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On some optimization problems that can be solved in $O(n)$ time

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Abstract. We consider nine elementary problems in optimization. We simply explore the conditions for optimality as known from the duality theory for convex optimization. This yields a quite straightforward solution method for each of these problems. The main contribution of this paper is that we show that even in the harder cases the solution needs only $O(n)$ time.

Keywords: optimization problems, linear time methods, optimality conditions

1 Introduction

This paper was inspired by a result in [2]. In that paper we needed the optimal objective value of the minimization problem

$$\min_{y,z,\beta}\left\{\|z\| \;:\; y \geq 0,\, \mathbf{1}^T y = 1,\, y = z + \beta v,\, z^T v = 0\right\},$$

where v is a given vector and $\mathbf{1}$ the all-one vector in $\mathbf{R}^n$; the variables are the scalar β and the vectors y and z in $\mathbf{R}^n$. It is a so-called second-order cone problem [1]. It turned out that the problem can be solved analytically in $O(n \log n)$ time. To obtain this result the entries of v must be ordered; this explains the factor $\log n$. The approach that led us to this surprising result is quite straightforward. It simply explores the conditions for optimality as known from the duality theory for convex optimization.
It is a natural question whether there are more nontrivial problems that can be solved analytically in a similar way. In this paper we show this true for problems of the following form:

$$\min_x \left\{\|a - x\|_{p_1} \;:\; \|x\|_{p_2} \leq 1\right\},$$

where a denotes a given vector in $\mathbf{R}^n$, and p_1 and p_2 are 1, 2 or ∞. In words, given a point $a \in \mathbf{R}^n$, we look for a point x in the unit sphere

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– with respect to the p_2-norm – that has minimal distance to a – with respect to the p_1-norm. Figure 1 provides a graphical illustration of the solution of each of the nine problems considered in this paper when $n = 2$, and $a = [1.3;\, 0.8]$.

Obviously, there are nine different (ordered) pairs (p_1, p_2). For each of these nine pairs we show that the above problem can be solved in linear time. In doing so, we always assume without saying that the vector a is ordered nonincreasingly:

$$a_1 \geq a_2 \geq \ldots \geq a_n.$$

It turns out that in some cases (specifically, if $p_1 = p_2$ or $p_2 = \infty$) the solution is trivial, or almost trivial; in other cases this is certainly not obviously the case. But as we show, in each case the problem can be solved in linear time. As far as the authors know, the method leading to this result is new; at least we are not aware of any such result in the existing literature.

In our analysis duality plays a crucial role. As a consequence we also need the so-called dual norm of $\|.\|_p$, for $p \in \{1,\, 2,\, \infty\}$, which is defined by

$$\|y\|_{p^*} = \max_x \left\{ x^T y \;:\; \|x\|_p = 1 \right\},$$

where x and y are vectors in $\mathbf{R}^n$. For future use we also recall an important consequence of this definition, namely the so-called Hölder inequality:

$$\|x\|_p \, \|y\|_{p^*} \geq x^T y, \qquad \forall x, y \in \mathbf{R}^n.$$

The outline of the paper is as follows.

Section 2 is preliminary. It consists of four subsections. Section 2.1 describes the fundamental role of duality in our approach. It recalls the so-called *vanishing gap condition* for optimality. For the problems that we consider in this paper this condition implies the primal and dual feasibility conditions, which is quite exceptional. Section 2.2 contains three lemmas dealing with the question of when the Hölder inequality holds with equality, for each of the three values of p considered in this paper. Section 2.3 serves to show that we may restrict our investigations to the case where the given vector a is nonnegative (cf. Lemma 4), and in Section 2.4 we distinquish easy types from harder types (p_1, p_2).

Section 3 contains the analysis of the nine problems, each in a separate subsection. Finally, Section 4 contains some recommendations for further research.

2 Preliminaries

2.1 Duality

As announced in the previous section, we consider problems of the following form:

$$\min_x \left\{ \|a - x\|_{p_1} \;:\; \|x\|_{p_2} \leq 1 \right\}, \tag{1}$$

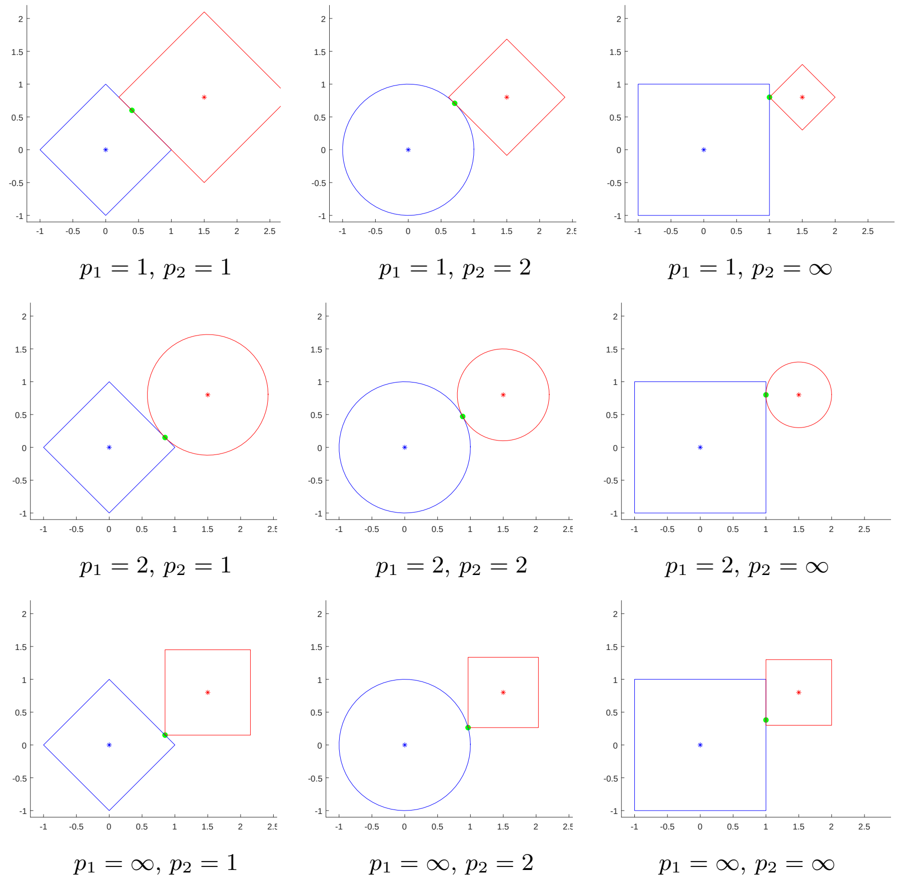

Fig. 1. Illustration of the optimal solutions of the nine problems considered in this paper, for $n = 2$ and $a = [1.3; 0.8]$. The blue dot represents the origin, the red dot a and the green dot the (or sometimes 'an') optimal solution x. The blue curve surrounds the region where the p_2−norm is less than 1, whereas the red curve depicts the p_1−neighborhood of a that just touches the blue region.

where a denotes a given vector in $\mathbf{R}^n$, and p_1 and p_2 are 1, 2 or ∞. The dual problem of (1) is given by

$$\max_{y} \left\{ a^T y - \|y\|_{p_2^*} \; : \; \|y\|_{p_1^*} \leq 1 \right\}, \tag{2}$$

where $\|.\|_{p_1^*}$ refers to the dual norm of $\|.\|_{p_1}$, and similarly for p_2.
In one case the solutions of problem (1) and problem (2) are immediate, namely if a is feasible for the primal problem, i.e., $\|a\|_{p_2} \leq 1$. Then $x = a$ solves the primal problem, because then the objective value equals zero, which is minimal. On the other hand, $y = 0$ is feasible for the dual

problem, yielding zero as dual objective value. Hence, if we take $x = a$ and $y = 0$ then the feasibility conditions are satisfied and the primal and dual objective values are equal. This means that we have solved the problem in case $\|a\|_{p_2} \le 1$. We call this the trivial case of the problem. In the sequel we only consider the nontrivial case, i.e., $\|a\|_{p_2} > 1$. In that case any optimal solution x will satisfy $x \neq a$. Since then $\|a - x\|_{p_1} > 0$, the optimal value of the primal problem will be positive. As a consequence, $y = 0$ does not close the duality gap. Therefore, at optimality we also have $y \neq 0$.
Now let x and y be primal and dual feasible, respectively. Then the duality gap can be reduced as follows:

$$
\begin{aligned}
\|a-x\|_{p_1} - \left(a^T y - \|y\|_{p_2^*}\right) &= \|a-x\|_{p_1} - a^T y + \|y\|_{p_2^*} \\
&\ge \|a-x\|_{p_1} \|y\|_{p_1^*} - a^T y + \|y\|_{p_2^*} \|x\|_{p_2} \\
&\ge (a-x)^T y - a^T y + y^T x \\
&= 0.
\end{aligned}
$$

where the second inequality follows by using the Hölder inequality twice. Thus we see that the duality gap vanishes if and only if

$$
\|a-x\|_{p_1} = \|a-x\|_{p_1} \|y\|_{p_1^*} = (a-x)^T y \tag{3}
$$

and

$$
\|y\|_{p_2^*} = \|y\|_{p_2^*} \|x\|_{p_2} = y^T x. \tag{4}
$$

Since $x \neq a$, (3) implies $\|y\|_{p_1^*} = 1$, whence $y \neq 0$. The latter implies $\|y\|_{p_2^*} > 0$. But then (4) implies $\|x\|_{p_2} = 1$. We conclude that in the nontrivial case the duality gap vanishes if and only if

$$
\|x\|_{p_2} = 1 = \|y\|_{p_1^*} \tag{5}
$$

$$
\|y\|_{p_2^*} = y^T x \tag{6}
$$

$$
\|a-x\|_{p_1} = y^T (a-x). \tag{7}
$$

Obviously (5) implies that the feasibility conditions in (1) and (2) are satisfied. Therefore, it suffices to solve the above system, under the assumption that $x \neq a$.
As stated before, we assume $p_1, p_2 \in \{1, 2, \infty\}$. For the sake of convenience we call the problems (1) and problem (2) of type (p_1, p_2).
Next we include a section with some lemmas that enable us to restate the conditions (6) and (7) in a way that is more tractable.

2.2 Basic lemmas

For future use we deal in this section with three elementary lemmas; they deal with the question when Hölder's inequality holds with equality. The first lemma concerns the well-known lemma of Cauchy-Schwarz, where $p^* = p = 2$.

Lemma 1. *The inequality* $\|x\|_2 \|y\|_2 \geq x^T y$ *holds with equality if and only if* $x = \lambda y$ *or* $y = \lambda x$ *for some* $\lambda \geq 0$.

Proof. We omit the proof, because the result is well-known. □

Less well-known are the next two lemmas that deal with the cases $p = 1$ and $p = \infty$.

Lemma 2. *The inequality* $\|x\|_1 \|y\|_\infty \geq x^T y$ *holds with equality if and only if* $x_i y_i \geq 0$ *for each* i *and* $x_i \neq 0$ *implies* $|y_i| = \|y\|_\infty$.

Proof. We may write

$$\|x\|_1 \|y\|_\infty = \sum_{i=1}^{n} |x_i| \|y\|_\infty \geq \sum_{i=1}^{n} |x_i| |y_i| \geq \sum_{i=1}^{n} x_i y_i = x^T y.$$

For each i, the i-th terms in the three subsequent summations are not increasing. Hence it follows that $\|x\|_1 \|y\|_\infty = x^T y$ holds if and only if these terms are mutually equal. In other words,

$$|x_i| \|y\|_\infty = |x_i| |y_i| = x_i y_i, \quad 1 \leq i \leq n.$$

The first equality holds if and only if $x_i \neq 0$ implies $|y_i| = \|y\|_\infty$. The second equality holds if and only if $|x_i y_i| = x_i y_i$, which is equivalent to $x_i y_i \geq 0$. □

Lemma 3. *The inequality* $\|x\|_\infty \|y\|_1 \geq x^T y$ *holds with equality if and only if* $x_i y_i \geq 0$ *for each* i *and* $y_i \neq 0$ *implies* $|x_i| = \|x\|_\infty$.

Proof. This lemma follows from the previous lemma by interchanging x and y. □

2.3 Simpifying observations

In this section we mention some properties of optimal solutions x and y of respectively (1) and (2) that are easy to understand. They lead us to the conclusion that in the following nine sections we only need to consider the case where a is a nonnegative vector, and also that we may safely assume that the optimal solutions x and y are nonnegative.

First we note that the contribution of x_i to $\|x\|_{p_2}$, with $p_2 = 1$, 2 or ∞, is determined completely by the absolute value $|x_i|$ of x_i. As a consequence, if x is feasible for (1) this will remain so if we change the sign of one or more of the entries in x.

Now consider the expression that we want to minimize: $\|a - x\|_{p_1}$. The contribution of x_i to this expression depends monotonically on $|a_i - x_i|$. If $x_i a_i \geq 0$ then $|a_i + x_i| \geq |a_i - x_i|$. Therefore, we may safely assume that each x_i has the same sign as a_i. A similar argument makes clear that we may assume that each entry y_i has the same sign as a_i, because changing the sign of y_i leaves $\|y\|_{p_1^*}$ and $\|y\|_{p_2^*}$ invariant in (2). On the other hand, the contribution of the product $a_i y_i$ to the dual objective value is maximal if the sign of y_i is the same as that of a_i. Therefore, if y is optimal then $a_i y_i \geq 0$.

We use the above observations as a preparation for the following lemma that makes clear that in the analysis of the system (5)-(7) we may safely assume $a \geq 0$. In this lemma we use a map f_S, where S is a subset of the indices 1 to n, which is defined as follows: for each vector $z \in \mathbf{R}^n$, $f_S(z)$ is the vector that arises from z by changing the signs of the entries z_i, $i \in S$. Obviously, when S is fixed, f_S is one-to-one, and idempotent, i.e., $f_S^2 = f_S$.

Lemma 4. *Let x and y denote solutions of the system (5) – (7) and $S \subseteq \{1, 2\ldots, n\}$. Then $f_S(x)$ and $f_S(y)$ solve the system when a is replaced by $f_S(a)$.*

Proof. Let x, y and S be as in the lemma. It is obvious that $\|x\|_{p_2}$ does not change if x is replaced by $f_S(x)$, because the norm of a vector does only depend on the absolute values of its entries. So, the same holds for the other norms in the system, in particularly also for $\|a - x\|_{p_1}$, since if $i \in S$ then also $a_i - x_i$ changes sign, because $(-a_i) - (-x_i) = -(a_i - x_i)$. Also the inner products do not change, because, e.g., $(-x_i)(-y_i) = x_i y_i$ for each $i \in S$. Hence the lemma follows. $\square$

We apply this lemma as follows. If the vector a has negative entries we define the index set $S = \{i \,:\, a_i < 0\}$. Then $f_S(a) \geq 0$. We then solve the system (5)-(7) with a replaced by $f_S(a)$. Let the solution be denoted as x' and y'. Then it follows from Lemma 4 that $x = f_S(x')$ and $y = f_S(y')$ are the solutions of the original system. As a consequence, below we may always assume that the vector a is nonnegative.

2.4 Easy and harder cases

In the following sections we deal with each of the nine types separately. It will turn out that for five of the nine problem-types a specific solution of (1) can be expressed nicely in a, as shown in Table 1. These are the types with $p_1 = p_2$ or $p_2 = \infty$. We call these types for the moment *easy*. It maybe worth pointing out that $x = \min(a, \mathbf{1})$ solves the primal problem in all cases with $p_2 = \infty$, also if $p_1 = \infty$. From Figure 1 one easily understands that – at least in some cases – multiple solutions exist. In general, we are not satisfied with the specific solution in Table 1 alone, but we intend to describe the whole set of optimal solutions.

$p_1 \backslash p_2$	1	2	∞
1	$\frac{a}{\|a\|_1}$	$\min(a, \alpha\mathbf{1})$	$\min(a, \mathbf{1})$
2	$(a - \alpha\mathbf{1})^+$	$\frac{a}{\|a\|_2}$	$\min(a, \mathbf{1})$
∞	$(a - \alpha\mathbf{1})^+$	$(a - \alpha\mathbf{1})^+$	$\frac{a}{\|a\|_\infty}$

Table 1. A specific solutions of (1) for each of the nine cases.

For the remaining four cases Table 1 also shows a specific solution of (1), but their descriptions need besides the vector a also a parameter α. Below we describe in more detail how α can be obtained, for each of the four *hard* cases. The notation x^+ is used to denote the vector that arises from a vector x by replacing its negative entries by zero. In other words, $x^+ = \max(x, 0)$.

Table 2 shows that in all cases one specific dual optimal solution can be expressed in a alone or in a and an arbitrary primal optimal solution x; this will become apparent in the related sections below. In this table $x > 0$ is used to denote the set of indices i for which x_i is positive. In a similar way $a \geq \mathbf{1}$ denotes the index set $\{i \,:\, a_i \geq 1\}$ and $a = \max(a)$ the index set $\{i \,:\, a_i = \max(a)\}$. For any index set I, we use a_I to denote the vector that arises from a by putting $a_i = 0$ if $i \notin I$. This explains the meaning of the notations $\mathbf{1}_{a\geq\mathbf{1}}$ and $\mathbf{1}_{a=\max(a)}$ in Table 2. It may be verified that if $p_1 = 2$ and $p_2 = \infty$ the dual optimal solution can be expressed in a alone; this follows by substitution of the primal optimal solution in Table 1 into $a - x$, which yields the vector $(a - \mathbf{1})_{a>\mathbf{1}}$.

$p_1 \backslash p_2$	1	2	∞
1	$\mathbf{1}$	$\frac{x}{\|x\|_\infty}$	$\mathbf{1}_{a\geq\mathbf{1}}$
2	$\frac{a-x}{\|a-x\|_2}$	$\frac{a}{\|a\|_2}$	$\frac{a-x}{\|a-x\|_2}$
∞	$\frac{\mathbf{1}_{x>0}}{\|\mathbf{1}_{x>0}\|_1}$	$\frac{x}{\|x\|_1}$	$\frac{\mathbf{1}_{a=\max(a)}}{\|\mathbf{1}_{a=\max(a)}\|_1}$

Table 2. Solutions of (2) for the all cases.

As far as the authors know, up till now problems that are not ‘easy’ in the above sense, can be solved only algorithmically. The main motivation of this paper, however, is to show that these problems are also easy in the sense that they can be solved analytically in $O(n)$ time. So, formally, in terms of computational complexity all nine types belong to the same class. Nevertheless, we will refer to the four types that are not ‘easy’ in the above sense as the harder-types, just to separate them from the ‘easy’ types.

The $O(n)$ time solution method for each of the four harder-type problems is achieved by introducing the parameter α that was mentioned before. It divides the index set $\{1, \ldots, n\}$ into two classes I and J, according to

$$I = \{i \,:\, a_i > \alpha\}, \qquad J = \{i \,:\, a_i \leq \alpha\}.$$

The number α is uniquely determined by a linear or quadratic equation $f(\alpha) = 0$, with $f(\alpha)$ as in Table 3. We use $|I|$ to denote the cardinality of the index set I. The number α and hence also I can be computed in linear time. After this the solution of the problem at hand needs $O(n)$ additional time. For the details we refer to the related sections below.

type	$f(\alpha)$	α
$(1,2)$	$1-\|a_J\|_2^2-\|I\|\,\alpha^2$	$\|x\|_\infty$
$(2,1)$	$1-\|a_I\|_1+\|I\|\,\alpha$	$\|a-x\|_\infty$
$(\infty,1)$	$1-\|a_I\|_1+\|I\|\,\alpha$	$\|a-x\|_\infty$
$(\infty,2)$	$1-\|a_I\|_2^2+2\alpha\,\|a_I\|_1-\|I\|\,\alpha^2$	$\|a-x\|_\infty$

Table 3. Definition of the number α.

3 Analysis of the nine problems

3.1 Problems of type $(1,1)$

With $\|a\|_1 > 1$, the primal problem is given by

$$\min_u \left\{ \|a-x\|_1 \;:\; \|x\|_1 \leq 1 \right\}, \tag{8}$$

and the dual problem by

$$\max_{y,z} \left\{ a^T y - \|y\|_\infty \;:\; \|y\|_\infty \leq 1 \right\}. \tag{9}$$

We recall from (5) – (7) the optimality conditions for x and y:

$$\|x\|_1 = 1 = \|y\|_\infty \tag{10}$$

$$\|y\|_\infty = y^T x. \tag{11}$$

$$\|a-x\|_1 = (a-x)^T y \tag{12}$$

As explained in Section 2.3 we may assume that a, and also x and y are nonnegative. According to Lemma 2, if (10) holds, then (11) is equivalent to
(i) for each i: $x_i \neq 0$ implies $y_i = \|y\|_\infty = 1$.
Similarly, by Lemma 3, if (10) holds, then (12) is equivalent to
(ii) for each i: $(a_i - x_i)\, y_i \geq 0$ and $a_i - x_i \neq 0$ implies $y_i = \|y\|_\infty = 1$,
Next we derive properties from the above conditions. Suppose that $a_i > 0$ for some i. Then either $x_i \neq 0$ or $a_i - x_i \neq 0$. Hence, by (i), (ii) and (10), $y_i = \|y\|_\infty = 1$. But then (ii) also implies $x_i \leq a_i$. This justifies the first line in Table 4.

a_i	x_i	y_i
> 0	$\leq a_i$	1
0	0	$0 \leq y_i \leq 1$

Table 4. Optimal solutions for type $(1,1)$.

The second line deals with the case where $a_i = 0$. If $x_i > 0$, we get from (i) that $y_i = 1$. As in the previous case, then (ii) gives $x_i \leq a_i$, whence $x_i = 0$. Since $\|y\|_\infty = 1$, this justifies the second line in Table 4.
This is all the information we can extract from the system (10) – (12). It means that the two (lower) lines in Table 4 represent all the possibilities for the triples (a_i, x_i, y_i), provided that $\|x\|_1 = 1$. In general multiple optimal solutions for problem (8) exist, because every vector x satisfying

$$0 \leq x \leq a, \qquad \|x\|_1 = 1$$

is optimal. Since $\|a\|_1 > 1$, one of these vectors is $x = a/\|a\|_1$, as given in Table 1.
If a has only positive entries then (9) has only one optimal solution, namely $y = \mathbf{1}$. If a has zero entries, then also other solutions exist. Then any vector y satisfying

$$\mathbf{1}_{a>0} \leq y \leq \mathbf{1},$$

is optimal, where $\mathbf{1}_{a>0}$ denotes the vector whose entries are 1 where a is positive and zero elsewhere.

3.2 Problems of type $(1, 2)$

In this section the primal problem is

$$\min_x \left\{ \|a - x\|_1 \ : \ \|x\|_2 \leq 1 \right\}, \tag{13}$$

where $\|a\|_2 > 1$. Its dual problem is

$$\max_y \left\{ a^T y - \|y\|_2 \ : \ \|y\|_\infty \leq 1 \right\}, \tag{14}$$

According to (5) – (7), x is optimal for (13) and y for (14) if and only if

$$\begin{aligned} \|x\|_2 &= 1 = \|y\|_\infty && (15)\\ \|y\|_2 &= y^T x && (16)\\ \|a - x\|_1 &= (a - x)^T y. && (17) \end{aligned}$$

As established in Section 2.3, we may take for granted that $a \geq 0$, $x \geq 0$ and $y \geq 0$.
We have $\|y\|_\infty = 1$, by (15). So $y \neq 0$. Also, $\|x\|_2 = 1$. As a consequence, (16) holds if and only if $\|x\|_2 \|y\|_2 = y^T x$. This in turn is equivalent with
(i) $x = \dfrac{y}{\|y\|_2}$,
by Lemma 1. Moreover, by Lemma 3 (17) holds if and only if
(ii) for each i: $(a_i - x_i)y_i \geq 0$ and if $a_i - x_i \neq 0$ then $y_i = \|y\|_\infty = 1$.
Since $y = \|y\|_2 \, x$, by (i), and $y \neq 0$, we may conclude that x_i and y_i have the same sign, for each i, and they vanish at the same time. Therefore, (ii) implies $x_i \leq a_i$, for each i. We define

$$I := \{i \ : \ x_i < a_i\}, \qquad J := \{i \ : \ x_i = a_i\}, \tag{18}$$

Now let $i \in I$ and $j \in J$. Since $a_i > x_i$, (ii) implies $y_i = \|y\|_\infty$. Since $y = \|y\|_2\, x$, we also have $x_i = \|x\|_\infty$. It follows that

$$a_i > x_i = \|x\|_\infty \geq x_j = a_j, \quad i \in I, j \in J. \tag{19}$$

This shows that the entries in a_I are strictly larger than those in a_J. Recall that we always assume that the entries of a are ordered nonincreasingly. Therefore, (19) implies the existence of an index q such that

$$I = \{i \,:\, i \leq q\}, \qquad J = \{i \,:\, i > q\}. \tag{20}$$

Putting $\alpha = \|x\|_\infty$, we see that (19) holds if and only if

$$a_q > \alpha \geq a_{q+1}, \tag{21}$$

Moreover, when knowing q and α, x uniquely follows from (18) and (19), according to

$$x_i = \begin{cases} \alpha, & \text{if} \quad i \leq q \\ a_i, & \text{if} \quad i > q. \end{cases} \tag{22}$$

Since x is nonzero and $\|y\|_\infty = 1$, we deduce from $y = \|y\|_2\, x$ that

$$y = \frac{x}{\|x\|_\infty}. \tag{23}$$

Next we arrive at the main objective of this paper, namely to show that in the current case q and also α can be found in $O(n)$ time. Because of (22) and (23) we may therefore conclude that (13) and (14) can be solved in $O(n)$ time.
From (15) we get $\|x\|_2 = 1$. Also using (22) we may write

$$1 = \|x\|_2^2 = \sum_{i=1}^n x_i^2 = \sum_{i\leq q} {x_i}^2 + \sum_{i>q} {x_i}^2 = q\alpha^2 + \sum_{i>q} a_i^2.$$

Since $q = |I|$ and $\sum_{i>q} a_i^2 = \|a_J\|_2^2$, we recognize at this stage that α satisfies $f(\alpha) = 0$, with $f(\alpha)$ as defined in Table 3 for type (1,2). Since α is nonnegative, α uniquely follows from q, because $f(\alpha) = 0$ holds if and only if

$$\alpha^2 = \frac{1 - \sum_{i>q} a_i^2}{q}.$$

As the next lemma reveals, q uniquely follows from (21). In order to prove this we define the vector τ as follows:

$$\tau_k = \frac{1 - \sum_{i>k} a_i^2}{k}, \quad 1 \leq k \leq n. \tag{24}$$

We then must find q such that $\alpha^2 = \tau_q$, with τ_q satisfying

$$a_q^2 > \tau_q \geq a_{q+1}^2. \tag{25}$$

Lemma 5. *q is the first index such that*

$$\tau_q = \max_k \{\tau_k \,:\, 1 \leq k \leq n\}. \tag{26}$$

Proof. For $k < n$ the definitions of τ_k and τ_{k+1} imply

$$(k+1)\tau_{k+1} = 1 - \sum_{i>k+1} a_i^2 = a_{k+1}^2 + 1 - \sum_{i>k} a_i^2 = a_{k+1}^2 + k\tau_k. \qquad (27)$$

This can be rewritten in the following two ways:

$$\begin{aligned} (k+1)\left(\tau_{k+1} - \tau_k\right) &= a_{k+1}^2 - \tau_k \\ k\left(\tau_{k+1} - \tau_k\right) &= a_{k+1}^2 - \tau_{k+1}. \end{aligned}$$

From this we deduce

$$\tau_{k+1} > \tau_k \quad \Leftrightarrow \quad a_{k+1}^2 > \tau_k \quad \Leftrightarrow \quad a_{k+1}^2 > \tau_{k+1}. \qquad (28)$$

So, τ is (strictly!) increasing at k if and only if $a_{k+1}^2 > \tau_k$ and this holds if and only if $a_{k+1}^2 > \tau_{k+1}$, for each $k < n$. From this we draw two conclusions. First that (25) holds if and only if τ is increasing at $k = q-1$ and nonincreasing at $k = q$. Second, if τ is nonincreasing at some $k < n$ it remains nonincreasing if k increases. This can be understood as follows. Suppose that τ is nonincreasing at some $k < n$, i.e., $\tau_{k+1} \leq \tau_k$. Then $a_{k+1}^2 \leq \tau_{k+1}$. Since $0 \leq a_{k+2} \leq a_{k+1}$, it follows that also $a_{k+2}^2 \leq \tau_{k+1}$. This in turn implies $\tau_{k+2} \leq \tau_{k+1}$, which proves the claim. The above two properties imply the statement in the lemma. □

The vector τ can be computed in $O(n)$ time by first computing τ_1 and then using (27), which gives:***

$$\tau_1 = 1 + a_1^2 - \|a\|^2, \qquad \tau_{k+1} = \frac{a_{k+1}^2 + k\tau_k}{k+1}, \quad 1 \leq k < n. \qquad (29)$$

Then (26) yields the value of q, still in $O(n)$ time. As mentioned before, this means that the current approach solves problem (13) and problem (14) in $O(n)$ time. Obviously, both solutions are unique.

Example 1. Table 5 shows the outcome of our analysis for a randomly generated vector a. It shows that τ is maximal at $k = 5$. So $I = \{1, \ldots, 5\}$, and $\alpha = \sqrt{\tau_5} = 0.3554$. So $x_i = 0.3554$ for $i \in I$ and $x_i = a_i$ for $i > 5$.

3.3 Problems of type $(1, \infty)$

With $\|a\|_\infty > 1$, we consider the problem

$$\min_x \left\{ \|a - x\|_1 \ : \ \|x\|_\infty \leq 1 \right\}. \qquad (30)$$

The dual of this problem is

$$\max_y \left\{ a^T y - \|y\|_1 \ : \ \|y\|_\infty \leq 1 \right\}. \qquad (31)$$

*** It may be worth mentioning that (29) reveals that τ_{i+1} is a convex combination of a_{i+1}^2 and τ_i.

i	a_i	τ_i	x_i	$a-x$	y_i
1	0.9293	−1.0048	0.3554	0.5739	1.0000
2	0.8308	−0.1573	0.3554	0.4754	1.0000
3	0.6160	0.0216	0.3554	0.2606	1.0000
4	0.5853	0.1019	0.3554	0.2299	1.0000
$5=q$	0.4733	0.1263	0.3554	0.1179	1.0000
6	0.3517	0.1259	0.3517	0.0000	0.9897
7	0.3500	0.1254	0.3500	0.0000	0.9849
8	0.2511	0.1176	0.2511	0.0000	0.7066
9	0.2435	0.1111	0.2435	0.0000	0.6852
10	0.0000	0.1000	0.0000	0.0000	0.0000

Table 5. Numerical illustration type $(1,2)$.

The conditions for optimality are

$$\begin{aligned} \|x\|_\infty &= 1 = \|y\|_\infty && (32)\\ \|y\|_1 &= y^T x && (33)\\ \|a-x\|_1 &= (a-x)^T y. && (34) \end{aligned}$$

As always we assume that $a \geq 0$, $x \geq 0$ and $y \geq 0$. According to Lemma 3, if (32) holds, then (33) holds if and only if

(i) for each i: $y_i \neq 0$ implies $x_i = \|x\|_\infty$;

and, by the same lemma, if (32) holds, then (34) holds if and only if

(ii) for each i: $(a_i - x_i)\, y_i \geq 0$ and $a_i - x_i \neq 0$ implies $y_i = \|y\|_\infty$.

We consider three cases, according to the value of a_i.

Let $a_i > 1$. Since $x_i \leq \|x\|_\infty = 1$, we then have $a_i - x_i > 0$. Then (ii) implies $y_i = \|y\|_\infty = 1$, and because of this (i) implies $x_i = \|x\|_\infty = 1$, where we also used (32). So, if $a_i > 1$, then $x_i = 1$ and $y_i = 1$.

If $a_i < 1$, we must have $y_i = 0$. Because otherwise $y_i > 0$, and then (i) would give $x_i = 1$ again. But then $a_i - x_i < 0$. This would imply $(a_i - x_i)\, y_i < 0$, contradicting (ii). So $y_i = 0$. But then we have $y_i < \|y\|_\infty$, which implies $x_i = a_i$, by (ii).

Finally, let $a_i = 1$. Suppose $x_i \neq a_i$. Then (ii) implies $y_i = 1$. Then, as before, (i) implies $x_i = 1$, whence $x_i = a_i$. Note that in that case (i) and (ii) are satisfied.

We conclude that at optimality x and y are as given in Table 6.

The primal solution is unique, and as given in Table 1, namely $x = \min(a, \mathbf{1})$. On the other hand, if all entries of a differ from 1, y is also unique. More precisely, then $y = \mathbf{1}_{a>\mathbf{1}}$. Otherwise there are multiple optimal solution. Every vector y such that

$$\mathbf{1}_{a>\mathbf{1}} \leq y \leq \mathbf{1}_{a\geq\mathbf{1}}$$

is dual optimal.

a_i	x_i	y_i
> 1	1	1
$= 1$	1	$\in [0,1]$
< 1	a_i	0

Table 6. Optimal solutions for type $(1,\infty)$.

3.4 Problems of type $(2,1)$

The problem that we consider in this section is

$$\min_x \left\{ \|a-x\|_2 \ : \ \|x\|_1 \le 1 \right\}, \tag{35}$$

where $\|a\|_1 > 1$. Its dual problem is

$$\max_{y,z} \left\{ a^T y - \|y\|_\infty \ : \ \|y\|_2 \le 1 \right\}. \tag{36}$$

According to (5) – (7), x is optimal for (35) and y for (36) if and only if

$$\|x\|_1 = 1 = \|y\|_2 \tag{37}$$
$$\|y\|_\infty = y^T x \tag{38}$$
$$\|a-x\|_2 = (a-x)^T y. \tag{39}$$

As before, under reference to Section 2.3, we assume that a, x and y are nonnegative. Then Lemma 2, (37) and (38) imply
(i) for each i: $x_i \neq 0$ implies $y_i = \|y\|_\infty$,
whereas, by Lemma 1, (37) and (39) imply
(ii) $y = \dfrac{a-x}{\|a-x\|_2}$.
We define

$$I := \{i \ : \ x_i > 0\}, \qquad J := \{i \ : \ x_i = 0\}. \tag{40}$$

Let $i \in I$. Then (i) implies $y_i = \|y\|_\infty$. Due to (37), $y \neq 0$. Hence $y_i > 0$. Because of (ii) we thus obtain $a_i > x_i$. From $y_i = \|y\|_\infty$ and (ii) we deduce that $a_i - x_i = \|a-x\|_\infty$. Now defining

$$\alpha = \|a-x\|_\infty, \tag{41}$$

we get $a_i - x_i = \alpha > 0$, whence

$$x_i = a_i - \alpha, \quad i \in I. \tag{42}$$

Hence

$$\|x\|_1 = \sum_{i\in I} (a_i - \alpha) = \|a_I\|_1 - |I|\,\alpha.$$

Since $\|x\|_1 = 1$, we obtain $f(\alpha) = 0$, where $f(\alpha) = 1 - \|a_I\|_1 + |I|\,\alpha$, as announced in Table 3 for type (2,1). This gives

$$\alpha = \frac{\|a_I\|_1 - 1}{|I|}. \tag{43}$$

Thus we find that if the index set I is known, then we can compute x and y: first one computes α from (43), and then x_I from (42). Since $x_J = 0$, we then know x, and y follows from (ii).
The question remains how we can find I. For that purpose we first observe that if $i \in I$ and $j \in J$ then

$$a_i = x_i + \alpha > \alpha = \|a - x\|_\infty \geq a_j - x_j = a_j. \tag{44}$$

This shows that the entries in a_I are strictly larger than those in a_J. Since the entries of a are ordered nonincreasingly, there must exist an index q such that

$$I = \{i \,:\, i \leq q\}, \quad J = \{i \,:\, i > q\}.$$

Then (44) holds if and only if

$$a_q > \alpha \geq a_{q+1}, \tag{45}$$

with α as in (43). We define the vector τ according to

$$\tau_k = \frac{\sum_{i \leq k} a_i - 1}{k}, \quad 1 \leq k \leq n. \tag{46}$$

Then (45) holds if and only if

$$a_q > \tau_q \geq a_{q+1}, \tag{47}$$

and then we necessarily have $\alpha = \tau_q$. We are now in a similar situation as in Section 3.2, and we proceed accordingly with the next lemma.

Lemma 6. *q is the first index such that*

$$\tau_q = \max_k \{\tau_k \,:\, 1 \leq k \leq n\}. \tag{48}$$

Proof. For $k < n$ the definition of τ_k implies

$$(k+1)\tau_{k+1} = \sum_{j \leq k+1} a_j - 1 = a_{k+1} + \sum_{j \leq k} a_j - 1 = a_{k+1} + k\tau_k. \tag{49}$$

This can be rewritten in the following two ways:

$$\begin{aligned} (k+1)\left(\tau_{k+1} - \tau_k\right) &= a_{k+1} - \tau_k \\ k\left(\tau_{k+1} - \tau_k\right) &= a_{k+1} - \tau_{k+1}. \end{aligned}$$

From this we deduce

$$\tau_{k+1} > \tau_k \quad \Leftrightarrow \quad a_{k+1} > \tau_k \quad \Leftrightarrow \quad a_{k+1} > \tau_{k+1}, \tag{50}$$

which proves that τ is increasing at k if and only if $a_{k+1} > \tau_k$ and this holds if and only if $a_{k+1} > \tau_{k+1}$, for each $k < n$. From here on we can use the same arguments as in the proof of Lemma 5. From (47) we conclude that τ is increasing at $k = q - 1$ and nonincreasing at $k = q$. Next, if τ is nonincreasing at some $k < n$ it remains nonincreasing if k increases, because if $\tau_{k+1} \leq \tau_k$ then $a_{k+1} \leq \tau_{k+1}$. Since $0 \leq a_{k+2} \leq a_{k+1}$, it follows that also $a_{k+2} \le \tau_{k+1}$. This in turn implies $\tau_{k+2} \le \tau_{k+1}$, proving the claim. From this the lemma follows. $\square$

As in Section 3.4, the vector τ can be computed in $O(n)$ recursively from[†]

$$\tau_1 = a_1 - 1, \qquad \tau_{k+1} = \frac{a_{k+1} + k\tau_k}{k+1}, \quad 1 \le k < n. \tag{51}$$

Then (48) yields the value of q, still in $O(n)$ time. Due to (42) this means that problem (35) and it dual problem can be solved time in $O(n)$ time. Obviously, the solutions of (35) and (36) are unique.

Example 2. Table 7 demonstrates our analysis for a randomly generated vector a. It shows that τ is maximal at $q = 5$. So $I = \{1, \ldots, 5\}$, and $\tau = 1.2799$.

i	a_i	τ_i	x_i	$a_i - x_i$	y_i
1	1.6363	0.6363	0.3564	1.2799	0.4181
2	1.6351	1.1357	0.3552	1.2799	0.4181
3	1.4449	1.2388	0.1650	1.2799	0.4181
4	1.3639	1.2701	0.0841	1.2799	0.4181
$5 = q$	1.3192	1.2799	0.0393	1.2799	0.4181
6	1.0433	1.2405	0.0000	1.0433	0.3409
7	0.2997	1.1061	0.0000	0.2997	0.0979
8	0.0000	0.9678	0.0000	0.0000	0.0000
9	0.0000	0.8603	0.0000	0.0000	0.0000
10	0.0000	0.7742	0.0000	0.0000	0.0000

Table 7. Numerical illustration type $(2, 1)$.

3.5 Problems of type $(2, 2)$

The primal problem is

$$\min_x \left\{ \|a - x\|_2 \;:\; \|x\|_2 \le 1 \right\}, \tag{52}$$

with $\|a\|_2 > 1$, and its dual problem

$$\max_y \left\{ a^T y - \|y\|_2 \;:\; \|y\|_2 \le 1 \right\}. \tag{53}$$

According to (5) – (7) the optimality conditions are

$$\|x\|_2 = 1 = \|y\|_2 \tag{54}$$
$$\|y\|_2 = y^T x \tag{55}$$
$$\|a - x\|_2 = (a - x)^T y. \tag{56}$$

[†] It may be worth mentioning that (51) reveals that τ_{k+1} is a convex combination of a_{k+1} and τ_k.

According to Lemma 1, (54) and (55) hold if and only if

(i) $x = \dfrac{y}{\|y\|_2}$,

and by the same lemma, (54) and (56) hold if and only if

(ii) $y = \dfrac{a-x}{\|a-x\|_2}$.

From (i) we derive that x and y have the same direction. Since x and y are both unit vectors, we must have $y = x$. By (ii), the vectors y and $a - x$ have the same direction. Since $y \neq 0$ this implies $a - x = \alpha y$ for some $\alpha > 0$. Thus we obtain $(1+\alpha)x = a$. This proves that x has the same direction as a. Since x is a unit vector, it follows that $x = \frac{a}{\|a\|_2}$, as in Table 1. Since $y = x$, we have solved (52) and (53). In this case both the primal and the dual solution are unique.

3.6 Problems of type $(2, \infty)$

The problem can then be stated as

$$\min_u \left\{ \|a-x\|_2 \;:\; \|x\|_\infty \leq 1 \right\}. \tag{57}$$

The dual problem is

$$\max_{y,z} \left\{ a^T y - \|y\|_1 \;:\; \|y\|_2 \leq 1 \right\}. \tag{58}$$

As in previous sections, we assume $\|a\|_\infty > 1$ and that x, y and a are nonnegative. According to (5) – (7), x is optimal for (57) and y for (58) if and only if

$$\|x\|_\infty = 1 = \|y\|_2 \tag{59}$$

$$\|y\|_1 = y^T x \tag{60}$$

$$\|a-x\|_2 = (a-x)^T y. \tag{61}$$

Let us assume (59). Then Lemma 3 states that (60) holds if and only if

(i) for each i: $y_i \neq 0$ implies $x_i = \|x\|_\infty = 1$,

whereas Lemma 1 states that (61) holds if and only if

(ii) $y = \dfrac{a-x}{\|a-x\|_2}$.

At optimality $\|a-x\|_2 > 0$, whence $x \neq a$. Let i be such that $y_i > 0$. Then (i) implies $x_i = 1$. Since y_i and $a_i - x_i$ have the same sign, we get $a_i > x_i = 1$.

We just showed that $y_i > 0$ implies $a_i > 1$. As a consequence we have $y_i = 0$ if $a_i \leq 1$. By (ii) we then have $x_i = a_i$. We conclude that at optimality x and y are as given in Table 8. It follows that both x and y are unique, with x as in Table 1: $x = \min(a, \mathbf{1})$.

a_i	x_i	y_i
>1	1	$(a_i - x_i)/\left\|a-x\right\|_2$
≤ 1	a_i	0

Table 8. Optimal solutions for type $(2, \infty)$.

3.7 Problems of type $(\infty, 1)$

With $\|a\|_1 > 1$, we consider the problem

$$\min_x \left\{ \|a-x\|_\infty \ : \ \|x\|_1 \leq 1 \right\}. \tag{62}$$

The dual of this problem is

$$\max_y \left\{ a^T y - \|y\|_\infty \ : \ \|y\|_1 \leq 1 \right\}. \tag{63}$$

As before, we only consider the case where a, x and y are nonnegative. The optimality conditions are

$$\|x\|_1 = 1 = \|y\|_1 \tag{64}$$
$$\|y\|_\infty = y^T x \tag{65}$$
$$\|a-x\|_\infty = (a-x)^T y. \tag{66}$$

According to Lemma 2, if (64) holds, then (65) is equivalent to
(i) for each i: $x_i \neq 0$ implies $y_i = \|y\|_\infty$;
and, for the same reason, then (66) is equivalent to
(ii) for each i: $y_i \neq 0$ implies $a_i - x_i = \|a-x\|_\infty$.
We partition the index set in the same way as in Section 3.4. So

$$I = \{i \ : \ x_i > 0\}, \quad J = \{i \ : \ x_i = 0\}.$$

Then (i) implies

$$y_i = \|y\|_\infty, \quad i \in I. \tag{67}$$

Since $y \neq 0$, by (64), we get $y_i > 0$. So (ii) applies, which implies $a_i - x_i = \|a-x\|_\infty$. Defining

$$\alpha = \|a-x\|_\infty, \tag{68}$$

it follows that

$$x_i = a_i - \alpha, \quad i \in I, \tag{69}$$

and hence we may write

$$\|x\|_1 = \sum_{i \in I} (a_i - \alpha) = \|a_I\|_1 - |I|\,\alpha.$$

Since $\|x\|_1 = 1$ we obtain

$$\alpha = \frac{\|a_I\|_1 - 1}{|I|}. \tag{70}$$

So, when we know I we can compute α from (70), and then the nonzero entries of x follows from (68). An interesting observation is that the formula for α is the same as (43) in Section 3.4. Like there, we also may write

$$a_i = x_i + \alpha > \alpha = \|a - x\|_\infty \geq a_j - x_j = a_j, \quad i \in I, j \in J, \tag{71}$$

Hence we have, for some index q,

$$I = \{i \,:\, i \leq q\}, \quad J = \{i \,:\, i > q\}.$$

Then (71) holds if and only if

$$a_q > \alpha \geq a_{q+1}, \tag{72}$$

with α as in (69). Thus the problem of finding q is the exactly the same as in Section 3.4. So we may state without further proof the following lemma.

Lemma 7. *One has $\alpha = \tau_q$, where q is the first index such that*

$$\tau_q = \max_k \{\tau_k \,:\, 1 \leq k \leq n\}, \tag{73}$$

and where the vector τ is defined recursively by

$$\tau_1 = a_1 - 1, \qquad \tau_{k+1} = \frac{a_{k+1} + k\tau_k}{k+1}, \quad 1 \leq k < n. \tag{74}$$

This means that problem (62) can be solved in $O(n)$ time, and the solution is unique.

In Section 3.4 the dual vector y was uniquely determined by x. This is now different, as becomes clear below. We derived from (i) that for indices $i \in I$, where x is positive, the entries y_i are positive and equal to $\|y\|_\infty$. If $i \in J$, where x is zero, (ii) requires that if $a_i \neq \|a - x\|_\infty$ then $y_i = 0$. So, if $a_i = \alpha$ then condition (ii) is void, and hence the only condition on y_i becomes $0 \leq y_i \leq \|y\|_\infty$. This can happen only if $a_{q+1} = \alpha$. Since $\alpha = \tau_q$ this is equivalent to $\tau_{q+1} = \tau_q$, by (74). Stated otherwise, we can have $0 \leq y_{q+1} \leq \|y\|_\infty$ if and only if τ is not decreasing at q. More generally, if q' is the highest index at which τ is maximal, with $q' \geq q$, i.e., if

$$\tau_q = \tau_{q+1} = \ldots = \tau_{q'} = \alpha,$$

which happens if and only if

$$a_q = a_{q+1} = \ldots = a_{q'} = \alpha. \tag{75}$$

then for any i such that $q \leq i \leq q'$ we can have $0 \leq y_i \leq \|y\|_\infty$. Any such vector y is obtained by first defining a vector z as follows:

$$z_i = \begin{cases} 1 & \text{if} \quad i \leq q, \\ \in [0, 1] & \text{if} \quad q < i \leq q', \\ 0 & \text{if} \quad i > q', \end{cases} \tag{76}$$

and then taking

$$y = \frac{z}{\|z\|_1}. \tag{77}$$

We then have $\|y\|_1 = 1$ and, because of (69) and (75), for each positive y_i that $a_i - x_i = \alpha = \|a - x\|_\infty$. This implies that y is dual feasible and also optimal.

Example 3. Table 9 demonstrates our analysis for a given vector a. It shows that τ is maximal at $q = 5$. So $I = \{1, \ldots, 5\}$, and $\alpha = \tau_5 = \tau_6 = 0.5362$.

i	a_i	τ_i	x_i	$a_i - x_i$	y_i
1	0.9174	−0.0826	0.3812	0.5362	0.1772
2	0.7655	0.3415	0.2293	0.5362	0.1772
3	0.7384	0.4738	0.2022	0.5362	0.1772
4	0.6834	0.5262	0.1472	0.5362	0.1772
$5 = q$	0.5762	0.5362	0.0400	0.5362	0.1772
$6 = q'$	0.5362	0.5362	0.0000	0.5362	0.1142
7	0.2691	0.4980	0.0000	0.2691	0.0000
8	0.2428	0.4661	0.0000	0.2428	0.0000
9	0.1526	0.4313	0.0000	0.1526	0.0000
10	0.0000	0.3882	0.0000	0.0000	0.0000

Table 9. Numerical illustration type $(\infty, 1)$.

3.8 Problems of type $(\infty, 2)$

With $\|a\|_2 > 1$, we consider the problem

$$\min_x \left\{ \|a - x\|_\infty \ : \ \|x\|_2 \leq 1 \right\}. \tag{78}$$

The dual of this problem is

$$\max_y \left\{ a^T y - \|y\|_2 \ : \ \|y\|_1 \leq 1 \right\}. \tag{79}$$

As always, $a \geq 0$, $x \geq 0$ and $y \geq 0$. The conditions for optimality are

$$\|x\|_2 = 1 = \|y\|_1 \tag{80}$$

$$\|y\|_2 = y^T x \tag{81}$$

$$\|a - x\|_\infty = (a - x)^T y \tag{82}$$

According to Lemma 1, if (80) holds, then (81) is equivalent to
(i) $x = \dfrac{y}{\|y\|_2}$;

and, by Lemma 2, (82) is equivalent to
(ii) for each i: $y_i \neq 0$ implies $a_i - x_i = \|a - x\|_\infty$.
With

$$I = \{i \,:\, x_i > 0\}\,, \quad J = \{i \,:\, x_i = 0\}\,,$$

the pair (I, J) is a partition of the index set. Let $i \in I$. So, $x_i > 0$. Now (i) implies $y_i > 0$. Therefore, (ii) implies $a_i - x_i = \|a - x\|_\infty$. Since $\|a - x\|_\infty > 0$, we get $x_i < a_i$. To simplify the presentation we define

$$\alpha = \|a - x\|_\infty\,. \tag{83}$$

Then we have

$$x_i = a_i - \alpha, \quad i \in I. \tag{84}$$

Now let $j \in J$. Then using $x_j = 0$, we may write

$$a_i = x_i + \alpha > \alpha = \|a - x\|_\infty \geq a_j - x_j = a_j. \tag{85}$$

This proves that the entries in a_I are strictly larger than those in a_J. Since the entries of a are ordered nonincreasingly, we get, for some q,

$$I = \{1,\, 2,\, \ldots,\, q\}\,, \qquad J = \{q + 1,\, \ldots,\, n\}\,.$$

Assuming that J is not empty, (85) implies

$$a_q > \alpha \geq a_{q+1}. \tag{86}$$

Otherwise, i.e., when $q = n$, we define $a_{n+1} = 0$; so we can always work as if (86) holds. Because of (80) we may write

$$1 = \|x\|_2^2 = \sum_{i \in I} x_i^2 = \sum_{i \in I} (a_i - \alpha)^2 = \|a_I\|_2^2 - 2\alpha \|a_I\|_1 + |I|\,\alpha^2.$$

Thus we obtain that α is one of the two roots of the equation $f(\alpha) = 0$, where

$$f(\alpha) := 1 - \|a_I\|_2^2 + 2\alpha \|a_I\|_1 - |I|\,\alpha^2.$$

Before proceeding, it will be convenient to introduce the notation

$$\sigma_{jk} := \sum_{i=1}^{k} a_i{}^j, \quad j \in \{1,\, 2\}\,,\, k \in \{1,\, \ldots,\, n\}\,. \tag{87}$$

Then $\|a_I\|_1 = \sigma_{1q}$ and $\|a_I\|_2^2 = \sigma_{2q}$, and hence $f(\alpha)$ can be rewritten as

$$f(\alpha) \quad = \quad 1 - \sigma_{2q} + 2\alpha\sigma_{1q} - q\alpha^2. \tag{88}$$

Now the sum of the two roots equals $2\sigma_{1q}/q$. So their average value is σ_{1q}/q. By (86) we have $a_q > \alpha$. Combining this with $a_1 \geq a_2 \geq \ldots \geq a_q$, we conclude that $\sigma_{1q} > q\alpha$, whence $\sigma_{1q}/q > \alpha$. It thus follows that the root α is smaller than the other root. This means that the discriminant of the equation $f(\alpha) = 0$ is positive. In other words

$$\sigma_{1q}^2 - q\,(\sigma_{2q} - 1) > 0. \tag{89}$$

Motivated by the solution technique developed in some of the preceding sections, we define

$$f_k(\xi) \qquad = \qquad 1 - \sigma_{2k} + 2\xi\sigma_{1k} - k\xi^2, \qquad 1 \le k \le n. \tag{90}$$

and

$$\omega_k := \sigma_{1k}^2 - k\left(\sigma_{2k} - 1\right), \qquad \tau_k := \frac{1}{k}\left(\sigma_{1k} - \sqrt{\omega_k}\right), \qquad 1 \le k \le n. \tag{91}$$

Obviously, ω_k is just the discriminant of the equation $f_k(\xi) = 0$ and if $\omega_k \ge 0$ then τ_k is its smallest root. In particular, $\tau_q = \alpha$. Hence, according to (86) we need to find q such that

$$a_q > \tau_q \ge a_{q+1}. \tag{92}$$

When knowing q, α follows from $\alpha = \tau_q$, and then x follows from (84). The question remains how much time it takes to solve q from (92) and similarly for x and y. We claim that all this can be done in $O(n)$ time. This can be understood as follows.
Clearly, $\omega_1 = 1$ and $\tau_1 = a_1 - 1$. For $j = 1, 2$, the recursive computation of $\sigma_{j1}, \dots, \sigma_{jq}$ requires $O(q)$ time, and so does the computation of ω_q and τ_q. If we have found q such that τ_q satisfies (92), then we also know α, because $\alpha = \tau_q$. Then x follows from $x_i = a_i - \alpha$ if $i \le q$ and $x_i = 0$ otherwise. Finally, from (i) we derive that

$$y = \frac{x}{\|x\|_1}.$$

Thus we have shown that problem (78) and its dual problem can be solved in $O(n)$ time.

Example 4. Table 10 shows the outcome of our analysis for a randomly generated vector a. Because of (92) the optimal value of q is 6 in this example. Note that the sequence τ_k is increasing, until it becomes undefined due to $\omega_k < 0$.

In Table 10 one may observe that the vector ω shows behaviour that we recognize from the vector τ in preceding sections: (i) when ω_k is nonincreasing at some k it remains nonincreasing when k grows, and (ii) the optimal index q occurs at the moment when ω attains its maximal value. It turns out that this surprising observation can be turned into the next lemma. A consequence of this lemma is that the index q can be obtained without computing τ. One only needs to compute the first $q + 1$ entries of the vector ω.

Lemma 8. *q is the first index such that*

$$\omega_q = \max_k \left\{\omega_k \;:\; 1 \le k \le n\right\}. \tag{93}$$

Proof. We first derive from (92) that the index q satisfies

$$f_q(a_q) > 0 \ge f_q(a_{q+1}). \tag{94}$$

k	a_k	$f_k(a_k)$	ω_k	τ_k	x_k	$a_k - x_k$	y_k
1	2.9667	1.0000	1.0000	1.9667	0.6736	2.2932	0.3023
2	2.7888	0.9683	1.9683	2.1763	0.4957	2.2932	0.2224
3	2.6370	0.8683	2.8366	2.2361	0.3439	2.2932	0.1543
4	2.5963	0.8241	3.6607	2.2689	0.3031	2.2932	0.1360
5	2.5521	0.7629	4.4236	2.2876	0.2590	2.2932	0.1162
$6 = q$	2.4462	0.5415	4.9651	2.2932	0.1530	2.2932	0.0687
7	2.0900	-1.1531	3.8120	2.3035	0.0000	2.0900	0.0000
8	1.7484	-4.3254	-0.5134	$-$	0.0000	1.7484	0.0000
9	1.6817	-5.1398	-5.6532	$-$	0.0000	1.6817	0.0000
10	0.0000	-52.0241	-57.6773	$-$	0.0000	0.0000	0.0000

Table 10. Numerical illustration for a problem of type $(\infty, 2)$.

Recall that τ_q is the smallest roots of the quadratic equation $f_q(\xi) = 0$. For the moment, let τ_q' denote the second (i.e., largest) root. By its definition (90), $f_q(\xi)$ is concave. We therefore have

$$f_q(\xi) > 0 \quad \Leftrightarrow \quad \tau_q < \xi < \tau_q', \tag{95}$$

where τ_q and τ_q' are such that

$$\begin{aligned} q\tau_q &= \sigma_{1q} - \sqrt{\omega_q} \\ q\tau_q' &= \sigma_{1q} + \sqrt{\omega_q}. \end{aligned}$$

By (92), $a_{q+1} \leq \tau_q < a_q$. The first inequality makes clear that a_{q+1} does not belong to (τ_q, τ_q'). Therefore, we immediately get the second inequality in (94): $f_q(a_{q+1}) \leq 0$. According to (95), the first inequality in (94) holds if and only if $\tau_q < a_q < \tau_q'$. We already have $a_q > \tau_q$. So it remains to prove $a_q < \tau_q'$. Since the entries of a are ordered nonincreasingly, we have $qa_q \leq \sigma_{1q}$. Since $\sigma_{1q} < \sigma_{1q} + \sqrt{\omega_q} = q\tau_q'$, we obtain $a_q < \tau_q'$, as desired. Thus (94) has now been proven.

We proceed by showing that the sequence $f_k(a_k)$ is nonincreasing for $1 \leq k \leq n$. By the definition (90) of $f_k(\xi)$ we may write

$$f_k(\xi) = 1 - \sum_{i \leq k} a_i^2 + 2\xi \sum_{i \leq k} a_i - k\xi^2 = 1 - \sum_{i \leq k} (a_i - \xi)^2. \tag{96}$$

Since $a_k \geq a_{k+1}$, we get $a_i - a_{k+1} \geq a_i - a_k$ for each i. So one also has

$$\sum_{i \leq k+1} (a_i - a_{k+1})^2 = \sum_{i \leq k} (a_i - a_{k+1})^2 \geq \sum_{i \leq k} (a_i - a_k)^2 .$$

Thus it follows from (96) that, for any $k < n$,

$$f_{k+1}(a_{k+1}) = f_k(a_{k+1}) \leq f_k(a_k). \tag{97}$$

This proves that the sequence $f_k(a_k)$ is nonincreasing when k increases. Because of this, (94) implies that $f_k(a_k)$ is positive if and only if $k \le q$. This has important consequences for the sequence ω_k, $1 \le k \le n$. This becomes clear by considering $\omega_{k+1} - \omega_k$. This expression can be reduced as follows.

$$\begin{aligned}
\omega_{k+1} - \omega_k &= \sigma_{1,k+1}^2 - (k+1)\left(\sigma_{2,k+1} - 1\right) - \left(\sigma_{1k}^2 - k\left(\sigma_{2k} - 1\right)\right) \\
&= \sigma_{1,k+1}^2 - (k+1)\sigma_{2,k+1} + (k+1) - \sigma_{1k}^2 + k\sigma_{2k} - k \\
&= 1 + \sigma_{1,k+1}^2 - \sigma_{1k}^2 - k\left(\sigma_{2,k+1} - \sigma_{2k}\right) - \sigma_{2,k+1} \\
&= 1 + \left(\sigma_{1,k+1} - \sigma_{1k}\right)\left(\sigma_{1,k+1} + \sigma_{1k}\right) - k\, a_{k+1}^2 - \sigma_{2,k+1} \\
&= 1 + a_{k+1}\left(a_{k+1} + 2\sigma_{1k}\right) - k\, a_{k+1}^2 - \sigma_{2,k+1} \\
&= 1 - \sigma_{2k} + 2a_{k+1}\sigma_{1k} - k a_{k+1}^2 \\
&= f_k(a_{k+1}) = f_{k+1}(a_{k+1}).
\end{aligned}$$

We may conclude that $\omega_{k+1} > \omega_k$ holds if and only if $f_{k+1}(a_{k+1}) > 0$. Since we have $f_k(a_k) \ge f_{k+1}(a_{k+1})$ for each k and because of (94) it follows that $f_{k+1}(a_{k+1}) > 0$ holds if and only if $k + 1 \le q$. So, when k runs from 1 to n then ω increases at k if and only if $k \le q - 1$, and from $k = q$ on ω is nonincreasing. Hence the lemma follows. □

3.9 Problems of type (∞, ∞)

While assuming $\|a\|_\infty > 1$ we consider the problem

$$\min_x \left\{\|a - x\|_\infty \;:\; \|x\|_\infty \le 1\right\}. \tag{98}$$

The dual of this problem is

$$\max_y \left\{a^T y - \|y\|_1 \;:\; \|y\|_1 \le 1\right\}. \tag{99}$$

As in the previous eight sections, we assume $a \ge 0$, $x \ge 0$ and $y \ge 0$. The conditions for optimality are given by

$$\|x\|_\infty = 1 = \|y\|_1 \tag{100}$$

$$\|y\|_1 = y^T x \tag{101}$$

$$\|a - x\|_\infty = (a - x)^T y. \tag{102}$$

According to Lemma 3, (100) and (101) imply
(i) for each i: $y_i \ne 0$ implies $x_i = \|x\|_\infty$;
and, by Lemma 2, (100) and (102) imply
(ii) for each i: $y_i \ne 0$ implies $a_i - x_i = \|a - x\|_\infty$.
Let $y_i > 0$. Then (i) and (ii) imply $x_i = \|x\|_\infty$ and $a_i - x_i = \|a - x\|_\infty$. By adding these two equalities we obtain

$$a_i = \|x\|_\infty + \|a - x\|_\infty.$$

Hence, for any other $j \ne i$, since $x_j \le \|x\|_\infty$ and $a_j - x_j \le \|a - x\|_\infty$, we get

$$a_i \ge x_j + (a_j - x_j) = a_j.$$

Hence, since the entries in a are ordered nonincreasingly, $a_i = a_1$. So, y_i is zero for each i with $a_i < a_1$ and maybe also for one or more indices i with $a_i = a_1$. Therefore, if I denotes the set of indices with $y_i > 0$ and J its complement, then

$$I \subseteq \{i \,:\, a_i = a_1\}\,, \qquad J = \{i \,:\, i \notin I\} \supseteq \{i \,:\, a_i < a_1\}\,,$$

with I nonempty, whereas $y_I > 0$ with $\|y_I\|_1 = 1$ and $y_J = 0$. The dual objective value at y equals $a^T y - \|y\|_1 = a_1 - 1$. Since the optimal primal objective value has the same value, this implies $\|a - x\|_\infty = a_1 - 1$. This value is positive, because $\|a\|_\infty = a_1 > 1$.
For x we are left with the following conditions. By (i), $x_i = 1$ for $i \in I$; then (ii) also holds because $a_i - x_i = a_1 - 1 = \|a - x\|_\infty$. For the remaining indices i ($i \in J$) there is a lot of freedom. The only condition for each $i \in J$ is that the value of x_i does not change the given values of $\|x\|_\infty$ $(= 1)$ and $\|a - x\|_\infty$ $(= a_1 - 1)$. So, with $\alpha = a_1 - 1$, we must have

$$\begin{aligned} 0 \leq x_J &\leq \mathbf{1}_J \\ -\alpha \mathbf{1}_J \leq a_J - x_J &\leq \alpha \mathbf{1}_J. \end{aligned}$$

Summarizing, a vector x is optimal for problem (98) if and only if

$$x_I = \mathbf{1}_I, \quad \max(0, a_J - \alpha \mathbf{1}_J) \leq x_J \leq \min(\mathbf{1}_J, a_J + \alpha \mathbf{1}_J). \tag{103}$$

Example 5. For a randomly generated vector a, Table 11 shows two solutions, $x^{(1)}$ and $x^{(2)}$. In $x^{(1)}$ we took for each entry the smallest possible value, and in $x^{(2)}$ the largest possible value, according to (103). All other optimal vectors x are obtained by taking for each x_i a value between these two extreme values. One of these solutions is $x = a/\|a\|_\infty$, as mentioned in Table 1, and as easily can be verified. Another 'nice' solution is $x = \min(\mathbf{1}, a)$.

4 Concluding remarks

This paper was inspired by a result in [2], where a nontrivial second-order cone optimization problem was solved analytically in linear time. This raised the question whether there exist other (classes of) problems that can be solved in linear time, by a variant of the same method. In this paper we consider a class of nine potentially important, easily stated and fundamental problems that form such a class. It is worth noting that in the four harder cases an important characteristic of the new method is that it first yields an 'optimal partition' of the variables in the problem. After this the values of the variables can be easily found. Though the problems considered in this paper are quite basic, hopefully it will inspire further research in this direction.
Figure 1 and the tables in this paper were generated by using Matlab. On the web site xxx one my find a Matlab file with the name `easy_problems.m`. It generates the results depicted in the Figure 1 if one calls this file by typing `easy_problems(a)`, with $a = [1.3; 0.8]$. One

i	a_i	$x_i^{(1)}$	$a_i - x_i^{(1)}$	$x_i^{(2)}$	$a_i - x_i^{(2)}$	y_i
1	1.9154	1.0000	0.9154	1.0000	0.9154	0.5354
2	1.9154	1.0000	0.9154	1.0000	0.9154	0.0000
3	1.9154	1.0000	0.9154	1.0000	0.9154	0.4646
4	1.2754	0.3600	0.9154	1.0000	0.2754	0
5	1.0543	0.1389	0.9154	1.0000	0.0543	0
6	1.0361	0.1207	0.9154	1.0000	0.0361	0
7	0.9148	0	0.9148	1.0000	−0.0852	0
8	0.8802	0	0.8802	1.0000	−0.1198	0
9	0.5620	0	0.5620	1.0000	−0.4380	0
10	0	0	0	0.9154	−0.9154	0

Table 11. Two numerical solutions for a problem of type (∞, ∞).

may use any vector a of any length ≥ 2 as input. It then generates the nine tables for this particular vector. The tables in the paper are obtained by typing `easy_problems(1)`. When typing `easy_problems(n)` with $n \geq 2$, it solves each of the nine problems, for each problem with a newly randomly generated vector of length n.

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