b(X)$ included in their domain of definition.

Lemma 4.3. *Suppose that $R(\lambda)$ with domains D_λ is a pseudo-resolvent that is also locally strictly equi-continuous on bounded sets. Suppose that*

(a) $R(\lambda)$ restricted to $D_\lambda \cap C_b(X)$ maps into $C_b(X)$,
(b) $D_\lambda \cap C_b(X)$ is quasi-dense in $C_b(X)$.

Then the restriction of $R(\lambda)$ to $C_b(X)$ can be extended to a pseudo-resolvent that is locally strictly equi-continuous on bounded sets such that for each λ the domain of this extension includes $C_b(X)$.

Proof. By Lemma 3.10, the restriction of $R(\lambda)$ to $D_\lambda \cap C_b(X)$ can be extended to an operator $\hat{R}(\lambda) : C_b(X) \to C_b(X)$ that satisfies property (b) of Proposition 3.8 (with the same choice of constants and compact set as in Definition 4.2). The pseudo-resolvent property follows by continuity. $\square$

4.1. Pseudo-resolvents give solutions to the Hamilton-Jacobi equation

Proposition 4.4. *Let B be a collection of functions such that $C_b(X) \subseteq B \subseteq M(X)$. Suppose that $R(\lambda) : B \to B$ is a contractive pseudo-resolvent. In addition, suppose that we have two operators $H_\dagger \subseteq LSC_l(X) \cap B \times USC_u(X) \cap B$ and $H_\ddagger \subseteq USC_u(X) \cap B \times LSC_l(X) \cap B$.*

Fix $\lambda > 0$ and $h \in M(X)$.

(a) Let $h \in B$ and let $H_\dagger$ be such that for all $(f,g) \in H_\dagger$ and $0 < \varepsilon < \lambda$ we have $f \geq R(\varepsilon)(f - \varepsilon g)$. Then $(R(\lambda)h)^*$ is a viscosity sub-solution to $f - \lambda H_\dagger f = h^*$.
(b) Let $h \in B$ and let $H_\ddagger$ be such that for all $(f,g) \in H_\ddagger$ and $0 < \varepsilon < \lambda$, we have $f \leq R(\varepsilon)(f - \varepsilon g)$. Then $(R(\lambda)h)_*$ is a viscosity super-solution to $f - \lambda H_\ddagger f = h_*$.

The proof of this result follows key steps in the proof of Theorem 8.27 of [22]. To establish this result, we use an auxiliary lemma.

Lemma 4.5 *(Lemma 7.8 in [22]). Let X be a some space and let $f, g : X \to [-\infty, \infty]$ be two functions.*

(a) Suppose there is some $\varepsilon_0 > 0$ such that $f - \varepsilon g \in M(X)$ and $\sup_x f(x) \leq \sup_x f(x) - \varepsilon g(x) < \infty$ for all $0 < \varepsilon < \varepsilon_0$, then there is a sequence x_n in X such that $\lim_n f(x_n) = \sup_x f(x)$ and $\limsup_n g(x_n) \leq 0$.
(b) Suppose there is some $\varepsilon_0 > 0$ such that $f - \varepsilon g \in M(X)$ and $\inf_x f(x) \geq \inf_x f(x) - \varepsilon g(x) > -\infty$ for all $0 < \varepsilon < \varepsilon_0$, then there is a sequence x_n in X such that $\lim_n f(x_n) = \inf_x f(x)$ and $\liminf_n g(x_n) \geq 0$.

Remark 4.6. The proof in Lemma 7.8 of [22] is not correct. In establishing $\lim_n f(x_n) = \sup_x f(x)$ for (a), it is used that $\inf_x g(x) > -\infty$. This claim, however, is not true in general. Consider the following example. Let $X = [0,1]$ and set

$$f(x) = \begin{cases} \log x & \text{if } x \neq 0, \\ -\infty & \text{if } x = 0, \end{cases} \qquad g(x) = \begin{cases} \log x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For all $0 < \varepsilon < 1$ we have $\sup_x f(x) = \sup_x f(x) - \varepsilon g(x) = 0$. However, it does not hold that $\inf_x g(x) > -\infty$.

Proof. We only prove (a). Let $0 < \varepsilon_n < \varepsilon_0$ and $\varepsilon_n \to 0$. For each n pick x_n such that

$$f(x_n) \leq \sup_x f(x) \leq \sup_x f(x) - \varepsilon_n g(x) \leq f(x_n) - \varepsilon_n g(x_n) + \varepsilon_n^2. \tag{4.1}$$

Combining the outer two-terms leads to $\limsup_n g(x_n) \leq \lim_n \varepsilon_n = 0$.

We now establish that $\lim_n f(x_n) = \sup_x f(x)$. To do so, we first prove $\liminf_n g(x_n) > -\infty$. Fix $\varepsilon \in (0, \varepsilon_0)$. For each n, using (4.1), we have the following chain of inequalities:

$$\begin{aligned}\sup_x f(x) - \varepsilon g(x) &\geq f(x_n) - \varepsilon g(x_n) \\ &= f(x_n) - \varepsilon_n g(x_n) + \varepsilon_n^2 - (\varepsilon - \varepsilon_n) g(x_n) - \varepsilon_n^2 \\ &\geq \left(\sup_x f(x)\right) - (\varepsilon - \varepsilon_n) g(x_n) - \varepsilon_n^2.\end{aligned}$$

Suppose there is a subsequence $x_{n(k)}$ such that $\lim_k g(x_{n(k)}) = -\infty$, then clearly the right-hand side would diverge to ∞. This contradicts the boundedness of the left-hand side.

From the second and final term in (4.1), we obtain

$$\left(\sup_x f(x)\right) + \varepsilon_n g(x_n) - \varepsilon_n^2 \leq f(x_n).$$

Taking a lim inf on both sides, using that $\liminf_n g(x_n) > -\infty$, we find $\liminf_n f(x_n) \geq \sup_x f(x)$, which establishes $\lim_n f(x_n) = \sup_x f(x)$. □

Proof of Proposition 4.4. We prove (a). Fix $\lambda > 0$ and $h \in USC_u(X) \cap B$. We prove that $(R(\lambda)h)^*$ is a viscosity solution to $f - \lambda H_\dagger f = h^*$. Fix $(f, g) \in H_\dagger$.

We use the pseudo-resolvent property of R with $0 < \varepsilon < \lambda$, to re-express $R(\lambda)h(x)$. For $f(x)$ we use the assumption in (a). We obtain:

$$\begin{aligned}&\sup_x R(\lambda)h(x) - f(x) \\ &\leq \sup_x R(\varepsilon)\left(R(\lambda)h(x) - \varepsilon \frac{R(\lambda)h(x) - h(x)}{\lambda}\right) - R(\varepsilon)\left(f(x) - \varepsilon g(x)\right) \\ &\leq \sup_x R(\lambda)h(x) - \varepsilon \frac{R(\lambda)h(x) - h(x)}{\lambda} - (f(x) - \varepsilon g(x)),\end{aligned}$$

where we have used the contractivity of $R(\varepsilon)$ to obtain the final inequality. Note that we use that the domain and range of $H_\dagger$ are contained in B to be able to write down $R(\lambda)$ and $R(\varepsilon)$ applied to h and $f - \varepsilon g$. Next, we take the upper semi-continuous regularization. As $\varepsilon < \lambda$ and $f, f - \varepsilon g \in LSC_l(X)$, we obtain

$$\sup_x \left(R(\lambda)h\right)^*(x) - f(x) \leq \sup_x \left(R(\lambda)h\right)^*(x) - \varepsilon \frac{\left(R(\lambda)h\right)^*(x) - h^*(x)}{\lambda} - (f(x) - \varepsilon g(x))$$

This establishes (a) of Lemma 4.5 which then yields that $(R(\lambda)h)^*$ is a viscosity subsolution to $f - \lambda H_\dagger f = h^*$. □

4.2. Solutions to the Hamilton-Jacobi equation give a pseudo-resolvent

In this section we prove the converse statement, namely that viscosity solutions give rise to a pseudo-resolvent. We start with an auxiliary lemma that will be key to recovering the pseudo-resolvent property for solutions of viscosity solutions.

Lemma 4.7. *Let $H_\dagger \subseteq LSC_l(X) \times USC_u(X)$ and $H_\ddagger \subseteq USC_u(X) \times LSC_l(X)$. Fix $\lambda > \varepsilon > 0$ and $h \in M(X)$. Then*

(a) A subsolution u to $f - \lambda H_\dagger f = h$ is a subsolution to $f - \varepsilon H_\dagger f = u - \varepsilon \frac{u-h}{\lambda}$.
(b) A supersolution v to $f - \lambda H_\dagger f = h$ is a supersolution to $f - \varepsilon H_\dagger f = v - \varepsilon \frac{v-h}{\lambda}$.

Proof. We prove (a) only. Let $(f, g) \in H_\dagger$ and let x_n be sequence such that $\lim_n u(x_n) - f(x_n) = \sup_x u(x) - f(x)$ and $\limsup_n u(x_n) - \lambda g(x_n) - h(x_n) \leq 0$. Because

$$u(x_n) - \varepsilon g(x_n) - \left(u(x_n) - \varepsilon \frac{u(x_n) - h(x_n)}{\lambda} \right) = \frac{\varepsilon}{\lambda} \left(u(x_n) - \lambda g(x_n) - h(x_n) \right),$$

it follows that u is a viscosity subsolution to $f - \varepsilon H_\dagger f = u - \varepsilon \frac{u-h}{\lambda}$. $\square$

Proposition 4.8. *Let $H_\dagger \subseteq LSC_l(X) \times USC_u(X)$ and $H_\ddagger \subseteq USC_u(X) \times LSC_l(X)$. Suppose that for each $\lambda > 0$ and $h \in C_b(X)$ the comparison principle holds for*

$$f - \lambda H_\dagger f = h, \qquad f - \lambda H_\ddagger f = h. \tag{4.2}$$

Suppose that there exists a viscosity solution to (4.2) for all $\lambda > 0$ and $h \in C_b(X)$. Denote this unique solution by $R(\lambda)h$. Then $R(\lambda)$ forms a contractive pseudo-resolvent on $C_b(X)$.

Remark 4.9. The proposition has the drawback that we have to assume the comparison principle for all $h \in C_b(X)$ and $\lambda > 0$. In practice, often one only has the comparison principle for a dense set of functions. The result does show that we should expect the pseudo-resolvent property to hold. Thus, if one has a method to produce viscosity solutions $R(\lambda)h$ via some other means, like a limiting procedure, or an explicit formula, one can aim to prove the pseudo-resolvent property directly.

Proof of Proposition 4.8. Let $0 < \varepsilon < \lambda$ and $h \in C_b(X)$. Let $R(\lambda)h$ be the unique viscosity solution to $f - \lambda H_\dagger f = h$ and $f - \lambda H_\ddagger f = h$. Note that by the comparison principle $R(\lambda)h \in C_b(X)$.

By the comparison principle $R(\lambda)$ is contractive. We next establish that $R(\lambda)$ is a pseudo-resolvent. By Lemma 4.7, we find for $h \in D$ that $R(\lambda)h$ is a viscosity solution to

$$f - \varepsilon H_\dagger f = R(\lambda)h - \varepsilon \frac{R(\lambda)h - h}{\lambda_0}, \qquad f - \varepsilon H_\ddagger f = R(\lambda)h - \varepsilon \frac{R(\lambda)h - h}{\lambda}.$$

As h and $R(\lambda)h \in C_b(X)$, we find by the comparison principle for (4.2) with ε instead of λ that

$$R(\lambda)h = R(\varepsilon)\left(R(\lambda)h - \varepsilon\frac{R(\lambda)h - h}{\lambda}\right). \quad \square$$

4.3. Defining a Hamiltonian using a pseudo-resolvent

We proceed by showing that a pseudo-resolvent can be used to define a Hamiltonian, so that the pseudo-resolvent gives both classical and viscosity solutions to the associated Hamilton-Jacobi equation. Thus, this newly defined operator satisfies the conditions for the Crandall-Liggett theorem, and can be used for approximation arguments.

Proposition 4.10. *Let $R(\lambda) : C_b(X) \to C_b(X)$ be a contractive pseudo-resolvent. Define the operator*

$$\widehat{H} = \left\{\left(R(\lambda)h, \frac{R(\lambda)h - h}{\lambda}\right) \,\middle|\, \lambda > 0, h \in C_b(E)\right\}$$

For all $h \in C_b(E)$ and $\lambda > 0$:

(a) For all $\lambda > 0$ and $h \in C_b(X)$ the comparison principle holds for $f - \lambda\widehat{H}f = h$, and $R(\lambda)h$ is the unique viscosity and classical solution.
(b) $\widehat{H}$ is dissipative and satisfies the range condition.

The proposition is mainly useful in combination with Proposition 4.8. Namely, viscosity solutions for (4.2) can be used to define a contractive pseudo-resolvent. Which by this proposition can be used to define a new operator that satisfies the conditions for the Crandall-Liggett result. Alternatively, one constructs a contractive pseudo-resolvent via an approximation argument as we will do below in Section 5.

We start by establishing a natural property for sub- and supersolutions of Hamilton-Jacobi equations. We mention it separately for later use.

Lemma 4.11. *Let $H_\dagger \subseteq LSC_l(X) \times USC_u(X)$ and $H_\ddagger \subseteq USC_u(X) \times LSC_l(X)$. Fix $\varepsilon > 0$.*

(a) Let $(f_0, g_0) \in H_\dagger$. Suppose that $\hat{f}$ is a viscosity subsolution to $f - \varepsilon H_\dagger f = f_0 - \varepsilon g_0$, then $\hat{f} \leq f_0$.
(b) Let $(f_0, g_0) \in H_\ddagger$. Suppose that $\hat{f}$ is a viscosity supersolution to $f - \varepsilon H_\ddagger f = f_0 - \varepsilon g_0$, then $\hat{f} \geq f_0$.

Proof. We only prove (a). Fix $\varepsilon > 0$, $(f_0, g_0) \in H_\dagger$ and let $\hat{f}$ be a viscosity subsolution to $f - \varepsilon H_\dagger f = f_0 - \varepsilon g_0$. Then there is a sequence x_n such that

$$\lim_n \hat{f}(x_n) - f_0(x_n) = \sup_x \hat{f}(x) - f_0(x)$$

and

$$\limsup_n \hat{f}(x_n) - \varepsilon g_0(x_n) - (f_0 - \varepsilon g_0)(x_n) \leq 0.$$

We obtain $\sup_x \hat{f}(x) - f_0(x) = \limsup_n \hat{f}(x_n) - f_0(x_n) \leq 0$ establishing the claim. □

Proof of Proposition 4.10. We prove the comparison principle. Fix $\lambda > 0$ and $h_1, h_2 \in D$. By construction $R(\lambda)h_1$ and $R(\lambda)h_2$ solve $f - \lambda \widehat{H} f = h_i$ classically. Let u be a subsolution to $f - \lambda \widehat{H} f = h_1$ and v a supersolution to $f - \lambda \widehat{H} f = h_2$. By Lemma 4.11 (a) for $\widehat{H}$ instead of $H_\dagger$ and $(f_0, g_0) = \left(R(\lambda)h_1, \frac{R(\lambda)h_1 - h_1}{\lambda}\right)$, we find $u \leq R(\lambda)h_1$. Because $R(\lambda)$ is contractive, we find

$$\sup_x u(x) - v(x) \leq \sup_x R(\lambda)h_1(x) - R(\lambda)h_2(x) \leq \sup_x h_1(x) - h_2(x),$$

establishing the comparison principle for $f - \lambda \widehat{H} f = h$.

Next, we prove that $R(\lambda)h$ is a viscosity subsolution to $f - \lambda \widehat{H} f = h$ with an argument similar to that of Proposition 4.4. Pick $R(\mu)h_0 \in \mathcal{D}(\widehat{H})$. By the pseudo-resolvent property of R, see Proposition 4.8, and the contractivity of R, we find for $0 < \varepsilon < \lambda \wedge \mu$ that

$$\begin{aligned}
&\sup_x R(\lambda)h(x) - R(\mu)h_0(x) \\
&= \sup_x R(\varepsilon)\left(R(\lambda)h - \varepsilon \frac{R(\lambda)h - h}{\lambda}\right)(x) - R(\varepsilon)\left(R(\mu)h_0 - \varepsilon \frac{R(\mu)h_0 - h_0}{\mu}\right)(x) \\
&\leq \sup_x R(\lambda)h(x) - \varepsilon \frac{R(\lambda)h(x) - h(x)}{\lambda} - \left(R(\mu)h_0(x) - \varepsilon \frac{R(\mu)h_0(x) - h_0(x)}{\mu}\right).
\end{aligned}$$

By Lemma 4.5, we conclude that $R(\lambda)h$ is a viscosity subsolution to $f - \lambda \widehat{H} f = h$. The super-solution property follows similarly.

Thus, by the comparison principle, $R(\lambda)h$ is the unique viscosity solution to $f - \lambda H f = h$. Finally, suppose that f_0 is another classical solution to $f - \lambda \widehat{H} f = h$. Thus, there is a g_0 such that $(f_0, g_0) \in \widehat{H}$ and $f_0 - \lambda g_0 = h$. As $R(\lambda)f$ is a viscosity solution to $f - \lambda \widehat{H} f = h$, and hence to $f - \lambda \widehat{H} f = f_0 - \lambda g_0$, we find again by Lemma 4.11 that $f_0 = R(\lambda)h$.

Finally, we establish (b). Note that the range condition for $\widehat{H}$ is satisfied by construction. We establish dissipativity. By construction, there is some $\lambda > 0$ and $h \in C_b(X)$ such that $f_1 = R(\lambda)h$ and $g_1 = \lambda^{-1}(f_1 - h)$. As $R(\lambda)h$ is a viscosity subsolution to $f - \lambda \widehat{H} f = h$ by (a), there are x_n such that

$$\lim_n f_1(x_n) - f_2(x_n) = \sup_x f_1(x) - f_2(x),$$

$$\limsup_n g_1(x_n) - g_2(x_n) \leq 0.$$

This implies for all $\mu > 0$ that

$$\begin{aligned}
&\sup_x f_1(x) - \mu g_1(x) - (f_2(x) - \mu g_2(x)) \\
&\quad \geq \limsup_n f_1(x_n) - \mu g_1(x_n) - (f_2(x_n) - \mu g_2(x_n)) \\
&\quad = \lim_n f_1(x_n) - f_2(x_n) = \sup_x f_1(x) - f_2(x).
\end{aligned}$$

A similar argument using the supersolution property yields the other inequality for the infima. We conclude that for all $\mu > 0$:

$$\|f_1 - \mu g_1 - (f_2 - \mu g_2)\| \geq \|f_1 - f_2\|$$

establishing dissipativity of $\widehat{H}$. □

4.4. The pseudo-resolvent yields viscosity solutions via a density argument

We introduce a final tool in the study of pseudo-resolvents $R(\lambda)$ and viscosity solutions to Hamilton-Jacobi equations

$$f - \lambda H_\dagger f = h_1, \qquad f - \lambda H_\ddagger f = h_2. \tag{4.3}$$

In Proposition 4.8, we showed that if we can solve (4.3) in the viscosity sense for all $\lambda > 0$ and $h \in C_b(X)$, then the comparison principle is sufficient to establish that the solutions $R(\lambda)$ form a contractive pseudo-resolvent.

Often, however, one can construct a pseudo-resolvent $R(\lambda)$ such that $R(\lambda)h$ solves (4.3) in the viscosity sense for $\lambda > 0$ and $h = h_1 = h_2 \in D$, where $D \subseteq C_b(X)$ is quasi-dense. Thus, the main step to establish the pseudo-resolvent property in the proof cannot be carried out. This happens for example in the construction in Theorem 5.7. Even though there we can establish the pseudo-resolvent property by approximation, this situation is not completely satisfying.

The result below gives an alternative that does not need an explicit form for the resolvent, or that it is the limit of a sequence of pseudo-resolvents.

A second reason to establish that $R(\lambda)h$ gives viscosity solutions for all $h \in C_b(X)$ is that this property can be used as input for follow-up arguments, see e.g. Condition 5.5 below.

The argument below is based on compactness and quasi-density of D in $C_b(X)$.

Proposition 4.12. *Let X be a space in which compact sets are metrizable. For each $\lambda > 0$ let $R(\lambda) : C_b(X) \to C_b(X)$ be an operator that is strictly continuous on bounded sets. Let D be quasi-dense in $C_b(X)$.*

Then (i) and (ii) hold.

(i) Let $H_\dagger, \widetilde{H}_\dagger \subseteq LSC_l(X) \times USC_u(X)$ be two operators such that

(a) For each $h \in C_b(X)$ and $\lambda > 0$, if f is a viscosity sub-solution to $f - \lambda H_\dagger f = h$ then it is a viscosity subsolution to $f - \lambda \widetilde{H}_\dagger f = h$.
(b) Each function $f \in \mathcal{D}(\widetilde{H}_\dagger)$ has compact sub-levelsets.

Suppose that $R(\lambda)h$ is a viscosity subsolution to $f - \lambda H_\dagger f = h$ for all $h \in D$ and $\lambda > 0$. Then $R(\lambda)$ is a viscosity subsolution to $f - \lambda \widetilde{H}_\dagger f = h$ for all $h \in C_b(X)$ and $\lambda > 0$.

(ii) Let $H_\ddagger, \widetilde{H}_\ddagger \subseteq USC_u(X) \times LSC_l(X)$ be two operators such that

(a) For each $h \in C_b(X)$ and $\lambda > 0$, if f is a viscosity super-solution to $f - \lambda H_\ddagger f = h$ then it is a viscosity supersolution to $f - \lambda \widetilde{H}_\ddagger f = h$.
(b) Each function $f \in \mathcal{D}(\widetilde{H}_\ddagger)$ has compact super-levelsets.

Suppose that $R(\lambda)h$ is a viscosity supersolution to $f - \lambda H_\ddagger f = h$ for all $h \in D$ and $\lambda > 0$. Then $R(\lambda)$ is a viscosity supersolution to $f - \lambda \widetilde{H}_\ddagger f = h$ for all $h \in C_b(X)$ and $\lambda > 0$.

Remark 4.13. The conditions sketched above are satisfied in a wide range of situations. Consider for example a Hamiltonian $\mathcal{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in terms of location $x \in \mathbb{R}$ and momentum $p \in \mathbb{R}$ (the argument easily extends to e.g. manifolds). We assume that $p \mapsto \mathcal{H}(x,p)$ is convex for all x.

Assume there is a continuously differentiable function Υ that has compact sub-level sets and is such that $\sup_x \mathcal{H}(x, \Upsilon'(x)) \leq c$ for some $c \in \mathbb{R}$.

The condition in (i) then holds for the operator $(H_\dagger, \mathcal{D}(H_\dagger))$ defined by

$$\mathcal{D}(H_\dagger) := C_b^1(\mathbb{R}), \qquad \forall\, f \in C_b^1(\mathbb{R}) : H_\dagger f(x) = \mathcal{H}(x, f'(x)),$$

and the operator $(\widetilde{H}_\dagger, \mathcal{D}(\widetilde{H}_\dagger))$ defined by

$$\mathcal{D}(\widetilde{H}_\dagger) := \bigcup_{\varepsilon \in (0,1)} \left\{ (1-\varepsilon) f + \varepsilon \Upsilon \,\middle|\, f \in C_b^1(\mathbb{R}) \right\}$$

$$\forall\, \widetilde{f} := (1-\varepsilon) f + \varepsilon \Upsilon : \qquad \widetilde{H}_\dagger \widetilde{f}(x) = (1-\varepsilon) H_\dagger f(x) + \varepsilon c.$$

For a proof of (a) one uses convexity and e.g. methods like Lemmas 7.6 and 7.7 in [22]. For an application of these lemmas, see e.g. Section A.2 in [7].

Proof of Proposition 4.12. We only prove (a). Fix $h \in C_b(X)$ and $\lambda > 0$.

Pick $(f,g) \in \widetilde{H}_\dagger$. As f has compact sub-level sets, the set $K := \{x \mid f(x) \leq 2\|h\|\}$ is compact. By quasi-density of D in $C_b(X)$, there are h_n such that $\sup_n \|h_n\| \leq 2\|h\|$ and $\sup_{x\in K} |h(x) - h_n(x)| \leq n^{-1}$.

By the viscosity subsolution property, the fact that f has compact level-sets, and $\sup_n \|h_n\| \leq 2\|h\|$, there are $x_n \in K$ such that

$$\begin{aligned} R(\lambda)h_n(x_n) - f(x_n) &= \sup_x R(\lambda)h_n(x) - f(x), \\ R(\lambda)h_n(x_n) - \lambda g(x_n) &\leq h_n(x_n). \end{aligned} \tag{4.4}$$

Because K is compact and metrizable, we can assume without loss of generality that x_n converges to x_0. h_n converges uniformly to h on K. As $R(\lambda)$ is strictly continuous on bounded sets also $R(\lambda)h_n$ converges uniformly on K to $R(\lambda)h$. Thus, we can take limit in the first equation and lim sup in the second equation of (4.4) to obtain

$$\begin{aligned} R(\lambda)h(x_0) - f(x_0) &= \sup_x R(\lambda)h(x) - f(x) \\ R(\lambda)h_n(x_0) - \lambda g(x_0) &\leq h(x_0). \end{aligned}$$

Note that we used that g is upper semi-continuous to obtain the correct inequality. These two equations establish that $R(\lambda)h$ is a viscosity solution $f - \lambda\widetilde{H}_\dagger f = h$. □

5. Convergence of resolvents

We now turn to the main question of the paper: that of approximation. Our first goal is to establish that viscosity solutions to $f - \lambda H_n f = h$ converge to a viscosity solution of the equation $f - \lambda H f = h$. All arguments will be based in the context of converging spaces.

As mentioned in the introduction some problems, e.g. slow-fast or multi-scale systems lead to natural limiting Hamiltonians that are multi-valued as a graph $H \subseteq C_b(X) \times C_b(Y)$, where Y is some larger space that takes into account the fast variable or the additional scales. The notions of convergence of functions that are relevant have been introduced in Sections 3.5 and 3.7.

5.1. *From convergence of Hamiltonians to convergence of resolvents*

A first notion of a limit of Hamiltonians is given by the notion of an extended limit. This notion is essentially the extension of the convergence condition for generators from the setting of the Trotter-Kato approximation theorem to a more general context. The generalization is made to include operators defined on different spaces, and is also applicable to non-linear operators as well. See e.g. the works of Kurtz and co-authors [16,48,49,22].

We define this notion for the setting in which $X = Y$.

Definition 5.1. Consider the setting of Assumptions 3.11 and 3.13. Suppose that for each n we have an operator $H_n \subseteq M_b(X_n) \times M_b(X_n)$. The *extended limit* $ex - \mathrm{LIM}_n H_n$ is defined by the collection $(f,g) \in M_b(X) \times M_b(X)$ such that there exist $(f_n, g_n) \in H_n$ with the property that $\mathrm{LIM}_n f_n = f$ and $\mathrm{LIM}_n g_n = g$.

We aim to have a more flexible notion of convergence by replacing all operators H_n and H by operators $(H_{n,\dagger}, H_{n,\ddagger}, H_\dagger, H_\ddagger)$ that intuitively form natural upper and lower bounds for H_n and H. We will also generalize by considering limiting Hamiltonians that take values in the set of functions on Y instead of X.

Definition 5.2. Consider the setting of Assumptions 3.26 and 3.27. Suppose that for each n we have two operators $H_{n,\dagger} \subseteq LSC_l(X_n) \times USC_u(X_n)$ and $H_{n,\ddagger} \subseteq USC_u(X_n) \times LSC_l(X_n)$.

(a) The *extended* lim sup $ex - \mathrm{LIM\,SUP}_n H_{n,\dagger}$ is defined by the collection $(f,g) \in H_\dagger \subseteq LSC_l(X) \times USC_u(Y)$ such that there exist $(f_n, g_n) \in H_{n,\dagger}$ satisfying

$$\mathrm{LIM}\, f_n \wedge c = f \wedge c, \qquad \forall c \in \mathbb{R}, \tag{5.1}$$

$$\sup_n \sup_{x \in X_n} g_n(x) < \infty, \tag{5.2}$$

and if for any $q \in \mathcal{Q}$ and sequence $z_{n(k)} \in K^q_{n(k)}$ (with $k \mapsto n(k)$ strictly increasing) such that $\lim_k \widehat{\eta}_{n(k)}(z_{n(k)}) = \widehat{\eta}(y)$ in $\mathcal{Y}$ with $\lim_k f_{n(k)}(z_{n(k)}) = f(\gamma(y)) < \infty$ we have

$$\limsup_{k \to \infty} g_{n(k)}(z_{n(k)}) \leq g(y). \tag{5.3}$$

(b) The *extended* lim inf $ex - \mathrm{LIM\,INF}_n H_{n,\ddagger}$ is defined by the collection $(f,g) \in H_\ddagger \subseteq USC_u(X) \times LSC_l(Y)$ such that there exist $(f_n, g_n) \in H_{n,\ddagger}$ satisfying

$$\mathrm{LIM}\, f_n \vee c = f \vee c, \qquad \forall c \in \mathbb{R}, \tag{5.4}$$

$$\inf_n \inf_{x \in X_n} g_n(x) > -\infty, \tag{5.5}$$

and if for any $q \in \mathcal{Q}$ and sequence $z_{n(k)} \in K^q_{n(k)}$ (with $k \mapsto n(k)$ strictly increasing) such that $\lim_k \widehat{\eta}_{n(k)}(z_{n(k)}) = \widehat{\eta}(y)$ in $\mathcal{Y}$ with $\lim_k f_{n(k)}(z_{n(k)}) = f(\gamma(y)) > -\infty$ we have

$$\liminf_{k \to \infty} g_{n(k)}(z_{n(k)}) \geq g(y). \tag{5.6}$$

Remark 5.3. The conditions in (5.2) and (5.3) are implied by $\mathrm{LIM\,SUP}_n g_n \leq g$ and (5.5) and (5.6) are implied by $\mathrm{LIM\,INF}_n g_n \geq g$.

It is not clear to the author whether a weakened symmetric statement in which (5.1) is replaced by $\mathrm{LIM\,INF}\, f_n \geq f$ and (5.4) by $\mathrm{LIM\,SUP}\, f_n \leq f$ is possible.

Remark 5.4. The notion of $ex-\mathrm{LIM\,SUP}$ and $ex-\mathrm{LIM\,INF}$ follows closely Condition 7.11 [22]. Note, however, that our definition does away with the first conditions in (7.19) and (7.22), which in [22] are used in a crucial way in controlling the approximation of H_n by operators H_n^ε that are constructed from the Yosida approximant A_n^ε of the linear operator A_n.

Given our main conditions on upper and lower bounds for sequences of Hamiltonians, we can state the main condition for our approximation result. Note that in the typical case $B_n = C_b(X_n)$ a couple of statements simplify.

Condition 5.5. Consider the setting of Assumptions 3.26 and 3.27.

There are sets $B_n \subseteq M(X_n)$, contractive pseudo-resolvents $R_n(\lambda) : B_n \to B_n$, $\lambda > 0$, operators

$$H_{n,\dagger} \subseteq LSC_l(X_n) \cap B_n \times USC_u(X_n) \cap B_n,$$
$$H_{n,\ddagger} \subseteq USC_u(X_n) \cap B_n \times LSC_l(X_n) \cap B_n,$$

and

$$H_\dagger \subseteq LSC_l(X) \times USC_u(Y), \qquad H_\ddagger \subseteq USC_u(X) \times LSC_l(Y).$$

These spaces and operators have the following properties:

(a) There is a $M > 0$ such that for each $h \in C_b(X)$ there are $h_n \in B_n$ such that $\mathrm{LIM}\, h_n = h$ and $\sup_n \|h_n\| \leq M\,\|h\|$
(b) $H_\dagger \subseteq ex - \mathrm{LIM\,SUP}\, H_{n,\dagger}$ and $H_\ddagger \subseteq ex - \mathrm{LIM\,INF}\, H_{n,\ddagger}$;
(c) For each $n \geq 1$, $\lambda > 0$ and $h \in B_n$ the function $(R_n(\lambda)h)^*$ is a viscosity subsolution to $f - \lambda H_{n,\dagger} = h$. Similarly, $(R_n(\lambda)h)_*$ is a viscosity supersolution to $f - \lambda H_{n,\ddagger} f = h$.
(d) We have local strict equi-continuity on bounded sets: for all $q \in \mathcal{Q}$, $\delta > 0$ and $\lambda_0 > 0$, there is a $\hat{q} \in \mathcal{Q}$ such that for all n and $h_1, h_2 \in B_n$ and $0 < \lambda \leq \lambda_0$ that

$$\sup_{y \in K_n^q} \left\{R_n(\lambda)h_1(y) - R_n(\lambda)h_2(y)\right\} \leq \delta \sup_{x \in X_n} \left\{h_1(x) - h_2(x)\right\} + \sup_{y \in K_n^{\hat{q}}} \left\{h_1(y) - h_2(y)\right\}.$$

Remark 5.6. As a follow up on Remark 3.19, note that (b) and (d) reflect a careful balance. Proving (d) will be relatively easy if one chooses large sets for K_n^q, which leads to a difficulties in (b). Context specific knowledge is needed for a proper choice.

We briefly discuss the relevance of our four conditions and a sketch of the proof.

- Conditions (b) and (c) are aimed at showing that $\mathrm{LIM\,SUP}_n\, R_n(\lambda)h_n$ yields a viscosity subsolution to $f - \lambda H_\dagger f = h$, whereas $\mathrm{LIM\,INF}_n\, R_n(\lambda)h_n$ yields a viscosity supersolution to $f - \lambda H_\ddagger f = h$ if $\mathrm{LIM}\, h_n = h$. In combination with the comparison principle for h in a quasi-dense set D, we obtain a viscosity solution for $h \in D$ that we call $R(\lambda)h$. In addition, we obtain that $\mathrm{LIM}\, R_n(\lambda)h_n = R(\lambda)h$.

- Using Proposition 3.24, the operator $R(\lambda)$ extends to $C_b(X)$ on which it is strictly continuous on bounded sets. In particular the operator is contractive.
- The operator $R(\lambda)$ is a pseudo-resolvent as it is the limit of pseudo-resolvents.

Some technical difficulties need to be settled. The main idea for our the first step of our strategy is to apply the method that was also used in the proof of Proposition 4.8. Here we used contractivity of the resolvent and Lemma 4.5. In this setting, we need to take care of our special notion of LIM. Thus, we need to replace contractivity by control along compact subsets K_n^q for a fixed q. This is the main aim of Condition (d).

Theorem 5.7. *Let Condition 5.5 be satisfied. Let $D \subseteq C_b(X)$ be quasi-dense in $C_b(X)$. Suppose that for each $\lambda > 0$ and $h \in D$ the comparison principle holds for*

$$f - \lambda H_\dagger f = h, \qquad f - \lambda H_\ddagger f = h. \tag{5.7}$$

Then there is a collection of operators $R(\lambda) : C_b(X) \to C_b(X)$ such that

(a) For each $h \in D$ and $\lambda > 0$ the function $R(\lambda)h$ is a viscosity subsolution to $f - \lambda H_\dagger f = h$ and a viscosity supersolution to $f - \lambda H_\ddagger f = h$.
(b) The operators are locally strictly equi-continuous on bounded sets.
(c) The operators form a pseudo-resolvent.
(d) For $\lambda > 0$, $h_n \in B_n$ and $h \in C_b(X)$ such that $\mathrm{LIM}\, h_n = h$, we have $\mathrm{LIM}\, R_n(\lambda)h_n = R(\lambda)h$.

We state the main argument for the theorem as a separate proposition as it is valid in a context that goes slightly beyond the theorem.

Proposition 5.8. *Let Condition 5.5 be satisfied.*

(a) Let $h_n \in B_n$, $h \in USC_u(X)$ and suppose that $\mathrm{LIM\,SUP}_n\, h_n \leq h$. Define

$$\overline{F} := \underset{n}{\mathrm{LIM\,SUP}}\, R_n(\lambda)h_n,$$

then $\overline{F}^*$ is a viscosity subsolution to $f - \lambda H_\dagger f = h$.

(b) Let $h_n \in B_n$, $h \in LSC_l(X)$ and suppose that $\mathrm{LIM\,INF}_n\, h_n \geq h$. Define

$$\underline{F} := \underset{n}{\mathrm{LIM\,INF}}\, R_n(\lambda)h_n,$$

then $\underline{F}^*$ is a viscosity supersolution to $f - \lambda H_\ddagger f = h$.

It should be noted that the proof of this proposition does not use Condition 5.5 (a).

The main idea of the proof of the proposition is based on the proof of Lemma 7.14 in [22], but improves on this result in terms of the three properties mentioned in the introduction: applicability outside of the context of large deviations, operators $H_{n,\dagger}, H_{n,\ddagger}$ instead of H_n and the possibility to work in $\mathcal{X}$ instead of in X.

Proof of Proposition 5.8. We only prove (a). Note first of all that by contractivity of $R_n(\lambda)$, we have

$$\sup_n \sup_{x \in X_n} R_n(\lambda) h_n(x) \leq \sup_n \sup_{x \in X_n} h_n(x) < \infty, \tag{5.8}$$

so that we can indeed write down $\overline{F} := \mathrm{LIM\,SUP}\, R_n(\lambda) h_n$.

We prove that $\overline{f} := \overline{F}^*$ is a viscosity subsolution of $f - \lambda H_\dagger f = h$. First of all, $\overline{f}$ is upper semi-continuous by construction. Second, $\overline{f}$ is bounded from above as seen above as a consequence of (5.8). We will prove that $\overline{f}$ also satisfies the final property of the definition of subsolutions. As in the proof of Proposition 4.4, we use Lemma 4.5(a).

Thus, for $(f_0, g_0) \in H_\dagger$ it suffices to prove that for $0 < \varepsilon < \lambda$

$$\begin{aligned} &\sup_y \left\{ \overline{F}(\gamma(y)) - f_0(\gamma(y)) \right\} \\ &\leq \sup_y \left\{ \left(\overline{F}(\gamma(y)) - \varepsilon \left(\frac{\overline{F}(\gamma(y)) - h(\gamma(y))}{\lambda} \right) \right) - \left(f_0(\gamma(y)) - \varepsilon g_0(y) \right) \right\} < \infty, \end{aligned} \tag{5.9}$$

as we can replace $\overline{F}$ by its upper semi-continuous regularization $\overline{f}$ first on the right and then on the left-side of the inequality. Note that the lemma indeed suffices to establish the sub-solution property as for any function ϕ on X, we have $\sup_x \phi(x) = \sup_y \phi(\gamma(x))$ because γ is surjective.

We extend the proof of Proposition 4.4. In that proof, we used that the pseudo-resolvent is contractive. In this case, we have to pass to the limit using the adapted notion of convergence. Thus, we replace contractivity by strict equi-continuity on bounded sets, Condition 5.5 (d).

For every $n \geq 1$ set $f_n := R_n(\lambda) h_n$ and $g_n := \frac{f_n - h_n}{\lambda}$. Note that f_n is well defined as $h_n \in B_n$. By assumption there are $(f_{n,0}, g_{n,0}) \in H_{n,\dagger}$ such that (5.1), (5.2) and (5.3) are satisfied. Let $\varepsilon \in (0, \lambda)$ and define

$$h_n^\varepsilon := f_n - \varepsilon g_n = f_n - \varepsilon \frac{f_n - h_n}{\lambda}, \qquad h_{n,0}^\varepsilon := f_{n,0} - \varepsilon g_{n,0}. \tag{5.10}$$

Note the following

(1) $f_n = R_n(\varepsilon) h_n^\varepsilon$ because R_n is a pseudo-resolvent;
(2) As the domain and range of $H_{n,\dagger}$ are contained in B_n, we can apply $R_n(\varepsilon)$ to $h_{n,0}^\varepsilon$. By Condition 5.5 (c) and Lemma 4.11 we find $f_{n,0} \geq (R_n(\varepsilon) h_{n,0}^\varepsilon)^* \geq R_n(\varepsilon) h_{n,0}^\varepsilon$.

(3) Finally

$$\begin{aligned} &\sup_n h_n^\varepsilon(x) \leq \sup_n h_n(x) =: M_1 < \infty, \\ &\inf_n h_{n,0}^\varepsilon(x) \geq \inf_n f_{n,0} - \varepsilon \sup_n g_{n,0} =: M_2 > -\infty. \end{aligned} \tag{5.11}$$

The first equality follows by $\operatorname{LIMSUP}_n h_n \leq h$ and $\varepsilon < \lambda$, whereas the second inequality follows by (5.1) and (5.2). Denote by $M := M_1 - M_2$.

Pick $x \in X$. Pick any $q \in \mathcal{Q}$ such that $x \in K^q$ and such that there are $x_n \in K_n^q$ which satisfy $\eta_n(x_n) \to \eta(x)$ in $\mathcal{X}$ (There is at least one such q by Assumption 3.27 (c)). Now take $\delta' > 0$ arbitrary. By (5.11) and Condition 5.5 (d), we find $\hat{q} \in \mathcal{Q}$ such that for all n, functions $\phi_1, \phi_2 \in B_n$ satisfying $\sup_{y \in X_n} \phi_1(y) - \phi_2(y) \leq M$

$$\sup_{y \in K_n^q} R_n(\varepsilon)\phi_1(y) - R_n(\varepsilon)\phi_2(y) \leq \delta' M + \sup_{y \in K_n^{\hat{q}}} \phi_1(y) - \phi_1(y)$$

Fix $\delta = M\delta'$, which we can choose arbitrarily small by choosing δ' small. In addition, we can find $z \in K_n^{\hat{q}}$ such that

$$\sup_{y \in K_n^q} R_n(\varepsilon)\phi_1(y) - R_n(\varepsilon)\phi_2(y) \leq \delta + \sup_{y \in K_n^{\hat{q}}} \phi_1(y) - \phi_1(y) \leq 2\delta + \phi_1(z) - \phi_2(z). \tag{5.12}$$

For next computation, we use (1) and (2) in line 3, Equation (5.10) in line 5 and for line 4 we find $z_n \in K_n^{\widehat{q}}$ such that Equation (5.12) holds for all n and $h_n^\varepsilon, h_{n,0}^\varepsilon$ instead of ϕ_1, ϕ_2. This gives

$$\begin{aligned} &f_n(x_n) - f_{n,0}(x_n) \\ &\quad \leq \sup_{y \in K_n^q} f_n(y) - f_{n,0}(y) \\ &\quad \leq \sup_{y \in K_n^q} R_n(\varepsilon)h_n^\varepsilon(y) - R_n(\varepsilon)h_{n,0}^\varepsilon(y) \\ &\quad \leq 2\delta + h_n^\varepsilon(z_n) - h_{n,0}^\varepsilon(z_n) \\ &\quad \leq 2\delta + \left(f_n(z_n) - \varepsilon \frac{f_n(z_n) - h_n(z_n)}{\lambda}\right) - \left(f_{n,0}(z_n) - \varepsilon g_{n,0}(z_n)\right). \end{aligned} \tag{5.13}$$

Recall that our aim is to prove (5.9). Our next step is to take a $\limsup_n$ on both sides of the inequality. To study this limsup, we see that only the term $g_{n,0}$ is not yet understood in terms of its limiting behavior. Our aim is to apply (5.3), for which we need to construct a subsequence $n(k)$ for which $\widehat{\eta}_{n(k)}(z_{n(k)}) \to \widehat{\eta}(y)$ in $\mathcal{Y}$ with $y \in \widehat{K}^{\hat{q}}$ satisfying $f_{n(k),0}^{\hat{q}}(z_{n(k)}) \to f_0(\gamma(y))$.

Without loss of generality, we assume that

$$\limsup_n f_n(x_n) - f_{n,0}(x_n) > -\infty,$$

as there is nothing to prove otherwise. Again without loss of generality, we restrict to a subsequence $n(k)$ of n such that

- the lim sup on the left-hand side of (5.13) is achieved as a limit:

$$\limsup_n f_n(x_n) - f_{n,0}(x_n) = \lim_k f_{n(k)}(x_{n(k)}) - f_{n(k),0}(x_{n(k)}) > -\infty, \tag{5.14}$$

- there is some $y \in \widehat{K}^{\hat{q}}$ with $\lim_k \hat{\eta}_{n(k)}(z_{n(k)}) = \hat{\eta}(y)$. This is possible due to Definition 3.27 (b). Note that by continuity of $\hat{\gamma} : \mathcal{Y} \to \mathcal{X}$ also $\lim_k \eta_{n(k)}(z_{n(k)}) = \eta(\gamma(y))$ with $\gamma(y) \in K^q$ (Assumption 3.27 (d)).

We first consider the $\liminf_k$ over both sides of the inequality (5.13). By our assumption (5.14) on $x_{n(k)}$ we find

$$\liminf_k \left(f_n(z_n) - \varepsilon \frac{f_n(z_n) - h_n(z_n)}{\lambda} \right) - (f_{n,0}(z_n) - \varepsilon g_{n,0}(z_n)) > -\infty.$$

By (5.8) the sequences $\{f_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ are uniformly bounded from above and by assumption (5.2) we have a uniform upper bound on $\{g_{n,0}\}_{n\geq 1}$. This leads to

$$\limsup_k f_{n(k),0}(z_{n(k)}) < \infty.$$

By (5.1), choosing c larger than this lim sup, we find $\lim_k f_{n(k),0}(z_{n(k)}) = f_0(\gamma(y))$, which established also the condition for the application of (5.3). Taking the $\limsup_k$ over both sides of (5.13), we find

$$\begin{aligned}
\limsup_n f_n(x_n) - f_{n,0}(x_n) &= \lim_k f_{n(k)}(x_{n(k)}) - f_{n(k),0}(x_{n(k)}) \\
&\leq 2\delta + \left(\overline{F}(\gamma(y)) - \varepsilon \frac{\overline{F}(\gamma(y)) - h(\gamma(y))}{\lambda} \right) - (f_0(\gamma(y)) - \varepsilon g_0(y)) .
\end{aligned}$$

Now we take the supremum over y on the right-hand side. Afterwards, we send $\delta \to 0$. This gives the correct right-hand side for (5.9). Next, we work on the left-hand side. We take a supremum over all q and sequences $\eta_n(x_n) \to \eta(x)$ in $\mathcal{X}$ with $x_n \in K_n^q$, followed by a supremum over x This establishes (5.9) which concludes the proof. □

Proof of Theorem 5.7. Fix $\lambda > 0$, $h \in D$ and $h_n \in B_n$ with $\mathrm{LIM}\, h_n = h$. Let $\overline{F}$ and $\underline{F}$ be as in Proposition 5.8. By construction, it follows that $\overline{F} \geq \underline{F}$. By the comparison principle, we also have $\underline{F}_* \leq \overline{F}^*$. The combination of these inequalities yields $\underline{F} = \underline{F}_* = \overline{F} = \overline{F}^*$. Denote this function by $\hat{R}(\lambda)h$, which is therefore the unique viscosity solution to (5.7).

Following Proposition 3.24, define

$$\begin{aligned}&\mathcal{D}(R(\lambda))\\&\quad=\{h\in C_b(X)\,|\,\exists\, h_n\in B_n : \mathrm{LIM}\, h_n=h,\ \mathrm{LIM}\, R_n(\lambda)h_n \text{ exists and is continuous}\}\end{aligned}$$

and $R(\lambda)h = \mathrm{LIM}\, R_n(\lambda)h_n$. By the argument above, $R(\lambda)$ extends $\hat{R}(\lambda)$. By Proposition 3.24 $\mathcal{D}(R(\lambda))$ is quasi-closed, and as D is quasi-dense in $C_b(X)$ by assumption we find $\mathcal{D}(R(\lambda)) = C_b(X)$. In addition Proposition 3.24 $\mathcal{D}(R(\lambda))$ yields that $R(\lambda)$ is strictly continuous on bounded sets.

The pseudo-resolvent property follows by continuity from that of the operators R_n. The local strict continuity on bounded sets can be proven by making the estimates in the proof of Proposition 3.24 (a) uniform for λ with $0 < \lambda \leq \lambda_0$, using the uniform choice of $\hat{q}$ as in Condition 5.5 (d). □

6. Convergence of semigroups

Theorem 6.1. *Let Condition 5.5 be satisfied. Suppose in addition that for all n we have a collection of functions B_n such that: $C_b(X_n) \subseteq B_n \subseteq M(X_n)$ and suppose that $R_n(\lambda)C_b(X_n) \subseteq C_b(X_n)$.*

Let $D \subseteq C_b(X)$ be quasi-dense in $C_b(X)$. Suppose that for each $\lambda > 0$ and $h \in D$ the comparison principle holds for

$$f - \lambda H_\dagger f = h, \qquad f - \lambda H_\ddagger f = h.$$

Denote by $R(\lambda) : C_b(X) \to C_b(X)$ the operators constructed in Theorem 5.7

Consider the operators

$$\widehat{H}_n := \bigcup_\lambda \left\{ \left(R_n(\lambda)h, \frac{R_n(\lambda)h - h}{\lambda} \right) \middle|\, h \in C_b(X_n) \right\}, \tag{6.1}$$

$$\widehat{H} := \bigcup_\lambda \left\{ \left(R(\lambda)h, \frac{R(\lambda)h - h}{\lambda} \right) \middle|\, h \in C_b(X) \right\}, \tag{6.2}$$

as in Proposition 4.10.

Let $V_n(t)$ and $V(t)$ be the operator semigroups on the uniform closures of $\mathcal{D}(\widehat{H}_n)$ and $\mathcal{D}(\widehat{H})$ generated by $\widehat{H}_n$ and $\widehat{H}$ as in the Crandall-Ligget theorem, see Theorem 3.3.

Suppose that the semigroups $V_n(t)$ are strictly equi-continuous on bounded sets: for all $q \in \mathcal{Q}$, $\delta > 0$ and $t_0 > 0$, there is a $\hat{q} \in \mathcal{Q}$ such that for all n and $h_1, h_2 \in B_n$ and $0 \leq t \leq t_0$ that

$$\sup_{y \in K_n^q} \{V_n(t)h_1(y) - V_n(t)h_2(y)\} \leq \delta \sup_{x \in X_n} \{h_1(x) - h_2(x)\} + \sup_{y \in K_n^{\hat{q}}} \{h_1(y) - h_2(y)\}.$$

Denote by $\mathcal{D}$ the quasi-closure of $\mathcal{D}(\widehat{H})$. Then

(a) *We have* $\widehat{H} \subseteq ex - \mathrm{LIM}\, \widehat{H}_n$ *as in Definition 3.15. That is, for all* $(f,g) \in \widehat{H}$ *there are* $(f_n, g_n) \in \widehat{H}_n$ *such that* $\mathrm{LIM}\, f_n = f$ *and* $\mathrm{LIM}\, g_n = g$.
(b) *The semigroup* $V(t)$ *extends to the quasi-closure* $\mathcal{D}$ *of* $\mathcal{D}(\widehat{H})$ *on which it is locally strictly equi-continuous on bounded sets.*
(c) *For each* $f \in \mathcal{D}$ *there are* f_n *in the uniform closures of* $\mathcal{D}(\widehat{H}_n)$ *such that* $\mathrm{LIM}\, f_n = f$.
(d) *If* f_n *are in the uniform closures of* $\mathcal{D}(\widehat{H}_n)$ *and* $f \in \mathcal{D}$ *such that* $\mathrm{LIM}\, f_n = f$ *and* $t_n \to t$ *then* $\mathrm{LIM}\, V_n(t_n) f_n = V(t) f$.

Remark 6.2. For applications it is of interest to know whether $\mathcal{D}(\widehat{H})$ is quasi-dense in $C_b(X)$. If the resolvent is obtained as in Theorem 5.7 and $\mathrm{LIM}\, R_n(\lambda)h = h$ as $\lambda \downarrow 0$ for all n, then this can sometimes be established directly from an approximation procedure, see for example Lemma 7.19 in [22]. This is indeed what one would expect from a Crandall-Liggett theorem for the strict topology. Another possibility is to find an explicit expression for the resolvent and verify this property directly. We will pursue a third possibility below, see Proposition 7.1, that is based on a comparison principle.

The main step to go from the result of Theorem 5.7 to that of above theorem is an approximation argument by Kurtz: Theorem 3.2 of [49]. The key argument in the approximation result is the embedding of all spaces and semigroups in a common product space. The notion of LIM is embedded into this product space as a closed subspace. We study these spaces in next proposition.

Proposition 6.3. *Let Assumptions 3.11 and 3.13 be satisfied. The space*

$$\mathfrak{L} := \left\{ \langle f, \{f_n\} \rangle \,\middle|\, f_n \in M_b(X_n), f \in M_b(X), \sup_n \|f_n\| < \infty \right\},$$

equipped with the norm $\|\langle f, \{f_n\}\rangle\| = \|f\| \vee \sup_n \|f_n\|$ *is a Banach space. Set*

$$\mathfrak{P} := \left\{ (\langle f, \{f_n\}\rangle, f) \in \mathfrak{L} \times M_b(X) \,|\, \mathrm{LIM}\, f_n = f \right\}.$$

The set $\mathfrak{P}$ *is a closed linear subspace* $\mathfrak{L} \times M_b(X)$ *and* $\mathfrak{P}$ *interpreted as an operator from* $\mathfrak{L}$ *to* $M_b(X)$ *satisfies* $\|\mathfrak{P}\| \leq 1$.

In the proposition, we do not consider for which f there are f_n such that $f = \mathrm{LIM}\, f_n$. We assume this e.g. in Condition 5.5 (a). In particular cases, however, surjectivity of $\mathfrak{P}$ can be established directly.

Lemma 6.4. *Suppose that* $\mathcal{X}$ *is a normal space and that the maps* $\eta_n : X_n \to \mathcal{X}$ *are continuous and that* $\eta : X \to \mathcal{X}$ *is a homeomorphism onto its image. Then for each* $f \in M_b(X)$ *there are* $f_n \in M_b(X_n)$ *such that* $\mathrm{LIM}\, f_n = f$. *If* $f \in C_b(X)$, *then* f_n *can be chosen in* $C_b(X_n)$.

Proof. First let $f \in C_b(X)$. The function $g := f \circ \eta^{-1}$ is a continuous function with norm $\|f\|$ on $\eta(X) \subseteq \mathcal{X}$. By the Tietze extension Theorem, it extends to a continuous function g on $\mathcal{X}$ with norm $\|g\| = \|f\|$. We then define $f_n := g \circ \eta_n$, which leads to $\|f_n\| \leq \|f\|$. Next, let $x_n \in K_n^q$ and $x \in K^q$ such that $\eta_n(x_n) \to \eta(x)$. Because g is continuous, we find that $g(\eta_n(x_n)) \to g(\eta(x))$ implying that $f_n(x_n) \to f(x)$. Thus $\mathrm{LIM}\, f_n = f$. The result for $M_b(X)$ then follows from the monotone class theorem, see e.g. Theorem 2.12.9 in [3]. $\square$

Remark 6.5. Note that in the use of the Monotone class theorem, we establish the result for functions that are bounded and measurable with respect to the σ-algebra generated by all bounded and continuous functions. For a general topological space this implies the final result holds for the set of bounded and measurable functions with respect to the Baire σ-algebra. In the case that $\mathcal{X}$ is Polish, the Baire and Borel σ algebra's coincide. More general, this holds for perfectly normal spaces, see Proposition 6.3.4 in [3].

Proof of Proposition 6.3. That $\mathfrak{L}$ is a Banach space, as that $\mathfrak{P}$ is linear, is immediate. We establish that $\mathfrak{P}$ is norm closed. Let $f_n^k, f_n \in M_b(X_n)$ and $f^k, f \in M_b(X)$ such that for all k: $\mathrm{LIM}\, f_n^k = f^k$ and $\lim_k \left(\|f - f^k\| \vee \sup_n \|f_n^k - f_n\|\right) = 0$. We prove $\mathrm{LIM}\, f_n = f$.

First of all, let k be such that $\sup_n \|f_n^k - f_n\| \leq 1$. Then $\|f_n\| \leq \|f_n - f_n^k\| + \|f_n^k\| \leq 1 + \|f_n^k\|$. The final term is bounded as $\mathrm{LIM}\, f_n^k = f^k$. For the second property, fix $q \in \mathcal{Q}$ and $x_n \in K_n^q$ converging to $x \in K^q$. We have

$$
\begin{aligned}
|f_n(x_n) - f(x)| &\leq \left|f_n(x_n) - f_n^k(x_n)\right| + \left|f_n^k(x_n) - f^k(x)\right| + \left|f^k(x) - f(x)\right| \\
&\leq \sup_m \left\|f_m - f_m^k\right\| + \left|f_n^k(x_n) - f^k(x)\right| + \left\|f^k - f\right\|.
\end{aligned}
$$

The first and third term on the right-hand side can be made arbitrarily small by choosing k large. For fixed k the term final term converges to 0 as $\mathrm{LIM}_n\, f_n^k = f^k$. Thus, we find $f_n(x_n) \to f(x)$. Contractivity of $\mathfrak{P}$ follows by assumption. $\square$

The proof of the theorem is based on a general semigroup approximation result [49, Theorem 3.2].

Proof of Theorem 6.1. For the proof of (a), pick $f \in \mathcal{D}(\widehat{H})$. By definition there are $\lambda > 0$ and $h \in D$ such that $f = R(\lambda)h$. By the assumption in Theorem 5.7, there are $h_n \in B_n \cap M_b(X_n)$ such that $\mathrm{LIM}\, h_n = h$. By Theorem 5.7, we obtain $\mathrm{LIM}\, R_n(\lambda)h_n = R(\lambda)h = f$. By construction $R_n(\lambda)h_n \in \mathcal{D}(\widehat{H}_n)$ establishing (a).

We proceed with the proof of (b), (c) and (d) for which we will use Theorem 3.2 of [49]. Recall the set $\mathfrak{L}$ and the closed subset $\mathfrak{P}$ of Proposition 6.3. Denote also

$$
\mathcal{H} := \left\{ (\langle f, \{f_n\}\rangle, \langle g, \{g_n\}\rangle) \in \mathfrak{L} \times \mathfrak{L} \,\middle|\, (f_n, g_n) \in \widehat{H}_n, (f, g) \in \widehat{H} \right\}.
$$

Note that $\mathcal{H}$ is dissipative and satisfies the range condition because the operators $\widehat{H}_n$ and $\widehat{H}$ do as well. The semigroup $\mathcal{V}(t)$ generated by $\mathcal{H}$ equals $\mathcal{V}(t)\left(\langle f, \{f_n\}\rangle\right) =$

$\langle V(t)f, \{V_n(t)f_n\}\rangle$ on the uniform closure in $\mathfrak{L}$ of $\mathcal{D}(\widehat{H}) \times \prod_n \mathcal{D}(\widehat{H}_n)$ (which might be smaller than the product over the uniform closures).

By (a), we have

$$\widehat{H} = \{(f,g) \,|\, ((\langle f, \{f_n\}\rangle, f), (\langle g, \{g_n\}\rangle, g)) \in \mathcal{H} \cap (\mathcal{D}(\mathfrak{P}) \times \mathcal{D}(\mathfrak{P}))\},$$

so that all conditions for Theorem 3.2 of [49] are satisfied. From Equation (3.4) in [49], we infer that if $t \geq 0$, $f_n \in \mathcal{D}(\widehat{H}_n)$ and $f \in \mathcal{D}(\widehat{H})$ such that $\mathrm{LIM}\, f_n = f$, then $\mathrm{LIM}\, V_n(t)f_n = V(t)f$.

Fix $t > 0$ and define

$$\begin{aligned} &\mathcal{D}(V(t)) := \\ &\quad \{h \in C_b(X) \,|\, \exists\, h_n \in B_n : h = \mathrm{LIM}\, h_n, \mathrm{LIM}\, V_n(t)h_n \text{ exists and is continuous}\}. \end{aligned} \tag{6.3}$$

By the argument above $\mathcal{D}(V(t))$ contains $\mathcal{D}(\widehat{H})$. By Proposition 3.24 the set $\mathcal{D}(V(t))$ is quasi-closed and the operator $V(t)$ extends to $\mathcal{D}(V(t))$ on which it is strictly continuous on bounded sets.

Thus, the quasi-closure $\mathcal{D}$ of $\mathcal{D}(\widehat{H})$ is contained in $\mathcal{D}(V(t))$ for all t. Thus, (b), (c) and (d) (for $t_n = t$) all follow from Proposition 3.24.

We now extend (d) to the context of t_n converging to t. Thus, let $t_n \to t$, $f_n \in \mathcal{D}(\widehat{H}_n)$ and $f \in \mathcal{D}(\widehat{H})$ such that $\mathrm{LIM}\, f_n = f$. We have seen above that $\mathrm{LIM}\, V_n(t)f_n = V(t)f$. Using the decomposition

$$V(t)f - V_n(t_n)f_n = [V(t)f - V_n(t)f_n] + [V_n(t)f_n - V_n(t_n)f_n]$$

and the uniform continuity of $\mathcal{V}(t)$ on $\mathcal{D}(\widehat{H}) \times \prod_n \mathcal{D}(\widehat{H}_n)$ we find that also $\mathrm{LIM}\, V_n(t_n)f_n = V(t)f$.

Repeating the argument above for

$$\begin{aligned} &\mathcal{D}_{\{t_n\}}(V(t)) := \\ &\quad \{h \in C_b(X) \,|\, \exists\, h_n \in B_n : h = \mathrm{LIM}\, h_n, \mathrm{LIM}\, V_n(t_n)h_n \text{ exists and is continuous}\}, \end{aligned} \tag{6.4}$$

we find by Proposition 3.24 a second extension of $V(t)$ with the correct properties. However, as we have seen the extensions based on (6.3) and (6.4) agree on the quasi-dense subset $\mathcal{D}(\widehat{H}) \subseteq \mathcal{D}$ and therefore must be the same on $\mathcal{D}$. This establishes (d). □

7. Density of the domain

In Theorem 6.1, we obtained a semigroup that was defined on the quasi-closure of $\mathcal{D}(\widehat{H})$. In applications, often it is of interest to know whether this quasi-closure is in fact equal to $C_b(X)$.

A key method to verify this quasi-density, is the verification that as $\lambda \downarrow 0$, we have $\mathrm{LIM}\, R(\lambda)h = h$ for the buc topology. For this there are two possible strategies:

1. One finds a explicit characterization of R, i.e. a control representation, and verifies this property directly,
2. In the context of Theorem 5.7, one knows that $\lambda \downarrow 0$, we have $\mathrm{LIM}\, R_n(\lambda)h = h$ for each n and h and establishes that such statements can be lifted to the limit, see e.g. Lemma 7.19 in [22].

We will introduce a new method, that bootstraps the procedure of Section 5.

We proceed with an informal discussion. Consider the setting in which $R(\lambda)h$ is the viscosity solution to $f - \lambda H f = h$. In the linear theory, it is generally known that $R(\lambda)h \to h$ as $\lambda \downarrow 0$ in an appropriate topology. Indeed, as $R(\lambda)h \in \mathcal{D}(H)$, this establishes density of $\mathcal{D}(H)$. We expect the same result to hold true in the non-linear case. Consider the operators $A_n = \frac{1}{n} H$ and resolvents $\mathcal{R}_n(\lambda) = R\left(\frac{\lambda}{n}\right)$. Formally, the operator A_n converges to the zero-operator $0 \cdot H$, so that we expect that the relaxed lim sup and lim inf of $\mathcal{R}_n(1)h$ yield a viscosity sub- and supersolution to $f - 0 \cdot Hf = h$, or informally written, to $f = h$. Clearly, we expect these limits to equal h. To obtain this result rigorously, we need a comparison principle.

Informally, we need that $\mathcal{D}(H)$ that is 'large enough to uniquely identify functions'.

We make this intuition rigorous.

Proposition 7.1. *Let X be a space with metrizable compact sets and let $H_\dagger \subseteq C_b(X) \times C_b(X)$ and $H_\ddagger \subseteq C_b(X) \times C_b(X)$. Let $R(\lambda) : C_b(X) \to C_b(X)$ be a collection of operators that is locally strictly equi-continuous on bounded sets as in Definition 4.2.*

Let D be a quasi-dense set in $C_b(X)$ and suppose that for $\lambda > 0$ and $h \in D$, the function $R(\lambda)h$ is a viscosity solution to

$$f - \lambda H_\dagger f = h, \qquad f - \lambda H_\ddagger f = h.$$

Denote $A_\dagger := 0 \cdot \mathcal{H}_\dagger$ and $A_\ddagger := 0 \cdot \mathcal{H}_\ddagger$. Suppose that the comparison principle holds for $f - A_\dagger f = h_1$ and $f - A_\ddagger f = h_2$ for $h_1, h_2 \in D$. Let $\lambda_n \downarrow 0$. Then for all $h \in D$ we have $\mathrm{LIM}_n\, R(\lambda_n)h = h$. In particular, the domain $\mathcal{D}(\widehat{H})$ is quasi-dense in $C_b(X)$.

Remark 7.2. Generally, proofs that establish the comparison principle for $f - \lambda H_\dagger f = h$ and $f - \lambda H_\ddagger f = h$ can be adapted in a straightforward way to also establish the comparison principle for $f - \lambda A_\dagger f = h$ and $f - \lambda A_\ddagger f = h$.

We start by proving the seemingly trivial fact that h solves the equation $f = h$.

Lemma 7.3. *Let $H_\dagger \subseteq LSC_l(X) \times USC_u(X)$ and $H_\ddagger \subseteq USC_u(X) \times LSC_l(X)$ and define $A_\dagger := 0 \cdot \mathcal{H}_\dagger$ and $A_\ddagger := 0 \cdot \mathcal{H}_\ddagger$.*

For any $h \in C_b(X)$ the function h is a subsolution to $f - A_\dagger f = h$ and a supersolution to $f - A_\ddagger f = h$.

Proof. We show that h is a viscosity subsolution to $f - A_\dagger f = h$ and a supersolution to $f - A_\ddagger f = h$.

We establish that h is a viscosity subsolution of $f - A_\dagger f = h$ by using Lemma 4.5 (a). Let $(f, g) \in A_\dagger$. Thus $g = 0 \cdot \hat{g}$ with $(f, \hat{g}) \in H_\dagger$. Note that $\hat{g}$ is bounded from above. Thus $0 \cdot \hat{g} \leq 0$. Thus, for all $\varepsilon > 0$ we have

$$\sup_x h(x) - f(x) \leq \sup_x h(x) - f(x) - \varepsilon\left((h(x) - h(x)) - 0 \cdot \hat{g}(x)\right).$$

This inequality, in combination with the fact that f is bounded from below, implies that the condition of Lemma 4.5 (a) is satisfied (note that the g's in the Lemma and here are different). As a consequence, we find $x_n \in X$ such that

$$\lim_n h(x_n) - f(x_n) = \sup_x h(x) - f(x)$$

$$\limsup_n h(x_n) - f(x_n) - 0 \cdot \hat{g}(x_n) \leq 0,$$

that is, h is a subsolution to $f - A_\dagger f = h$. Similarly, we prove that h is a supersolution to $f - A_\ddagger f = h$, which concludes the proof. □

In the proof below, the notion of LIM refers to buc convergence. Thus, $\mathcal{Q}$ is the set of compact sets $\mathcal{K}$ in X with $K_n^{\mathcal{K}} = K^{\mathcal{K}} = \mathcal{K}$. See Example 3.16. Note that we need metrizable compacts to extract converging subsequences from sequences in compact sets.

Proof of Proposition 7.1. If we can establish that $\mathrm{LIM}\, R(\lambda_n) h = h$, then we have that $\mathcal{D}(\widehat{H})$ is quasi-dense in D. As D is quasi-dense in $C_b(X)$ by assumption, this establishes the claim.

By Lemma 7.3 and uniqueness of viscosity solutions, it suffices to apply Proposition 5.8 for $A_{n,\dagger} = \lambda_n H_\dagger$, $A_{n,\ddagger} := \lambda_n H_\ddagger$ and $\mathcal{R}_n := R(\lambda_n)$.

Doing so, we obtain that $\mathcal{R}_n h$ is a viscosity subsolution to $f - A_{n,\dagger} f = h$ and a supersolution to $f - A_{n,\ddagger} f = h$. Thus, it suffices to verify Condition 5.5.

We work with $B_n = C_b(X)$. Then (a) is immediate. Condition (c) follows by assumption. Also (d) is immediate by local strict equi-continuity on bounded sets of the resolvent $R(\lambda)$.

Next, we establish (b), i.e.:

$$A_\dagger \subseteq ex - \mathrm{LIM\,SUP}\, A_{n,\dagger}, \qquad A_\ddagger \subseteq ex - \mathrm{LIM\,INF}\, A_{n,\ddagger}.$$

We only prove the first claim. Suppose $(f, g) \in A_\dagger$. Then there is a $\hat{g}$ such that $g = 0 \cdot \hat{g}$ and $(f, \hat{g}) \in H_\dagger$. Set $f_n = f$ and $g_n = \lambda_n \cdot \hat{g}$. It follows that $(f_n, g_n) \in A_{n,\dagger}$. It is immediate that $\mathrm{LIM}\, f_n \wedge c = f$ for all c and as $\hat{g}$ is bounded above also $\sup_n \sup_x g_n(x) \leq 0 \vee \sup_x g(x) < \infty$. Finally, we establish (5.3). Note that as $x_n \to x$ in some compact set $\mathcal{K} \subseteq X$, we have

$$\limsup_n \widehat{g}(x_n) \leq \widehat{g}(x)$$

as $\widehat{g}$ is upper semi-continuous. It follows that

$$\limsup_n g_n(x_n) = \limsup_n \lambda_n \widehat{g}(x_n) \leq 0 \cdot g(x).$$

Thus, we conclude by Proposition 5.5 and the comparison principle that $\mathrm{LIM}\, \mathcal{R}_n h$ as $n \to \infty$ is the unique viscosity solution h to $f - A_\dagger f = h$ and $f - A_\ddagger f = h$. $\square$

8. Two applications

In this section, we consider two one-dimensional examples that are meant to illustrate working with embedding maps $\eta_n, \hat{\eta}_n$ that are not the identity map. They are not meant to introduce new results, nor to illustrate the most advanced uses of the results of this paper, but to give an idea of how to apply the techniques.

In particular, we consider two distinct settings, in the first one we consider the convergence of semigroups that correspond to the homogenization of a one dimensional first-order Hamilton-Jacobi equation. In the second case, we consider the semigroups corresponding the large deviations of a discrete mean-field model.

8.1. A homogenisation result

Consider the semigroups corresponding to the equations

$$\partial_t f(t,x) - a_\varepsilon(x) \left(\partial_x f(t,x)\right)^2 = 0, \qquad t \geq 0, x \in \mathbb{R}. \tag{8.1}$$

Here a is roughly of the form $a_\varepsilon(x) = a_1(x) + a_2(x/\varepsilon)$, where a_2 is one-periodic. We assume a_ε to be uniformly bounded and bounded away from 0 in x and ε. It is known [37], that these semigroups converge as $\varepsilon \downarrow 0$ to the solution of the homogenized equation

$$\partial_t f(t,x) - \overline{a}(x)(\partial_x f(t,x))^2 = 0, \qquad t \geq 0, x \in \mathbb{R}, \tag{8.2}$$

where $\overline{a}$ can be determined exactly, see Theorem 8.1 below, due to the on-dimensional nature of the problem.

We consider this problem from the point of view of the techniques of this paper (Choosing e.g. $\varepsilon = 1/n$). We thus turn our view towards the convergence of viscosity solutions of the Hamilton-Jacobi equations

$$f - \lambda H_\varepsilon f = h, \quad \text{ to the solution of } \quad f - \lambda \overline{H} f = h \tag{8.3}$$

where $H_\varepsilon, \overline{H} \subseteq C_b(\mathbb{R}) \times \mathbb{R}$ with domains $C_c^2(\mathbb{R})$ given by $H_\varepsilon f(x) = \mathcal{H}_\varepsilon(x, f'(x))$ and $\overline{H} f(x) = \overline{\mathcal{H}}(x, f'(x))$

$$\mathcal{H}_\varepsilon(x,p) = \frac{1}{2} a_\varepsilon(x) p^2, \qquad \overline{\mathcal{H}}(x,p) = \frac{1}{2}\overline{a}(x)p^2. \tag{8.4}$$

Standard control theory, see e.g. [2] or Chapter 8 of [22], yields that the associated semigroups and pseudo-resolvents are given in terms of Lagrangians obtained by taking a Legendre transform of the Hamiltonians $\mathcal{H}_\varepsilon, \mathcal{H}$:

$$\mathcal{L}_\varepsilon(x,v) = \frac{1}{2a_\varepsilon(x)} v^2, \qquad \mathcal{L}_\varepsilon(x,v) = \frac{1}{2\overline{a}(x)} v^2.$$

Let $\mathcal{AC}$ be the space of absolutely continuous trajectories on $\mathbb{R}$. We then have

$$V_\varepsilon(t) f(x) := \sup_{\gamma \in \mathcal{AC}, \gamma(0)=x} f(\gamma(t)) - \int_0^t \mathcal{L}_\varepsilon(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s,$$

$$V(t) f(x) := \sup_{\gamma \in \mathcal{AC}, \gamma(0)=x} f(\gamma(t)) - \int_0^t \mathcal{L}(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s,$$

and

$$R_\varepsilon(\lambda) h(x) := \sup_{\gamma \in \mathcal{AC} : \gamma(0) = x} \left\{ \int_0^\infty \lambda^{-1} e^{-\lambda^{-1} t} \left[h(\gamma(t)) - \int_0^t \mathcal{L}_\varepsilon(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s \right] \mathrm{d}t \right\},$$

$$R(\lambda) h(x) := \sup_{\gamma \in \mathcal{AC} : \gamma(0) = x} \left\{ \int_0^\infty \lambda^{-1} e^{-\lambda^{-1} t} \left[h(\gamma(t)) - \int_0^t \mathcal{L}(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s \right] \mathrm{d}t \right\}.$$

In particular, we have that $R_\varepsilon(\lambda)h$ and $R(\lambda)h$ are viscosity solutions to the two equations in (8.3) respectively. Working e.g. along $\varepsilon_n := 1/n$, we will establish the following result. Here $S^1 := \mathbb{R}/\mathbb{Z}$ is the 1d torus.

Theorem 8.1. *Let H_ε and $\overline{H}$ be given as in (8.4). Suppose that there exists a function $a \in C(\mathbb{R} \times S^1)$ that is bounded and bounded away from 0 such that for any sequence $x_n \in \mathbb{R}$ and $(x,z) \in \mathbb{R} \times S^1$ satisfying $(x_n, x_n/\varepsilon_n \text{ mod } 1) \to (x,z)$ we have*

$$\lim_n a_{\varepsilon_n}(x_n) = a(x,z). \tag{8.5}$$

Then we have the following convergence statements. Let $f_n, f \in C_b(\mathbb{R})$ such that $buc - \lim f_n = f$. Furthermore, let $\lambda, t > 0$.

Then $buc - \lim R_{\varepsilon_n}(\lambda) f_n = R(\lambda) f$ and $buc - \lim V_{\varepsilon_n}(t) f_n = V(t) f$.

To prove this result with the framework of Theorem 2.1, we need to find appropriate functions f_ε converging to f, so that also $H_\varepsilon f_\varepsilon$ converges. This can be done by finding the so-called corrector and using the perturbed test function method, see e.g. [37]. To apply this method directly, however, one needs to establish regularity of the corrector. It turns out that we can avoid establishing regularity by performing an analysis on the basis of 5.7 and 6.1. The use of the corrector is then postponed to the proof of the comparison principle.

We analyze the limiting behavior of H_ε. Let $f \in C_c^1(\mathbb{R})$ and $\phi \in C^1(S^1)$. Set $f_\varepsilon(x) = f(x) + \varepsilon\phi(x/\varepsilon)$. Then

$$H_\varepsilon f_\varepsilon(x) = \frac{1}{2} a_\varepsilon(x) \left(f'(x) + \phi'(x/\varepsilon)\right)^2, \tag{8.6}$$

and a limiting result for $H_\varepsilon f_\varepsilon$ is immediate from (8.5) if the limit is expressed in terms of the extended set of variables $(x, x/\varepsilon)$.

This motivates working in the context of Assumptions 3.26 and 3.27. In this context, we can identify $X = \mathcal{X} = \mathbb{R}$ and $Y = \mathcal{Y} = \mathbb{R} \times S^1$ ($\eta, \widehat{\eta}$ both being the identity map) and work with $\eta_n : \mathbb{R} \to \mathbb{R}$, $\eta_n(x) = x$ and $\widehat{\eta}_n(x) = (x, x/\varepsilon_n)$.

For Assumption 3.27, we will work with a simple variant of the set-up of Example 3.29. We take $\mathcal{Q}$ to be the set

$$\mathcal{Q} := \{\mathcal{K} \,|\, \mathcal{K} \subseteq \mathbb{R} \text{ compact}\},$$

and $K_n^{\mathcal{K}} = K^{\mathcal{K}} = \mathcal{K}$, $\widehat{K}^{\mathcal{K}} = \mathcal{K} \times S^1$. This way the notion LIM equals $buc - \lim$, i.e. the notion of bounded and uniform convergence on compact sets.

The formula in (8.6) motivates the definition of the following limiting operator H. For $f \in C_c^1(\mathbb{R}), \phi \in C^1(S^1)$, set

$$G_{f,\phi}(x,z) := \frac{1}{2} a(x,z) \left[f'(x) + \phi'(z)\right]^2,$$

and

$$H := \left\{(f, G_{f,\phi}) \in C_b(\mathbb{R}) \times C_b(\mathbb{R} \times S^1) \,\middle|\, f \in C_c^1(\mathbb{R}), \phi \in C^1(S^1)\right\},$$

so that we immediately obtain $H \subseteq ex - \mathrm{LIM}\, H_\varepsilon$.

For the application of Theorem 5.7 and 6.1, we need to verify the comparison principle for H, and verify that we have the appropriate conditions for convergence. We start with the latter.

Lemma 8.2. *Condition 5.5 is satisfied with $B_n = C_b(\mathbb{R})$. We also have for any compact set $K \subseteq \mathbb{R}$, $\delta > 0$, $t_0 > 0$, that there exists a compact set $\widehat{K} \subseteq \mathbb{R}$ such that for all $h_1, h_2 \in C_b(\mathbb{R})$ and $\varepsilon > 0$ and t such that $0 \leq t \leq t_0$:*

$$\sup_{x \in K} V_\varepsilon(t)h_1(x) - V_\varepsilon(t)h_2(x) \leq \delta \sup_y \{h_1(y) - h_2(y)\} + \sup_{y \in \widehat{K}} \{h_1(y) - h_2(y)\}.$$

The proof of the joint equi-coercivity estimates for the resolvents and semigroups will make use of a Lyapunov argument. As we will use the Lyapunov function at various later stages, we introduce it here for later reference. Set

$$\Upsilon(x) = \log\left(1 + \frac{1}{2}x^2\right) \tag{8.7}$$

and note that

$$c_\Upsilon := \sup_\varepsilon \sup_x \frac{1}{2} a_\varepsilon(x)(\Upsilon'(x))^2 < \infty. \tag{8.8}$$

Proof of Lemma 8.2. We start with the verification of Condition 5.5. (a) is immediate, (b) has just been established and (c) follows from general theory on viscosity solutions, see e.g. [2, Section 3.2] or [22, Chapter 8]. We prove (d) using a Lyapunov function technique.

Let $h_1, h_2 \in C_b(\mathbb{R})$, fix a compact set $K \subseteq \mathbb{R}$, $\delta > 0$ and $\lambda_0 > 0$, and λ such that $0 < \lambda \leq \lambda_0$. Finally, let γ_x be the optimizer for $R_\varepsilon(\lambda)h_1(x)$. We find that

$$\sup_{x \in K} R_\varepsilon(\lambda)h_1(x) - R_\varepsilon(\lambda)h_2(x) \leq \int_0^\infty \lambda^{-1} e^{-\lambda^{-1}t} \left[h_1(\gamma_x(t)) - h_2(\gamma_x(t))\right] \mathrm{d}t. \tag{8.9}$$

Choose T such that

$$\sup_{0<\lambda\leq\lambda_0} \int_T^\infty \lambda^{-1} e^{-\lambda^{-1}t} \mathrm{d}t \leq \delta.$$

Inserting this estimate into (8.9), we find

$$\begin{aligned} &\sup_{x \in K} R_\varepsilon(\lambda)h_1(x) - R_\varepsilon(\lambda)h_2(x) \\ &\qquad \leq \delta \sup_y \{h_1(y) - h_2(y)\} + \int_0^T \lambda^{-1} e^{-\lambda^{-1}t} \left[h_1(\gamma_x(t)) - h_2(\gamma_x(t))\right] \mathrm{d}t. \end{aligned} \tag{8.10}$$

We now analyze the time integral on the right-hand side. Note that for every $\varepsilon > 0$, the path that does not move has 0 Lagrangian cost. Thus,

$$R_\varepsilon(\lambda)h_1(x) \geq h_1(x),$$

which implies that the optimizer γ_x for $R_\varepsilon(\lambda)h_1(x)$ satisfies

$$\int_0^\infty \lambda^{-1} e^{-\lambda^{-1}t} \int_0^t \mathcal{L}_\varepsilon(\gamma_x(s), \dot{\gamma}_x(s)) \mathrm{d}s \, \mathrm{d}t \leq 2 \|h_1\|.$$

As $\mathcal{L}_\varepsilon \geq 0$ and outer integral is taken over a measure with mass 1, we also find that

$$\int_0^t \mathcal{L}_\varepsilon(\gamma_x(s), \dot{\gamma}_x(s)) \mathrm{d}s \leq 2 \|h_1\|$$

for all $t \geq 0$ and $\varepsilon > 0$. Recall Υ defined in (8.7) and the estimate (8.8). It follows from the Fenchel-Young inequality that for all $t \leq T$:

$$\begin{aligned}\Upsilon(\gamma_x(t)) &= \Upsilon(\gamma_x(0)) + \int_0^t \langle \dot{\gamma}_x(s), \Upsilon'(\gamma_x(s)) \rangle \mathrm{d}s \\ &\leq \Upsilon(\gamma_x(0)) + tc_\Upsilon + \int_0^t \mathcal{L}_\varepsilon(\gamma_x(s), \dot{\gamma}_x(s)) \mathrm{d}s \\ &\leq \Upsilon(x) + Tc_\Upsilon + 2\|h_1\|.\end{aligned} \tag{8.11}$$

As K is compact, Υ is bounded on K. This implies that the set

$$\widehat{K} := \left\{ y' \mid \Upsilon(y') \leq \sup_{y \in K} \Upsilon(y) + Tc_\Upsilon + 2\|h_1\| \right\},$$

is a compact set. By the definition of $\hat{K}$ and (8.11), we find that for all $x \in K$, $\varepsilon > 0$, λ such that $0 < \lambda \leq \lambda_0$ and $t \leq T$ we have $\gamma_x(t) \in \widehat{K}$. Plugging this into (8.10), we obtain

$$\sup_{x \in K} R_\varepsilon(\lambda) h_1(x) - R_\varepsilon(\lambda) h_2(x) \leq \delta \sup_y \{h_1(y) - h_2(y)\} + \sup_{y \in \widehat{K}} \{h_1(y) - h_2(y)\}.$$

The joint equi-coercivity for the semigroups follows analogously. $\square$

The second key ingredient for the application of Theorems 5.7 and 6.1 is the comparison principle for the Hamilton-Jacobi equation $f - \lambda H f = h$.

Proposition 8.3. *The comparison principle holds for $f - \lambda H f = h$ and $f - \lambda \overline{H} f = h$ and the solutions, if they exist, to both equations are the same.*

The proof of this proposition is inspired by the methods of [22, Chapter 11] and [35] and is carried out by establishing the diagram of Fig. 1. The operators H and $\overline{H}$ are the ones that were introduced above, the further four operators are defined below and are constructed out of H and $\overline{H}$ by the addition of a Lyapunov function in the domain that takes care of localization arguments.

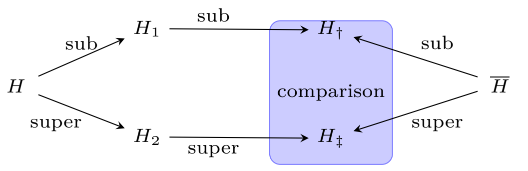

Fig. 1. An arrow connecting an operator A with operator B with subscript 'sub' means that viscosity subsolutions of $f - \lambda A f = h$ are also viscosity subsolutions of $f - \lambda B f = h$. Similarly for arrows with a subscript 'super'. The box around the operators $H_\dagger$ and $H_\ddagger$ indicates that the comparison principle holds for subsolutions of $f - \lambda H_\dagger f = h$ and supersolutions of $f - \lambda H_\ddagger f = h$.

Let $f \in C_b^1(\mathbb{R})$, $\phi \in C_b^1(S^1)$, $\eta \in (0,1)$, set

$$\begin{aligned}
f^\dagger(x) &:= (1-\eta) f(x) + \eta \Upsilon(x), \\
G^\dagger_{f,\phi,\eta}(x,z) &:= (1-\eta)\left(\frac{1}{2} a(x,z)\left[f'(x) + \phi'(z)\right]^2\right) + \eta c_\Upsilon, \\
f^\ddagger(x) &:= (1+\eta) f(x) - \eta \Upsilon(x), \\
G^\ddagger_{f,\phi,\eta}(x,z) &:= (1+\eta)\left(\frac{1}{2} a(x,z)\left[f'(x) + \phi'(z)\right]^2\right) - \eta c_\Upsilon,
\end{aligned}$$

and set

$$\begin{aligned}
H_1 &:= \left\{ (f^\dagger_\eta, G^\dagger_{f,\phi,\eta}) \,\middle|\, f \in C_b^1(\mathbb{R}), \phi \in C_b^1(S^1), \eta \in (0,1) \right\}, \\
H_2 &:= \left\{ (f^\ddagger_\eta, G^\ddagger_{f,\phi,\eta}) \,\middle|\, f \in C_b^1(\mathbb{R}), \phi \in C_b^1(S^1), \eta \in (0,1) \right\}.
\end{aligned}$$

Also set

$$\begin{aligned}
f^\dagger_\eta(x) &:= (1-\eta) f(x) + \eta \Upsilon(x), \\
g^\dagger_\eta(x) &:= (1-\eta) \overline{\mathcal{H}}(x, f'(x)) + \eta c_\Upsilon \\
f^\ddagger_\eta(x) &:= (1+\eta) f(x) - \eta \Upsilon(x), \\
g^\ddagger_\eta(x) &:= (1+\eta) \overline{\mathcal{H}}(x, f'(x)) - \eta c_\Upsilon,
\end{aligned}$$

and finally

$$\begin{aligned}
H_\dagger &:= \left\{ (f^\dagger_\eta, g^\dagger_\eta) \,\middle|\, f \in C_b^1(\mathbb{R}), \eta \in (0,1) \right\}, \\
H_\ddagger &:= \left\{ (f^\ddagger_\eta, g^\ddagger_\eta) \,\middle|\, f \in C_b^1(\mathbb{R}), \eta \in (0,1) \right\}.
\end{aligned}$$

The operator H and $\mathbf{H}$ are related to H_1, H_2 and $H_\dagger, H_\ddagger$ by the following two Lemma's respectively.

Lemma 8.4. *Fix $\lambda > 0$ and $h \in C_b(\mathbb{R})$.*

(a) *Every subsolution to* $f - \lambda H f = h$ *is also a subsolution to* $f - \lambda H_1 f = h$.
(b) *Every supersolution to* $f - \lambda H f = h$ *is also a supersolution to* $f - \lambda H_2 f = h$.

Lemma 8.5. *Fix* $\lambda > 0$ *and* $h \in C_b(\mathbb{R})$.

(a) *Every subsolution to* $f - \lambda \overline{H} f = h$ *is also a subsolution to* $f - \lambda H_\dagger f = h$.
(b) *Every supersolution to* $f - \lambda \overline{H} f = h$ *is also a supersolution to* $f - \lambda H_\ddagger f = h$.

Analogous results to Lemma's 8.4 and 8.5 has been proven in Lemma 3.3 of [35], so we refrain from carrying out the proofs here. We next establish the two horizontal arrows in Fig. 1, in which the corrector plays a key role.

Lemma 8.6. *Fix* $\lambda > 0$ *and* $h \in C_b(\mathbb{R})$.

(a) *Every subsolution to* $f - \lambda H_1 f = h$ *is also a subsolution to* $f - \lambda H_\dagger f = h$.
(b) *Every supersolution to* $f - \lambda H_2 f = h$ *is also a supersolution to* $f - \lambda H_\ddagger f = h$.

Remark 8.7. In the proof of the lemma, it essential that we can indeed find an optimizer x_0 for the definition of viscosity solutions, as this allows us to introduce the corrector ϕ_{x_0} in way that does away with establishing regularity on the corrector as a function of x. To find such an optimizer, we need compact level sets of the test functions. This motivates the construction of $H_1, H_2, H_\dagger, H_\ddagger$ using the Lyapunov function Υ.

Proof of Lemma 8.6. Let u be a subsolution to $f - \lambda H_1 f = h$. We show it is also a subsolution to $f - \lambda H_\dagger f = h$. Pick $f \in C_b^1(\mathbb{R})$ and $\eta \in (0,1)$.

First we show there exists x_0 such that

$$u(x_0) - f_\eta^\dagger(x_0) = \sup_x \left\{ u(x) - f_\eta^\dagger(x) \right\}. \tag{8.12}$$

First of all, note that u and $-f_\eta^\dagger$ are upper semi-continuous. As Υ has compact sub-level sets, there exists x_0 such that

$$u(x_0) - f_\eta^\dagger(x_0) = \sup_x \left\{ u(x) - f_\eta^\dagger(x) \right\}.$$

Next, let $\hat{f} \in C_b^1(\mathbb{R})$ be a smooth function such that $\hat{f}(x_0) = f(x_0)$ and $\hat{f}(x) > f(x)$ if $x \neq x_0$ so that x_0 is the unique optimizer in

$$u(x_0) - \hat{f}_\eta^\dagger(x_0) = \sup_x \left\{ u(x) - \hat{f}_\eta^\dagger(x_0) \right\} \tag{8.13}$$

and in addition $f'(x_0) = \hat{f}'(x_0)$.

Next, we consider the corrector. The corrector ϕ_{x_0} is such that

$$G_{\hat{f},\phi_{x_0},\eta}(x_0,z)$$

does not depend on z. This way, if we consider the subsolution property in (x_0,z), the constant z does not play any role. Due to the one-dimensional setting, we can solve for ϕ_{x_0}, and find

$$\phi_{x_0}(z) := -f'(x_0)\left[z - \frac{\int_0^z \frac{1}{a^{1/2}(x_0,y)}\mathrm{d}y}{\int_0^1 \frac{1}{a^{1/2}(x_0,y)}\mathrm{d}y}\right] \tag{8.14}$$

and

$$G_{\hat{f},\phi_{x_0},\eta}(x_0,z) = (1-\eta)\frac{1}{2}\overline{a}(x_0)(f'(x_0))^2 + \eta c_\Upsilon.$$

Note that $\phi_{x_0}(1) = \phi_{x_0}(0)$, so that indeed $\phi \in C^1(S^1)$.

As u is a subsolution to $f - \lambda H_\dagger f = h$, there are (x_n,z_n) such that

$$\lim_n u(x_n) - \hat{f}_\eta^\dagger(x_n) = \sup_x u(x) - \hat{f}_\eta^\dagger(x)$$

and

$$\limsup_n u(x_n) - \lambda G^\dagger_{\hat{f},\phi_{x_0},\eta}(x_n,z_n) - h(x_n) \leq 0.$$

As x_0 is the unique optimizer of $\sup(u - \hat{f}_\eta^\dagger)$, and as S^1 is compact, there exists $z_0 \in S^1$ such that along a subsequence we have $(x_n,z_n) \to (x_0,z_0)$. We conclude that

$$\begin{aligned}
&u(x_0) - \lambda\left((1-\eta)\frac{1}{2}\overline{a}(x_0)(f'(x_0))^2 + \eta c_\Upsilon\right)(x_0,z_0) - h(x_0) \\
&= u(x_0) - \lambda G^\dagger_{\hat{f},\phi_{x_0},\eta}(x_0,z_0) - h(x_0) \\
&\leq 0.
\end{aligned}$$

So that in combination with (8.12), we have obtained the two desired properties for each pair of functions in $H_\dagger$. We conclude that u is a subsolution to $f - \lambda H_\dagger f = h$. □

Proof of Proposition 8.3. By Lemma's 8.4, 8.5, and 8.6, viscosity subsolutions to $f - \lambda H f = h$ or $f - \lambda \overline{H} f = h$ are also viscosity subsolutions to $f - \lambda H_\dagger f = h$. A similar statement holds for the supersolutions.

The comparison principle for sub- and supersolutions for $f - \lambda H_\dagger f = h$ and $f - \lambda H_\ddagger f = h$ follows from standard comparison principles (See e.g. the proof of Theorem II.3.5 of [2]). Note that proofs are typically written down in terms of a single equation, but in the first step of the doubling-of-variables method, a penalization function is added to localize the argument. The penalization function is now already contained in the domains of $H_\dagger, H_\ddagger$, and the comparison principle proof does not change, see e.g. [7] where this different point of view is worked out. □

Proof of Theorem 8.1. By Lemma 8.2 and Proposition 8.3 all conditions for Theorems 5.7 and 6.1 are satisfied.

Thus, there exists a pseudo-resolvent $R_0(\lambda)$ and semigroup $V_0(t)$ such that $R_\varepsilon(t)$ and $V_\varepsilon(t)$ converge to $R_0(\lambda)$ and $V_0(t)$. What is left to prove is that $V_0(t) = V(t)$ and $R_0(\lambda) = R(\lambda)$. For the pseudo-resolvents this is immediate by Proposition 8.3. For the semigroups, note that by Theorem 3.3, the pseudo-resolvents $R_0(\lambda)$ approximate the semigroup $V_0(t)$. The same statement holds for the pseudo-resolvents $R(\lambda)$ and semigroup $V(t)$ by e.g. Lemma 8.18 in [22]. We conclude that as the pseudo-resolvents agree, the semigroups agree also.

This establishes the claim. □

8.2. Lumping together of variables: the dynamic Curie-Weiss model

We consider the dynamic Curie-Weiss model: a naive description for the microscopic causes of ferromagnetism. We refrain from giving a full description, but rather give the necessary definitions that are needed to follow the application of our main results. For a thorough discussion of the static Curie-Weiss model and related models, see e.g. [25]. The large deviation analysis of the dynamic model leads to a Hamilton-Jacobi equation, see also e.g. [14,30].

Let $X_n = \{-1,1\}^n$. We denote elements in X_n by $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ which are interpreted as 'spins' or magnetic moments. On X_n we consider a Markov process that we interpret as the thermal fluctuations of the spins. Let $\beta \geq 0$ be a parameter that models the inverse of the temperature.

On X_n, we define a Markov process $t \mapsto \sigma^n(t)$ that has infinitesimal generator $A_n \subseteq C_b(X_n) \times C_b(X_n)$ where for each $f \in C_b(X_n)$ we set

$$A_n f(\sigma) := \sum_{i=1}^n e^{-\beta \sigma_i m_n(\sigma)} \left[f(\sigma^i) - f(\sigma) \right] \tag{8.15}$$

where the magnetization m_n is defined by

$$m_n(\sigma) := \frac{1}{n} \sum_{i=1}^n \sigma_i$$

and where

$$\sigma_j^i = \begin{cases} -\sigma_j & \text{if } j = i, \\ \sigma_j & \text{if } j \neq i, \end{cases}$$

is the effect of changing the sign of the i-th spin.

Denote by $\mathbb{P}_\sigma^n$ the measure on the Skorokhod space $D_{X_n}(\mathbb{R}^+)$, see e.g. [16], of trajectories on X_n that are right-continuous and have limits, that describes the law of the Markov process $t \mapsto \sigma^n(t)$ when started in the configuration $\sigma^n(0) = \sigma$.

In the theory of large deviations for Markov processes, a key role is played by the semigroup of conditional log exponential generating functions:

$$V_n(t)f(\sigma) := \sup_{\mathbb{Q} \in \mathcal{P}(D_{X_n}(\mathbb{R}^+))} \left\{ \int f(\eta)\mathbb{Q}(\mathrm{d}\eta) - \frac{1}{n} S(\mathbb{Q} \,|\, \mathbb{P}_\sigma^n) \right\}$$

where $\mathcal{P}(D_{X_n}(\mathbb{R}^+))$ is the space of probability measures on $D_{X_n}(\mathbb{R}^+)$ and where S is the relative entropy defined by

$$S(\nu \,|\, \mu) = \begin{cases} \int \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \log \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\mu & \text{if } \nu \ll \mu, \\ \infty & \text{otherwise.} \end{cases}$$

It is shown in [32] that $V_n(t) : C_b(X_n) \to C_b(X_n)$ is a semigroup that corresponds to the pseudo-resolvent

$$R_n(\lambda)h(\sigma) = \sup_{\mathbb{Q} \in \mathcal{P}(D_{X_n}(\mathbb{R}^+))} \left\{ \int_0^\infty \lambda^{-1} e^{-\lambda^{-1} t} \left[\int f(\eta)\mathbb{Q}(\mathrm{d}\eta) - \frac{1}{n} S_{[0,t]}(\mathbb{Q} \,|\, \mathbb{P}_\sigma^n) \right] \mathrm{d}t \right\},$$

where $S_{[0,t]}$ is the relative entropy of the measures when they are restricted to the σ-algebra of information up to time t. In addition, it is shown in [32] that the pseudo-resolvent $R_n(\lambda)$ yields viscosity solutions to the Hamilton-Jacobi equations in terms of the Hamiltonians H_n given by

$$\left\{ \left(f, \frac{1}{n} e^{-nf} g \right) \,\middle|\, \left(e^{nf}, g \right) \in A_n \right\} \subseteq H_n \tag{8.16}$$

and where A_n is the operator defined in (8.15).

As $n \to \infty$, we aim to study the limiting behavior of the magnetization $t \mapsto m_n(t)$. This motivates us to introduce $X = \mathcal{X} = [-1, 1]$ and $\eta_n(x) : X_n \to X$ defined by $\eta_n(\sigma) = m_n(\sigma)$.

For $f \in C_b^2(X)$, we have

$$\begin{aligned} H_n(f \circ \eta_n)(\sigma) &= \frac{1}{n} \sum_{i=1}^n e^{-\beta \sigma_i m_n(\sigma)} \left[e^{n\left(f(m_n(\sigma^i)) - f(m_n(\sigma))\right)} - 1 \right] \\ &= \frac{1}{n} \sum_{i:\sigma_i = -1} e^{\beta m_n(\sigma)} \left[e^{n\left(f(m_n(\sigma) + \frac{2}{n}) - f(m_n(\sigma))\right)} - 1 \right] \\ &\quad + \frac{1}{n} \sum_{i:\sigma_i = 1} e^{-\beta m_n(\sigma)} \left[e^{n\left(f(m_n(\sigma) - \frac{2}{n}) - f(m_n(\sigma))\right)} - 1 \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1-m_n(\sigma)}{2} e^{\beta m_n(\sigma)} \left[e^{n\left(f(m_n(\sigma)+\frac{2}{n})-f(m_n(\sigma))\right)} - 1 \right] \\
&\qquad + \frac{1+m_n(\sigma)}{2} e^{-\beta m_n(\sigma)} \left[e^{n\left(f(m_n(\sigma)-\frac{2}{n})-f(m_n(\sigma))\right)} - 1 \right] \\
&= Hf(\eta_n(\sigma)) + o(1) \qquad (8.17)
\end{aligned}
$$

where $Hf(x) = \mathcal{H}(x, f'(x))$ and where

$$
\mathcal{H}(x,p) = \frac{1-x}{2} e^{\beta x} \left[e^{2p} - 1 \right] + \frac{1+x}{2} e^{-\beta x} \left[e^{-2p} - 1 \right] \qquad (8.18)
$$

and where $o(1)$ is an error that is uniformly small as $f \in C_b^2(X)$.

Denote $\mathcal{L}(x,v) := \sup_p pv - \mathcal{H}(x,p)$. The limiting Hamiltonian corresponds to the semigroup and resolvent

$$
\begin{aligned}
V(t)f(x) &:= \sup_{\gamma \in \mathcal{AC}: \gamma(0)=x} \left\{ f(\gamma(t)) - \int_0^t \mathcal{L}(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s \right\}, \\
R(\lambda)h(x) &:= \sup_{\gamma \in \mathcal{AC}: \gamma(0)=x} \left\{ \int_0^\infty \lambda^{-1} e^{-\lambda^{-1} t} \left[h(\gamma(t)) - \int_0^t \mathcal{L}(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s \right] \mathrm{d}t \right\}.
\end{aligned}
$$

Theorem 8.8. *Let H_n and H be given as in (8.16) and (8.18) respectively.*

Then we have the following convergence statements. Let $f_n, f \in C_b(\mathbb{R})$ such that $buc - \lim f_n = f$. Furthermore, let $\lambda, t > 0$.

Then $buc - \lim R_{\varepsilon_n}(\lambda) f_n = R(\lambda) f$ and $buc - \lim V_{\varepsilon_n}(t) f_n = V(t) f$.

Proof. As seen above, we can work in this setting with $X = \mathcal{X} = Y = \mathcal{Y} = [0,1]$ and $X_n = \{-1,1\}^n$. As all these sets are compact, it suffices to work with $\mathcal{Q}$ a singleton set, corresponding to taking all compact sets equal to the full set. If $f_n \in C_b(X_n)$ and $f \in C_b(X)$ then $\mathrm{LIM}\, f_n = f$ is equivalent to $\lim_n \|f_n - f \circ \eta_n\| = 0$.

The convergence of the operators H_n to H has been established in (8.17). The equicontinuity estimates on the resolvent and semigroup can be verified by Lemmas 7.15 7.16 in [32], if the assumption of exponential tightness for the processes is satisfied. As the spaces X_n and X are all compact, this follows immediately from the convergence of H_n to H and Corollary 4.17 in [22]. The comparison principle for the Hamilton-Jacobi equation in terms of H is established in Theorem 3 of [30].

Thus, the convergence of resolvents and semigroups follows from Theorems 5.7 and 6.1. Note the large deviation perspective on this convergence statement has been worked out in further detail in [22] and [32]. □

Appendix A. Proof of Proposition 3.8

Proof of Proposition 3.8. We prove (a) to (b). Fix a compact set $K \subseteq X$ and $r, \delta > 0$.

Because the semi-norm $p(f) = \sup_{x \in K} |f(x)|$ is continuous for the strict topology, and T is strictly continuous, there is a semi-norm $q(f) = \sup_n a_n \sup_{x \in K_n} |f(x)|$ such that $p(Tf - Tg) \leq q(f - g)$.

Set $C_0(r) = 2r$, $C_1(\delta, r) = a_1$, and

$$\hat{K}(K, \delta, r) := \bigcup_{i : a_i > \delta} K_i.$$

As $a_i \downarrow 0$, $\hat{K}$ is indeed a compact set. Let n_0 such that for $n \geq n_0$ we have $a_n \leq \delta$. Then, if $\|f\| \vee \|g\| \leq r$:

$$\begin{aligned} p(Tf - Tg) \leq q(f - g) &\leq \sup_{n < n_0} a_n \sup_{x \in K_n} |f(x) - g(x)| + \sup_{n \geq n_0} a_n \sup_{x \in K_n} |f(x) - g(x)| \\ &\leq a_1 \sup_{x \in \hat{K}} |f(x) - g(x)| + \delta \|f - g\| \\ &\leq C_1(\delta, r) \sup_{x \in \hat{K}} |f(x) - g(x)| + \delta C_0(r), \end{aligned}$$

establishing (b).

We prove (b) to (c). Let f_α be a bounded net that converges to f. To prove that Tf_α converges strictly to f, we need to establish that Tf_α is bounded and that for each compact set $K \subseteq X$ and $\varepsilon > 0$ there is a α_0 such that for $\alpha \geq \alpha_0$, we have

$$\sup_{x \in K} |Tf_\alpha(x) - Tf(x)| \leq \varepsilon.$$

First of all, as the net is bounded there is some r such that $\sup_n \|f_n\| \leq r$ for some r. By (b), using $\delta = 1$, we find that $\|Tf_\alpha - Tf\| \leq C_0(r) + 2rC_1(1, r)$. Next, fix a compact set $K \subseteq X$ and $\varepsilon > 0$. By (b), with $\delta = \frac{1}{2}\varepsilon C_0(r)^{-1}$, we find a compact set $\widehat{K}$ such that

$$\sup_{x \in K} |Tf_\alpha(x) - Tf(x)| \leq \frac{1}{2}\varepsilon + C_1(\delta, r) \sup_{x \in \widehat{K}} |f_\alpha(x) - f(x)|$$

Thus, there is some α_0 such that for $\alpha \geq \alpha_0$ the left hand side is bounded by ε. We conclude that T is strictly continuous on bounded sets. □

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