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# Macroscopic Noisy Bounded Confidence Models With Distributed Radical Opinions

Mohamad Amin Sharifi Kolarijani , Anton V. Proskurnikov , *Senior Member, IEEE*,  
and Peyman Mohajerin Esfahani 

**Abstract**—In this article, we study the nonlinear Fokker–Planck (FP) equation that arises as a mean-field (macroscopic) approximation of bounded confidence opinion dynamics, where opinions are influenced by environmental noises and opinions of radicals (stubborn individuals). The distribution of radical opinions serves as an infinite-dimensional exogenous input to the FP equation, visibly influencing the steady opinion profile. We establish mathematical properties of the FP equation. In particular, we, first, show the well-posedness of the dynamic equation, second, provide existence result accompanied by a quantitative global estimate for the corresponding stationary solution, and, third, establish an explicit lower bound on the noise level that guarantees exponential convergence of the dynamics to stationary state. Combining the results in second and third readily yields the *input–output stability* of the system for sufficiently large noises. Next, using Fourier analysis, the structure of opinion clusters under the uniform initial distribution is examined. The results of analysis are validated through several numerical simulations of the continuum-agent model (partial differential equation) and the corresponding discrete-agent model (interacting stochastic differential equations) for a particular distribution of radicals.

**Index Terms**—Distributed parameter systems, nonlinear systems, opinion dynamics, stability of nonlinear systems, stochastic systems.

## I. INTRODUCTION

RECENT decades have witnessed enormous progress in study of complex systems and their system-theoretic properties [1], [2]. The main effort has been invested into the study of “self-organization” and “spontaneous order” phenomena [3] that

have inspired the development of synchronization and consensus theory [4], [5]. Paradoxically, these regular behaviors arising from local interactions between subsystems (agents, nodes) of a complex system are studied much better than various “irregular” dynamic effects such as persistent disagreement and clustering, exhibited by many real-world systems. Although some culprits of this asynchrony and dissent (e.g., symmetries and other special structures in the coupling mechanisms, exogenous forces acting on some nodes, heterogeneous dynamics of nodes, etc.) have been revealed in the literature [6]–[10], only a few mathematical models have been proposed that are sufficiently “rich” to capture the diversity of clustering behaviors in real-world networks and, at the same time, admit rigorous analysis. Long before the recent “boom” in complex systems, the lack of such models was realized in mathematical sociology. The problem of disclosing mechanisms preventing consensus and maintaining enduring disagreement between individuals [11] is nowadays referred to as the community cleavage problem or Abelson’s diversity puzzle [12], [13]. The interdisciplinary area of sociodynamical modeling [13]–[20] has attracted enormous attention of the research community and is primarily concerned with mechanisms of *opinion formation* under social influence.

Only few models, proposed in the literature to describe opinion formation processes, have been secured by experimental evidence. Such models, however, play an important role and contribute, in various aspects, in comprehending complex systems’ behaviors such as birth, death, and evolution of clusters in systems of interacting particles, and in developing algorithms for control of these behaviors. This explains explosion of interest in models of “opinion formation” in the systems and control literature. From the control-theoretic prospect, most of these models are simply networks of interacting agents, obeying the first-order integrator model. However, the term “opinion” is now widespread and used to denote the scalar or multidimensional state of an agent, even if this state does not have a clear sociological interpretation<sup>1</sup> (belonging, e.g., to an abstract manifold [21]). The opinion is, thus, some value of interest, held by an agent and updated, based on displayed opinions of the other agents.

Nowadays, linear models of opinion dynamics, extending the classical French–DeGroot system in various directions (allowing, e.g., stubborn agents, asynchronous interactions, and

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<sup>1</sup>From the sociological viewpoint, mechanisms of social influence alter *cognitive orientations* of individuals toward some objects or topics [13], and the “opinion” is a quantitative characteristics of such an orientation.

repulsion of opinions [13], [17], [22], [23]) have been thoroughly studied. These models are sufficient to explain consensus and disagreement in social groups, as well as formation of special opinion profiles (e.g., bimodal distributions, standing for opinion polarization), however, general mechanisms leading to emergence and destruction of unequal clusters are still far from being well understood. To explain them, more complicated *nonlinear* models have been proposed, mimicking some important features of social influence. One feature observed in social and biological systems is the *homophily* [24], or tendency of individuals to bond with similar ones. Homophily is related to *biased assimilation* [25] effects: individuals readily accept opinions consistent with their views and tend to dismiss and discount opinions contradicting to their own views. Mathematically, coupling between close opinions is stronger than that of distant opinions, which is modeled by introducing *opinion-dependent* influence weights. Although the possibility of such nonlinearities in opinion dynamics models was mentioned in the pioneering work [11], substantial progress has been primarily achieved in the analysis of *bounded confidence* models proposed several decades later as extensions of the deterministic [26] and randomized gossip-based [27] consensus algorithms for multiagent networks. Bounded confidence models stipulate that a social actor is insensitive to opinions beyond its bounded *confidence set* (usually, this set is an open or closed ball, centered at the actor's own opinion), which makes the graph of interactions among the agents *distance-dependent*. A detailed survey of bounded confidence models and relevant mathematical results can be found in [18]. Bounded confidence models exhibit convergence of the opinions to some steady values, which can reach consensus or split into several disjoint clusters. If the state-dependent interaction graph of the system is symmetric, this follows from general properties of iterative averaging procedures, and can alternatively be proved by exploring a special Lyapunov function ("kinetic energy") [18], [28], [29]. In the general case of *asymmetric* interaction graphs, such a convergence has been proved only in special situations [29], [30], but seems to be a generic behavior [30]–[32].

Opinions in real social groups, however, usually do not terminate at steady values yet oscillate, which is usually explained by two factors. The first reason explaining opinion fluctuation is exogenous influence, which can be interpreted as some "truth" available to some individuals [33] or a position shared by a group of close-minded opinion leaders or stubborn individuals ("radicals") [34]–[36]. Important results on stability of the Hegselmann–Krause model with radicals and more general "inertial" bounded confidence models were obtained in [30]. Typically, the exogenous signal is supposed to change slowly compared to the opinion evolution and is, thus, replaced by a constant; the main concern is the dependence between the constant input and the resulting opinion profile. Numerical results, reported in [34] and [35] demonstrate high sensitivity of the opinion clusters to the radical's opinion, and reveal some counter-intuitive effects; e.g., an increase in the number of radicals sometimes *decreases* the number of their followers. The second culprit of persistent opinion fluctuation is uncertainty in the opinion dynamics, usually modeled as a random drift of each

opinion. The presence of a random excitation can be interpreted as "free will" and unpredictability of a human's decision [37]; besides this, randomized opinion dynamics models are broadly adopted in statistical physics [38]–[41] to study phase transitions in systems of interacting particles.

Even for the classical models from [26], [27], disclosing the relation between the initial and the terminal opinion profiles remains a challenging problem (including, e.g., the *2R*-conjecture [42], [43]). In presence of noise, the analysis becomes even more difficult; some progress in the study of the interplay between confidence range and noise level has been achieved in recent works [44], [45]. One of the important directions in analysis of bounded confidence models is examination of their *asymptotic* properties as the number of social actors becomes very large  $N \rightarrow \infty$  and their individual opinions are replaced by infinitesimal "elements." The arising *macroscopic* approximations of agent-based models describe the evolution of the distribution of opinion (usually supposed to have a density) and are referred to as density-based [46], continuum-agent [47], [48], Eulerian [49], [50], kinetic [51], hydrodynamical [28], or mean-field [43], [52] models of opinion formation. In the continuous-time situation, the density obeys a nonlinear Fokker–Planck (FP) equation. To study clustering behavior of the macroscopic bounded confidence models, efficient numerical methods have been proposed that are based on Fourier analysis [40], [43], [53].

From practical viewpoint, it is convenient to consider opinions staying in a predefined interval, e.g.,  $[0,1]$ . The Hegselmann–Krause and Deffuant–Weisbuch models, as well as their continuous-time counterparts [18], imply that starting within the interval, opinions never escape from it. This property, however, is destroyed by arbitrarily small noises. To keep the opinions bounded, some "boundary conditions" are usually introduced. The *absorbing* boundary condition assumes that the opinions are saturated at the extreme values 0 and 1 [40], [45]; an important result from [45] demonstrates that arbitrarily small noises in this situation destroy clusters and lead to approximate consensus (the maximal deviation of opinions is proportional to the noise level). More interesting are opinion dynamics with the *periodic* boundary condition, wrapping the interval  $[0,1]$  into a circle. The opinion density on the circle corresponds to a *1-periodic* solution of the FP equation on the real line [43], [53], [54]. A disadvantage of the simple periodic boundary condition is the merging of two extreme opinion values 0 and 1. To distinguish between these extreme opinions, we incorporate an "almost reflective" (precisely, an *even 2-periodic*) boundary condition. Dealing with the macroscopic FP equation, the opinion density is then conveniently represented by an even 2-periodic solution on the real line. This article is primarily concerned with mathematical properties of such solutions.

*Main contributions:* In this article, we advance the theory of macroscopic modeling of bounded confidence dynamics. We consider a bounded confidence model with environmental noise that also includes radical opinions, which are not concentrated at a single point (as in [33], [34], and [49]) but rather distributed over the interval  $[0, 1]$ . The FP equation acquires an (infinite-dimensional) exogenous *input*, describing the density

and the total mass of the radical opinions. This setup allows us to consider the interplay between the noise and the distributed radicals concerning the behavior of the system. In particular, for the macroscopic FP equation,

- 1) the criteria for the existence, uniqueness, and regularity of an even periodic solution are established (see Theorem II.1);
- 2) the existence of stationary solution is studied and a global estimate is provided that bounds the deviation of the stationary state from the uniform distribution (see Theorem II.2);
- 3) a sufficient condition is presented for exponential convergence of the dynamics to stationary state (see Theorem II.3), entailing [in combination with 2)] also the input–output stability of the system (see Corollary II.4),

Developing ideas from [40], [43], [53], we then use Fourier analysis to characterize the clustering behavior of the system under the uniform initial distribution and some particular distributions of radical opinions. Specifically,

- 4) a numerical scheme is provided to analyze the impact of the noise and the radical opinions density on the number and timing of the initial clustering behavior (see Section V-B).

In the preliminary version of this article [55], we reported the results of Theorems II.2 (Estimate) and II.3 without detailed technical proofs. In this article, we provide the details along with several necessary preparatory lemmas. In [55, Sec. IV], we also used Fourier Analysis to study the interplay between the relative number (mass) of radical agents with respect to (w.r.t.) normal agents and the critical noise level for order-disorder transition.

This article is organized as follows. Section II introduces the macroscopic opinion dynamics model in question. Here, we also present our main theoretical results regarding well-posedness and stability of the model. The next two sections are concerned with technical proofs of these results. Section III is devoted to the proof of well-posedness of the dynamics. In Section IV, we examine the properties of the corresponding stationary equation and provide the technical proofs for theoretical results on stability of stationary state. In Section V, Fourier analysis is used for characterization of the clustering behavior of the model. This general scheme is then used in Section VI for a particular distribution of the radical opinions. These results are accompanied by numerical simulations of both the agent-based and the macroscopic models.

*Notations:* The convolution of two functions  $f$  and  $g$  is denoted by  $f \star g$ . We note that in our case, one of the functions has a compact support, so the convolution integral always exists. For a function  $f(t, x)$ , we use  $f_x$  (resp.  $f_t$ ) to denote the (partial) derivative w.r.t.  $x$  (resp.  $t$ ), so that  $f_{xx}$  is the second derivative w.r.t.  $x$ . We also use  $\partial_x^i f$  for the  $i$ th order derivative w.r.t.  $x$ . Let  $X = [0, 1]$  and  $\tilde{X} = [-1, 1]$ . We use  $\mathcal{P}(X)$  to denote the space of probability densities on  $X$ . That is,  $\rho \in \mathcal{P}(X)$  if  $\int_X \rho(x) dx = 1$  and  $\rho(x) \geq 0$  for all  $x \in X$ . We also use  $\mathcal{P}_e(\tilde{X})$  to denote the space of probability densities on  $\tilde{X}$ , extended evenly to  $\tilde{X}$ . That is,  $\mathcal{P}_e(\tilde{X})$  is the space of all functions  $\rho : \tilde{X} \rightarrow [0, \infty)$  such that  $\int_{\tilde{X}} \rho(x) dx = 1$  and  $\rho(x) = \rho(-x) \geq 0$  for all  $x \in \tilde{X}$ .  $L^p(\tilde{X})$  denotes the Banach space of all measurable functions

$f : \tilde{X} \rightarrow \mathbb{R}$  for which the  $L_p$ -norm  $\|f\|_{L^p(\tilde{X})}$  is finite.  $H^k(\tilde{X})$  for  $k \in \mathbb{N}$  is used to denote the Sobolev space  $W^{k,2}(\tilde{X})$ . We use the subscripts *per* (resp. *ep*) to denote the closed subspace of *periodic* (resp. *even periodic*) functions in the corresponding function space. We denote the dual space of  $H_{\text{per}}^1(\tilde{X})$  by  $H_{\text{per}}^{-1}(\tilde{X})$  and we use  $\langle \cdot, \cdot \rangle$  to denote the corresponding pairing of  $H_{\text{per}}^1(\tilde{X})$  and  $H_{\text{per}}^{-1}(\tilde{X})$ . We use  $\rightarrow$  and  $\rightharpoonup$  to denote strong and weak convergences, respectively, in an appropriate Banach space. A brief overview of function spaces relevant to this article is provided in Appendix A.

## II. MODEL DESCRIPTION AND MAIN RESULTS

### A. Macroscopic Model of Opinion Formation

The conventional bounded confidence model describes opinion formation process in a network of  $N > 1$  agents. All agents have the same *confidence range*  $R > 0$ . Agent  $i$ 's opinion at time  $t \geq 0$ , denoted by  $x_i(t) \in \mathbb{R}$ , is (directly) influenced only by the opinions of agents  $j$ , such that  $|x_j(t) - x_i(t)| \leq R$ . One of the simplest continuous-time bounded confidence models is as follows [28]:

$$\dot{x}_i(t) = \sum_{j=1}^N \frac{w(x_j(t) - x_i(t))}{N}, \quad w(\xi) = \begin{cases} \xi, & |\xi| \leq R \\ 0, & |\xi| > R. \end{cases} \quad (1)$$

It can be shown [18] that the opinions obeying model (1) always converge  $x_i(t) \rightarrow x_i^s$  as  $t \rightarrow \infty$ , with  $w(x_i^s - x_j^s) = 0$  for all  $i, j$ , corresponding to either consensus or fragmentation of opinions into multiple clusters. Dynamics of real opinions (as well as physical processes, portrayed by ‘‘opinion dynamics’’ models) often do not exhibit convergence to steady values, and the fluctuation of opinions persists. In order to capture this effect, random uncertainties can be introduced into the model mimicking ‘‘free will’’ and unpredictability of a human’s decision [37]. The simplest of these uncertainties is an additive random noise. Model (1) is then replaced by the system of nonlinear stochastic differential equations (SDEs)

$$dx_i(t) = \frac{1}{N} \sum_{j=1}^N w(x_j(t) - x_i(t)) dt + \sigma dW_i(t) \quad (2)$$

where  $W_i$  are independent standard Wiener processes and  $\sigma > 0$  characterizes the noise level.

Since the dynamics of the stochastic system (2) becomes quite complicated as the number of agents grows, the standard approach to examine it is the *mean-field* (or macroscopic) approximation, considering the opinion profile  $(x_i(t))_{i=1}^N$  as a *random sampling* drawn from some (time-varying) probability distribution of the opinion. Precisely, it can be shown [56]–[58] that empirical distributions  $N^{-1} \sum_{i=1}^N \delta_{x_i(t)}$  converge (in the weak sense) as  $N \rightarrow \infty$  to a distribution, whose density  $\rho(t, x)$  obeys the FP equation

$$\rho_t = (\rho(w \star \rho))_x + \frac{\sigma^2}{2} \rho_{xx}, \quad t \geq 0, \quad x \in \mathbb{R}. \quad (3)$$

A natural extension of the bounded confidence dynamics allows the presence of  $N_r \geq 1$  *radicals* (stubborn agents, zealots)

that do not assimilate others' opinions, however, influence them directly or indirectly. Typically, the radicals' opinions are supposed to be constant (or changing very slowly compared to the opinion formation of "normal" agents). Indexing the "normal" individuals by  $i \in \mathcal{I}_n = \{1, \dots, N\}$  and the radicals by  $i \in \mathcal{I}_r = \{N+1, \dots, N+N_r\}$ , the opinion dynamics becomes as follows:

$$\begin{aligned} dx_i(t) &= \sum_{j=1}^{N+N_r} \frac{w(x_j(t) - x_i(t))}{N} dt + \sigma dW_i(t), \quad i \in \mathcal{I}_n \\ \dot{x}_i(t) &= 0, \quad i \in \mathcal{I}_r. \end{aligned} \quad (4)$$

Often it is supposed that the radicals share a common opinion  $x_i \equiv T$  for  $i \in \mathcal{I}_r$ , which may also be considered as "truth" perceived by some individuals [33] or some exogenous signal [34]. The ratio  $M = N_r/N$  can be treated as the relative "weight" or "strength" of this external opinion. More generally, one can assume that the radicals' opinions are spread over  $\mathbb{R}$ . Supposing that  $N, N_r \rightarrow \infty$ , the relative mass of the radicals  $M$  remains constant, and their empirical distribution  $N_r^{-1} \sum_{i=1}^{N_r} \delta_{x_{N+i}}$  converges (in the weak sense) to a distribution with sufficiently smooth density  $\rho_r$ , the density of the "normal" opinions obeys the modified FP equation

$$\rho_t = (\rho(w \star (\rho + M\rho_r)))_x + \frac{\sigma^2}{2} \rho_{xx}, \quad t \geq 0, \quad x \in \mathbb{R}. \quad (5)$$

Note that the classical bounded confidence dynamics (1), being a special case of continuous-time consensus protocol, has an important property: the minimal and maximal opinions  $\min_i x_i(t)$  and  $\max_i x_i(t)$  are, respectively, nondecreasing and nonincreasing. In particular, if the initial opinions are confined to some predefined interval, e.g.,  $x_i(0) \in [0, 1]$ , then one has  $x_i(t) \in [0, 1]$  for all  $t \geq 0$ . The additive noise leads to random drift of the opinion profile, thus destroying the latter important property. Since in practice bounded ranges of opinions are usually considered, the dynamics (2) and (4) are usually complemented by *boundary conditions* [40], preventing the opinions from leaving the predefined range.

A typical boundary condition is the *periodic* condition, where the opinion domain  $[0, 1]$  is wrapped on a circle of circumference 1 (formally, replacing a real opinion value  $x \in \mathbb{R}$  by its fractional part  $\{x\} = x \bmod 1$ ). A disadvantage of the periodic boundary condition is that there is no distinction between the extreme opinions 0 and 1. In this article, we address this issue by considering another type of boundary condition, which we call *even 2-periodic*. Precisely, a real opinion  $x \in \mathbb{R}$  is replaced by  $f(x)$ , where  $f$  is an even 2-periodic function, such that  $f(x) = x$  on  $[0, 1]$  (and hence  $f(x) = -x$  for  $x \in [-1, 0]$ ,  $f(x) = 2 - x$  for  $x \in [1, 2]$  and so on). In other words, we first *evenly* extend the opinion domain  $[0, 1]$  into the interval  $[-1, 1]$  and then wrap it on a circle of circumference 2 so that the extreme opinions 0 and 1 correspond to the antipodes of this circle. We note that with this even 2-periodic extension, the "effective" boundary condition experienced by the agents is an *almost reflective* one, that is, when an agent leaves the opinion domain from one end, it is reflected back into the domain from the same end. This is different from the behavior under simple periodic boundary

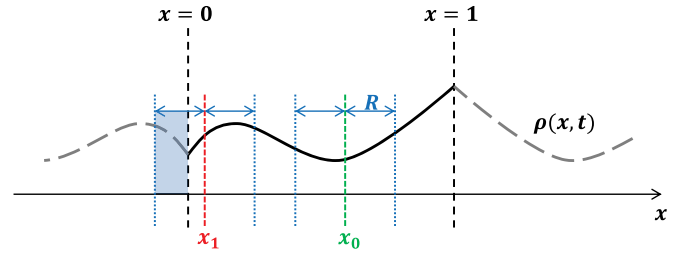


Fig. 1. Even 2-periodic extension of the system. The opinion value  $x_0 \in [R, 1 - R]$  effectively experiences a reflective boundary condition, while for the opinion value  $x_1 \in [0, R]$  there is also a boundary effect due to the even extension. In particular, the influence of more extreme neighbors of  $x_1$  is reinforced by introducing artificial ones (the shaded area in blue). The same boundary effect exists for opinion values in  $[1 - R, 1]$ .

condition where the agents leaving the domain from one end, enter the domain from the other end. However, the introduced boundary condition is "almost" reflective since the even extension causes some *boundary effects*: the influence of more extreme neighbors of opinion values in the  $R$ -neighborhood of extreme opinions 0 and 1 is reinforced. This is due to the even extension that introduces more extreme "artificial" neighbors; see Fig. 1.

As discussed in [43], [53], and [54], the FP equation (3) under the periodic conditions retains its validity, however,  $\rho(t, x)$  is not a probability density on  $\mathbb{R}$  but a 1-periodic function  $\rho(t, x+1) = \rho(t, x) \geq 0$ , such that  $\int_0^1 \rho(t, x) dx = 1$  (that is,  $\rho(t, \cdot)$  serves as a density on the interval  $[0, 1]$ ). Similarly, for the even 2-periodic boundary condition, (3) retains its validity when we replace the probability density  $\rho(t, x)$  with an even 2-periodic function, that is,  $\rho(t, -x) = \rho(t, x)$  and  $\rho(t, x+2) = \rho(t, x)$ . On the interval  $[0, 1]$ , the function  $\rho(t, \cdot)$  again serves as a probability density:  $\int_0^1 \rho(t, x) dx = 1$ . We also assume that the initial density  $\rho_0(x) = \rho(0, x)$  and the density of radical opinions  $\rho_r(x)$ , defined on  $[0, 1]$ , are extended (in the unique possible way) to even 2-periodic functions on  $\mathbb{R}$ .

In this article, without loss of generality, we take  $X = [0, 1]$  and  $\tilde{X} = [-1, 1]$  to be the bounded opinion domain and its even extension, respectively. To summarize the discussion mentioned above, the macroscopic model for opinion dynamics considered in this article is fully described by the following partial differential equation (PDE)

$$\begin{aligned} \rho_t &= (\rho G_\rho)_x + \frac{\sigma^2}{2} \rho_{xx} \quad \text{in } \tilde{X} \times (0, T) \\ \rho(\cdot + 2, t) &= \rho(\cdot, t) \quad \text{on } \partial\tilde{X} \times (0, T) \\ \rho(x, \cdot) &= \rho_0(x) \quad \text{on } \tilde{X} \times \{t = 0\} \end{aligned} \quad (6)$$

where

$$G_\rho(x, t) := w(x) \star (\rho(x, t) + M\rho_r(x)). \quad (7)$$

Note that in (6), we are considering the dynamics over a finite time horizon  $T$  for the sake of analysis, however,  $T$  can be chosen arbitrarily large. We again emphasize that the initial density  $\rho_0$  and the radical density  $\rho_r$  are the unique even 2-periodic extensions of the corresponding densities from

$X$  to  $\tilde{X}$ . In essence, we are considering the same dynamics as in [54] with the extra requirement for  $\rho_0$  (and the newly introduced density  $\rho_r$ ) to be even. Finally, we note that [59] also provides a detailed treatment of this dynamics (without radicals) for a class of interaction potentials on a torus in higher dimensions.

## B. Main Theoretical Results

To recapitulate, we are interested in even 2-periodic solutions of PDE (6), where  $\rho_0$  and  $\rho_r$  are even 2-periodic. A natural question arises as to whether the model is well-posed in the sense that every (sufficiently smooth) initial condition  $\rho_0$  and input  $\rho_r$  correspond to a unique solution. The affirmative answer is given in the following theorem.

**Theorem II.1 (Well-posedness of dynamics):** Let the initial density of normal opinions and the radical opinions density satisfy  $\rho_0 \in H_{\text{ep}}^3(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  and  $\rho_r \in H_{\text{ep}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$ , respectively. Then, PDE (6) has a unique, even, strictly positive, classical solution  $\rho \in C^1(0, \infty; C_{\text{ep}}^2(\tilde{X}))$  such that  $\rho(t) \in \mathcal{P}_e(\tilde{X})$  for all  $t > 0$ .

This result implies that  $\rho(t) := \rho(t, \cdot)$  is a (strictly positive) probability density on  $X = [0, 1]$  for all  $t > 0$ , as required. For the autonomous systems (without radicals), the authors in [54] and [59] provide a sufficient condition for exponential convergence of the dynamics toward uniform distribution  $\rho = 1$  as an equilibrium of the system. Unlike those studies, the uniform distribution is not an equilibrium of the model considered in this article. However, it is possible to extend this stability result to our model. To this end, we first consider the stationary equation corresponding to PDE (6) given by

$$\frac{\sigma^2}{2} \rho_{xx} + (\rho G_\rho)_x = 0. \quad (8)$$

We are particularly interested in even stationary solutions  $\rho^s \in \mathcal{P}_e(\tilde{X})$  of (8). Our next result characterizes the stationary state of the system.

**Theorem II.2 (Stationary behavior):** Let  $\rho_r \in H_{\text{ep}}^1(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  be the radical opinions density.

- 1) Existence: the stationary equation (8) has an even, strictly positive, classical solution  $\rho^s \in C_{\text{ep}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$ .
- 2) Estimate: for any  $\eta > 0$ , if  $\sigma^2 > \sigma_b^2 + \eta c_b$ , then

$$\|\rho^s - 1\|_{L^2} \leq \frac{1}{\eta} \|\rho_r\|_{L^2}$$

where

$$\sigma_b^2 := \frac{4R}{\pi} \left( M + \frac{R}{\sqrt{3}} + 2 \right) \text{ and } c_b := \frac{4R^2 M}{\pi \sqrt{3}}. \quad (9)$$

Notice how the global estimate in Theorem II.2 bounds the difference between the stationary solution and the uniform distribution. This result shows that, even in presence of radical opinions, the stationary solution can be made arbitrarily close to the uniform distribution by increasing the noise level beyond a minimum level  $\sigma_b$ . We note that the minimum noise level  $\sigma_b$  is directly related to the confidence range  $R$ —similar to the results previously reported for the autonomous system in [54] and [59].

As expected,  $\sigma_b$  is also directly related to the mass  $M$  of radicals. Moreover, as the “energy”  $M \|\rho_r\|_{L^2}$  of the radicals increases, in order to counteract their effect and keep the stationary profile in a somewhat uniform state, one must increase the noise level further beyond  $\sigma_b$ .

With this result in hand, we can now consider the asymptotic stability of stationary state. The next result provides a sufficient condition for exponential convergence of the dynamics to stationary state for arbitrary (and sufficiently smooth) initial density  $\rho_0$  and radical density  $\rho_r$ .

**Theorem II.3 (Stability):** Let  $\rho_0 \in H_{\text{ep}}^3(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  be the initial density of normal opinions and  $\rho_r \in H_{\text{ep}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  be the radical opinions density. Also, let  $\rho \in C^1(0, \infty; C_{\text{ep}}^2(\tilde{X}))$  with  $\rho(t) \in \mathcal{P}_e(\tilde{X})$  be the solution to the dynamic equation (6). Then,  $\rho(t)$  converges to a stationary state  $\rho^s \in C_{\text{ep}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  exponentially in  $L^2$  as  $t \rightarrow \infty$  if  $\sigma > \sigma_s$ , where  $\sigma_s > 0$  uniquely solves

$$\sigma_s^2 = \frac{4R(3+M)}{\pi} + \frac{4R^2}{\pi\sqrt{3}} \exp\left(\frac{8R(1+M)}{\sigma_s^2}\right). \quad (10)$$

An immediate result of Theorems II.2 and II.3 is that for sufficiently large noises, the dynamics will converge to a stationary state that can be made arbitrarily close to uniform distribution by increasing the noise level.

**Corollary II.4 (Input–output stability):** For any  $\eta > 0$ , if  $\sigma^2 > \max\{\sigma_b^2 + \eta c_b, \sigma_s^2\}$ , where  $\sigma_b$  and  $c_b$  are defined in (9) and  $\sigma_s > 0$  uniquely solves (10), then it holds that

$$\|\rho(t) - 1\|_{L^2} \leq \beta e^{-\lambda t} + \frac{1}{\eta} \|\rho_r\|_{L^2} \quad (11)$$

where the constant  $\beta > 0$  depends on  $\rho_0$  and  $\rho_r$ , and the convergence rate  $\lambda > 0$  depends on  $\sigma$ ,  $R$ , and  $M$ .

**Remark II.5 (Connection to existing works):** The stability result of Corollary II.4 corresponds to the result reported in [54, Th. 2.3] on global stability of uniform distribution  $\rho = 1$  for sufficiently large noises in the autonomous system *without radicals*. In particular, by setting  $M = 0$  in the estimate given in Theorem II.2, one has  $c_b = 0$ , hence  $\rho^s = 1$  is the unique stationary state of the system for  $\sigma^2 > \sigma_b^2 = \frac{4R}{\pi} (2 + R/\sqrt{3})$ . We note that  $\sigma_b$  is the same minimum noise level given in [54, Th. 2.3], taking into account a multiplicative factor of two due to the even extension considered in the current article. However, direct application of Theorem II.3 for stability of  $\rho^s = 1$  leads to a sufficient minimum noise level  $\sigma_s > \sigma_b$ . This is due to the fact that this result is based on conservative estimates for  $\rho^s$ . Indeed, if we incorporate the fact that  $\rho^s = 1$  (and modify some of the arguments provided in the proof of Theorem II.3 in Section IV-C), then we can show that, in the absence of radical agents, the uniform distribution  $\rho^s = 1$  is also globally exponentially stable for  $\sigma > \sigma_b$ , and reproduce the result of [54, Th. 2.3].

Finally, we note that, based on the results provided in [59], the input–output stability result of Corollary II.4 can be generalized to multidimensional first-order stochastic interacting particle systems for a particular class of interaction potentials.

The following two sections are mainly concerned with the technical proofs of the theoretical results listed earlier.

### III. WELL-POSEDNESS OF DYNAMICS

This section is devoted to the proof of Theorem II.1 concerning the well-posedness of the dynamics (6). Throughout this section, all the norms are w.r.t.  $\tilde{X} = [-1, 1]$  (as opposed to  $X = [0, 1]$ ), unless indicated otherwise. We use  $C, C_0, C_1, \dots$  to represent a generic constant (depending on the model parameters), which actual values may change from line to line. In case these constants depend on a particular object of interest, say  $\theta$ , this dependence is explicitly indicated by  $C[\theta]$ .

Let us first note that because of periodicity, the mass is preserved in (6), that is,  $\int_{\tilde{X}} \rho(x, t) dx = \int_{\tilde{X}} \rho_0(x) dx = 2$ , for all  $t \geq 0$ . In particular, we have

$$\|\rho(t)\|_{L^1} \geq \int_{\tilde{X}} \rho(x, t) dx = 2 > 0.$$

We will be using this property in the sequel.

We begin with presenting some useful estimates for the object  $G_\rho$  defined in (7) that make it possible to extend the results provided by [54] to our model.

*Lemma III.1 (Estimates for  $G_\rho$ ):* Let  $G_\rho$  be the function defined in (7) with  $\rho_r \in \mathcal{P}_e(\tilde{X})$ . If  $\rho(t) \in L^1_{\text{per}}(\tilde{X})$ , then

$$\|G_\rho\|_{L^\infty} \leq R(\|\rho(t)\|_{L^1} + 2M). \quad (12)$$

If, moreover,  $\|\rho(t)\|_{L^1} > 0$ , then

$$\|G_\rho\|_{L^\infty} \leq C \|\rho(t)\|_{L^1} \leq C \|\rho(t)\|_{L^2}. \quad (13)$$

*Proof:* See the proof of [60, Lemma 3.1].  $\blacksquare$

Using the estimate (13) in Lemma III.1, one can follow similar arguments as in [54, Lemma 2.1] to show  $\|\rho(t)\|_{L^1} = 2$  and  $\rho(t) \geq 0$  for all  $t \geq 0$  (see also [54, Corollary 2.2]). Specifically, assuming PDE (6) has a solution  $\rho \in C^1(0, T; C^2_{\text{per}}(\tilde{X}))$ , one can derive *a priori* estimate, which in turn implies that the solution is *nonnegative* so that  $\rho(t)$  is a probability distribution on  $X = [0, 1]$  for all  $t \geq 0$ . We will be using these *a priori* properties in the sequel.

*Lemma III.2 (Estimates for  $\partial_x^k G_\rho$ ):* Let  $G_\rho$  be the function defined in (7) with  $\rho_r \in \mathcal{P}_e(\tilde{X})$ .

i) For  $1 \leq p \leq \infty$ , if  $\rho(t), \rho_r \in L^p_{\text{per}}(\tilde{X})$  with  $\|\rho(t)\|_{L^1} > 0$ , then

$$\begin{aligned} \|(G_\rho)_x\|_{L^p} &\leq C_1 \|\rho(t)\|_{L^p} + C_2 \|\rho_r\|_{L^p} \\ &\leq C[\|\rho_r\|_{L^p}] \|\rho(t)\|_{L^p}. \end{aligned} \quad (14)$$

ii) For  $k \geq 2$ , if  $\rho(t), \rho_r \in H^{k-1}_{\text{per}}(\tilde{X})$  with  $\|\rho(t)\|_{L^1} > 0$ , then

$$\|\partial_x^k G_\rho\|_{L^2} \leq C[\|\rho_r\|_{H^{k-1}}] \|\rho(t)\|_{H^{k-1}}. \quad (15)$$

*Proof:* See the proof of [60, Lemma 3.2].

The following Lemma extends [54, Proposition 4.1] for our system with exogenous input.

*Lemma III.3 (More estimates for  $G_\rho$ ):* Let  $\nu \in H^k_{\text{per}}(\tilde{X})$ ,  $\rho_r \in H^{k-1}_{\text{per}}(\tilde{X}) \cap \mathcal{P}_e(X)$ , and  $\rho(t) \in H^{k-1}_{\text{per}}(\tilde{X})$  with  $\|\rho(t)\|_{L^1} > 0$ . Then, for  $k \geq 2$ ,

$$\|\nu G_\rho\|_{H^k} \leq C[\|\rho_r\|_{H^{k-1}}] \|\nu\|_{H^k} \|\rho(t)\|_{H^{k-1}}. \quad (16)$$

*Proof:* See the proof of [60, Lemma 3.3].

With these estimates in hand, we can follow the same arguments as in [54] to show well-posedness of the dynamics described by PDE (6). In the sequel, we provide the sketch of the proof of Theorem II.1. For a more detailed version, see [60, Sec. 3].

*Proof sketch for Theorem II.1:* Consider the PDE sequence

$$\begin{aligned} (\rho_n)_t &= (\rho_n G_{\rho_{n-1}})_x + \frac{\sigma^2}{2} (\rho_n)_{xx} \quad \text{in } \tilde{X} \times (0, T) \\ \rho_n(\cdot + 2, t) &= \rho_n(\cdot, t) \quad \text{on } \partial\tilde{X} \times (0, T) \\ \rho_n(x, \cdot) &= \rho_0(x) \quad \text{on } \tilde{X} \times \{t = 0\} \end{aligned} \quad (17)$$

with smooth data  $\rho_0, \rho_r \in C^\infty_{\text{per}}(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  for now. By standard results on linear parabolic PDEs [61, Ch. 7], there exists a sequence  $\{\rho_n : n \geq 0\}$  in  $C^\infty(0, T; C^\infty_{\text{per}}(\tilde{X}))$  that satisfies (17). Furthermore, using the estimate (13) in Lemma III.1, one can follow the same procedure provided in [54, Proposition 3.1] to show  $\|\rho_n(t)\|_{L^1} = \|\rho_n(0)\|_{L^1} = 2$ , and hence,  $\rho_n(t) \geq 0$  for all  $n \geq 1$  and  $t \geq 0$  (see also [54, Corollary 3.2]).

*Remark III.4 (Evenness of  $\rho_n$ ):* One can use the evenness of  $\rho_0$  and  $\rho_r$  to show that the unique solutions  $\rho_n$  to PDEs (17) are also even in  $x$  for all  $t \geq 0$ . However, since this property will not be used for existence, uniqueness, and regularity results provided as follows, we will postpone this argument to later when we deal with the evenness of the unique solution to PDE (6).

*Existence with smooth data.* Using Lemmas III.1 and III.2 and following a similar idea as in [54, Lemmas 3.5 and 3.7], we can obtain the following convergence results for a limiting object  $\bar{\rho}$

$$\begin{aligned} \rho_n &\rightarrow \bar{\rho} \quad \text{in } L^1(0, T; L^1_{\text{per}}(\tilde{X})) \\ \rho_{n_k} &\rightarrow \bar{\rho} \quad \text{in } L^2(0, T; H^1_{\text{per}}(\tilde{X})) \\ \partial_t \rho_{n_k} &\rightarrow \bar{\rho}_t \quad \text{in } L^2(0, T; H^{-1}_{\text{per}}(\tilde{X})) \end{aligned} \quad (18)$$

where  $n_k$  denotes a subsequence. Moreover, we have the following estimate for  $\{\rho_n : n \geq 1\}$  and  $\bar{\rho}$

$$\|\rho\|_{L^\infty(0, T; L^2)} + \|\rho\|_{L^2(0, T; H^1)} + \|\rho_t\|_{L^2(0, T; H^{-1})} \leq C[T] \|\rho_0\|_{L^2}. \quad (19)$$

We claim that  $\bar{\rho}$  is the unique weak solution to (6). That is,  $\bar{\rho}$  solves the weak formulation of (6) defined as

$$\int_0^T \langle \eta, \rho_t \rangle dt + \int_0^T \int_{\tilde{X}} \left( \frac{\sigma^2}{2} \rho_x + \rho G_\rho \right) \eta_x dx dt = 0 \quad (20)$$

for any  $\eta \in L^2(0, T; H^1_{\text{per}}(\tilde{X}))$ . To this end, we multiply (17) by  $\eta$  with  $n = n_k$  and integrate to obtain

$$\begin{aligned} \int_0^T \langle \eta, \partial_t \rho_{n_k} \rangle dt + \frac{\sigma^2}{2} \int_0^T \int_{\tilde{X}} \partial_x \rho_{n_k} \eta_x dx dt \\ + \int_0^T \int_{\tilde{X}} \rho_{n_k} G_{\rho_{n_k-1}} \eta_x dx dt = 0. \end{aligned} \quad (21)$$

Using the convergence results (18) and the estimate (19), one can show that the limits of the three terms in (21) are zero as  $k \rightarrow \infty$ , that is,  $\bar{\rho}$  indeed satisfies the weak formulation (20). It remains to show  $\bar{\rho}(x, 0) = \rho_0(x)$ . Note that this condition makes sense since  $\bar{\rho} \in C(0, T; L^2_{\text{per}}(\tilde{X}))$  by [54, Th. 3.8] and the last two items in convergence results (18). Pick some



$\eta \in C^1(0, T; H_{\text{per}}^1(\tilde{X}))$  with  $\eta(T) = 0$  and rewrite the weak formulation (20) as

$$\begin{aligned} & - \int_0^T \langle \bar{\rho}, \eta_t \rangle dt + \int_0^T \int_{\tilde{X}} \left( \frac{\sigma^2}{2} \bar{\rho}_x + \bar{\rho} G_{\bar{\rho}} \right) \eta_x dx dt \\ & = \int_{\tilde{X}} \bar{\rho}(x, 0) \eta(x, 0) dx. \end{aligned} \quad (22)$$

Similarly, since  $\rho_{n_k}(x, 0) = \rho_0(x)$ , we have

$$\begin{aligned} & - \int_0^T \langle \rho_{n_k}, \eta_t \rangle dt + \int_0^T \int_{\tilde{X}} \left( \frac{\sigma^2}{2} \partial_x \rho_{n_k} + \rho_{n_k} G_{\rho_{n_k}} \right) \eta_x dx dt \\ & = \int_{\tilde{X}} \rho_0(x) \eta(x, 0) dx. \end{aligned} \quad (23)$$

Let  $k \rightarrow \infty$  in (23). Then, for arbitrary  $\eta(x, 0)$ , we obtain from (23) and (22) that

$$\int_{\tilde{X}} \bar{\rho}(x, 0) \eta(x, 0) dx = \int_{\tilde{X}} \rho_0(x) \eta(x, 0) dx.$$

This implies that  $\bar{\rho}(x, 0) = \rho_0(x)$ .

*Relaxed regularity on data:* In order to relax regularity assumption on data to  $\rho_0, \rho_r \in L_{\text{per}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$ , we can use the mollified version of the distributions  $\rho_0^\epsilon = \phi_\epsilon \star \rho_0$  and  $\rho_r^\epsilon = \phi_\epsilon \star \rho_r$  with the standard positive mollifier  $\phi_\epsilon$ , follow the same procedure and take the limit  $\epsilon \rightarrow 0$  at the end (see also [54, Th. 3.12] for the details of this process).

*Uniqueness:* Let  $\xi = \bar{\rho}_1 - \bar{\rho}_2$  where  $\bar{\rho}_1$  and  $\bar{\rho}_2$  are two weak solutions to (6) with  $\rho_0, \rho_r \in L_{\text{per}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$ . Then, for every  $\eta \in L^2(0, T; H_{\text{per}}^1(\tilde{X}))$  we have

$$\begin{aligned} & \int_0^T \langle \eta, \xi_t \rangle dt + \frac{\sigma^2}{2} \int_0^T \int_{\tilde{X}} \xi_x \eta_x dx dt \\ & + \int_0^T \int_{\tilde{X}} (\bar{\rho}_1 G_{\bar{\rho}_1} - \bar{\rho}_2 G_{\bar{\rho}_2}) \eta_x dx dt = 0. \end{aligned}$$

Setting  $\eta = \xi$  and using Lemma III.1, we can follow similar arguments to ones provided in [54, Th. 3.10] to obtain

$$\int_0^T \langle \xi, \xi_t \rangle dt \leq (C_1 + C_2[T] \|\rho_0\|_{L^2}^2) \|\xi\|_{L^2(0, T; L^2)}^2.$$

By [54, Th. 3.8], we know  $\langle \xi, \xi_t \rangle = \frac{1}{2} \frac{d}{dt} \|\xi(t)\|_{L^2}^2$ . Thus, for all  $T$ , we have

$$\frac{1}{2} \int_0^T \frac{d}{dt} \|\xi(t)\|_{L^2}^2 dt \leq (C_1 + C_2[T] \|\rho_0\|_{L^2}^2) \int_0^T \|\xi(t)\|_{L^2}^2 dt.$$

This implies that, for a.e.  $t \in [0, T]$

$$\frac{d}{dt} \|\xi(t)\|_{L^2}^2 \leq C[T, \rho_0] \|\xi(t)\|_{L^2}^2.$$

Hence, by Grönwall's inequality

$$\|\xi(t)\|_{L^2}^2 \leq C[T, \rho_0] \|\xi(0)\|_{L^2}^2 = C[T, \rho_0] \|\rho_0 - \rho_0\|_{L^2}^2 = 0.$$

That is,  $\|\xi(t)\|_{L^2} = \|\bar{\rho}_1(t) - \bar{\rho}_2(t)\|_{L^2} = 0$ . Then, from continuity of  $\bar{\rho}_1$  and  $\bar{\rho}_2$  in time (by [54, Th. 3.8]), we have  $\bar{\rho}_1 = \bar{\rho}_2$  for all  $t \in [0, T]$ .

*Regularity:* Employing Lemma III.3, we can extend the improved regularity results in space in [54, Th. 4.2]. To be precise,

for  $\rho_0 \in H_{\text{per}}^k(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  and  $\rho_r \in H_{\text{per}}^{k-1}(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$ , we have

$$\bar{\rho} \in L^2(0, T; H_{\text{per}}^{k+1}(\tilde{X})) \cap L^\infty(0, T; H_{\text{per}}^k(\tilde{X})). \quad (24)$$

Moreover, since  $\rho_r$  is constant in time, we can also employ the results on improved regularity in time provided by [54, Th. 4.3] for our model. That is, for  $\rho_0 \in H_{\text{per}}^{2k}(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  and  $\rho_r \in L_{\text{per}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$ , we have for  $i \leq k$

$$\partial_t^i \bar{\rho} \in L^2(0, T; H_{\text{per}}^{2k-2i+1}(\tilde{X})) \cap L^\infty(0, T; H_{\text{per}}^{2k-2i}(\tilde{X})) \quad (25)$$

and

$$\partial_t^{k+1} \bar{\rho} \in L^2(0, T; H_{\text{per}}^{-1}(\tilde{X})). \quad (26)$$

With these regularity results in space and time, we can derive the required regularity on the solution as stated in Theorem II.1. Let  $\rho_0 \in H_{\text{per}}^3(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  and  $\rho_r \in H_{\text{per}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  and also let  $\bar{\rho}$  be the unique weak solution to PDE (6). By (24), we have  $\bar{\rho} \in L^\infty(0, T; H_{\text{per}}^3(\tilde{X}))$ . Hence, by the Sobolev embedding theorem [62, Sec. 4.12], we have  $\bar{\rho}(t) \in C_{\text{per}}^2(\tilde{X})$  (after possibly being redefined on a set of measure zero). This gives the required regularity in space. Also, (25) and (26) imply that  $\bar{\rho}_t \in L^2(0, T; H_{\text{per}}^1(\tilde{X}))$  and  $\bar{\rho}_{tt} \in L^2(0, T; H_{\text{per}}^{-1}(\tilde{X}))$ . Hence, by [54, Th. 3.8], we have  $\bar{\rho}_t \in C(0, T; L_{\text{per}}^2(\tilde{X}))$  (after possibly being redefined on a set of measure zero). This gives the required regularity in time. Putting these results together, we have  $\bar{\rho} \in C^1(0, T; C_{\text{per}}^2(\tilde{X}))$ .

*Evenness:* The evenness imposed on  $\rho_0$  and  $\rho_r$  implies that if  $\rho(x, t)$  is a solution of (6), then  $\rho(-x, t)$  is also a solution. Then, assuming  $\rho_0 \in H_{\text{ep}}^3(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  and  $\rho_r \in H_{\text{ep}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  (notice that  $H_{\text{ep}}^k(\tilde{X}) \subset H_{\text{per}}^k(\tilde{X})$ ), the *uniqueness* of the solution  $\bar{\rho} \in C^1(0, T; C_{\text{per}}^2(\tilde{X}))$  to PDE (6) implies that the solution is even, that is,  $\bar{\rho} \in C^1(0, T; C_{\text{ep}}^2(\tilde{X}))$ .

*Positivity:* Using the same approach as in [59], we consider the following version of (6) in the unknown function  $\rho$  with  $\bar{\rho}$  being the nonnegative weak solution

$$\rho_t = (\rho G_{\bar{\rho}})_x + \frac{\sigma^2}{2} \rho_{xx}.$$

This is a linear parabolic PDE with smooth and bounded coefficients (by Lemmas III.1 and III.2) for which  $\bar{\rho}$  is a classic nonnegative solution. Thus, by parabolic Harnack inequality [61, Sec. 7.1.4, Th. 10], we have  $\sup_{x \in \tilde{X}} \bar{\rho}(x, t_1) \leq c \cdot \inf_{x \in \tilde{X}} \bar{\rho}(x, t_2)$  for  $0 < t_1 < t_2 < \infty$  and some constant  $c > 0$ . Nonnegativity of  $\bar{\rho}(x, t)$  implies that  $\inf_{x \in \tilde{X}} \bar{\rho}(x, t)$ , and thus,  $\bar{\rho}(x, t)$  are positive for all  $t > 0$ . ■

## IV. STATIONARY BEHAVIOR AND STABILITY

### A. Existence of Stationary Solution

This section mainly concerns the proof of *existence* result in Theorem II.2 for stationary equation (8). Throughout this section, the norms are w.r.t.  $X = [0, 1]$  (as opposed to  $\tilde{X} = [-1, 1]$ ), unless indicated otherwise. We note that norms on the even 2-periodic spaces computed w.r.t. to  $X$  and  $\tilde{X}$  differ by a multiplicative constant, e.g.,  $\|u\|_{L^p(\tilde{X})} = 2^{\frac{1}{p}} \|u\|_{L^p(X)}$ . We

again use  $C, C_0, C_1, \dots$  to represent a generic constant whose actual values may change from line to line. Also,  $C[\theta]$  denotes the dependence of the constant  $C$  on a particular object  $\theta$ .

Let us begin with providing a fixed point characterization of the solution to stationary equation (8). We note that, corresponding to the solution to dynamic equation (6), we are particularly interested in *even* solutions  $\rho^s \in \mathcal{P}_e(\tilde{X})$  of stationary equation (8).

*Lemma IV.1 (Fixed point characterization):*  $\rho^s \in C_{\text{ep}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$  is a solution of stationary equation (8) if and only if  $\rho^s$  is a fixed point of the operator  $\mathcal{T} : \mathcal{P}_e(\tilde{X}) \rightarrow \mathcal{P}_e(\tilde{X})$  defined as

$$\mathcal{T}\rho := \frac{1}{K} \exp\left(-\frac{2}{\sigma^2} \int_0^x G_\rho(z) dz\right) \quad (27)$$

where the constant  $K$  is determined by the normalizing condition

$$K = \int_0^1 \exp\left(-\frac{2}{\sigma^2} \int_0^x G_\rho(z) dz\right) dx.$$

*Proof:* See the proof of [60, Lemma 4.1]. ■

This characterization allows us to use tools from operator theory, in particular, Schauder fixed point theorem, to show existence of stationary solution. Before that, we present some preliminary results for the operator  $\mathcal{T}$ .

*Lemma IV.2 (Estimates for  $\mathcal{T}$ ):* Let  $\mathcal{T}$  be the operator on  $\mathcal{P}_e(\tilde{X})$  defined by (27).

1) If  $\rho, \rho_r \in \mathcal{P}_e(\tilde{X})$ , then

$$\|\mathcal{T}\rho\|_{L^\infty} \leq e^{8R(1+M)/\sigma^2} \quad (28)$$

and

$$\|\partial_x \mathcal{T}\rho\|_{L^\infty} \leq \frac{4R(1+M)}{\sigma^2} e^{8R(1+M)/\sigma^2}. \quad (29)$$

2) For  $k \geq 3$ , if  $\rho, \rho_r \in H_{\text{ep}}^{k-2}(\tilde{X}) \cap \mathcal{P}_e(\tilde{X})$ , then

$$\|\mathcal{T}\rho\|_{H^k} \leq \sum_{i=1}^{k-1} C_i \|\rho_r\|_{H^{k-2}} \|\rho\|_{H^{k-2}}^i. \quad (30)$$

*Proof:* See the proof of [60, Lemma 4.2]. ■

*Proposition IV.3 (Lipschitz continuity of  $\mathcal{T}$ ):* Let  $\mathcal{T}$  be the operator on  $\mathcal{P}_e(\tilde{X})$  defined by (27) with  $\rho_r \in \mathcal{P}_e(\tilde{X})$ . Then,  $\mathcal{T}$  is Lipschitz continuous in  $L^p$  for  $1 \leq p < \infty$  with Lipschitz constant

$$L_{\mathcal{T}} = \frac{1}{2} \exp\left(\frac{8R(1+M)(p-1)}{\sigma^2 p}\right) \left(e^{16R/\sigma^2} - 1\right). \quad (31)$$

*Proof:* See the proof of [60, Proposition 4.3]. ■

With this preliminary results in hand, we next move on to the proof of existence of stationary solution.

*Proof of Theorem II.2 (Existence):* Following the same argument as in [59, Th. 2.3] and using Lemma IV.1, we can present the existence result for the stationary solution as the fixed point of the operator  $\mathcal{T}$ . First note that using the estimate (28) in Lemma IV.2, we have  $\|\mathcal{T}\rho\|_{L^2} \leq C \|\mathcal{T}\rho\|_{L^\infty} \leq c$  for some positive constant  $c$ . Thus, for the purpose of finding the fixed points of  $\mathcal{T}$ , we can restrict  $\mathcal{T}$  to act on the closed and convex set  $E := \{\rho \in L_{\text{ep}}^2(\tilde{X}) \cap \mathcal{P}_e(\tilde{X}) : \|\rho\|_{L^2} \leq c\}$ . Now, from

inequalities (28) and (29) in Lemma IV.2, we have, for all  $\rho \in E$

$$\begin{aligned} \|\mathcal{T}\rho\|_{H^1}^2 &\leq \|\mathcal{T}\rho\|_{L^2}^2 + \|\partial_x \mathcal{T}\rho\|_{L^2}^2 \\ &\leq C_1 \|\mathcal{T}\rho\|_{L^\infty}^2 + C_2 \|\partial_x \mathcal{T}\rho\|_{L^\infty}^2 \leq c' \end{aligned} \quad (32)$$

for some constant  $c' > 0$ . That is,  $\mathcal{T}(E) \subset E$  is uniformly bounded in  $H_{\text{ep}}^1(\tilde{X})$ . Thus, by the Rellich–Kondrachov compactness theorem [61, Sec. 5.7, Th. 1],  $\mathcal{T}(E)$  is precompact in  $L_{\text{ep}}^2(\tilde{X})$ . Since  $E \subset L_{\text{ep}}^2(\tilde{X})$  is closed, this implies  $\mathcal{T}(E)$  is also precompact in  $E$ . Also,  $\mathcal{T}$  is Lipschitz continuous by Proposition IV.3. Hence, by Schauder fixed point theorem [61, Sec. 9.2.2, Th. 3], it has a fixed point  $\rho^s \in E$ , which by (32) belongs to  $H_{\text{ep}}^1(\tilde{X})$ .

*Regularity:* The estimate (30) in Lemma (IV.2) implies that if  $\rho_r \in H_{\text{ep}}^{k-2}(\tilde{X})$ , then the fixed point  $\rho^s = \mathcal{T}\rho^s \in H_{\text{ep}}^k(\tilde{X})$ . In particular, if  $\rho_r \in H_{\text{ep}}^1(\tilde{X})$ , then  $\rho^s \in H_{\text{ep}}^3(\tilde{X})$ . Hence, by Sobolev embedding theorem,  $\rho^s \in C_{\text{ep}}^2(\tilde{X})$  (after possibly being redefined on a set of measure zero).

*Positivity:* Strict positivity of the fixed point (stationary solution) immediately follows from the representation (27). ■

*Remark IV.4 (Uniqueness):* By Proposition IV.3,  $\mathcal{T}$  is Lipschitz continuous in  $L^p$  with Lipschitz constant  $L_{\mathcal{T}}$  given by (31), and thus, is a contraction for  $L_{\mathcal{T}} < 1$ . Hence, by Banach fixed-point theorem [61, Sec. 9.2.1, Th. 1],  $\mathcal{T}$  has a unique fixed point for  $L_{\mathcal{T}} < 1$ . Setting  $p = 1$  in (31) gives the sufficient condition  $\sigma^2 > 16R/\ln 3$  for *uniqueness* of stationary solution. This result corresponds to the sufficient condition provided in [52, Th. 2].

## B. Global Estimate for Stationary Solution

This section is devoted to the proof of the *estimate* given in Theorem II.2. In this section, all the norms are w.r.t. the domain  $\tilde{X} = [-1, 1]$ , unless indicated otherwise.

*Proof of Theorem II.2 (Estimate):* Let  $\psi = \rho^s - 1$  so that  $\int_{\tilde{X}} \psi(x) dx = 0$ . From the stationary equation (8), we obtain

$$-\frac{\sigma^2}{2} \psi_{xx} = [(\psi + 1) G_{\psi+1}]_x = [\psi G_\psi]_x + [G_\psi]_x$$

where we used the fact that  $w * 1 = 0$ . Next, we multiply this last equation by  $\psi$  and integrate by part over  $\tilde{X}$  to derive (extra terms are zero due to periodicity)

$$\frac{\sigma^2}{2} \|\psi_x\|_{L^2}^2 = - \int_{\tilde{X}} \psi_x \psi G_\psi dx - \int_{\tilde{X}} \psi_x G_\psi dx.$$

Thus,

$$\begin{aligned} \frac{\sigma^2}{2} \|\psi_x\|_{L^2}^2 &\leq \left| \int_{\tilde{X}} \psi_x \psi G_\psi dx \right| + \left| \int_{\tilde{X}} \psi_x G_\psi dx \right| \\ &\leq \|G_\psi\|_{L^\infty} \|\psi_x\|_{L^2} \|\psi\|_{L^2} + \|\psi_x\|_{L^2} \|G_\psi\|_{L^2}. \end{aligned} \quad (33)$$

Now, using inequality (12) in Lemma III.1, we obtain

$$\begin{aligned} \|G_\psi\|_{L^\infty} &\leq 2R (\|\psi\|_{L^1(X)} + M) \\ &\leq 2R (\|\rho\|_{L^1(X)} + 1 + M) \leq 2R(M + 2). \end{aligned} \quad (34)$$

Also, we have

$$\begin{aligned} |G_\psi(x)|^2 &= \left( \int_{x-R}^{x+R} (x-y) (\psi(y) + M\rho_r(y)) dy \right)^2 \\ &\leq \int_{x-R}^{x+R} (x-y)^2 dy \int_{x-R}^{x+R} (\psi(y) + M\rho_r(y))^2 dy \\ &\leq (2/3)R^3 \int_{x-R}^{x+R} (\psi(y) + M\rho_r(y))^2 dy. \end{aligned} \quad (35)$$

Hence,

$$\begin{aligned} \|G_\psi\|_{L^2}^2 &\leq (2/3)R^3 \int_{\tilde{X}} \int_{x-R}^{x+R} (\psi(y) + M\rho_r(y))^2 dy dx \\ &= (2/3)R^3 \int_{\tilde{X}} \int_{-R}^R (\psi(x+y) + M\rho_r(x+y))^2 dy dx \\ &= (4/3)R^4 \|\psi + M\rho_r\|_{L^2}^2. \end{aligned} \quad (36)$$

Using estimates (34) and (36) in (33), we have (recall that uniform distribution is not an equilibrium of the system and hence  $\|\psi_x\|_{L^2} \neq 0$ )

$$\begin{aligned} \frac{\sigma^2}{2} \|\psi_x\|_{L^2} &\leq 2R(M+2)\|\psi\|_{L^2} + (2R^2/\sqrt{3})\|\psi + M\rho_r\|_{L^2} \\ &\leq 2R(M+2)\|\psi\|_{L^2} + (2R^2/\sqrt{3})(\|\psi\|_{L^2} + M\|\rho_r\|_{L^2}) \\ &= 2R \left( M + \frac{R}{\sqrt{3}} + 2 \right) \|\psi\|_{L^2} + \frac{2R^2M}{\sqrt{3}} \|\rho_r\|_{L^2}. \end{aligned} \quad (37)$$

Now, since  $\int_{\tilde{X}} \psi(x) dx = 0$ , Poincaré inequality [61, Sec. 5.8.1, Th. 1] implies that  $\|\psi\|_{L^2} \leq C \|\psi_x\|_{L^2}$ . The optimal value for the Poincaré constant for  $\tilde{X} = [-1, 1]$  is  $C = 1/\pi$ . Using this for inequality (37), we have

$$\left( \sigma^2 - \frac{4R}{\pi} \left( M + \frac{R}{\sqrt{3}} + 2 \right) \right) \|\psi\|_{L^2} \leq \frac{4R^2M}{\pi\sqrt{3}} \|\rho_r\|_{L^2}. \quad (38)$$

Defining  $\sigma_b$  and  $c_b$  as in (9) gives the desired inequality  $\|\psi\|_{L^2} \leq \frac{1}{\eta} \|\rho_r\|_{L^2}$ , where  $\eta = (\sigma^2 - \sigma_b^2)/c_b$ . ■

### C. Stability of Stationary State

This section is devoted to the proof of Theorem II.3 concerning the stability of stationary state. All the norms in this section are w.r.t. the domain  $\tilde{X} = [-1, 1]$  (as opposed to  $X = [0, 1]$ ), unless indicated otherwise.

*Proof of Theorem II.3:* We follow a similar argument as in [54], except we consider a general stationary state  $\rho^s$ , instead of the uniform distribution considered in [54]. Let  $\psi = \rho - \rho^s$  so that  $\int_{\tilde{X}} \psi(x) dx = 0$ . Using the fact that  $\rho^s$  is a solution to the stationary equation (8), that is,

$$[\rho^s G_{\rho^s}]_x + \frac{\sigma^2}{2} \rho^s_{xx} = 0$$

and inserting  $\rho = \psi + \rho^s$  into equation (6), we obtain

$$\psi_t = [\psi(w \star \psi + G_{\rho^s})]_x + [\rho^s(w \star \psi)]_x + \frac{\sigma^2}{2} \psi_{xx}.$$

Multiplying this equation by  $\psi$  and integrating by part over  $\tilde{X}$ , we obtain (the extra terms are zero due to periodicity)

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\psi\|_{L^2}^2 + \frac{\sigma^2}{2} \|\psi_x\|_{L^2}^2 &\leq \left| \int_{\tilde{X}} \psi_x \psi (w \star \psi + G_{\rho^s}) dx \right| + \left| \int_{\tilde{X}} \psi_x \rho^s (w \star \psi) dx \right| \\ &\leq (\|w \star \psi\|_{L^\infty} + \|G_{\rho^s}\|_{L^\infty}) \|\psi_x\|_{L^2} \|\psi\|_{L^2} \\ &\quad + \|\rho^s\|_{L^\infty} \|\psi_x\|_{L^2} \|w \star \psi\|_{L^2}. \end{aligned} \quad (39)$$

Now, from inequality (12) in Lemma III.1, we have

$$\begin{aligned} \|w \star \psi\|_{L^\infty} &\leq 2R \|\psi\|_{L^1(X)} \\ &\leq 2R (\|\rho\|_{L^1(X)} + \|\rho^s\|_{L^1(X)}) = 4R \end{aligned}$$

and

$$\|G_{\rho^s}\|_{L^\infty} \leq 2R (\|\rho^s\|_{L^1(X)} + M) = 2R(1 + M).$$

Also, following a similar procedure as in (35) and (36) with  $M = 0$ , we obtain  $\|w \star \psi\|_{L^2} \leq (2R^2/\sqrt{3}) \|\psi\|_{L^2}$ . Finally, from (28) in Lemma IV.2, we have  $\|\rho^s\|_{L^\infty} \leq e^{8R(1+M)/\sigma^2}$ . To save space, let us define

$$\alpha := 2R(3 + M) + (2R^2/\sqrt{3}) e^{8R(1+M)/\sigma^2}. \quad (40)$$

Using these estimates and the Young's inequality, we can rewrite (39) as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\psi\|_{L^2}^2 + \frac{\sigma^2}{2} \|\psi_x\|_{L^2}^2 &\leq \alpha \|\psi_x\|_{L^2} \|\psi\|_{L^2} \\ &\leq \frac{\alpha^2}{\sigma^2} \|\psi\|_{L^2}^2 + \frac{\sigma^2}{4} \|\psi_x\|_{L^2}^2. \end{aligned}$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|\psi\|_{L^2}^2 \leq \frac{\alpha^2}{\sigma^2} \|\psi\|_{L^2}^2 - \frac{\sigma^2}{4} \|\psi_x\|_{L^2}^2.$$

Once again, since  $\int_{\tilde{X}} \psi(x) dx = 0$ , we can use the Poincaré inequality  $\|\psi\|_{L^2} \leq C \|\psi_x\|_{L^2}$  with optimal Poincaré constant  $C = 1/\pi$ , to obtain

$$\frac{d}{dt} \|\psi\|_{L^2}^2 \leq \left( \frac{2\alpha^2}{\sigma^2} - \frac{\pi^2\sigma^2}{2} \right) \|\psi\|_{L^2}^2.$$

Then, by Grönwall's inequality, we have

$$\|\psi(t)\|_{L^2}^2 \leq \|\psi(0)\|_{L^2}^2 \exp \left\{ \left( \frac{2\alpha^2}{\sigma^2} - \frac{\pi^2\sigma^2}{2} \right) t \right\}.$$

Now, notice that  $\|\psi(0)\|_{L^2} \leq \|\rho_0\|_{L^2} + \|\rho^s\|_{L^2}$  is finite. Thus, if the constant factor in the exponential is negative, then  $\|\psi(t)\|_{L^2}^2 \rightarrow 0$  as  $t \rightarrow \infty$ . Negativity of the this constant factor corresponds to the condition  $\sigma > \sigma_s$ , where  $\sigma_s > 0$  solves (10) – the object  $\alpha$  is defined in (40). ■

## V. CLUSTERING BEHAVIOR: FOURIER ANALYSIS

In this section, we exploit the periodic nature of the system and use Fourier analysis to study the behavior of the solution to the PDE (6) with *uniform initial condition*  $\rho_0 = 1$ . To this end, we derive a system of ordinary differential equations (ODEs)

describing the evolution of Fourier coefficients of the normal opinion density  $\rho$ . These ODEs are then used to provide an approximation scheme for characterizing the initial clustering behavior of the system including the number and the timing of possible clusters. This numerical scheme is in essence similar to the linear stability analysis previously employed by [37], [40], [43], [53], [63] for analysis of noisy bounded confidence models without radicals.

### A. Finite-Dimensional Approximation in Frequency Domain

Notice that the set  $\{\cos(\pi n x)\}_{n=0}^{\infty}$  is an orthogonal basis for the space  $L_{\text{ep}}^2(\tilde{X})$  containing even 2-periodic functions on  $\tilde{X} = [-1, 1]$ . Then, the even 2-periodic extension of the probability densities in the model allows us to consider the Fourier expansions of  $\rho$  and  $\rho_r$  in the form of

$$\begin{cases} \rho(x, t) = \sum_{n=0}^{\infty} p_n(t) \cos(\pi n x) \\ \rho_r(x) = \sum_{n=0}^{\infty} q_n \cos(\pi n x). \end{cases} \quad (41)$$

By inserting the expansions (41) into (6) and setting the inner product of the residual with elements of the basis to zero (in other words, taking inverse Fourier transform), we can obtain a system of quadratic ODEs describing the evolution of Fourier coefficients  $p_n(t)$ . Considering only the first frequency components  $n = 1, \dots, N_f$  of the expansion, these ODEs are expressed as

$$\dot{p}_n = c_n + b_n^T p + p^T Q_n p \quad (42)$$

where  $p = (p_1, p_2, \dots, p_{N_f})^T$ . Note that for  $n = 0$ , i.e., the constant term in the Fourier expansion, we obtain  $\dot{p}_0 = 0$ . This is due to the periodic nature of the system that preserves the zeroth moment. The coefficients in (42) are given by

$$\begin{aligned} c_n &= 2MRf_n q_n \\ (b_n)_k &= \begin{cases} 2Rf_n + \frac{MR}{2} f_{2n} q_{2n} - \frac{\pi^2 \sigma^2 n^2}{2}, & k = n \\ nMR(g_{n+k} + g_{n-k}), & k \neq n \end{cases} \\ (Q_n)_{k,l} &= \begin{cases} nRf_k/k, & l = n - k > 1 \\ nR \left( \frac{f_k}{k} + \frac{f_{n-k}}{n-k} \right), & l = k - n > 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (43)$$

where  $f_i := -\cos(\pi i R) + \text{sinc}(\pi i R)$  and  $g_i := q_{|i|} f_i / i$ , with  $\text{sinc } x = \sin x / x$ . Recall that  $q_n, n \in \mathbb{N}$  are the Fourier coefficients of  $\rho_r$ .

### B. Initial Clustering Behavior

The possible clustering behavior of noisy interacting particle systems is known to mainly depend on the noise level. That is, for noises larger than a critical level, the random drifts due to noise are expected to overcome the attractive forces among the agent and suppress clustering behaviors (see Fig. 2 for numerical simulations of the model under study for increasing levels of noise). A detailed discussion on this phenomenon commonly known as order-disorder transition for the system under study is provided in [60, Secs. 5.2 and 6.2].

For noises smaller than the critical level corresponding to order-disorder transition, agents start to form clusters. Here, we

are interested in the characterization of this initial clustering behavior. To this end, we make use of the *exponential growth rate*  $\gamma_n := (b_n)_n$  and *linear growth rate*  $c_n$  given in (43). The proposed numerical scheme is as follows. We ignore the interactions between different frequencies in (42), that is, for each frequency  $n = 1, \dots, N_f$ , we consider the equation  $\dot{p}_n = c_n + \gamma_n p_n$  with  $p_n(0) = 0$  for the initial evolution of the Fourier coefficient  $p_n$ . Then, for a given set of model parameters  $(\sigma, R, M)$  and radical opinions density  $\rho_r$ , we numerically compute the *dominant wave-number*  $n^* := \arg \max_{n \in \mathbb{N}} \gamma_n$  with  $\gamma_{n^*} > 0$ , that is, the unstable mode with the largest exponential growth rate. We speculate that the corresponding trigonometric term  $p_{n^*} \cos(\pi n^* x)$  is the dominant component of the initial clustering behavior and, hence, determines the number of initially formed clusters. The sign of  $p_{n^*}$  depends on the linear growth rate  $c_{n^*}$ ; precisely,  $p_{n^*} > 0$  if  $c_{n^*} > 0$ , and  $p_{n^*} < 0$  otherwise.

Considering the even 2-periodic extension of the model, the dominant wave-form must be interpreted on the interval  $\tilde{X} = [-1, 1]$ . Then, the *number of initial clusters*  $n_{\text{clu}}$  in the interval  $X = [0, 1]$ , resulting from the wave-form  $1 + p_{n^*} \cos(\pi n^* x)$ , is given by

$$n_{\text{clu}} := \begin{cases} \lfloor \frac{n^*}{2} \rfloor + 1, & c_{n^*} > 0 \\ \lceil \frac{n^*}{2} \rceil, & c_{n^*} < 0. \end{cases} \quad (44)$$

We also expect that the timing of this initial clustering behavior to be inversely related to the corresponding exponential growth rate  $\gamma_{n^*}$ . Indeed, by solving for the time for which the solution to the equation  $\dot{p}_n = c_n + \gamma_n p_n$  is equal to  $\pm 1$ , we can approximate the *time to initial clustering*  $t_{\text{clu}}$  as

$$t_{\text{clu}} := \frac{1}{\gamma_{n^*}} \ln \left( 1 + \frac{\gamma_{n^*}}{|c_{n^*}|} \right). \quad (45)$$

A similar approximation has been used in [53] in order to derive the time to the initial clustering using fluctuation theory.

## VI. NUMERICAL STUDY

In this final section, we provide a numerical study of the model at hand for a particular distribution of radical agents/opinions through simulations of the corresponding discrete- and continuum-agent models. In particular, we validate the result of Fourier analysis for characterization of initial clustering behavior presented in Section V-B.

### A. Setup

The particular radical distribution considered in this section is a triangular distribution with average  $A$  and width  $2S$

$$\rho_r(x) = \frac{S - |x - A|}{S^2} \text{ for } |x - A| \leq S, \text{ 0 otherwise.} \quad (46)$$

Although this choice may seem specific, it is rich enough for our purposes. In particular, with this choice, the zeroth, first, and second moments of the radical opinions density are simply captured by the parameters  $M$ ,  $A$ , and  $S$ , respectively. Moreover, we assume that the radicals are concentrated around their average opinion, that is, we consider small values of  $S$  (w.r.t. the confidence range  $R$ ).

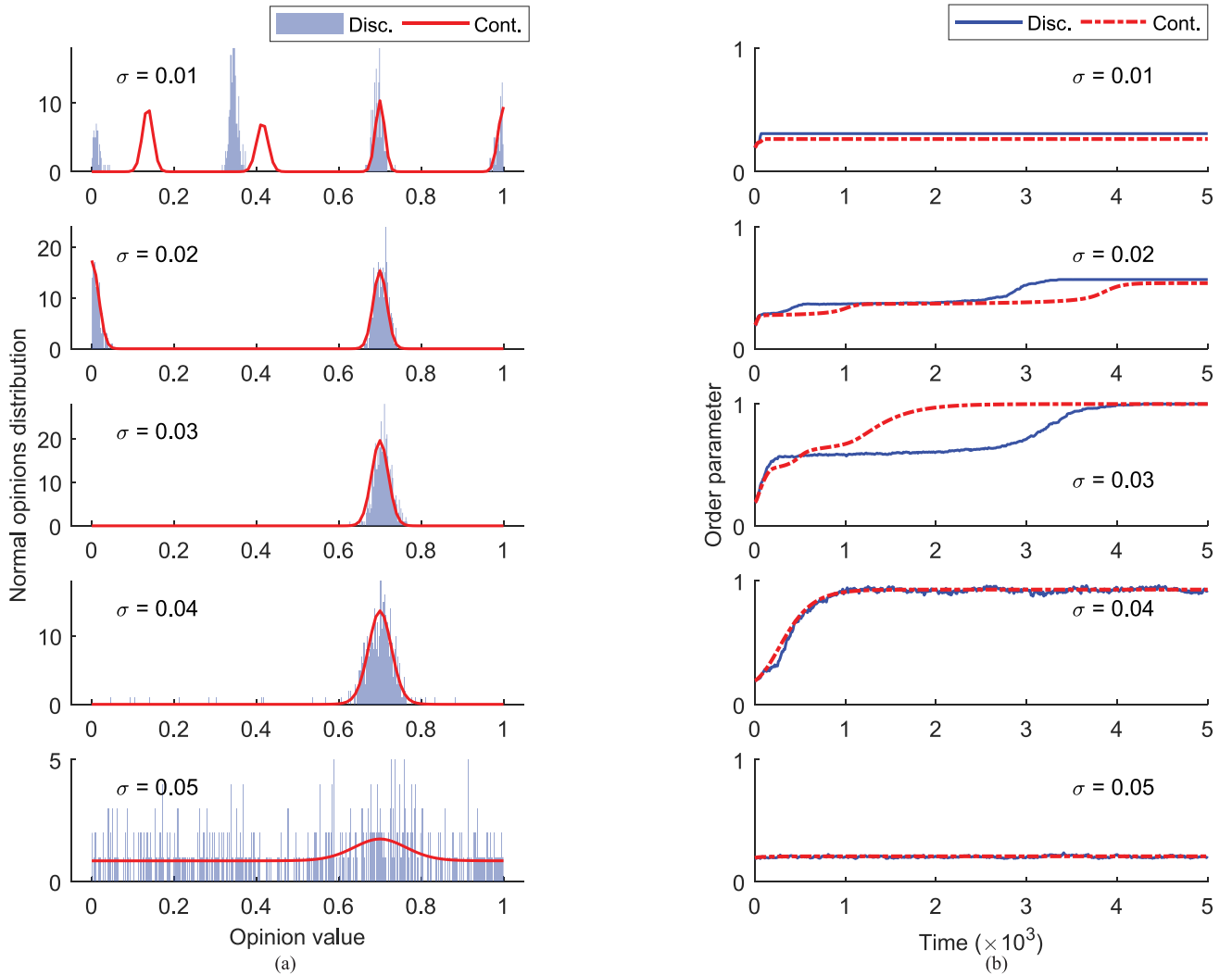


Fig. 2. Numerical simulation of the discrete-agent model (Disc.) and continuum-agent model (Cont.) for different values of noise  $\sigma$  with model parameters  $(R, M) = (0.1, 0.1)$ : (a) Distribution of opinions/agents at  $t = 10^4$  (b) Evolution of the order parameter. The initial distribution of normal agents/opinions is taken to be uniform and the distribution of radical agent/opinions is the triangular distribution of (46) with  $(A, S) = (0.7, 0.1)$ . As noise increases the number of clusters decreases so that for a large enough noise (e.g.,  $\sigma = 0.05$ ), the clustering behavior almost disappears (order-disorder transition). Moreover, in case of a clustered profile, the inverse of the order parameter is approximately equal to the number of clusters; e.g., for  $\sigma = 0.03$  the system first forms a two-cluster profile (the flat area with order parameter around 0.5), and then converges to a single-cluster profile (with order parameter equal to one).

For the discrete-agent model, the SDEs (4) are solved numerically using the Euler–Maruyama method for  $N = 500$  normal agents with time step  $\Delta t = 0.01$ . In particular, for the radical agents, we produce a random sample of size  $N_r = MN$  from the triangular distribution (46). The initial distribution of normal agents is taken to be uniform, that is, the initial opinions are randomly sampled from a uniform distribution on the interval  $X = [0, 1]$ . For complete correspondence between the discrete- and continuum-agent models, we also consider the effect of even 2-periodic extension in the simulations of the discrete-agent model.

For numerical simulation of the continuum-agent model described by PDF (6), we use the ODEs (42) to compute the coefficients of Fourier expansion of normal opinion density  $\rho$  using the first  $N_f = 128$  terms of the expansion. However, regarding the radical opinion density, one notices that the considered triangular

distribution does not satisfy the conditions of Theorem II.1 for well-posedness of PDE (6), that is  $\rho_r \notin H_{\text{ep}}^2(\tilde{X})$ . This will not be an issue since we will be working with the projection of the proposed  $\rho_r$  in the Hilbert space  $L_{\text{ep}}^2(\tilde{X})$ . That is, we use the Fourier coefficients of  $\rho_r$  in (42), which for the triangular distribution (46) are given by

$$q_n = 2 \cos(n\pi A) \text{sinc}^2(n\pi S/2). \quad (47)$$

To be precise, we need the Fourier coefficients  $q_n$  of  $\rho_r$  for  $1 \leq n \leq 2N_f$ , that is, twice the length of Fourier expansion of  $\rho$  [see the linear terms of (42)]. For the initial condition, we again consider uniform distribution  $\rho_0 = 1$ , which corresponds to  $p_0 = 1$  and  $p_n(0) = 0$  for the Fourier coefficients. A detailed description of the numerical scheme for simulation of the discrete- and continuum-agent models is provided in [60, Sec. 6].

In the sequel, we use the *order parameter*

$$Q_d(t) = N^{-2} \sum_{i,j=1}^N \mathbf{1}_{|x_i(t)-x_j(t)| \leq R}$$

introduced in [43] and its continuum counterpart

$$Q_c(t) = \int_{X^2} \rho(x,t)\rho(y,t)\mathbf{1}_{|x-y| \leq R} dx dy$$

to quantify orderedness in the clustering behavior of the model. In particular, we use the evolution of the order parameter for a better characterization of the *timing* of the clustering behavior in the simulation results. In words, the order parameter  $Q$  is the (normalized) number/mass of agents that are in  $R$ -neighborhood of and hence interacting with each other. In particular, in the continuum case,  $Q_c = 2R$  for a uniform distribution of opinions (absolute disorder), while  $Q_c = 1$  for a single-cluster distribution with all agents residing in an interval of width  $R$  or less (complete order). In case of a clustered behavior, roughly speaking, the inverse of the order parameter is equal to the *number of clusters*. For instance, the  $2R$ -conjecture [42], [43] states that for a noiseless system without radicals and starting from a uniform initial distribution ( $Q = 1$ ), the dynamics will converge to a clustered profile with the distance between clusters being (approximately)  $2R$ , or equivalently, with  $\frac{1}{2R}$  clusters ( $Q \approx 2R$ ). In the presence of noise, however, the system experiences a phase transition (order-disorder transition) depending on the noise level [43], [53]. In particular, for noises larger than a critical level (depending on  $R$ ), the clustering behavior disappears and the system converges to a (somewhat) uniform state with  $Q \approx 1$ . As shown in Fig. 2, the same transition occurs in noisy systems in presence of radicals, considered in this article. We refer the reader to [60, Secs. 5.2 and 6.2] for a detailed discussion on order-disorder transition for the system considered in this article.

In all the simulation results reported in this section, the width of radicals distribution and the confidence range are fixed at  $S = 0.1$  and  $R = 0.1$ , respectively.

## B. Initial Clustering Behavior

For sufficiently small noises, agents start to form clusters (see Figs. 2 and 3). In particular, we observe a cluster of normal agents around the average radical opinion  $A$  due to the force field generated by the radicals. Generally, the following three types of clusters may form:

- 1) the cluster at the average radical opinion  $A$ ;
- 2) the cluster(s) at the extreme opinions  $x = 0, 1$ ;
- 3) the cluster(s) around opinion values other than  $x = 0, 1, A$ .

The third type of clusters is expected to perform a random walk with their center of mass moving like a Brownian motion (assuming clusters do not interact). The effective diffusivity of these Brownian motions is inversely related to the size of the cluster, i.e., the number of agents in the cluster. This will result in a process of consecutive merging between these clusters until complete disappearance of them. Detailed descriptions of this process are provided in [43] and [53]. Notice, however, that

this description does not apply to cluster(s) formed at  $x = A$  and  $x = 0, 1$ . These clusters are affected by forces other than the normal attractions among the agents within the cluster. The cluster formed at  $x = A$  is under influence of radicals and the possible clusters at the extreme opinions  $x = 0, 1$  are reinforced due to the even 2-periodic extension considered in our model. The behavior of these clusters (survival or dissolution) depends on their size, the exogenous force acting on them, and the effect of other clusters in their neighborhood.

We now use the analysis scheme provided in Section V-B to investigate the effect of the zeroth and first moment of radicals ( $M$  and  $A$ , respectively) on the initial clustering behavior of the model for sufficiently small noises. In particular, we investigate the effect of  $M$  and  $A$  on the number, position, and timing of initial clusters for different values of  $\sigma$ . We again emphasize that we are considering a concentrated triangular distribution for radical agents and a uniform initial distribution for normal agents.

Following the provided scheme in Section V-B, we can compute the dominant wave-number  $n^*$ , number of initial clusters  $n_{\text{clu}}$ , and time to initial clustering  $t_{\text{clu}}$  for a general combination of model parameter  $(\sigma, M, A)$ ; recall that we fixed  $R = S = 0.1$ . A detailed illustrative example on this process is provided in [60, Sec. 6.3.1]. Fig. 4 shows the result of this analysis for different values of  $M$  and  $A$  at three different noise levels  $\sigma$ . Here, we only consider the values  $A < 1 - R = 0.9$  since for  $1 - R < A < 1$  the boundary effect due to even 2-periodic extension comes into play.

Comparing the left, middle, and right panels of Fig. 4 for different levels of noise, the analysis shows that as the level of noise increases, the number of clusters in the possible clustering behavior of the system is expected to decrease (see Fig. 4(b)), while the timing experiences a general increase (see Fig. 4(c)). In particular, with respect to the timing, we notice that as the level of noise decreases, the initial clustered profile is expected to emerge faster. Indeed, these effects can be seen in the simulation results depicted in Fig 2(b).

For low levels of noise, e.g.,  $\sigma = 0.01$  (see the left panels in Fig. 4), the analysis shows that the dominant wave-number does not depend on the  $M$  or  $A$ . However, both  $M$  and  $A$  are expected to affect the timing of the initial clustering behavior. In particular, as  $M$  increases,  $t_{\text{clu}}$  decreases. Fig. 3 shows the simulation results for  $\sigma = 0.01$  and compares the evolution of opinions for different values of  $M$  and  $A$ . For the continuum model in the top panels of Fig. 3, we observe that a four-cluster profile has emerged in all systems, while the analysis predicts a four-cluster profile for  $\mathcal{S}_1$  and  $\mathcal{S}_2$  and a five-cluster profile for  $\mathcal{S}_3$  for the initial clustering (see the left panel of Fig. 4). Moreover, comparing Fig. 3(a) and (b) shows that  $M$  only affects the timing of clustering behavior. This effect is better seen in Fig. 3(g), where we observe a faster convergence of order parameter for  $\mathcal{S}_2$  with larger  $M$ . On the other hand, comparing Fig. 3(b) and (c) corresponding to  $A = 0.85$  and  $A = 0.7$ , respectively, we observe that the first moment of radical opinions density  $A$  mainly affects the position of clusters. That is, the clustered profile emerges in a way that we observe a particular cluster formed at the average radical opinion  $A$ .

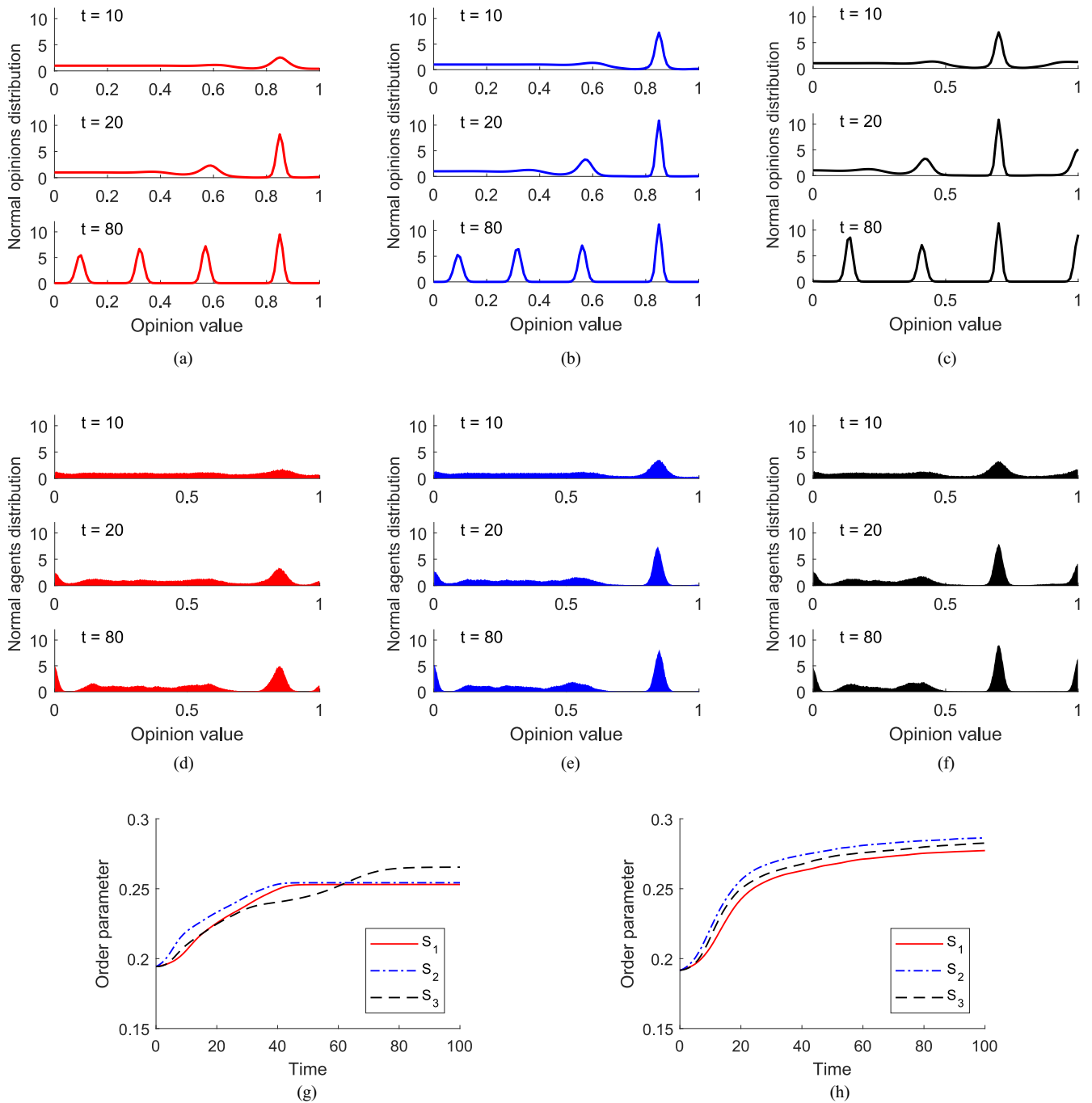


Fig. 3. Numerical simulation of the model with  $\sigma = 0.01$  for different values of  $(M, A)$ : (a,d)  $S_1 : (0.05, 0.85)$ , (b,e)  $S_2 : (0.15, 0.85)$ , and (c,f)  $S_3 : (0.15, 0.7)$ . The upper panels (a, b, and c) show the opinion distribution for continuum-agent model. The middle panels (d, e, and f) show the the result of Monte Carlo simulation (average of 300 realizations) of discrete-agent model. The lower panels show the evolution of order parameter for the corresponding (g) continuum-agent and (h) discrete-agent systems.

Monte Carlo simulations of the discrete-agent model reveal that the same general description also holds for this system. This is particularly seen in the time evolution of the order parameter in the discrete-agent model as depicted in Fig. 3(h). However, we note that there are differences between the behavior of the continuum- and discrete-agent models. First, we observe an almost uniform distribution of normal agents in the opinion range  $[0.1; 0.5]$  in the middle panels of Fig. 3 for the discrete-agent model. This is due to the fact that the exact position of the

corresponding clusters formed in the discrete-agent model varies within this range. Individual realizations of the discrete model show one, two, or three clusters in this range. This effect can be clearly seen in the top panel of Fig. 2 for  $\sigma = 0.01$ , and has also been reported by [37] in Monte Carlo simulations of a noisy Defuant model. Second, the evolution of order parameter in Fig. 3(g) shows that the continuum-agent model has seemingly converged to steady-state with four clusters, while this is clearly not the case for the discrete-agent model as can be seen in

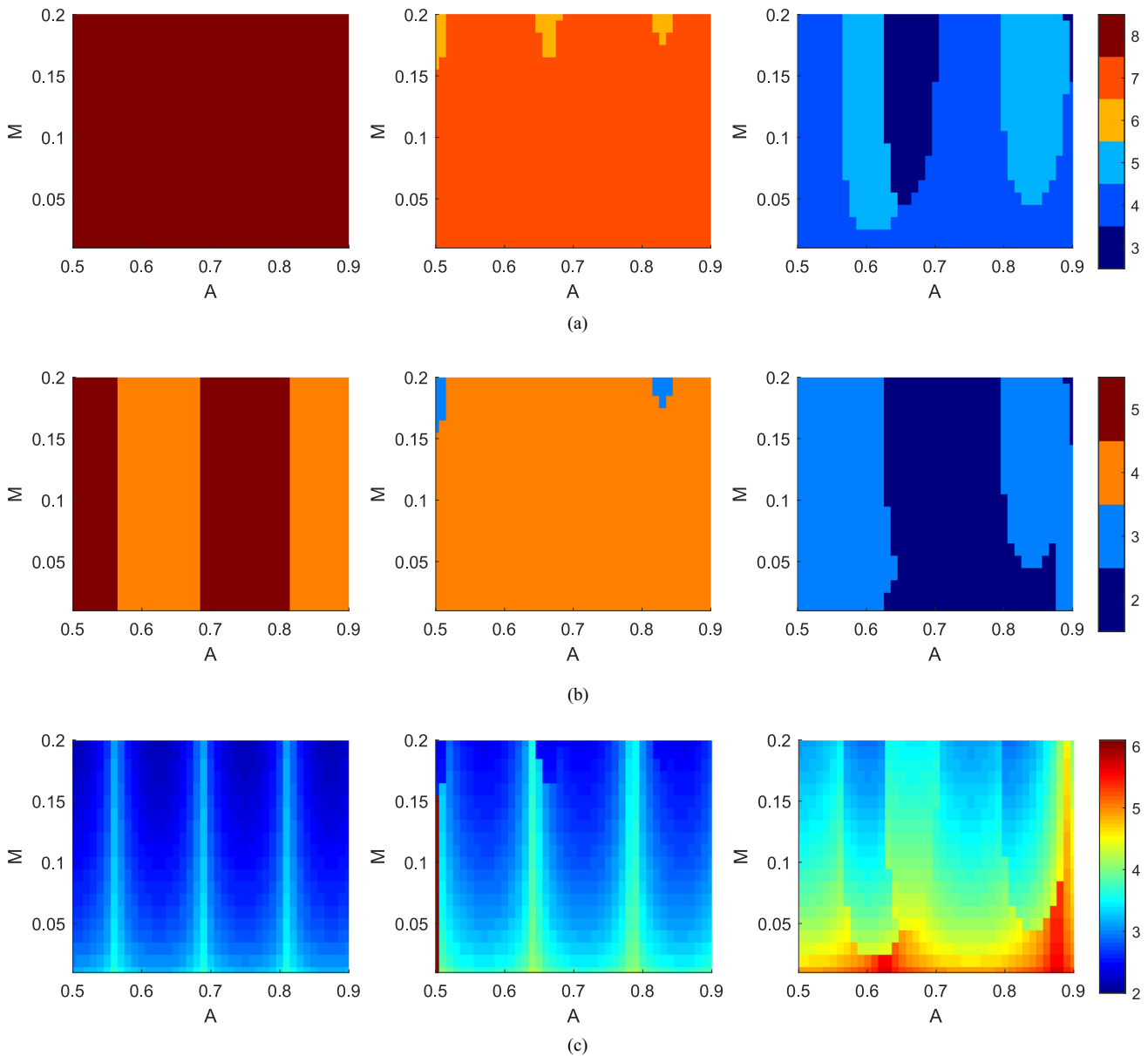


Fig. 4. Characterization of the initial clustering behavior based on the dominant wave-number in the Fourier expansion of the continuum-agent model for different values of  $M$  and  $A$  with noise levels  $\sigma = 0.01$  (left column),  $\sigma = 0.02$  (middle column), and  $\sigma = 0.03$  (right column) : (a) Dominant wave-number  $n^*$  (b) Number of initial clusters  $n_{\text{clu}}$  (c) Time to initial clustering  $\ln(t_{\text{clu}})$ .

Fig. 3(h). Indeed, in the discrete-agent model, as described in the beginning of this section, all the possible clusters formed around opinion values other than  $x = 0, 1, A$  will necessarily disappear in the steady-state profile, where the time required for their disappearance depends on the noise level and particularly the size of these clusters. Hence, unlike the discrete-agent model, for the continuum-agent model (in the limit  $N \rightarrow \infty$ ), the system may require *infinite time* for this merging of the clusters to occur. This, in turn, can lead to different behaviors in the discrete- and continuum-agent models over exponentially large times scales [43] (see also [60, Section 6]).

As shown in Fig. 4, for higher levels of noise, e.g.,  $\sigma = 0.03$ , we observe nonlinear effects. That is,  $M$  and  $A$  start to affect the dominant wave-number (see the middle and right panels of

Fig. 4). Nevertheless, these effects are limited as the number of clusters is still 3 or 4 for  $\sigma = 0.02$  and 2 or 3 for  $\sigma = 0.03$ . Besides, we still observe a general increase in the timing of the clustering behavior as  $M$  decreases. Numerical simulations of the model for  $\sigma = 0.03$  are also in general agreement with these predictions (results not shown here, see [60, Sec. 6.3.2]).

To summarize the discussions mentioned above, for concentrated distribution of radicals, the main effect of the zeroth and first moments of radical distribution is on the timing and positioning of the possible clustering behavior, respectively. The number of clusters (to be precise, the life-time of possible transient clustered profiles) is mainly determined by the noise level of the system. This is particularly the case for lower levels of noise.



## APPENDIX A PRELIMINARIES ON FUNCTION SPACES

The definitions provided in this section are mostly borrowed from [61]. Let  $\{f_k\}_{k=1}^\infty$  be a sequence in a Banach space  $B$  with norm  $\|\cdot\|_B$ . The *strong* convergence  $f_k \rightarrow f$  implies  $\|f_k - f\|_B \rightarrow 0$ , while the *weak* convergence  $f_k \rightharpoonup f$  implies  $g(f_k) \rightarrow g(f)$  for all bounded linear functionals  $g : B \rightarrow \mathbb{R}$ .

Let  $f : \tilde{X} \rightarrow \mathbb{R}$  be a measurable function on  $\tilde{X} = (-1, 1)$ . The  $L^p$ -norm of  $f$  is defined as follows:

$$\|f\|_{L^p(\tilde{X})} = \begin{cases} \left( \int_{\tilde{X}} |f(x)|^p dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \text{ess sup}_{\tilde{X}} |f(x)|, & p = \infty. \end{cases}$$

$L^p(\tilde{X})$  denotes the Banach space of all measurable functions  $f : \tilde{X} \rightarrow \mathbb{R}$  for which  $\|f\|_{L^p(\tilde{X})} < \infty$ .

Let  $f, g \in L^1_{\text{loc}}(\tilde{X})$  be locally summable functions (i.e.,  $f, g$  have a finite integral over every compact subset of  $\tilde{X}$ ). We say that  $g$  is the  $k$ th *weak* (partial) derivative of  $f$ , if

$$\int_{\tilde{X}} f \partial_x^k \phi dx = (-1)^k \int_{\tilde{X}} g \phi dx$$

for all test functions  $\phi \in C_c^\infty(\tilde{X})$  (infinitely differentiable functions  $\phi : \tilde{X} \rightarrow \mathbb{R}$  with compact support in  $\tilde{X}$ ).

$H^k(\tilde{X})$  for  $k \in \mathbb{N}$  is used to denote the Sobolev space  $W^{k,2}(\tilde{X})$  consisting of functions  $f \in L^2(\tilde{X})$  whose *weak* derivatives up to order  $k$  exist and belong to  $L^2(\tilde{X})$ . Note that  $H^k(\tilde{X})$  is a Hilbert space.

We use the subscript *per* to denote the closed subspace of *periodic* functions in the corresponding function space, e.g.,

$$L^p_{\text{per}}(\tilde{X}) = \{f \in L^p(\tilde{X}) : f(-1) = f(1)\}$$

$$H^k_{\text{per}}(\tilde{X}) = \{f \in H^k(\tilde{X}) : f(-1) = f(1)\}.$$

Similarly, we use the subscript *ep* to denote the closed subspace of *even periodic* functions in the corresponding function space, e.g.,

$$L^p_{\text{ep}}(\tilde{X}) = \{f \in L^p_{\text{per}}(\tilde{X}) : f(-x) = f(x) \quad \forall x \in \tilde{X}\}$$

$$H^k_{\text{ep}}(\tilde{X}) = \{f \in H^k_{\text{per}}(\tilde{X}) : f(-x) = f(x) \quad \forall x \in \tilde{X}\}.$$

We denote the dual space of  $H^1_{\text{per}}(\tilde{X})$  by  $H^{-1}_{\text{per}}(\tilde{X})$ , that is, the space of bounded linear functionals on  $H^1_{\text{per}}(\tilde{X})$ . Moreover, we use  $\langle \cdot, \cdot \rangle$  to denote the corresponding pairing of  $H^1_{\text{per}}(\tilde{X})$  and  $H^{-1}_{\text{per}}(\tilde{X})$ . That is, for  $f \in H^1_{\text{per}}(\tilde{X})$  and  $g \in H^{-1}_{\text{per}}(\tilde{X})$ , we use  $\langle g, f \rangle$  to denote the real number  $g(f)$ . Since periodic boundary condition allows for integration by parts without extra terms,  $H^{-1}_{\text{per}}(\tilde{X})$  has most of the properties of the space  $H^{-1}(\tilde{X})$ , the dual space of  $H^1_0(\tilde{X})$  (see [61, Sec. 5.9.1] for a detailed description of the space  $H^{-1}(\tilde{X})$ ).

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