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ORIGINAL PAPER

On resonances in a weakly nonlinear microbeam due to an electric actuation

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Abstract In this paper, the oscillations of an actuated, simply supported microbeam are studied for which it is assumed that the electric load is composed of a small DC polarization voltage and a small, harmonic AC voltage. Bending stiffness and mid-plane stretching are taken into account as well as small viscous or structural damping. No tensile axial force is assumed to be present. By using a multiple time-scales perturbation method, approximations of the solutions of the initial-boundary value problem for the microbeam equation are constructed. This analysis is performed without truncating the infinite series representation in advance as is usually done in the existing literature. It is shown in which cases truncation is allowed for this problem. Moreover, accurate and explicit approximations of the natural frequencies up to order ε^3 of the actuated microbeam are also obtained. Intriguing and new modal vibrations are found when the frequency of the harmonic AC voltage is (near) half or twice a natural frequency of the microbeam, i.e., near a superharmonic or a subharmonic resonance.

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1 Introduction

Electrically actuated micro- and nano-beams have been studied extensively in the literature, and have already been used in many micro- and nano-mechanical devices such as sensors and switches. One can find a comprehensive overview background on recent work on micro- and nano- beams in the Table A1 to A12 of [1]. The analytical study of these actuated beams roughly falls into two groups (see for instance [2–10]) for a small, but representative overview of the available approaches). In the first group one formulates a problem for a nonlinear beam equation, and one computes a one (or a two) mode response consisting of only the directly excited eigenmode(s). The applied frequency (or frequencies) in the harmonic AC voltage of the electric load is (or are) equal to the natural eigenfrequencies of the nonlinear beam. Or one studies for instance a three-to-one internal resonance by only considering the two modes which are involved in this three-to-one internal resonance. In fact, in this approach one truncates the solution of the problem to one mode (or two modes) of oscillation, and one studies a single (or a system of two) second order, nonlinear differential equation(s). In the second group of studies one formulates a (system of) second-order ordinary differential equation(s), which include linear and nonlinear terms with unknown parameters.

Based upon experiments, one fits the unknown parameter with different kind of methods to the obtained measurements, such that the experiments and the model equations give more or less the same bifurcations and amplitude-frequency responses. In this paper, it will be explicitly shown whether the truncation to a few oscillation modes is allowed or not. Moreover, in this paper accurate approximations of the eigenfrequencies of the actuated beam are presented, and the influence of damping and weak nonlinearities are described. This will improve the parameter fitting procedure significantly.

In this paper, we will consider an initial-boundary value problem for an actuated microbeam for which it is assumed that the electric load is composed of a small DC polarization voltage and a small, harmonic AC voltage. To simplify the analytical computations it is assumed that the microbeam is simply supported, but for other boundary conditions similar (but more complicated) computations can be performed. Bending stiffness and mid-plane stretching of the microbeam are taken into account as well as small viscous or structural damping. No axial, tensile force is assumed to be present. A multiple time-scales perturbation method (as the one for instance presented in [21,22] and used in [11,12]) will be used to construct accurate approximations of the solutions of the problem which are valid on long time-scales. In fact, we will study weakly nonlinearly perturbed beam equations involving a small, positive parameter ε. By following a proof as given in [13] it can be shown that all of our approximations are order ε accurate on time-scales of order $1/\varepsilon$. Moreover, it should be mentioned that Fourier series representations for the solution will be used without truncating these series in advance. So, no (unknown) truncation errors are introduced in the approximations of the solution. Truncations of the solution to a finite number of oscillation modes is quite common in solving initial-boundary value problem such as in the studies of actuated micro- and nano-beam problems. However, in some problems such as in [14,15], this truncation cannot be done due to the modes internal resonances. This in general cannot be known in advance. In this paper, it is shown whether the truncation approach can be used or not. Moreover, it will also turn out that due to some different scalings introduced in the modelling of the problem, we will find some new explicit resonances which are not equal or not close to the natural frequencies of the microbeam.

As is well known, these resonant frequencies and internal resonances are significant for those who work on actuated micro- and nano-beam problems for instance on mass sensing technique problems. In those problems, the resonance frequency shifts are tracked to measure the mass of nano particles as in [16–18].

This paper is organized as follows. In Sect. 2 of this paper we will formulate an initial- boundary value problem for the actuated microbeam equation by following partly the derivation as given in [2] and in [10] chapter 6. In Sect. 3 we will give accurate approximations of the natural frequencies of the microbeam, and we will study the influence of the electro-static and dynamic force on the oscillations of the microbeam. In Sect. 4, we will include mid-plane stretching and viscous or structural damping. We will investigate their influence on the oscillations, and we will describe some new resonances in the actuated microbeam. Finally, in Sect. 5, we will draw some conclusions, and we will make some remarks on future research.

2 Formulation of the problem

In this paper, we consider the oscillations of a simply supported microbeam which is actuated by an electric load consisting of a DC component V_0 and an AC component $v(t)$. Bending stiffness and midplane stretching are included in the model equation as well as viscous and/or structural damping. No tensile axial force is assumed to be present. Following [2,3,6,8,11,19,20] one arrives at the nowadays standard and nondimensional equation for the nondimensional displacement $u = u(x,t)$ of the microbeam:

$$u_{tt} + u_{xxxx} + \hat{c}_1 u_t + \hat{c}_2 u_{txxxx} = \hat{\alpha}_1 \int_0^1 u_x^2 \, dx \, u_{xx} + \hat{\alpha}_2 \frac{(\hat{V}_0 + \hat{v}(t))^2}{(1-u)^2}, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \tag{2}$$

and subject to the initial conditions

$$u(x,0) = \hat{f}(x), \text{ and } u_t(x,0) = \hat{g}(x), \tag{3}$$

where $\hat{f}$ and $\hat{g}$ represent the nondimensional initial displacement and initial velocity of the microbeam, respectively. For a schematic impression of the actuated microbeam the reader is referred to Fig. 1. In (1), (2), and (3) x is the nondimensional coordinate along

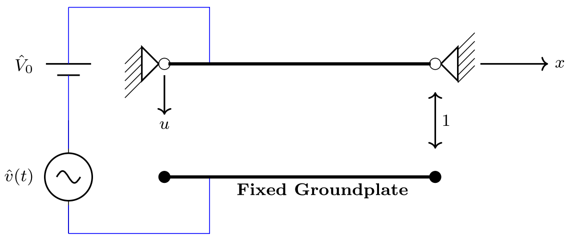


Fig. 1 Schematic impression of the fixed groundplate and the displacement u of the actuated microbeam which is simply supported at $x = 0$ and at $x = 1$. The electrostatic actuation is $\hat{V}_0 + \hat{v}(t)$, and the nondimensional distance between the microbeam and groundplate is 1

the beam with $0 < x < 1$, t is the nondimensional time, and $\hat{c}_1$ and $\hat{c}_2$ are nondimensional damping parameters.

Furthermore, $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{V}_0$ are nondimensional parameters, and $\hat{v}(t)$ is a nondimensional function. To derive (1) it is assumed that the electric field between the groundplate and the microbeam is perpendicular to the surfaces of both the fixed groundplate and the microbeam. This assumption implies that the deflection u of the microbeam is much smaller than the nondimensional distance 1 between the fixed plate and the microbeam. Furthermore, it will be assumed that the electric load is given by $\hat{V}_0 + \hat{V}_{AC}\sin(\omega t)$, where $\hat{V}_{AC}$ is the magnitude of the applied AC voltage and ω the excitation frequency. Based upon these assumptions, the following rescalings are used in this paper:

$$\begin{aligned} &\hat{V}_{AC} = \varepsilon V_{AC}, \quad \hat{V}_0 = V_0 V_{AC}, \\ &\hat{\alpha}_1 = \alpha, \quad \hat{\alpha}_2 = \frac{\varepsilon}{V_{AC}^2}, \\ &\hat{c}_1 = \varepsilon c_1, \quad \hat{c}_2 = \varepsilon^2 c_2, \\ &\hat{f}(x) = \varepsilon f(x), \quad \hat{g}(x) = \varepsilon g(x), \\ &\text{and } u(x,t) = \varepsilon v(x,t), \end{aligned} \tag{4}$$

where ε is a small positive parameter, that is, $0 < \varepsilon \ll 1$. By using the rescalings (4) it follows from (1)–(3) that we now have to study the following initial-boundary value problem for $v(x, t)$

$$\begin{aligned} v_{tt} + v_{xxxx} = &-\varepsilon c_1 v_t - \varepsilon^2 c_2 v_{txxxx} \\ &+\alpha\varepsilon^2 \int_0^1 v_x^2(x,t)dx\, v_{xx} \\ &+\frac{(V_0+\varepsilon\sin(\omega t))^2}{(1-\varepsilon v)^2}, 0 < x < 1, t > 0, \end{aligned} \tag{5}$$

$$v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0, \quad t \geq 0, \tag{6}$$

$$v(x,0) = f(x), v_t(x,0) = g(x), \quad 0 < x < 1. \tag{7}$$

Since ε is small the denominator in (5) can be expanded as

$$\frac{1}{(1-\varepsilon v)^2} = 1 + 2\varepsilon v + 3\varepsilon^2 v^2 + \mathcal{O}(\varepsilon^3). \tag{8}$$

Moreover, due to the boundary conditions (6) $v(x, t)$ can be expanded in the following Fourier series:

$$v(x,t) = \sum_{k=1}^{\infty} v_k(t)\sin(k\pi x). \tag{9}$$

In the next section, we will study the natural frequencies of the actuated beam in detail, i.e. $c_1 = c_2 = \alpha = 0$. By using a three-time-scales perturbation method (see [21,22] for a description of the multiple time-scales perturbation technique) we will construct $\mathcal{O}(\varepsilon^3)$ accurate approximations of these frequencies. In Sect. 4 of this paper we will first consider the case when the excitation frequency ω is close to or equal to a natural frequency of the actuated beam. Only a relatively large damping can reduce the amplitudes of the oscillations of the beam. For that reason we first consider $c_1 \neq 0$ (i.e. we take into account a large viscous damping (and $c_2 = \alpha = 0$)). Secondly, we will assume that the excitation frequency ω is not equal or not close to a natural frequency of the actuated beam. We will study the occurrence of super-or subharmonic resonances and we will take into account the influence of the nonlinear elastic forces in (5) as well as relatively small structural damping, i.e., $c_1 = 0$, $c_2 \neq 0$ and $\alpha \neq 0$.

3 The influence of the electrostatic force

In this section, we will study the influence of the electrostatic force on the oscillations of the microbeam. Damping and nonlinear elastic forces are neglected, that is, $c_1 = c_2 = \alpha = 0$ in Eq. (5). By substituting (9) into (5), and by using the orthogonality properties of the sine functions, it follows that $v_k(t)$ in (9) has to satisfy

$$\begin{aligned} v_{k_{tt}} + (k\pi)^4 v_k = &\, 2(V_0^2 + 2\varepsilon V_0 \sin\omega t + \varepsilon^2\sin^2\omega t) \\ &(H(k) + \varepsilon v_k + 3\varepsilon^2 L(k) + \mathcal{O}(\varepsilon^3)), \end{aligned} \tag{10}$$

where

$$H(k) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ \frac{2}{k\pi}, & \text{if } k \text{ is odd,} \end{cases} \tag{11}$$

and

$$L(k) = \sum_{m,n=1}^{\infty} c_{m,n}\, v_m v_n, \tag{12}$$

where $c_{m,n} = 0$ if $\pm m \pm n \pm k$ is even or zero and it is equal to $\frac{-4mnk}{\pi(k+m+n)(k+m-n)(k-m+n)(k-m-n)}$ if $\pm m \pm n \pm k$ is odd.

By substituting (9) into (7) it similarly follows that

$$v_k(0) = 2\int_0^1 f(x)\sin(k\pi x)dx;\ v_{k_t}(0) = 2\int_0^1 g(x)\sin(k\pi x)dx. \tag{13}$$

Since a naive perturbation expansion for $v_k(t)$ leads to secular terms in $v_k(t)$, we will now apply a three-time-scales perturbation method (with $t_0 = t$, $t_1 = \varepsilon t$, $t_2 = \varepsilon^2 t$) to obtain highly accurate approximations for $v_k(t)$ which are valid on a time-scale of order $1/\varepsilon$. So,

$$\begin{aligned} &v_k(t,\varepsilon) = v_k(t_0, t_1, t_2, \varepsilon),\\ &\tfrac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2,\\ &\tfrac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_1 D_0 + \varepsilon^2(D_1^2 + 2D_2D_0) + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{14}$$

with $D_i = \frac{\partial}{\partial t_i}$. By substituting (14) into (10) we obtain

$$\begin{aligned} &D_0^2 v_k + (k\pi)^4 v_k + 2\varepsilon D_1 D_0 v_k\\ &\quad +\varepsilon^2(D_1^2 + 2D_2D_0)v_k + \mathcal{O}(\varepsilon^3)\\ &\quad = 2(V_0^2 + 2\varepsilon V_0\sin(\omega t) + \varepsilon^2\sin^2(\omega t))\\ &\quad (H(k) + \varepsilon v_k + 3\varepsilon^2 L(k) + \mathcal{O}(\varepsilon^3)). \end{aligned} \tag{15}$$

By putting (ω_i is a constant independent of ε for $i = 1$, 2, and 3)

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2, \tag{16}$$

we can rewrite $\sin(\omega t)$ in (15) as

$$\sin(\omega t) = \sin(\omega_0 t_0 + \omega_1 t_1 + \omega_2 t_2). \tag{17}$$

As usual $v_k(t_0, t_1, t_2)$ is expanded in

$$\begin{aligned} v_k(t_0, t_1, t_2, \varepsilon) &= v_{k,0}(t_0, t_1, t_2) + \varepsilon v_{k,1}(t_0, t_1, t_2)\\ &\quad +\varepsilon^2 v_{k,2}(t_0, t_1, t_2) + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{18}$$

By substituting (18) into (15) and by collecting terms of $\mathcal{O}(1)$, terms of $\mathcal{O}(\varepsilon)$, and terms of $\mathcal{O}(\varepsilon^2)$, we obtain the usual $\mathcal{O}(1)$-, $\mathcal{O}(\varepsilon)$-, $\mathcal{O}(\varepsilon^2)$- problems: $\mathcal{O}(1)$-problem :

$$\mathcal{L}v_{k,0} = 2V_0^2 H(k), \tag{19}$$

$\mathcal{O}(\varepsilon)$-problem:

$$\mathcal{L}v_{k,1} = -2D_1D_0v_{k,0} + 2V_0^2 v_{k,0} + 4V_0H(k)\sin(\omega t), \tag{20}$$

$\mathcal{O}(\varepsilon^2)$-problem:

$$\begin{aligned} \mathcal{L}v_{k,2} &= -2D_1D_0v_{k,1} - D_1^2 v_{k,0} - 2D_2D_0v_{k,0} + 2V_0^2 v_{k,1}\\ &\quad +2H(k)\sin^2(\omega t) + 4V_0 v_{k,0}\sin(\omega t)\\ &\quad +6V_0^2 \sum_{\substack{m,n=1\\ \pm m\pm n\pm k \text{ odd}}}^{\infty} c_{m,n}\, v_{m,0}\, v_{n,0}, \end{aligned} \tag{21}$$

where $\mathcal{L}v = D_0^2 v + (k\pi)^4 v$ and $\sin(\omega t)$ is given by (17). The solution of the $\mathcal{O}(1)$-problem (19) is given by

$$\begin{aligned} v_{k,0}(t_0, t_1, t_2) &= A_{k,0}(t_1, t_2)\cos(k^2\pi^2 t_0)\\ &\quad +B_{k,0}(t_1, t_2)\sin(k^2\pi^2 t_0) + \frac{2V_0^2 H(k)}{k^4\pi^4}. \end{aligned} \tag{22}$$

After substituting (22) into the $\mathcal{O}(\varepsilon)$-problem (20), we obtain

$$\begin{aligned} \mathcal{L}v_{k,1} &= -2k^2\pi^2\left(-\tfrac{\partial A_{k,0}}{\partial t_1}\sin(k^2\pi^2 t_0) + \tfrac{\partial B_{k,0}}{\partial t_1}\cos(k^2\pi^2 t_0)\right)\\ &\quad +4V_0H(k)\sin(\omega_0 t_0 + \omega_1 t_1 + \omega_2 t_2)\\ &\quad +2V_0^2\Bigg(A_{k,0}\cos(k^2\pi^2 t_0) + B_{k,0}\sin(k^2\pi^2 t_0)\\ &\quad +\tfrac{2V_0^2 H(k)}{k^4\pi^4}\Bigg) \end{aligned} \tag{23}$$

Now we have to consider two cases:
Case 1 $\omega_0 \neq K^2\pi^2$ for all odd $K \in \mathbb{N}$, that is, $\sin(\omega_0 t_0 + \omega_1 t_1 + \omega_2 t_2)$ is not a resonant term in the right-hand side of Eq. (23) for $v_{k,1}$,
Case 2 $\omega_0 = K^2\pi^2$ for a certain, fixed, and odd $K \in \mathbb{N}$, that is, $\sin(\omega_0 t_0 + \omega_1 t_1 + \omega_2 t_2)$ is a resonant term in the right-hand side of Eq. (23) for $v_{k,1}$.
First we will study the problem for $v_{k,1}$ (given by (23)) in case 1.

Case 1 $\boldsymbol{\omega_0 \neq K^2\pi^2}$ **for all odd** $\boldsymbol{K} \in \mathbb{N}$.

In this case, it follows from (23) that in order to avoid secular terms in $v_{k,1}$ that $A_{k,0}$ and $B_{k,0}$ have to satisfy

$$\begin{aligned} &\tfrac{\partial A_{k,0}}{\partial t_1} + \tfrac{V_0^2}{k^2\pi^2}B_{k,0} = 0,\\ &\tfrac{\partial B_{k,0}}{\partial t_1} - \tfrac{V_0^2}{k^2\pi^2}A_{k,0} = 0. \end{aligned} \tag{24}$$

The solution for $A_{k,0}$ and $B_{k,0}$ is given by

$$\begin{aligned} A_{k,0}(t_1, t_2) &= C_{k,0}(t_2)\cos\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right)\\ &\quad +D_{k,0}(t_2)\sin\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right),\\ B_{k,0}(t_1, t_2) &= C_{k,0}(t_2)\sin\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right)\\ &\quad -D_{k,0}(t_2)\cos\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right). \end{aligned} \tag{25}$$

And so, in this case, we have

$$v_{k,0}(t_0,t_1,t_2) = \cos(k^2\pi^2 t_0)\Big(C_{k,0}(t_2)\cos\left(\frac{V_0^2}{k^2\pi^2}t_1\right) + D_{k,0}(t_2)\sin\left(\frac{V_0^2}{k^2\pi^2}t_1\right)\Big) + \sin(k^2\pi^2 t_0)\Big(C_{k,0}(t_2)\sin\left(\frac{V_0^2}{k^2\pi^2}t_1\right) - D_{k,0}(t_2)\cos\left(\frac{V_0^2}{k^2\pi^2}t_1\right)\Big) + \frac{2V_0^2H(k)}{k^4\pi^4} \quad (26)$$

and

$$v_{k,1}(t_0,t_1,t_2) = A_{k,1}(t8_1,t_2)\cos(k^2\pi^2 t_0) + B_{k,1}(t_1,t_2)\sin(k^2\pi^2 t_0) + \frac{4V_0H(k)}{k^4\pi^4-\omega_0^2}\sin(\omega_0 t_0+\omega_1 t_1+\omega_2 t_2) + \frac{4V_0^4H(k)}{k^8\pi^8}. \quad (27)$$

Case 2 $\boldsymbol{\omega_0 = K^2\pi^2}$ **for a certain, fixed, odd** $\boldsymbol{K \in \mathbb{N}}$. In this case it follows from (23) that no secular terms in $v_{k,1}$ will occur when $A_{k,0}$ and $B_{k,0}$ satisfy (24) when $k \neq K$, and satisfy for $k = K$

$$\frac{\partial A_{K,0}}{\partial t_1} + \frac{V_0^2}{K^2\pi^2}B_{K,0} = -\frac{2V_0H(K)}{K^2\pi^2}(\cos(\omega_1 t_1+\omega_2 t_2)),$$
$$\frac{\partial B_{K,0}}{\partial t_1} - \frac{V_0^2}{K^2\pi^2}A_{K,0} = \frac{2V_0H(K)}{K^2\pi^2}(\sin(\omega_1 t_1+\omega_2 t_2)). \quad (28)$$

It is obvious that for all $k \neq K$, $v_{k,0}$ and $v_{k,1}$ are given by (26) and (27), respectively. To determine $v_{K,0}$ and $v_{K,1}$ we first have to solve (28). The system (28) of two first-order ordinary differential equations can be rewritten as a second-order ordinary differential equation for $A_{K,0}$, i.e.,

$$\frac{\partial^2 A_{K,0}}{\partial t_1^2} + \frac{V_0^4}{K^4\pi^4}A_{K,0} = \frac{2V_0H(k)}{K^2\pi^2}\left(\omega_1 - \frac{V_0^2}{K^2\pi^2}\right)\sin(\omega_1 t_1+\omega_2 t_2). \quad (29)$$

To determine $A_{K,0}$ from (29) (and then $B_{K,0}$ from the first equation in (28)) we now have to consider three cases:

Case 2.1 $\boldsymbol{\omega_0 = K^2\pi^2}$ **for a certain, fixed, odd** $\boldsymbol{K \in \mathbb{N}}$**, and** $\boldsymbol{\omega_1 \neq \pm\frac{V_0^2}{K^2\pi^2}}$. In this case $A_{K,0}$ and $B_{K,0}$ are given by

$$A_{K,0}(t_1,t_2) = C_{K,0}(t_2)\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right) + D_{K,0}(t_2)\sin\left(\frac{V_0^2}{K^2\pi^2}t_1\right) - \frac{2V_0H(K)}{V_0^2+\omega_1K^2\pi^2}\sin(\omega_1 t_1+\omega_2 t_2),$$
$$B_{K,0}(t_1,t_2) = C_{K,0}(t_2)\sin\left(\frac{V_0^2}{K^2\pi^2}t_1\right) - D_{K,0}(t_2)\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right) - \frac{2V_0H(K)}{V_0^2+\omega_1K^2\pi^2}\cos(\omega_1 t_1 + \omega_2 t_2). \quad (30)$$

Case 2.2 $\boldsymbol{\omega_0 = K^2\pi^2}$ for a certain, fixed, odd $\boldsymbol{K \in \mathbb{N}}$**, and** $\boldsymbol{\omega_1 = \frac{V_0^2}{K^2\pi^2}}$. In this case $A_{K,0}$ and $B_{K,0}$ are given by

$$A_{K,0}(t_1,t_2) = C_{K,0}(t_2)\cos(\omega_1 t_1) + D_{K,0}(t_2)\sin(\omega_1 t_1),$$
$$B_{K,0}(t_1,t_2) = C_{K,0}(t_2)\sin(\omega_1 t_1) - D_{K,0}(t_2)\cos(\omega_1 t_1) - \frac{2H(K)}{V_0}\cos(\omega_1 t_1+\omega_2 t_2). \quad (31)$$

Case 2.3 $\boldsymbol{\omega_0 = K^2\pi^2}$ for a certain, fixed, odd $\boldsymbol{K \in \mathbb{N}}$ **and** $\boldsymbol{\omega_1 = -\frac{V_0^2}{K^2\pi^2}}$.
In this case $A_{K,0}$ and $B_{K,0}$ are given by

$$A_{K,0}(t_1,t_2) = C_{K,0}(t_2)\cos(\omega_1 t_1) - D_{K,0}(t_2)\sin(\omega_1 t_1) - \frac{2V_0H(K)}{K^2\pi^2}t_1\cos(\omega_1 t_1+\omega_2 t_2),$$
$$B_{K,0}(t_1,t_2) = -C_{K,0}(t_2)\sin(\omega_1 t_1) - D_{K,0}(t_2)\cos(\omega_1 t_1) + \frac{2V_0H(K)}{K^2\pi^2}t_1\sin(\omega_1 t_1+\omega_2 t_2). \quad (32)$$

It should be observed that we get in the solutions for $A_{K,0}$ and $B_{K,0}$ when $\omega_0 = K^2\pi^2$ and $\omega_1 = -\frac{V_0^2}{K^2\pi^2}$ (see case 2.3, and Eq. (32)) unbounded terms in t_1. However, on t time-scales of order $1/\varepsilon$ these terms remain bounded. So, for $t = \mathcal{O}(\frac{1}{\varepsilon})$ we still have bounded functions for $v_{K,0}(t_0,t_1,t_2)$ and $v_{K,1}(t_0,t_1,t_2)$, and these functions for the 3 cases (cases 2.1, case 2.2, and case 2.3) are given by

$$v_{K,0}(t_1,t_1,t_2) = A_{K,0}(t_1,t_2)\cos(K^2\pi^2 t_0) + B_{K,0}(t_1,t_2)\sin(K^2\pi^2 t_0) + \frac{2V_0^2H(K)}{K^4\pi^4}, \quad (33)$$

$$v_{K,1}(t_0,t_1,t_2) = A_{K,1}(t_1,t_2)\cos(K^2\pi^2 t_0) + B_{K,1}(t_1,t_2)\sin(K^2\pi^2 t_0) + \frac{4V_0^4H(K)}{K^8\pi^8}. \quad (34)$$

To determine $v_{k,0}(t_0,t_1,t_2)$ completely for all the cases (that is, for case 1, case 2.1, case 2.2, and case 2.3) we still have to determine $C_{k,0}(t_2)$ and $D_{k,0}(t_2)$ for all k by solving the $\mathcal{O}(\varepsilon^2)$-problem (21). In the right-hand side of Eq. (21) for $v_{k,2}$ we encounter the nonlinear term

$$\sum_{\substack{m,n=1 \\ \pm m\pm n\pm k \text{ odd}}}^{\infty} c_{m,n}\, v_{m,0}(t_0,t_1,t_2)\, v_{n,0}(t_0,t_1,t_2). \quad (35)$$

By substituting $v_{m,0}$ and $v_{n,0}$ into this term we will obtain products of $\sin(m^2\pi^2 t_0)$ (or $\cos(m^2\pi^2 t_2)$) with $\sin(n^2\pi^2 t_0)$ or $\cos(n^2\pi^2 t_2)$, and we will get products of sine or cosine functions with constants. For the products of the trigonometric functions we can use identities yielding $\cos((\pm m^2 \pm n^2)t_0)$ and $\sin((\pm m^2 \pm n^2)t_0)$. These functions can be resonant terms in the nonlinear term (35) of Eq. (21) when $k^2 = \pm m^2 \pm n^2$ and $\pm m \pm n \pm k$ is odd. It can easily be shown that this cannot occur. For instance, assume that both m and n are odd (or even) then from $k^2 = \pm m^2 \pm n^2$ it follows that k is even, but this contradicts the fact that $\pm m \pm n \pm k$ should be odd. Similarly, if one assumes m to be odd and n to be even, then from $k^2 = \pm m^2 \pm n^2$ it follows that k is odd, but this again contradicts the fact that $\pm m \pm n \pm k$ should be odd. So, the products of the trigonometric functions in (35) will not lead to resonant terms in Eq. (21), and the only resonant terms coming from (35) (that is, the last term in Eq. (21)) are

$$\left(A_{k,0}\cos\left(k^2\pi^2 t_0\right) + B_{k,0}\sin\left(k^2\pi^2 t_0\right)\right)\left(\sum_{n \text{ odd}} c_{k,n}\frac{2V_0^2 H(n)}{n^4\pi^4} + \sum_{m \text{ odd}} c_{m,k}\frac{2V_0^2 H(m)}{m^4\pi^4}\right).$$

Using the symmetry of $c_{m,n}$ (see (12)) this resonant term can be further simplified to

$$\left(A_{k,0}\cos\left(k^2\pi^2 t_0\right) + B_{k,0}\sin\left(k^2\pi^2 t_0\right)\right)S(k),$$

with

$$S(k) = \left(4V_0^2 \sum_{n \text{ odd}} c_{k,n}\frac{H(n)}{n^4\pi^4}\right). \tag{36}$$

Furthermore, it should be observed that we have in the right-hand side of Eq. (21) the term $2H(k)\sin^2(\omega t) = H(k)(\frac{1}{2} - \frac{1}{2}\cos(2\omega t))$ which can also be a resonant term in $v_{k,2}$ when $k = L$ is odd and $2\omega_0 = L^2\pi^2$. And also the term $4V_0 v_{k,0}\sin(\omega t)$ for $k = M$ in (21) can give rise to resonant terms when $\omega_0 = 2M^2\pi^2$ for a fixed $M \in \mathbb{N}$ (M can be even or odd). So, apart from the resonance case, i.e., case 2 with $\omega_0 = K^2\pi^2$ for a certain, fixed, and odd $K \in \mathbb{N}$, we have to consider three additional cases. In case 1.1 we will consider the case when ω_0 is not equal to the "pure" resonance frequency, and is not equal to first superharmonic and subharmonic resonance frequency. In case 1.2 we will consider the case when ω_0 is equal to the first superharmonic resonance frequency, and in case 1.3 we will study the case when ω_0 is equal to the first subharmonic resonance frequency.

Case 1.1 $\boldsymbol{\omega_0 \neq K^2\pi^2}$ **for all odd** $\boldsymbol{K \in \mathbb{N}}$**, and** $\boldsymbol{\omega_0 \neq \frac{1}{2}L^2\pi^2}$ **for all odd** $\boldsymbol{L \in \mathbb{N}}$**, and** $\boldsymbol{\omega_0 \neq 2M^2\pi^2}$ **for all** $\boldsymbol{M \in \mathbb{N}}$**.**

By substituting $v_{k,0}$ and $v_{k,1}$ as given by (26) and (27), respectively, into the $\mathcal{O}(\varepsilon^2)$-problem (21) for $v_{k,2}$, we obtain

$$\begin{aligned}\mathcal{L}v_{k,2} = {}& \cos(k^2\pi^2 t_0)\Big(-2k^2\pi^2\frac{\partial B_{k,1}}{\partial t_1} + \frac{V_0^4}{k^4\pi^4}A_{k,0} \\ & +2V_0^2 A_{k,1} - 2k^2\pi^2\frac{\partial B_{k,0}}{\partial t_2} + 6V_0^2 A_{k,0}S(k)\Big) \\ & +\sin(k^2\pi^2 t_0)\Big(2k^2\pi^2\frac{\partial A_{k,1}}{\partial t_1} + \frac{V_0^4}{k^4\pi^4}B_{k,0} \\ & +2V_0^2 B_{k,1} + 2k^2\pi^2\frac{\partial A_{k,0}}{\partial t_2} + 6V_0^2 B_{k,0}S(k)\Big) \\ & +NST\end{aligned} \tag{37}$$

where NST stands for non-secular terms. To avoid secular terms in $v_{k,2}$, it follows from (37) that $A_{k,1}$ and $B_{k,1}$ have to satisfy

$$\begin{aligned}\frac{\partial A_{k,1}}{\partial t_1} + \frac{V_0^2}{k^2\pi^2}B_{k,1} &= -\frac{V_0^4}{2k^6\pi^6}B_{k,0} \\ &\quad -\frac{\partial A_{k,0}}{\partial t_2} - \frac{3V_0^2 S(k)}{k^2\pi^2}B_{k,0},\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{\partial B_{k,1}}{\partial t_1} - \frac{V_0^2}{k^2\pi^2}A_{k,1} &= \frac{V_0^4}{2k^6\pi^6}A_{k,0} \\ &\quad -\frac{\partial B_{k,0}}{\partial t_2} + \frac{3V_0^2 S(k)}{k^2\pi^2}A_{k,0}.\end{aligned} \tag{39}$$

By differentiating (38) with respect to t_1 and by substituting $\frac{\partial B_{k,1}}{\partial t_1}$ from (39) into the so-obtained equation, we obtain

$$\begin{aligned}&\frac{\partial^2 A_{k,1}}{\partial {t_1}^2} + \left(\frac{V_0^2}{k^2\pi^2}\right)^2 A_{k,1} \\ &= -\cos\left(\frac{V_0^2}{k^2\pi^2}t_1\right)\left(\frac{V_0^6}{k^8\pi^8}C_{k,0} + \frac{2V_0^2}{k^2\pi^2}\frac{dD_{k,0}}{dt_2} + \frac{6V_0^4 S(k)}{k^4\pi^4}C_{k,0}\right) \\ &\quad -\sin\left(\frac{V_0^2}{k^2\pi^2}t_1\right)\left(\frac{V_0^6}{k^8\pi^8}D_{k,0} - \frac{2V_0^2}{k^2\pi^2}\frac{dC_{k,0}}{dt_2} + \frac{6V_0^4 S(k)}{k^4\pi^4}D_{k,0}\right).\end{aligned}$$

Since $A_{k,1}$ and $B_{k,1}$ have to be bounded, we have to avoid secular terms in $A_{k,1}$ and $B_{k,1}$, and so we obtain

$$\begin{aligned}&\frac{dC_{k,0}}{dt_2} - \left(\frac{V_0^4}{2k^6\pi^6} + \frac{3V_0^2 S(k)}{k^2\pi^2}\right)D_{k,0} = 0, \\ &\frac{dD_{k,0}}{dt_2} + \left(\frac{V_0^4}{2k^6\pi^6} + \frac{3V_0^2 S(k)}{k^2\pi^2}\right)C_{k,0} = 0.\end{aligned} \tag{40}$$

By solving (40) it follows that $C_{k,0}$ and $D_{k,0}$ are given by

$$C_{k,0}(t_2) = C_{k,0}(0)\cos\left(\left(\frac{V_0^4}{2k^6\pi^6}+\frac{3V_0^2S(k)}{k^2\pi^2}\right)t_2\right) +D_{k,0}(0)\sin\left(\left(\frac{V_0^4}{2k^6\pi^6}+\frac{3V_0^2S(k)}{k^2\pi^2}\right)t_2\right),$$
$$D_{k,0}(t_2) = -C_{k,0}(0)\sin\left(\left(\frac{V_0^4}{2k^6\pi^6}+\frac{3V_0^2S(k)}{k^2\pi^2}\right)t_2\right) +D_{k,0}(0)\cos\left(\left(\frac{V_0^4}{2k^6\pi^6}+\frac{3V_0^2S(k)}{k^2\pi^2}\right)t_2\right). \tag{41}$$

Hence,

$$v_{k,0}(t_0,t_1,t_2) = C_{k,0}(0)\cos(\hat{\omega}(t_0,t_1,t_2)) - D_{k,0}(0)\sin(\hat{\omega}(t_0,t_1,t_2)) + \frac{2V_0^2H(k)}{k^4\pi^4}, \tag{42}$$

where $\hat{\omega}(t_0,t_1,t_2) = k^2\pi^2 t_0 - \frac{V_0^2}{k^2\pi^2}t_1 - \left(\frac{V_0^4}{2k^6\pi^6}+\frac{3V_0^2S(k)}{k^2\pi^2}\right)t_2$. In this 'nonresonant' case it follows from (42) that the natural frequencies of the microbeam up to $\mathcal{O}(\varepsilon^3)$ are given by

$$k^2\pi^2 - \frac{V_0^2}{k^2\pi^2}\varepsilon - \left(\frac{V_0^4}{2k^6\pi^6}+\frac{3V_0^2S(k)}{k^2\pi^2}\right)\varepsilon^2 \tag{43}$$

for all $k \in \mathbb{N}$. Later on, we will see that if the frequency of the external force (the AC frequency) is in a neighborhood of a natural frequency of the vibration, it will give rise to resonance.

Case 1.2 $\boldsymbol{\omega_0 = \frac{1}{2}L^2\pi^2}$ **for a certain, fixed, and odd** $\boldsymbol{L \in \mathbb{N}}$**.**

In this case it follows from the $\mathcal{O}(\varepsilon^2)$-problem (21) for $v_{k,2}$ that all $v_{k,2}$ with $k \neq L$ have to satisfy (37) and that $v_{k,0}$ is given by (42). For $k = L$ it follows from (21) that $v_{L,2}$ has to satisfy

$$\begin{aligned}\mathcal{L}v_{L,2} = {} & \cos(L^2\pi^2 t_0)\Big(-2L^2\pi^2\frac{\partial B_{L,1}}{\partial t_1} + \frac{V_0^4}{L^4\pi^4}A_{L,0} + 2V_0^2A_{L,1} \\ & -2L^2\pi^2\frac{\partial B_{L,0}}{\partial t_2} + 6V_0^2S(L)A_{L,0} \\ & -H(L)\cos(2\omega_1t_1+2\omega_2t_2)\Big) \\ & +\sin(L^2\pi^2t_0)\Big(2L^2\pi^2\frac{\partial A_{L,1}}{\partial t_1} + \frac{V_0^4}{L^4\pi^4}B_{L,0} + 2V_0^2B_{L,1} \\ & +2L^2\pi^2\frac{\partial A_{L,0}}{\partial t_2} + 6V_0^2S(L)B_{L,0} \\ & +H(L)\sin(2\omega_1t_1+2\omega_2t_2)\Big) \\ & +NST.\end{aligned} \tag{44}$$

To avoid secular terms in $v_{L,2}$ it follows from (44) that $A_{L,1}$ and $B_{L,1}$ have to satisfy

$$\begin{aligned}\frac{\partial A_{L,1}}{\partial t_1} + \frac{V_0^2}{L^2\pi^2}B_{L,1} &= -\frac{V_0^4}{2L^6\pi^6}B_{L,0} - \frac{\partial A_{L,0}}{\partial t_2} - \frac{3V_0^2S(L)}{L^2\pi^2}B_{L,0} \\ &\quad -\frac{H(L)}{2L^2\pi^2}\sin(2\omega_1t_1+2\omega_2t_2), \\ \frac{\partial B_{L,1}}{\partial t_1} - \frac{V_0^2}{L^2\pi^2}A_{L,1} &= \frac{V_0^4}{2L^6\pi^6}A_{L,0} - \frac{\partial B_{L,0}}{\partial t_2} + \frac{3V_0^2S(L)}{L^2\pi^2}A_{L,0} \\ &\quad -\frac{H(L)}{2L^2\pi^2}\cos(2\omega_1t_1+2\omega_2t_2).\end{aligned} \tag{45}$$

Combining the two equations in (45), we obtain for $A_{L,1}$

$$\begin{aligned}\frac{\partial^2A_{L,1}}{\partial t_1^2} + \frac{V_0^4}{L^4\pi^4}A_{L,1} = {} & -\cos\left(\frac{V_0^2}{L^2\pi^2}t_1\right) \left[\frac{V_0^6}{L^8\pi^8}C_{L,0} + \frac{2V_0^2}{L^2\pi^2}\frac{dD_{L,0}}{dt_2} + \frac{6V_0^4S(L)}{L^4\pi^4}C_{L,0}\right] \\ & -\sin\left(\frac{V_0^2}{L^2\pi^2}t_1\right)\left[\frac{V_0^6}{L^8\pi^8}D_{L,0} - \frac{2V_0^2}{L^2\pi^2}\frac{dC_{L,0}}{dt_2} + \frac{6V_0^4S(L)}{L^4\pi^4}D_{L,0}\right] \\ & -\frac{H(L)}{2L^2\pi^2}\left[2\omega_1 - \frac{V_0^2}{L^2\pi^2}\right]\cos(2\omega_1t_1+2\omega_2t_2).\end{aligned} \tag{46}$$

Now we have to consider three cases, that is, $2\omega_1 = \frac{V_0^2}{L^2\pi^2}$, $2\omega_1 = -\frac{V_0^2}{L^2\pi^2}$, and $2\omega_1 \neq \pm\frac{V_0^2}{L^2\pi^2}$. If $2\omega_1 \neq \pm\frac{V_0^2}{L^2\pi^2}$, then we will obtain the same $C_{L,0}$ and $D_{L,0}$ as in (41). When $2\omega_1 = \frac{V_0^2}{L^2\pi^2}$, then the last term in the right-hand side of (46) will be zero, and we will have the same $C_{L,0}$ and $D_{L,0}$ as given in (41). So, both of these two cases will give us the same result as in case 1.1. When $2\omega_1 = -\frac{V_0^2}{L^2\pi^2}$, it follows from (45) that in order to avoid secular terms in $A_{L,1}$ that $C_{L,0}$ and $D_{L,0}$ have to satisfy

$$\begin{aligned}&\frac{dC_{L,0}}{dt_2} - \left(\frac{V_0^4}{2L^6\pi^6}+\frac{3V_0^2S(L)}{L^2\pi^2}\right)D_{L,0} = -\frac{H(L)}{2L^2\pi^2}\sin(2\omega_2t_2), \\ &\frac{dD_{L,0}}{dt_2} + \left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)C_{L,0} = \frac{H(L)}{2L^2\pi^2}\cos(2\omega_2t_2).\end{aligned} \tag{47}$$

Combining the two equations in (47), we obtain for $C_{L,0}$

$$\frac{d^2C_{L,0}}{dt_2^2} + \left(\frac{V_0^4}{2L^6\pi^6}+\frac{3V_0^2S(L)}{L^2\pi^2}\right)^2C_{L,0} + \left(2\omega_2 - \left(\frac{V_0^4}{2L^6\pi^6}+\frac{3V_0^2S(L)}{L^2\pi^2}\right)\right)\frac{H(L)}{2L^2\pi^2}\cos(2\omega_2t_2) = 0. \tag{48}$$

From (48) it follows that we have to consider 3 subcases, i.e., $2\omega_2 \neq \frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}$, $2\omega_2 = \frac{V_0^4}{2L^6\pi^6} +$

$\frac{3V_0^2S(L)}{L^2\pi^2}$, and $2\omega_2 = -\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)$. For $2\omega_2 = \frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}$, we obtain the same $C_{L,0}$ and $D_{L,0}$ as given in (41). While for $2\omega_2 \neq \pm\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)$, $C_{L,0}$ and $D_{L,0}$ are given by

$$\begin{aligned} C_{L,0}(t_2) &= K_1 \cos\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + K_2 \sin\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + \frac{L^4\pi^4H(L)}{2\mu L^6\pi^6 + V_0^4 + 6V_0^2L^4\pi^4S(L)}\cos(2\omega_2t_2), \\ D_{L,0}(t_2) &= -K_1 \sin\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + K_2 \cos\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + \frac{L^4\pi^4H(L)}{4\mu L^6\pi^6 + V_0^4 + 6V_0^2L^4\pi^4S(L)}\sin(2\omega_2t_2), \end{aligned} \tag{49}$$

where K_1 and K_2 are constants of integration. For the last subcase, when $2\omega_2 = -\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)$, we obtain

$$\begin{aligned} C_{L,0}(t_2) &= K_1 \cos\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + K_2 \sin\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + \frac{H(L)}{2L^2\pi^2}t_2 \sin\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right), \\ D_{L,0}(t_2) &= -K_1 \sin\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + K_2 \cos\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right) \\ &\quad + \frac{H(L)}{2L^2\pi^2}t_2 \cos\left(\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right)t_2\right). \end{aligned} \tag{50}$$

Having determined $C_{L,0}(t_2)$ and $D_{L,0}(t_2)$ we now have completely computed $v_{k,0}(t_0, t_1, t_2)$ as given by (26). It should be observed that $v_{k,0}$ is bounded on a t time-scale of order $1/\varepsilon$, but it contains terms for $k = L$ that become unbounded for times t larger than order $1/\varepsilon^2$ when the frequency of the AC voltage is $\mathcal{O}(\varepsilon^3)$ close to

$$\frac{1}{2}L^2\pi^2 - \frac{1}{2}\varepsilon\frac{V_0^2}{L^2\pi^2} - \frac{1}{2}\varepsilon^2\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2S(L)}{L^2\pi^2}\right). \tag{51}$$

The resonance for ω as given by (51) is usually referred to as the first superharmonic resonance.

Case 1.3 : $\boldsymbol{\omega_0 = 2M^2\pi^2}$ **for a certain, and fixed $\boldsymbol{M \in \mathbb{N}}$.**

In this case it follows from the $\mathcal{O}(\varepsilon^2)$-problem (21) for $v_{k,2}$ that all $v_{k,2}$ with $k \neq M$ have to satisfy (37) and that $v_{k,0}$ is given by (42). For $k = M$, it follows from (21) that $v_{M,2}$ has to satisfy

$$\begin{aligned} \mathcal{L}v_{M,2} = \cos(M^2\pi^2t_0)\Big(& -2M^2\pi^2\frac{\partial B_{M,1}}{\partial t_1} + \frac{V_0^4}{M^4\pi^4}A_{M,0} \\ & +2V_0^2A_{M,1} + 6V_0^2S(M)A_{M,0} \\ & -2M^2\pi^2\frac{\partial B_{M,0}}{\partial t_2} \\ & +2V_0[A_{M,0}\sin(\omega_1t_1 + \omega_2t_2) \\ & +B_{M,0}\cos(\omega_1t_1 + \omega_2t_2)]\Big) \\ & +\sin(M^2\pi^2t_0)\Big(2M^2\pi^2\frac{\partial A_{M,1}}{\partial t_1} + \frac{V_0^4}{M^4\pi^4}B_{M,0} \\ & +2V_0^2B_{M,1} + 6V_0^2S(M)B_{M,0} \\ & 2M^2\pi^2\frac{\partial A_{M,0}}{\partial t_2} \\ & +2V_0[A_{M,0}\cos(\omega_1t_1 + \omega_2t_2) \\ & -B_{M,0}\sin(\omega_1t_1 + \omega_2t_2)]\Big) \\ & +NST. \end{aligned} \tag{52}$$

To avoid secular term in $v_{M,2}$, it follows from (52) that $A_{M,1}$ and $B_{M,1}$ have to satisfy

$$\begin{aligned} \frac{\partial A_{M,1}}{\partial t_1} + \frac{V_0^2}{M^2\pi^2}B_{M,1} &= -\frac{V_0^4}{2M^6\pi^6}B_{M,0} - \frac{\partial A_{M,0}}{\partial t_2} - \frac{3V_0^2S(M)}{M^2\pi^2}B_{M,0} \\ &\quad -\frac{V_0}{M^2\pi^2}\big[A_{M,0}\cos(\omega_1t_1 + \omega_2t_2) \\ &\quad -B_{M,0}\sin(\omega_1t_1 + \omega_2t_2)\big], \\ \frac{\partial B_{M,1}}{\partial t_1} - \frac{V_0^2}{M^2\pi^2}A_{M,1} &= \frac{V_0^4}{2M^6\pi^6}A_{M,0} - \frac{\partial B_{M,0}}{\partial t_2} + \frac{3V_0^2S(M)}{M^2\pi^2}A_{M,0} \\ &\quad +\frac{V_0}{M^2\pi^2}\big[A_{M,0}\sin(\omega_1t_1 + \omega_2t_2) \\ &\quad +B_{M,0}\cos(\omega_1t_1 + \omega_2t_2)\big]. \end{aligned} \tag{53}$$

Combining the two equations in (53), we get

$$\begin{aligned} \frac{\partial^2 A_{M,1}}{\partial t_2^2} + \frac{V_0^4}{M^4\pi^4}A_{M,1} =& -\cos\left(\frac{V_0^2}{M^2\pi^2}t_1\right)\Big[\frac{V_0^6}{M^8\pi^8}C_{M,0} \\ & +\frac{2V_0^2}{M^2\pi^2}\frac{dD_{M,0}}{dt_2} + \frac{6V_0^4S(M)}{M^4\pi^4}C_{M,0}\Big] \\ & -\sin\left(\frac{V_0^2}{M^2\pi^2}t_1\right)\Big[\frac{V_0^6}{M^8\pi^8}D_{M,0} - \frac{2V_0^2}{M^2\pi^2}\frac{dC_{M,0}}{dt_2} \\ & +\frac{6V_0^4S(M)}{M^4\pi^4}D_{M,0}\Big] \\ & +\frac{\omega_1V_0}{M^2\pi^2}\big[C_{M,0}\sin(\omega_2t_2) - D_{M,0}\cos(\omega_2t_2)\big] \\ & \cos\left(\left(\omega_1 + \frac{V_0^2}{M^2\pi^2}\right)t_1\right) \\ & +\frac{\omega_1V_0}{M^2\pi^2}[C_{M,0}\cos(\omega_2t_2) + D_{M,0}\sin(\omega_2t_2)] \\ & \sin\left(\left(\omega_1 + \frac{V_0^2}{M^2\pi^2}\right)t_1\right). \end{aligned} \tag{54}$$

Now we have to consider three cases, that is, when $\omega_1 \neq 0$ and $\omega_1 \neq -\frac{2V_0^2}{M^2\pi^2}$, when $\omega_1 = 0$, and when $\omega_1 = -\frac{2V_0^2}{M^2\pi^2}$. In the first two cases we will obtain the same $C_{M,0}$ and $D_{M,0}$ as given by (41) and leading to the same $v_{k,0}$ as given by (42). When $\omega_1 = -\frac{2V_0^2}{M^2\pi^2}$ it follows from (54) that in order to avoid secular terms in $A_{M,1}$ that $C_{M,0}$ and $D_{M,0}$ have to satisfy

$$\begin{aligned}\frac{dC_{M,0}}{dt_2} - \left(\frac{V_0^4}{2M^6\pi^6} + \frac{3V_0^2 S(M)}{M^2\pi^2}\right) D_{M,0} &= -\frac{V_0}{M^2\pi^2}(C_{M,0}\cos(\omega_2 t_2) \\ &\quad + D_{M,0}\sin(\omega_2 t_2)), \\ \frac{dD_{M,0}}{dt_2} + \left(\frac{V_0^4}{2M^6\pi^6} + \frac{3V_0^2 S(M)}{M^2\pi^2}\right) C_{M,0} &= -\frac{V_0}{M^2\pi^2}(C_{M,0}\sin(\omega_2 t_2) \\ &\quad - D_{M,0}\cos(\omega_2 t_2)).\end{aligned} \tag{55}$$

System (55) can be rewritten as

$$\begin{aligned}\begin{pmatrix}\frac{dC_{M,0}}{dt_2} \\ \frac{dD_{M,0}}{dt_2}\end{pmatrix} &= \begin{pmatrix}0 & \gamma \\ -\gamma & 0\end{pmatrix}\begin{pmatrix}C_{M,0} \\ D_{M,0}\end{pmatrix} \\ &\quad - \frac{V_0}{M^2\pi^2}\begin{pmatrix}\cos(\omega_2 t_2) & \sin(\omega_2 t_2) \\ \sin(\omega_2 t_2) & -\cos(\omega_2 t_2)\end{pmatrix}\begin{pmatrix}C_{M,0} \\ D_{M,0}\end{pmatrix},\end{aligned} \tag{56}$$

where $\gamma = \frac{V_0^4}{2M^6\pi^6} + \frac{3V_0^2 S(M)}{M^2\pi^2} > 0$. By introducing the time-rescaling $\tau = \gamma t_2$ system (56) becomes

$$\begin{aligned}\begin{pmatrix}\frac{dC_{M,0}}{d\tau} \\ \frac{dD_{M,0}}{d\tau}\end{pmatrix} &= \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}C_{M,0} \\ D_{M,0}\end{pmatrix} \\ &\quad - \frac{V_0}{\gamma M^2\pi^2}\begin{pmatrix}\cos\left(\frac{\omega_2}{\gamma}\tau\right) & \sin\left(\frac{\omega_2}{\gamma}\tau\right) \\ \sin\left(\frac{\omega_2}{\gamma}\tau\right) & -\cos\left(\frac{\omega_2}{\gamma}\tau\right)\end{pmatrix}\begin{pmatrix}C_{M,0} \\ D_{M,0}\end{pmatrix}.\end{aligned} \tag{57}$$

System (57) is of the form $\dot{X} = AX + B(\tau)X$, where A is a constant matrix and where $B(\tau)$ is a continuous and periodic matrix. The fundamental matrix $\Phi(t)$ for $\dot{X} = AX$ is given by $\begin{pmatrix}\cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau)\end{pmatrix}$. Then, by using the method of variation of constants we can take $X(\tau) = \Phi(\tau)C(\tau)$ with $C(\tau) = (C_1(\tau), C_2(\tau))^T$, and we obtain $\dot{C}(\tau) = \Phi^{-1}(\tau)B(\tau)\Phi(\tau)C(\tau)$. When this method is applied to (57) we obtain

$$\begin{pmatrix}\frac{dC_1}{d\tau} \\ \frac{dC_2}{d\tau}\end{pmatrix} = -\frac{V_0}{\gamma M^2\pi^2}\begin{pmatrix}\cos\left(\left(2+\frac{\omega_2}{\gamma}\right)\tau\right) & \sin\left(\left(2+\frac{\omega_2}{\gamma}\right)\tau\right) \\ \sin\left(\left(2+\frac{\omega_2}{\gamma}\right)\tau\right) & -\cos\left(\left(2+\frac{\omega_2}{\gamma}\right)\tau\right)\end{pmatrix}\begin{pmatrix}C_1 \\ C_2\end{pmatrix}, \tag{58}$$

and $(C_{M,0}(\tau), D_{M,0}(\tau))^T$ is given by $\Phi(\tau)C(\tau)$. Let $\tau = -\frac{\gamma M^2\pi^2}{V_0}s$ and $\alpha = -\frac{\gamma M^2\pi^2}{V_0}\left(2 + \frac{\omega_2}{\gamma}\right)$. Then (58) becomes

$$\begin{pmatrix}\frac{dC_1}{ds} \\ \frac{dC_2}{ds}\end{pmatrix} = \begin{pmatrix}\cos(\alpha s) & \sin(\alpha s) \\ \sin(\alpha s) & -\cos(\alpha s)\end{pmatrix}\begin{pmatrix}C_1 \\ C_2\end{pmatrix}. \tag{59}$$

By introducing polar coordinates for $C_1(s)$ and $C_2(s)$, that is, $C_1(s) = r(s)\cos(\phi(s))$ and $C_2(s) = r(s)\sin(\phi(s))$, it follows from (59) that $r(s)$ and $\phi(s)$ have to satisfy

$$\begin{aligned}\frac{dr}{ds} &= r\cos(2\phi - \alpha s), \\ \frac{d\phi}{ds} &= -\sin(2\phi - \alpha s).\end{aligned} \tag{60}$$

By putting $\psi(s) = 2\phi(s) - \alpha s$ the system of differential equations (60) can be simplified to

$$\begin{aligned}\frac{dr}{ds} &= r\cos(\psi), \\ \frac{d\psi}{ds} &= -\alpha - 2\sin(\psi).\end{aligned} \tag{61}$$

It is not hard to see that the autonomous system (61) admits the following first integral

$$(\alpha + 2\sin(\psi))r^2 = \text{constant}. \tag{62}$$

From (62) it follows simply for $|\alpha| > 2$ that $\alpha + 2\sin(\psi)$ is sign definite and so, the function $r(s)$ is bounded for all s. And so, for $|\alpha| > 2$ system (55) has only bounded solutions $C_{M,0}(t_2)$ and $D_{M,0}(t_2)$. These functions can be computed from (61), but the complicated and long expressions will be ommited here for convenience. For $|\alpha| \leq 2$, we will study (59) by introducing $z(s) = C_1(s) + iC_2(s)$, where $i^2 = -1$. Then, it can easily be shown (by using (59)), and the complex notation for $\cos(\alpha s)$ and $\sin(\alpha s)$), that

$$\dot{z} = e^{i\alpha s}\bar{z}, \tag{63}$$

where $\bar{z}$ is the complex conjugate of z. Eq. (63) can be solved by looking for nontrivial solutions in the form

$$z(s) = (\xi_1 + i\xi_2)\exp((\lambda_1 + i\lambda_2)s), \tag{64}$$

where ξ_1, ξ_2, λ_1, and λ_2 are real constants. By substituting (64) into (63) one obtains

$$(\xi_1 + i\xi_2)(\lambda_1 + i\lambda_2) = e^{i(\alpha - 2\lambda_2)s}(\xi_1 - i\xi_2). \tag{65}$$

The left-hand side of (65) does not depend on s, whereas the right-hand side does. Moreover, we look for nontrivial solutions of (63) (that is, $(\xi_1, \xi_2) \neq (0, 0)$), and so, it follows from (65) that $\alpha - 2\lambda_2 = 0$, and that $(\xi_1 + i\xi_2)(\lambda_1 + i\lambda_2) = \xi_1 - i\xi_2$, which implies

$$\begin{aligned}&\lambda_2 = \frac{\alpha}{2}, \\ &\quad (\lambda_1 - 1)\xi_1 - \frac{\alpha}{2}\xi_2 = 0, \\ &\quad \frac{\alpha}{2}\xi_1 + (\lambda_1 + 1)\xi_2 = 0.\end{aligned} \tag{66}$$

To have a nontrivial solution for (63), that is, $(\xi_1, \xi_2) \neq (0, 0)$, it follows from the two last equations in (66) that the determinant of the coefficient matrix should be zero, implying $(\lambda_1 - 1)(\lambda_1 + 1) + \frac{\alpha^2}{4} = 0$, or equivalently

$$\lambda_1 = \tfrac{1}{2}\sqrt{4-\alpha^2} \text{ or } \lambda_1 = -\tfrac{1}{2}\sqrt{4-\alpha^2}. \tag{67}$$

For $|\alpha| < 2$ we find two different roots, and so two functionally independent solutions for (63) (and for (59)) can be found by solving and using (64) and (66), yielding

$$\begin{aligned} z_1(s) &= \exp\left(\frac{1}{2}\sqrt{4-\alpha^2}s\right)\left(\cos\left(\frac{\alpha s}{2}\right) + \frac{\alpha \sin\left(\frac{\alpha s}{2}\right)}{2+\sqrt{4-\alpha^2}}\right.\\ &\quad \left. + i\left(-\frac{\alpha\cos\left(\frac{\alpha s}{2}\right)}{2+\sqrt{4-\alpha^2}} + \sin\left(\frac{\alpha s}{2}\right)\right)\right),\\ z_2(s) &= \exp\left(-\frac{1}{2}\sqrt{4-\alpha^2}s\right)\left(\frac{\alpha \cos\left(\frac{\alpha s}{2}\right)}{2+\sqrt{4-\alpha^2}} + \sin\left(\frac{\alpha s}{2}\right)\right.\\ &\quad \left. + i\left(-\cos\left(\frac{\alpha s}{2}\right) + \frac{\alpha\sin\left(\frac{\alpha s}{2}\right)}{2+\sqrt{4-\alpha^2}}\right)\right). \end{aligned} \tag{68}$$

And so, the general solution of (59) can readily be obtained from $z(s) = C_1(s) + iC_2(s)$ and (68), yielding

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = k_1 \begin{pmatrix} \mathrm{Re}(z_1(s)) \\ \mathrm{Re}(z_2(s)) \end{pmatrix} + k_2 \begin{pmatrix} \mathrm{Im}(z_1(s)) \\ \mathrm{Im}(z_2(s)) \end{pmatrix}, \tag{69}$$

where k_1 and k_2 are constants of integration, and where Re and Im stand for the real and imaginary part, respectively.

For $\alpha = 2$ or for $\alpha = -2$ it follows from (67) that we find coinciding roots. So, we have only one solution for (63) (and for (59)). The other functionally independent solution, however, can easily be found by using the method of variation of constants. We will omit the elementary computations. For $\alpha = 2$, the general solution of (59) is given by

$$\begin{aligned} \begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} &= k_1 \begin{pmatrix} s(\cos(s) + \sin(s)) + \cos(s) \\ s(-\cos(s) + \sin(s)) + \sin(s) \end{pmatrix} \\ &\quad + k_2 \begin{pmatrix} \cos(s) + \sin(s) \\ -\cos(s) + \sin(s) \end{pmatrix}, \end{aligned} \tag{70}$$

and for $\alpha = -2$ the general solution of (59) is given by

$$\begin{aligned} \begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} &= k_1 \begin{pmatrix} s(\cos(s) + \sin(s)) + \cos(s) \\ s(\cos(s) - \sin(s)) - \sin(s) \end{pmatrix} \\ &\quad + k_2 \begin{pmatrix} \cos(s) + \sin(s) \\ \cos(s) - \sin(s) \end{pmatrix}, \end{aligned} \tag{71}$$

where k_1 and k_2 are constants of integration. From (68)-(71) the solutions $C_{M,0}(t_2)$ and $D_{M,0}(t_2)$ of (55) can now be easily obtained, and so $v_{M,0}(t_0, t_1, t_2)$ given by (26) is now completely determined. For $|\alpha| \le 2$ or equivalently for $-2\gamma - \frac{2V_0}{M^2\pi^2} \le \omega_2 \le -2\gamma + \frac{2V_0}{M^2\pi^2}$ $\left(\text{where } \gamma = \frac{V_0^4}{2M^6\pi^6} + \frac{3V_0^2 S(M)}{M^2\pi^2}\right)$ the solution $v_{M,0}(t_0, t_1, t_2)$ is unstable, else the solution is stable.

So far we studied up to $\mathcal{O}(\varepsilon^3)$ the cases for which $\omega_0 \ne K^2\pi^2$ for all odd $K \in \mathbb{N}$ and we found the natural frequencies of the microbeam (up to $\mathcal{O}(\varepsilon^3)$) which are given by $f_k = k^2\pi^2 - \varepsilon\frac{V_0^2}{k^2\pi^2} - \varepsilon^2\left(\frac{V_0^4}{2k^6\pi^6} + \frac{3V_0^2 S(k)}{k^2\pi^2}\right)$ for $k = 1, 2, \ldots$ We also found a superharmonic resonance when $\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 = \frac{1}{2} f_L$ for a fixed and odd $L \in \mathbb{N}$, and we found for a fixed $M \in \mathbb{Z}$ subharmonic resonance for $\omega = 2M^2\pi^2 - 2\varepsilon\frac{V_0^2}{M^2\pi^2} + \varepsilon^2\omega_2$, where $\omega_2 \in \left[-2\gamma - \frac{2V_0}{M^2\pi^2}, -2\gamma + \frac{2V_0}{M^2\pi^2}\right]$ with $\gamma = \frac{V_0^4}{2M^6\pi^6} + \frac{3V_0^2 S(M)}{M^2\pi^2}$. We still have to determine $v_{k,0}$ completely when $\omega_0 = K^2\pi^2$ for a certain, fixed, and odd $K \in \mathbb{N}$, that is, case 2.1, case 2.2, and case 2.3 have to be studied further by considering the $\mathcal{O}(\varepsilon^2)$-problem for $v_{k,2}$ for those cases.

Case 2.1 $\boldsymbol{\omega_0 = K^2\pi^2}$ **for a certain fixed, odd** $\boldsymbol{K \in \mathbb{N}}$**, and** $\boldsymbol{\omega_1 \ne \pm\frac{V_0^2}{K^2\pi^2}}$.

Following the earlier made remarks in case 2 we now only have to consider $v_{K,0}, v_{K,1}, v_{K,2}$. By substituting $v_{K,0}, v_{K,1}, A_{K,0}, B_{K,0}$ (as given in (33), (34), (30)) into the right-hand side of Eq. (23) and by taking apart those terms in this right-hand side that cause secular terms in $v_{K,2}$ it follows that no secular terms in $v_{K,2}$ will occur when $A_{K,1}$ and $B_{K,1}$ satisfy

$$\begin{aligned} \frac{\partial A_{K,1}}{\partial t_1} + \frac{V_0^2}{K^2\pi^2}B_{K,1} &= -\frac{V_0^4}{2K^6\pi^6}\Big[C_{K,0}\sin\Big(\frac{V_0^2}{K^2\pi^2}t_1\Big) \\ &\quad - D_{K,0}\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right)\Big] \\ &\quad - \Big[\frac{dC_{K,0}}{dt_2}\cos\Big(\frac{V_0^2}{K^2\pi^2}t_1\Big) \\ &\quad + \frac{dD_{K,0}}{dt_2}\sin\Big(\frac{V_0^2}{K^2\pi^2}t_1\Big) \\ &\quad - \frac{2\omega_2 V_0 H(K)}{V_0^2 + \delta K^2\pi^2}\cos(\omega_1 t_1 + \omega_2 t_2)\Big] \\ &\quad + \frac{\delta^2 V_0 H(K)}{K^2\pi^2(V_0^2 + \omega_1 K^2\pi^2)}\cos(\omega_1 t_1 + \omega_2 t_2) \\ &\quad - \frac{4V_0^3 H(K)}{K^6\pi^6}\cos(\omega_1 t_1 + \omega_2 t_2) \\ &\quad - \frac{3V_0^2 S(K)}{K^2\pi^2}\Big[C_{K,0}\sin\left(\frac{V_0^2}{K^2\pi^2}t_1\right) \\ &\quad - D_{K,0}\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right) \\ &\quad - \frac{2V_0 H(K)}{V_0^2 + \omega_1 K^2\pi^2}\cos(\omega_1 t_1 + \omega_2 t_2)\Big], \end{aligned}$$

$$\frac{\partial B_{K,1}}{\partial t_1} - \frac{V_0^2}{K^2\pi^2}A_{K,1} = \frac{V_0^4}{2K^6\pi^6}\Big[C_{K,0}\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right)$$

$$+D_{K,0}\sin\left(\frac{V_0^2}{K^2\pi^2}t_1\right)\Big]$$
$$-\Big[\frac{dC_{K,0}}{dt_2}\sin\left(\frac{V_0^2}{K^2\pi^2}t_1\right)$$
$$-\frac{dD_{K,0}}{dt_2}\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right)$$
$$+\frac{2\omega_2V_0H(K)}{V_0^2+\omega_1K^2\pi^2}\sin(\omega_1t_1+\omega_2t_2)\Big]$$
$$-\frac{\delta^2V_0H(K)}{K^2\pi^2(V_0^2+\delta K^2\pi^2)}\sin(\omega_1t_1+\omega_2t_2)$$
$$+\frac{4V_0^3H(K)}{K^6\pi^6}\sin(\omega_1t_1+\omega_2t_2)$$
$$+\frac{3V_0^2S(K)}{K^2\pi^2}\Big[C_{K,0}\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right)$$
$$+D_{K,0}\sin\left(\frac{V_0^2}{K^2\pi^2}t_1\right)$$
$$-\frac{2V_0H(K)}{V_0^2+\omega_1K^2\pi^2}\sin(\omega_1t_1+\omega_2t_2)\Big]. \quad (72)$$

Combining the two equations in (72), we obtain

$$\frac{\partial^2A_{K,1}}{\partial t_1^2}+\frac{V_0^4}{K^4\pi^4}A_{K,1}=-\cos\left(\frac{V_0^2}{K^2\pi^2}t_1\right)\Big[\frac{V_0^6}{K^8\pi^8}C_{K,0}+\frac{2V_0^2}{K^2\pi^2}\frac{dD_{K,0}}{dt_2}$$
$$+\frac{6V_0^4S(k)}{K^4\pi^4}C_{K,0}\Big]$$
$$-\sin\left(\frac{V_0^2}{K^2\pi^2}t_1\right)\Big[\frac{V_0^6}{K^8\pi^8}D_{K,0}-\frac{2V_0^2}{K^2\pi^2}\frac{dC_{K,0}}{dt_2}$$
$$+\frac{6V_0^4S(k)}{K^4\pi^4}D_{K,0}\Big]+NST.$$

To avoid secular terms in $A_{K,1}$ and $B_{K,1}$, $C_{K,0}$ and $D_{K,0}$ have to satisfy (40). Eventually, we will obtain to the same $C_{K,0}$ and $D_{K,0}$ as given by (41). In this case, $v_{K,0}$ is given by:

$$v_{K,0}(t_0,t_1,t_2)=K_1\cos(\bar\omega(t_0,t_1,t_2))-K_2\sin(\bar\omega(t_0,t_1,t_2))$$
$$-\frac{2V_0H(K)}{V_0^2+\omega_1K^2\pi^2}\sin(\omega_0t_0+\omega_1t_1+\omega_2t_2)$$
$$+\frac{2V_0^2H(K)}{K^4\pi^4},$$

where $\bar\omega(t_0,t_1,t_2)=K^2\pi^2t_0-\frac{V_0^2}{K^2\pi^2}t_1-\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)t_2$, and, K_1 and K_2 are constants of integration.

Case 2.2 $\boldsymbol{\omega_0=K^2\pi^2}$ **for a certain fixed, odd** $\boldsymbol{K\in\mathbb{N}}$**, and** $\boldsymbol{\omega_1=\frac{V_0^2}{K^2\pi^2}}$.

Similar to the previous case, we substitute $v_{K,0}$, $v_{K,1}$, $A_{K,0}$, $B_{K,0}$ (as given in (33), (34), (31)) into the $\mathcal{O}(\varepsilon^2)$-problem (21) and by collecting secular terms in $v_{K,2}$, it turns out that $A_{K,1}$ and $B_{K,1}$ have to satisfy

$$\frac{\partial A_{K,1}}{\partial t_1}+\omega_1B_{K,1}=-\frac{V_0^4}{2K^6\pi^6}\Big[C_{K,0}\sin(\omega_1t_1)$$
$$-D_{K,0}\cos(\omega_1t_1)$$
$$-\frac{2H(K)}{V_0}\cos(\omega_1t_1+\omega_2t_2)\Big]$$
$$-\Big[\frac{dC_{K,0}}{dt_2}\cos(\omega_1t_1)+\frac{dD_{K,0}}{dt_2}\sin(\omega_1t_1)\Big]$$
$$-\frac{3V_0^2S(K)}{K^2\pi^2}\Big[C_{K,0}\sin(\omega_1t_1)-D_{K,0}\cos(\omega_1t_1)$$
$$-\frac{2H(K)}{V_0}\cos(\omega_1t_1+\omega_2t_2)\Big]$$
$$-\frac{4V_0^3H(K)}{K^6\pi^6}\cos(\omega_1t_1+\omega_2t_2),$$
$$\frac{\partial B_{K,1}}{\partial t_1}-\omega_1A_{K,1}=\frac{V_0^4}{2K^6\pi^6}$$
$$\big(C_{K,0}\cos(\omega_1t_1)+D_{K,0}\sin(\omega_1t_1)\big)$$
$$-\Big[\frac{dC_{K,0}}{dt_2}\sin(\omega_1t_1)-\frac{dD_{K,0}}{dt_2}\cos(\omega_1t_1)$$
$$+\frac{2\omega_2H(K)}{V_0}\sin(\omega_1t_1+\omega_2t_2)\Big]$$
$$+\frac{3V_0^2S(K)}{K^2\pi^2}\big(C_{K,0}\cos(\omega_1t_1)+D_{K,0}\sin(\omega_1t_1)\big)$$
$$+\frac{4V_0^3H(K)}{K^6\pi^6}\sin(\omega_1t_1+\omega_2t_2). \quad (73)$$

Combining the two equations in (73), we obtain

$$\frac{\partial^2A_{K,1}}{\partial t_1^2}+\omega_1^2A_{K,1}=-\cos(\omega_1t_1)\Big(\frac{V_0^6}{K^8\pi^8}C_{K,0}+\frac{2V_0^2}{K^2\pi^2}\frac{dD_{K,0}}{dt_2}$$
$$-\frac{2V_0H(K)}{K^2\pi^2}\Big[\omega_2-\frac{V_0^4}{2K^6\pi^6}-\frac{3V_0^2S(K)}{K^2\pi^2}\Big]\sin(\omega_2t_2)$$
$$+\frac{6V_0^4S(K)}{K^4\pi^4}C_{K,0}\Big)$$
$$-\sin(\omega_1t_1)\Big(\frac{V_0^6}{K^8\pi^8}D_{K,0}-\frac{2V_0^2}{K^2\pi^2}\frac{dC_{K,0}}{dt_2}$$
$$-\frac{2V_0H(K)}{K^2\pi^2}\Big[\omega_2-\frac{V_0^4}{2K^6\pi^6}-\frac{3V_0^2S(K)}{K^2\pi^2}\Big]\cos(\omega_2t_2)$$
$$+\frac{6V_0^4S(K)}{K^4\pi^4}D_{K,0}\Big). \quad (74)$$

To avoid secular terms in (74), $C_{K,0}$ and $D_{K,0}$ have to satisfy

$$\frac{dC_{K,0}}{dt_2}-\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)D_{K,0}=-\frac{H(K)}{V_0}\cos(\omega_2t_2)$$
$$\Big[\omega_2-\frac{V_0^4}{2K^6\pi^6}-\frac{3V_0^2S(K)}{K^2\pi^2}\Big],$$
$$\frac{dD_{K,0}}{dt_2}+\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)C_{K,0}=\frac{H(K)}{V_0}\sin(\omega_2t_2)$$
$$\Big[\omega_2-\frac{V_0^4}{2K^6\pi^6}-\frac{3V_0^2S(K)}{K^2\pi^2}\Big]. \quad (75)$$

Combining the equations in (75) we obtain

$$\frac{d^2C_{K,0}}{dt_2^2}+\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)^2C_{K,0}=\frac{H(K)}{V_0}\sin(\omega_2t_2)$$
$$\left(\omega_2^2-\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)^2\right).$$

When $\omega_2=\pm\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)$, $C_{K,0}$ and $D_{K,0}$ are given by (41). In the other cases, we have

$$C_{K,0}(t_2)=K_1\cos\left(\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)t_2\right)$$

$$+K_2 \sin\left(\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)t_2\right) -\frac{H(K)}{V_0}\sin(\omega_2 t_2),$$

$$D_{K,0}(t_2) = -K_1 \sin\left(\left(\frac{V_0^4}{2K^6\pi^6}+\frac{3V_0^2S(K)}{K^2\pi^2}\right)t_2\right) +K_2 \cos\left(\left(\frac{V_0^4}{2K^6\pi^6} +\frac{3V_0^2S(K)}{K^2\pi^2}\right)t_2\right) - \frac{H(K)}{V_0}\cos(\omega_2 t_2),$$

where K_1 and K_2 are constants of integration.

Case 2.3 $\boldsymbol{\omega_0 = K^2\pi^2}$ for a certain fixed, odd $\boldsymbol{K \in \mathbb{N}}$, **and** $\boldsymbol{\omega_1 = -\frac{V_0^2}{K^2\pi^2}}$.

By substituting $v_{K,0}$, $v_{K,1}$, $A_{K,0}$, $B_{K,0}$ (as given in (33), (34), (32)) into the $\mathcal{O}(\varepsilon^2)$-problem (21) and by collecting the secular terms in $v_{K,2}$, it turns out that $A_{K,1}$ and $B_{K,1}$ have to satisfy

$$\begin{aligned}
\frac{\partial A_{K,1}}{\partial t_1} - \omega_1 B_{K,1} =& \frac{V_0^4}{2K^6\pi^6}\Big[C_{K,0}\sin(\omega_1 t_1) + D_{K,0}\cos(\omega_1 t_1)\Big] \\
&- \frac{6V_0^3H(K)}{K^6\pi^6}\cos(\omega_1 t_1+\omega_2 t_2) \\
&- \frac{2V_0H(K)}{K^2\pi^2}\left[\omega_2 + \frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right]t_1 \sin(\omega_1 t_1+\omega_2 t_2) \\
&- \left[\frac{dC_{K,0}}{dt_2}\cos(\omega_1 t_1) - \frac{dD_{K,0}}{dt_2}\sin(\omega_1 t_1)\right] \\
&+ \frac{3V_0^2S(K)}{K^2\pi^2}\Big[C_{K,0}\sin(\omega_1 t_1) + D_{K,0}\cos(\omega_1 t_1)\Big], \\
\frac{\partial B_{K,1}}{\partial t_1} + \omega_1 A_{K,1} =& \frac{V_0^4}{2K^6\pi^6}\Big[C_{K,0}\cos(\omega_1 t_1) - D_{K,0}\sin(\omega_1 t_1)\Big] \\
&+ \frac{6V_0^3H(K)}{K^6\pi^6}\sin(\omega_1 t_1+\omega_2 t_2) \\
&- \frac{2V_0H(K)}{K^2\pi^2}\left[\omega_2 + \frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right]t_1 \cos(\omega_1 t_1+\omega_2 t_2) \\
&+ \left[\frac{dC_{K,0}}{dt_2}\sin(\omega_1 t_1) + \frac{dD_{K,0}}{dt_2}\cos(\omega_1 t_1)\right] \\
&+ \frac{3V_0^2S(K)}{K^2\pi^2}\Big[C_{K,0}\cos(\omega_1 t_1) - D_{K,0}\sin(\omega_1 t_1)\Big].
\end{aligned} \tag{76}$$

Combining the equations in (76), we obtain

$$\begin{aligned}
\frac{\partial^2 A_{K,1}}{\partial t_1^2} + \omega_1^2 A_{K,1} =& -\cos(\omega_1 t_1)\Big(\frac{V_0^6}{K^8\pi^8}C_{K,0} + \frac{2V_0^2}{K^2\pi^2}\frac{dD_{K,0}}{dt_2} \\
&+ \frac{2V_0H(K)}{K^2\pi^2}\left[\omega_2 + \frac{13V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right]\sin(\omega_2 t_2) \\
&+ \frac{6V_0^4S(K)}{K^4\pi^4}C_{K,0}\Big) \\
&+ \sin(\omega_1 t_1)\Big(\frac{V_0^6}{K^8\pi^8}D_{K,0} - \frac{2V_0^2}{K^2\pi^2}\frac{dC_{K,0}}{dt_2} \\
&- \frac{2V_0H(K)}{K^2\pi^2}\left[\omega_2 + \frac{13V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right]\cos(\omega_2 t_2) \\
&+ \frac{6V_0^4S(K)}{K^4\pi^4}D_{K,0}\Big) \\
&+ \frac{4V_0^3H(K)}{K^4\pi^4}\left[\omega_2 + \frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right]t_1 \cos(\omega_1 t_1+\omega_2 t_2).
\end{aligned}$$

Neglecting the 'secular term' $t_1\cos(\omega_1 t_1+\omega_2 t_2)$, we can avoid additional secular terms in $A_{K,1}$ by setting

$$\begin{aligned}
\frac{dC_{K,0}}{dt_2} - \left(\frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right)D_{K,0} =& -\frac{H(K)}{V_0}\cos(\omega_2 t_2)\left[\omega_2 + \frac{13V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right], \\
\frac{dD_{K,0}}{dt_2} + \left(\frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right)C_{K,0} =& -\frac{H(K)}{V_0}\sin(\omega_2 t_2)\left[\omega_2 + \frac{13V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right].
\end{aligned} \tag{77}$$

Combining the equations in (77), we obtain

$$\begin{aligned}
&\frac{d^2C_{K,0}}{dt_2^2} + \left(\frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right)^2 C_{K,0} = \frac{H(K)}{V_0}\left(\omega_2 + \frac{13V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right) \\
&\left(\omega_2 - \left[\frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right]\right)\sin(\omega_2 t_2).
\end{aligned} \tag{78}$$

When $\omega_2 = \frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}$, the right hand side of (78) will be zero, and $C_{K,0}$ and $D_{K,0}$ are given by (41), but as $v_{K,1}$, and $v_{K,0}$ contain secular terms, the solution will be unbounded for increasing times. For $\omega_2 \neq \pm\left(\frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right)$, $C_{K,0}$ and $D_{K,0}$ will be bounded functions, but since $v_{K,0}$ contains secular terms, $v_{K,0}$ also becomes unbounded for increasing times. For the last subcase $\omega_2 = -\left(\frac{V_0^4}{2K^6\pi^6} + \frac{3V_0^2S(K)}{K^2\pi^2}\right)$, $C_{K,0}$ and $D_{K,0}$ will contain secular terms in t_2. Hence, both $v_{K,0}$ and $v_{K,1}$ have secular terms and the solution becomes unbounded for longer times. All these three subcases in case 2.3 lead to unboundedness of the solution for longer times. So, we can now conclude in case 2 that when $\omega_0 = K^2\pi^2$ for a certain, fixed, and odd $K \in \mathbb{N}$, then $v_K(t_0, t_1, t_2)$ given by (18) is stable when $\omega_1 \neq -\frac{V_0^2}{K^2\pi^2}$, and $v_K(t_0, t_1, t_2)$ is unstable when $\omega_1 = -\frac{V_0^2}{K^2\pi^2}$.

4 The influence of damping and weakly nonlinear elastic forces

In this section, we will study the interplay between the electrostatic force, the damping force, and the weakly nonlinear elastic forces. First we will consider the case for which the actuation frequency ω is equal to or close to a natural eigenfrequency of the actuated beam. In this case, the damping force needs to be sufficiently large (that is, needs to be of order ε) in order to stabilize the vibrations of the beam. In subsection 4.1 this case will be studied. Viscous damping of order ε is assumed to be present. The weakly nonlinear elastic forces and the structural damping force are of order ε^2, and are too small to play any role in the stabilization of the beam, and are for that reason neglected in subsection 4.1. Next we will consider the cases for which the actuation frequency ω is equal or close to a superharmonic or to a subharmonic frequency of the actuated beam. For these cases, the damping force cannot be too large else the vibration amplitudes of the actuated beam become too small to have any practical significance. For that reason we will not consider in the subsections 4.2, 4.3, and 4.4 the viscous damping force in Eq. (5). In these subsections, the structural damping force, the weakly nonlinear elastic forces, and the actuation forcing give rise to an intriguing behaviour of the vibration modes of the actuated beam. In subsection 4.2, we formulate the general problem for these super-and subharmonic cases, and in subsection 4.3 and in subsection 4.4 we will study in detail a superharmonic and a subharmonic case, respectively.

4.1 Viscous damping of $\mathcal{O}(\varepsilon)$ without nonlinear elastic forces

In this subsection, we will consider the actuated beam Eq. (5) with $\alpha = 0$, $c_2 = 0$, and $c_1 = \beta > 0$, that is, compared to the previous section viscous damping is added to the beam equation:

$$v_{tt} + v_{xxxx} = \frac{(V_0 + \varepsilon \sin(\omega t))^2}{(1-\varepsilon v)^2} - \varepsilon\beta v_t, \tag{79}$$

subject to the boundary and initial conditions (6) and (7). Here, the constant β is independent of ε. We follow the same steps as in the previous section using a two time-scales perturbation method, and obtain the same $\mathcal{O}(1)$-problem as in (19). While for the $\mathcal{O}(\varepsilon)$-problem, we have:

$$\begin{aligned}\mathcal{L}v_{k,1} =& -2D_1D_0v_{k,0} + 2V_0^2 v_{k,0}\\ &+4V_0H(k)\sin\omega t_0 - \beta D_0 v_{k,0}.\end{aligned} \tag{80}$$

Let $\omega = K^2\pi^2 - \frac{V_0^2}{K^2\pi^2}\varepsilon$, for a fixed and odd $K \in \mathbb{N}$, that is, ω is order ε^2 close to one of the natural frequencies of the actuated beam. Observe that the excited mode is just mode $k = K$. Substituting the same $v_{K,0}$ as given by (22) with $k = K$ into the right-hand side (RHS) of (80), we have

$$\begin{aligned}RHS =& \cos(K^2\pi^2t_0)\Big(-2K^2\pi^2\tfrac{dB_{K,0}}{dt_1} - \beta K^2\pi^2 B_{K,0}\\ &+2V_0^2A_{K,0} - 4V_0H(K)\sin\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right)\Big)\\ &+\sin(K^2\pi^2t_0)\Big(2K^2\pi^2\tfrac{dA_{K,0}}{dt_1} + \beta K^2\pi^2 A_{K,0}\\ &+2V_0^2B_{K,0} + 4V_0H(K)\cos\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right)\Big)\\ &+NST.\end{aligned}$$

To eliminate secular terms, $A_{K,0}$ and $B_{K,0}$ have to satisfy

$$\begin{aligned}\tfrac{dA_{K,0}}{dt_1} &= -\tfrac{\beta}{2}A_{K,0} - \tfrac{V_0^2}{K^2\pi^2}B_{K,0} - \tfrac{2V_0H(K)}{K^2\pi^2}\cos\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right),\\ \tfrac{dB_{K,0}}{dt_1} &= \tfrac{V_0^2}{K^2\pi^2}A_{K,0} - \tfrac{\beta}{2}B_{K,0} - \tfrac{2V_0H(K)}{K^2\pi^2}\sin\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right).\end{aligned}$$

This system of ODE can readily be solved, yielding

$$\begin{aligned}A_{K,0}(t_1) =& e^{-\beta t_1/2}\Big(C_{K,0}\cos\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right)\\ &+D_{K,0}\sin\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right)\Big)\\ &-\tfrac{4V_0H(K)}{\beta K^2\pi^2}\cos\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right),\\ B_{K,0}(t_1) =& e^{-\beta t_1/2}\Big(C_{K,0}\sin\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right)\\ &-D_{K,0}\cos\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right)\Big)\\ &-\tfrac{4V_0H(K)}{\beta K^2\pi^2}\sin\left(\tfrac{V_0^2}{K^2\pi^2}t_1\right).\end{aligned}$$

We see that adding viscous damping of order ε to the system stabilizes the solution when the applied frequency of the AC voltage is order ε^2 close to an eigenfrequency of the actuated beam.

4.2 Structural damping of $\mathcal{O}(\varepsilon^2)$ and a weakly nonlinear elastic force

In this subsection, we consider the actuated beam Eq. (5) with $\alpha \neq 0$, $c_1 = 0$, and $c_2 = \beta > 0$ (where β is a constant independent of ε), that is,

$$\begin{aligned}v_{tt} + v_{xxxx} =& -\varepsilon^2\beta v_{txxxx} + \alpha\varepsilon^2\Big(\textstyle\int_0^1 v_x^2(x,t)\,dx\Big)v_{xx}\\ &+\tfrac{(V_0+\varepsilon\sin(\omega t))^2}{(1-\varepsilon v)^2},\end{aligned}$$

subject to the boundary and initial conditions (6) and (7). Also we will consider two different values for the frequency ω (related to the superharmonic case, and related to the subharmonic case), that is, $2\omega = L^2\pi^2 + \mathcal{O}(\varepsilon)$ for a fixed and odd $L \in \mathbb{N}$, and $\omega = 2M^2\pi^2 + \mathcal{O}(\varepsilon)$ for a fixed $M \in \mathbb{N}$. By using a three time-scales perturbation method as in Sect. 3, we obtain the same $\mathcal{O}(1)$-problem (19), and the same $\mathcal{O}(\varepsilon)$-problem (20). Their solutions $v_{k,0}$ and $v_{k,1}$ are given by (22) and (27), respectively. By substituting $v_{k,0}$ and $v_{k,1}$, into the $\mathcal{O}(\varepsilon^2)$-problem we obtain

$$\begin{aligned}\mathcal{L}v_{k,2} &= -2D_1D_0v_{k,1} - D_1^2v_{k,0} - 2D_2D_0v_{k,0} + 2H(k)\sin^2(\omega t_0)\\ &\quad +4V_0v_{k,0}\sin(\omega t_0) + 6V_0^2\sum_{\substack{m,n=1\\ \pm m\pm n\pm k \text{ odd}}}^{\infty} c_{m,n}v_{m,0}v_{n,0}\\ &\quad +2V_0^2v_{k,1} - \beta k^4\pi^4D_0v_{k,0} - \frac{\alpha k^2\pi^4}{2}\sum_{n=1}^{\infty}n^2v_{n,0}^2v_{k,0}.\end{aligned} \tag{81}$$

In (81), we already see that the dynamics of $v_{k,2}$, and so the behaviour of $v_{k,0}$, are influenced by the electrostatic force, the structural damping, and the nonlinear elastic force. In the next subsection, the interplay between these three factors will be explained further. We will consider a superharmonic case, that is, $2\omega_0 = L^2\pi^2$ (with the excited mode $k = L$), and a subharmonic case, that is, $\omega_0 = 2M^2\pi^2$ (with the excited mode $k = M$), in the next two subsections.

4.3 The superharmonic case (case $2\omega_0 = L^2\pi^2$)

In this subsection, we will consider the superharmonic case $2\omega = L^2\pi^2 + \mathcal{O}(\varepsilon)$ (that is, $2\omega_0 = L^2\pi^2$) for a fixed and odd $L \in \mathbb{N}$. By substituting $v_{k,0}$ and $v_{k,1}$ into the $\mathcal{O}(\varepsilon^2)$ equation (81), we obtain

$$\begin{aligned}\mathcal{L}v_{k,2} &= \cos(k^2\pi^2t_0)\Big(-2k^2\pi^2\frac{\partial B_{k,1}}{\partial t_1} + \frac{V_0^4}{k^4\pi^4}A_{k,0} - 2k^2\pi^2\frac{\partial B_{k,0}}{\partial t_2}\\ &\quad +2V_0^2A_{k,1} - \delta_{k,L}H(k)\cos(2\omega_1t_1 + 2\omega_2t_2)\\ &\quad +6V_0^2S(k)A_{k,0} - \beta k^6\pi^6B_{k,0}\\ &\quad -\frac{\alpha k^2\pi^4}{2}A_{k,0}\Big[\frac{8k^2V_0^4H^2(k)}{k^8\pi^8} + \frac{k^2}{4}(A_{k,0}^2 + B_{k,0}^2)\\ &\quad +\sum_{n=1}^{\infty}\frac{n^2}{2}(A_{n,0}^2 + B_{n,0}^2) + \sum_{n=1}^{\infty}\frac{4n^2V_0^4H^2(n)}{n^8\pi^8}\Big]\Big),\\ &\quad +\sin(k^2\pi^2t_0)\Big(2k^2\pi^2\frac{\partial A_{k,1}}{\partial t_1} + \frac{V_0^4}{k^4\pi^4}B_{k,0} + 2k^2\pi^2\frac{\partial A_{k,0}}{\partial t_2}\\ &\quad +2V_0^2B_{k,1} + \delta_{k,L}H(k)\sin(2\omega_1t_1 + 2\omega_2t_2)\\ &\quad +6V_0^2S(k)B_{k,0} + \beta k^6\pi^6A_{k,0}\\ &\quad -\frac{\alpha k^2\pi^4}{2}B_{k,0}\Big[\frac{8k^2V_0^4H^2(k)}{k^8\pi^8} + \frac{k^2}{4}(A_{k,0}^2 + B_{k,0}^2)\\ &\quad +\sum_{n=1}^{\infty}\frac{n^2}{2}(A_{n,0}^2 + B_{n,0}^2) + \sum_{n=1}^{\infty}\frac{4n^2V_0^4H^2(n)}{n^8\pi^8}\Big]\Big)\\ &\quad +NST\end{aligned} \tag{82}$$

where

$$\delta_{k,L} = \begin{cases}0, & \text{for } k \neq L,\\ 1, & \text{for } k = L,\end{cases}$$

and

$$A_{k,0}(t_1,t_2) = C_{k,0}(t_2)\cos\left(\frac{V_0^2}{k^2\pi^2}t_1\right) + D_{k,0}(t_2)\sin\left(\frac{V_0^2}{k^2\pi^2}t_1\right),$$

$$B_{k,0}(t_1,t_2) = C_{k,0}(t_2)\sin\left(\frac{V_0^2}{k^2\pi^2}t_1\right) - D_{k,0}(t_2)\cos\left(\frac{V_0^2}{k^2\pi^2}t_1\right).$$

To avoid secular terms in $v_{k,2}$ it follows from (82) that $A_{k,1}$ and $B_{k,1}$ have to satisfy

$$\begin{aligned}\frac{\partial A_{k,1}}{\partial t_1} + \frac{V_0^2}{k^2\pi^2}B_{k,1} &= -\frac{V_0^4}{2k^6\pi^6}B_{k,0} - \frac{\partial A_{k,0}}{\partial t_2} - \frac{3V_0^2S(k)}{k^2\pi^2}B_{k,0}\\ &\quad -\frac{\beta k^4\pi^4}{2}A_{k,0} - \delta_{k,L}\frac{H(k)}{2k^2\pi^2}\sin(2\omega_1t_1 + 2\omega_2t_2)\\ &\quad +\frac{\alpha\pi^2}{4}B_{k,0}\Big[\frac{8k^2V_0^4H^2(k)}{k^8\pi^8} + \frac{k^2}{4}(C_{k,0}^2 + D_{k,0}^2)\\ &\quad +\sum_{n=1}^{\infty}\frac{n^2}{2}(C_{n,0}^2 + D_{n,0}^2) + \sum_{n=1}^{\infty}\frac{4n^2V_0^4H^2(n)}{n^8\pi^8}\Big],\\ \frac{\partial B_{k,1}}{\partial t_1} - \frac{V_0^2}{k^2\pi^2}A_{k,1} &= \frac{V_0^4}{2k^6\pi^6}A_{k,0} - \frac{\partial B_{k,0}}{\partial t_2} + \frac{3V_0^2S(k)}{k^2\pi^2}A_{k,0}\\ &\quad -\frac{\beta k^4\pi^4}{2}B_{k,0} - \delta_{k,L}\frac{H(k)}{2k^2\pi^2}\cos(2\omega_1t_1 + 2\omega_2t_2)\\ &\quad -\frac{\alpha\pi^2}{4}A_{k,0}\Big[\frac{8k^2V_0^4H^2(k)}{k^8\pi^8} + \frac{k^2}{4}(C_{k,0}^2 + D_{k,0}^2)\\ &\quad +\sum_{n=1}^{\infty}\frac{n^2}{2}(C_{n,0}^2 + D_{n,0}^2) + \sum_{n=1}^{\infty}\frac{4n^2V_0^4H^2(n)}{n^8\pi^8}\Big].\end{aligned}$$

Combining these two equations yields

$$\begin{aligned}\frac{\partial^2 A_{k,1}}{\partial t_1^2} + \frac{V_0^4}{k^4\pi^4}A_{k,1} &= -\cos\left(\frac{V_0^2}{k^2\pi^2}t_1\right)\Big(\frac{V_0^6}{k^8\pi^8}C_{k,0} + \frac{2V_0^2}{k^2\pi^2}\frac{dD_{k,0}}{dt_2}\\ &\quad +\frac{6V_0^4S(k)}{k^4\pi^4}C_{k,0} + \beta V_0^2k^2\pi^2D_{k,0}\\ &\quad -\frac{\alpha V_0^2X(k)}{2k^2}C_{k,0}\Big)\\ &\quad -\sin\left(\frac{V_0^2}{k^2\pi^2}t_1\right)\Big(\frac{V_0^6}{k^8\pi^8}D_{k,0} - \frac{2V_0^2}{k^2\pi^2}\frac{dC_{k,0}}{dt_2}\\ &\quad +\frac{6V_0^4S(k)}{k^4\pi^4}D_{k,0} - \beta V_0^2k^2\pi^2C_{k,0}\\ &\quad -\frac{\alpha V_0^2X(k)}{2k^2}D_{k,0}\Big)\\ &\quad -\delta_{k,L}\frac{H(k)}{2k^2\pi^2}\left[2\omega_1 - \frac{V_0^2}{k^2\pi^2}\right]\\ &\quad \cos(2\omega_1t_1 + 2\omega_2t_2),\end{aligned} \tag{83}$$

where

$$X(k) = \frac{k^2}{4}(C_{k,0}^2 + D_{k,0}^2) + \frac{8k^2 V_0^4 H^2(k)}{k^8\pi^8} + \sum_{n=1}^{\infty} \frac{n^2}{2}(C_{n,0}^2 + D_{n,0}^2) + \sum_{n=1}^{\infty} \frac{4n^2 V_0^4 H^2(n)}{n^8\pi^8}. \tag{84}$$

For all modes $k \neq L$ secular terms in $A_{k,1}$ and $B_{k,1}$ can be avoided when $C_{k,0}$ and $D_{k,0}$ satisfy (as follows from (83)):

$$\begin{aligned} \frac{dC_{k,0}}{dt_2} &= -\frac{\beta k^4\pi^4}{2} C_{k,0} + \left(\frac{V_0^4}{2k^6\pi^6} + \frac{3V_0^2 S(k)}{k^2\pi^2} - \frac{\alpha\pi^2 X(k)}{4}\right) D_{k,0}, \\ \frac{dD_{k,0}}{dt_2} &= -\left(\frac{V_0^4}{2k^6\pi^6} + \frac{3V_0^2 S(k)}{k^2\pi^2} - \frac{\alpha\pi^2 X(k)}{4}\right) C_{k,0} - \frac{\beta k^4\pi^4}{2} D_{k,0}. \end{aligned} \tag{85}$$

From (85) it follows that

$$C_{k,0}\frac{dC_{k,0}}{dt_2} + D_{k,0}\frac{dD_{k,0}}{dt_2} = -\frac{\beta k^4\pi^4}{2}(C_{k,0}^2 + D_{k,0}^2),$$

and so, $R_{k,0}^2 = C_{k,0}^2 + D_{k,0}^2$ satisfies

$$\frac{dR_{k,0}^2}{dt_2} = -\beta k^4\pi^4 R_{k,0}^2. \tag{86}$$

Hence, $R_{k,0}$, and so $C_{k,0}$ and $D_{k,0}$ are stable equilibria which all tend to zero for $t_2 \to \infty$.
For mode $k = L$, we have to consider three subcases, that is, when $2\omega_1 = \frac{V_0^2}{L^2\pi^2}$, $2\omega_1 = -\frac{V_0^2}{L^2\pi^2}$, and $2\omega_1 \neq \pm\frac{V_0^2}{L^2\pi^2}$. When $2\omega_1 = \frac{V_0^2}{L^2\pi^2}$, the last term in (83) will become 0, and $C_{L,0}$ and $D_{L,0}$ are satisfying (85)-(86). When $2\omega_1 \neq \pm\frac{V_0^2}{L^2\pi^2}$, the last term in (83) will not lead to secular terms, and $C_{L,0}$ and $D_{L,0}$ satisfy again (85)-(86). For the case $2\omega_1 = -\frac{V_0^2}{L^2\pi^2}$, $C_{L,0}$ and $D_{L,0}$ have to satisfy

$$\begin{aligned} \frac{dC_{L,0}}{dt_2} &= -\frac{\beta L^4\pi^4}{2} C_{L,0} + \left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2 S(L)}{L^2\pi^2} - \frac{\alpha\pi^2 X(L)}{4}\right) D_{L,0} \\ &\quad - \frac{H(L)}{2L^2\pi^2}\sin(2\omega_2 t_2), \\ \frac{dD_{L,0}}{dt_2} &= -\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2 S(L)}{L^2\pi^2} - \frac{\alpha\pi^2 X(L)}{4}\right) C_{L,0} - \frac{\beta L^4\pi^4}{2} D_{L,0} \\ &\quad + \frac{H(L)}{2L^2\pi^2}\cos(2\omega_2 t_2). \end{aligned} \tag{87}$$

By introducing polar coordinates

$$C_{L,0}(t_2) = R_{L,0}(t_2)\cos(\phi_{L,0}(t_2)),$$
$$D_{L,0}(t_2) = R_{L,0}(t_2)\sin(\phi_{L,0}(t_2)),$$

system (87) becomes

$$\begin{aligned} \frac{dR_{L,0}}{dt_2} &= -\frac{\beta L^4\pi^4}{2} R_{L,0} + \frac{H(L)}{2L^2\pi^2}\sin(\phi_{L,0} - 2\omega_2 t_2), \\ \frac{d\phi_{L,0}}{dt_2} &= -\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2 S(L)}{L^2\pi^2} - \frac{\alpha\pi^2 X(L)}{4}\right) \\ &\quad + \frac{H(L)}{2L^2\pi^2}\frac{\cos(\phi_{L,0} - 2\omega_2 t_2)}{R_{L,0}}. \end{aligned} \tag{88}$$

Let $\psi_{L,0}(t_2) = \phi_{L,0}(t_2) - 2\omega_2 t_2$, then the nonautonomous system (88) becomes the following system:

$$\begin{aligned} \frac{dR_{L,0}}{dt_2} &= -\frac{\beta L^4\pi^4}{2} R_{L,0} + \frac{H(L)}{2L^2\pi^2}\sin(\psi_{L,0}), \\ \frac{d\psi_{L,0}}{dt_2} &= -\left(\frac{V_0^4}{2L^6\pi^6} + \frac{3V_0^2 S(L)}{L^2\pi^2} - \frac{\alpha\pi^2 X(L)}{4}\right) \\ &\quad + \frac{H(L)}{2L^2\pi^2}\frac{\cos(\psi_{L,0})}{R_{L,0}} - 2\omega_2. \end{aligned} \tag{89}$$

Observe that system (89) is still a nonautonomous system due to term involving $X(L)$ as defined by (84). If we assume that there is no initial energy in mode k for all $k \neq L$, then we only have to consider $R_{L,0}$ and $\psi_{L,0}$. This will simplify the $X(L)$ function to

$$X(L) = \frac{8L^2 V_0^4 H^2(L)}{L^8\pi^8} + \frac{3L^2}{4} R_{L,0}^2 + \sum_{n=1}^{\infty} \frac{4n^2 V_0^4 H^2(n)}{n^8\pi^8},$$

and system (89) becomes an autonomus system and can be rewritten in:

$$\begin{aligned} \frac{dR_{L,0}}{dt_2} &= -\frac{\beta L^4\pi^4}{2} R_{L,0} + \frac{1}{L^3\pi^3}\sin(\psi_{L,0}), \\ \frac{d\psi_{L,0}}{dt_2} &= (C - 2\omega_2) + \frac{3\alpha L^2\pi^2}{16} R_{L,0}^2 + \frac{1}{L^3\pi^3}\frac{\cos(\psi_{L,0})}{R_{L,0}}, \end{aligned} \tag{90}$$

with $C = -\frac{V_0^4}{2L^6\pi^6} - \frac{3V_0^2 S(L)}{L^2\pi^2} + \frac{8\alpha V_0^4}{L^8\pi^8} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4\alpha V_0^4}{n^8\pi^8}$.

We will analyze the equilibria of system (90) and their stability. To find the equilibria of (90), we have to solve the following two equations:

$$\frac{1}{L^3\pi^3}\sin(\psi_{L,0}) = \frac{\beta L^4\pi^4}{2} R_{L,0}, \tag{91}$$

$$\frac{1}{L^3\pi^3}\cos(\psi_{L,0}) = -R_{L,0}\left((C - 2\omega_2) + \frac{3\alpha L^2\pi^2}{16} R_{L,0}^2\right). \tag{92}$$

To determine the stability of the equilibria we have to look at the jacobian matrix J of the vector field (90) around the solutions of (91)-(92), where J is given by

$$J = \begin{pmatrix} -\frac{\beta L^4\pi^4}{2} & \frac{\cos(\psi_{L,0})}{L^3\pi^3} \\ \frac{3\alpha L^2\pi^2}{2^3} R_{L,0} - \frac{\cos(\psi_{L,0})}{L^3\pi^3 R_{L,0}^2} & -\frac{\sin(\psi_{L,0})}{L^3\pi^3 R_{L,0}} \end{pmatrix}. \tag{93}$$

The eigenvalues λ of J satisfy:

$$\begin{aligned} \lambda^2 &+ \left(\frac{\beta L^4\pi^4}{2} + \frac{\sin(\psi_{L,0})}{L^3\pi^3 R_{L,0}}\right)\lambda + \frac{\beta L^4\pi^4 \sin(\psi_{L,0})}{2L^3\pi^3 R_{L,0}} \\ &- \frac{3\alpha R_{L,0}\cos(\psi_{L,0})}{2^3 L\pi} \\ &+ \frac{\cos^2(\psi_{L,0})}{L^6\pi^6 R_{L,0}^2} = 0, \end{aligned} \tag{94}$$

which can be further simplified by using (91) to

$$\lambda^2 + (\beta L^4\pi^4)\lambda + \frac{\beta^2 L^8\pi^8}{4} - \frac{3\alpha R_{L,0}\cos(\psi_{L,0})}{2^3 L\pi} + \frac{\cos^2(\psi_{L,0})}{L^6\pi^6 R_{L,0}^2} = 0, \tag{95}$$

By combining the equations (91) and (92), and by using trigonometric identities, we obtain for the equilibria

$$\frac{3^2}{2^8}\alpha^2 L^4\pi^4 R_{L,0}^6 + \frac{3\alpha L^2\pi^2(C-2\omega_2)}{2^3} R_{L,0}^4 + \left[(C-2\omega_2)^2 + \frac{\beta^2 L^8\pi^8}{4}\right] R_{L,0}^2 - \frac{1}{L^6\pi^6} = 0,$$

and by putting $R = R_{L,0}^2$, we finally obtain the cubic equation

$$\frac{3^2}{2^8}\alpha^2 L^4\pi^4 R^3 + \frac{3\alpha L^2\pi^2(C-2\omega_2)}{2^3} R^2 + \left[(C-2\omega_2)^2 + \frac{\beta^2 L^8\pi^8}{4}\right] R - \frac{1}{L^6\pi^6} = 0. \tag{96}$$

Of course we are only interested in the real and non-negative solutions of (96). First, we will consider the case when $\alpha = 0$, that is, we will first consider the case without nonlinear elastic forces.

4.3.1 *The case without weakly nonlinear elastic forces ($\alpha = 0$)*

For $\alpha = 0$ and $C = 2\omega_2$ and $\beta = 0$, there are no nontrivial $R_{L,0}$ for which equilibria exist for system (90) (see Fig. 2c). This case corresponds to case 1.2 as studied in Sect. 3 for which ω is up to $\mathcal{O}(\varepsilon^3)$ equal to a $\frac{1}{2}$ times a natural frequency of the actuated beam. For $\alpha = 0$ and $C = 2\omega_2$ and $\beta > 0$ a nontrivial $R_{L,0}$ exists for which stable equilibria $(R_{L,0}, \psi_{L,0})$ of system (90) occur with $R_{L,0} = \frac{2}{L^7\pi^7\beta}$ and $\psi_{L,0} = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$ (see Fig. 2f). In this case, the structural damping stabilizes the vibrations of the actuated beam for which the actuation frequency ω is $\mathcal{O}(\varepsilon^3)$ close to a $\frac{1}{2}$ times a natural frequency of the actuated beam. For $\alpha = 0$ and $\beta = 0$ and $2\omega_2 > C$, or $2\omega_2 < C$, we have as nontrivial equilibria for system (90): $R_{L,0} = \frac{1}{L^3\pi^3(2\omega_2 - C)}$ and $\psi_{L,0} = 2n\pi$, or $R_{L,0} = \frac{1}{L^3\pi^3(C-2\omega_2)}$ and $\psi_{L,0} = \pi + 2n\pi$, respectively (and $n \in \mathbb{Z}$). These two cases correspond to the nonresonant case 1.2 as studied in Sect. 3. The phase portraits for these two cases are given in Figs. 2a–b and 2d–e. For $\alpha = 0$ and $2\omega_2 \neq C$ and $\beta > 0$ we have as nontrivial equilibria for system (90): $R_{L,0} = \frac{2}{L^3\pi^3\sqrt{4(C-2\omega_2)^2+\beta^2 L^8\pi^8}}$, and $\psi_{L,0} = \psi^* + 2n\pi$ (with $n \in \mathbb{Z}$), and where ψ^* is a solution of

$$\sin(\psi^*) = \frac{\beta L^4\pi^4}{\sqrt{4(C-2\omega_2)^2+\beta^2 L^8\pi^8}}$$
$$\cos(\psi^*) = -\frac{2(C-2\omega_2)}{\sqrt{4(C-2\omega_2)^2+\beta^2 L^8\pi^8}} - \frac{3\alpha}{2L^4\pi^4(4(C-2\omega_2)^2+\beta^2 L^8\pi^8)^{3/2}}.$$

It can be checked that these equilibria are asymptotically stable, and that the phase portraits are similar to the one in Fig. 2f.

We can summarize this case $\alpha = 0$ (when no elastic forces are present) as follows. When damping β is present, all solutions will be asymptotically stable. While when no damping is present, we can have unbounded solutions when $2\omega_2 = C$, or equivalently when the actuation frequency ω is up to $\mathcal{O}(\varepsilon^3)$ equal to a $\frac{1}{2}$ times a natural frequency of the actuated beam (see also Eq. (51)).

The transitions of phase portraits when the nonlinear elastic force and the structural damping are not present, are shown in Figs. 2a to 2e as ω_2 gets larger and larger. In Figs. 2c, $\omega_2 = \frac{1}{2}C$, the resonance frequency, and so the solutions become unbounded. When damping is present all solution will be bounded but not completely damped out as in Figure 2 (f).

4.3.2 *The case with weakly nonlinear elastic forces ($\alpha > 0$)*

Since $\alpha > 0$, we can rewrite Eq. (96) by using the shift $\hat{R} = R + \frac{2^5(C-2\omega_2)}{3^2\alpha L^2\pi^2}$, and we obtain the depressed cubic form in $\hat{R}$:

$$\hat{R}^3 + \frac{2^6}{3^3\alpha^2 L^4\pi^4}\left(3\beta^2 L^8\pi^8 - 4(C-2\omega_2)^2\right)\hat{R} - \frac{2^{13}(C-2\omega_2)^3}{3^6\alpha^3 L^6\pi^6} - \frac{2^{11}\beta^2 L^2\pi^2(C-2\omega_2)}{3^4\alpha^3} - \frac{2^8}{3^2\alpha^2 L^{10}\pi^{10}} = 0. \tag{97}$$

Let the coefficient of $\hat{R}$ and the constant term of Eq. (97) be p and q, respectively. Now, we will consider the cases when $3\beta^2 L^8\pi^8 = 4(C-2\omega_2)^2$, and when $3\beta^2 L^8\pi^8 \neq 4(C-2\omega_2)^2$. **Case 1:** $3\beta^2 L^8\pi^8 = 4(C-2\omega_2)^2$

For this case, we have two subcases to consider, i.e. $(C-2\omega_2) = \pm\frac{1}{2}\sqrt{3}\beta L^4\pi^4$. We also have a special subcase, that is, when $C = 2\omega_2$ and $\beta = 0$. In this last special subcase, there are equilibria

$$(R_{L,0}, \psi_{L,0}) = \left(\frac{2}{3L\pi}\sqrt[3]{\frac{2.3^2}{\alpha L^2\pi^2}}, \pi + 2n\pi\right), \quad n \in \mathbb{Z},$$

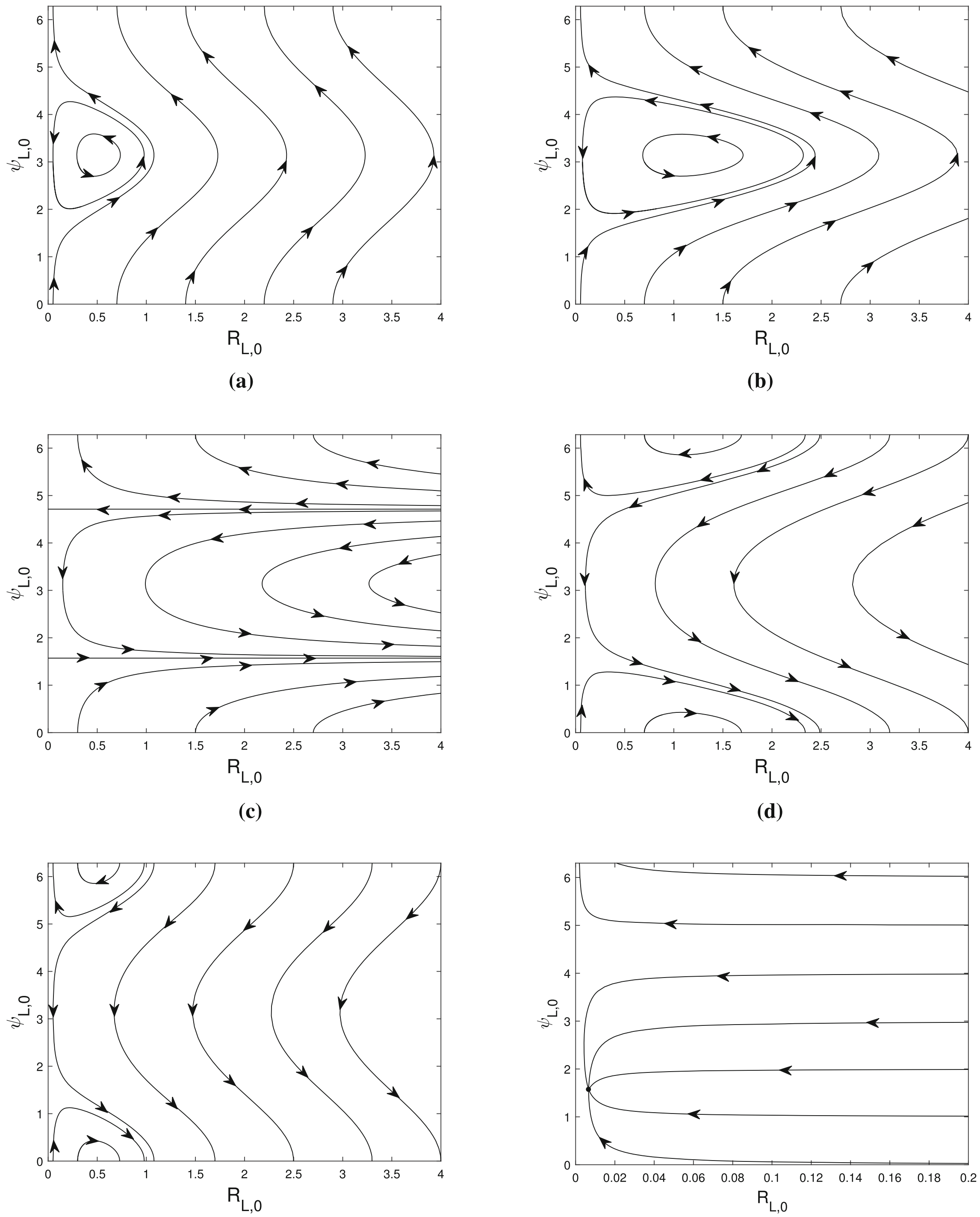


Fig. 2 The phase portrait of system (90) when $\alpha = 0$, $\beta = 0$, and for various ω_2, except in Figure (f) where $\beta > 0$. Figure (a) and (b) are the phase portraits when $2\omega_2 < C$. When $2\omega_2 = C$, the phase portrait is shown at Figure (c). Here, we have the pure resonance case, and so the solution will be unbounded. While figure (d) and (e) are for $2\omega_2 > C$. From Figure (a) to (e), we see a transition of phase portraits as ω_2 becomes larger and larger. All solutions are bounded except when $2\omega_2 = C$. When structural damping β is present, all solutions will be bounded as in Figure (f)

which are Lyapunov stable, see Fig. 3a. For the first subcase, $(C-2\omega_2)=\frac{1}{2}\sqrt{3}\beta L^4\pi^4$ (with $C>2\omega_2$ and $\beta>0$), we have asymptotically stable equilibria with

$$R_{L,0}=\frac{2}{3L\pi}\sqrt{\frac{1}{\alpha L\pi}}\sqrt{\sqrt[3]{(2^2\sqrt{3}\beta L^5\pi^5)^3+\frac{2^2 3^4\alpha}{L\pi}}-2^2\sqrt{3}\beta L^5\pi^5}$$

and $\psi_{L,0}=\psi^*+2n\pi$ and $\frac{\pi}{2}<\psi^*<\pi$, see Figure 3(b). For the second subcase, $(C-2\omega_2)=-\frac{1}{2}\sqrt{3}\beta L^4\pi^4$ (with $C<2\omega_2$ and $\beta>0$), there will also be asymptotically stable equilibria with $R_{L,0}$ given by

$$R_{L,0}=\frac{2}{3L\pi}\sqrt{\frac{1}{\alpha L\pi}}\sqrt{\sqrt[3]{-(2^2\sqrt{3}\beta L^5\pi^5)^3+\frac{2^2 3^4\alpha}{L\pi}}+2^2\sqrt{3}\beta L^5\pi^5}.$$

and with similar phase portraits as in the first subcase.

Case 2: $3\beta^2L^8\pi^8\neq 4(C-2\omega_2)^2$
When the discriminant of the cubic equation (97), $D=-(4p^3+27q^2)$, is equal to 0, we have as additional condition on the parameters that

$$\begin{aligned}&2^8\beta^2L^{16}\pi^{16}(C-2\omega_2)^4+2^6 3\alpha L^4\pi^4(C-2\omega_2)^3\\&\quad+2^7\beta^4L^{24}\pi^{24}(C-2\omega_2)^2\\&\quad+2^4 3^3\alpha\beta^2L^{12}\pi^{12}(C-2\omega_2)\\&\quad+2^4\beta^6L^{30}\pi^{30}+3^5\alpha=0.\end{aligned}\tag{98}$$

Satisfying the condition $D=0$, we will further divide this case into two subcases, that is, when $4(C-2\omega_2)^2-3\beta^2L^8\pi^8<0$, and when $4(C-2\omega_2)^2-3\beta^2L^8\pi^8>0$. For the first subcase, that is, when β is large, the discriminant of the cubic equation is negative. This means there will always be one equilibria of the system (97). For the second subcase, if the discriminant is 0 and $(C-2\omega_2)>\frac{1}{2}\sqrt{3}\beta L^4\pi^4$, that is, when β is small, then there can be only one stable equilibrium with $R_{L,0}$ given by

$$R_{L,0}=\sqrt{\frac{2^7 3\beta^2L^{12}\pi^{12}(C-2\omega_2)+2^2 3^4\alpha}{3^2\alpha L^6\pi^6[2^2(C-2\omega_2)^2-3\beta^2L^8\pi^8]}}.$$

If $(C-2\omega_2)<-\frac{1}{2}\sqrt{3}\beta L^4\pi^4$ and less than $-\frac{3^3\alpha}{2^5\beta^2L^{12}\pi^{12}}$, then there are no nontrivial equilibria. While if $(C-2\omega_2)$ is between $-\frac{3^3\alpha}{2^5\beta^2L^{12}\pi^{12}}$ and $-\frac{1}{2}\sqrt{3}\beta L^4\pi^4$, then there are at most three equilibria.

When the discriminant of the cubic equation (97) is zero, we can identify at most three equilibria, that is, two stable equilibria and one unstable saddle type equilibrium. We can also determine when the system has no nontrivial equilibria also.

When the structural damping β is relatively large, then all solutions will either be damped out or converge to a nontrivial stable equilibrium. While when the damping is relatively small, we have several cases to consider. When $\omega_2\leq\frac{1}{2}C-\frac{1}{4}\sqrt{3}\beta L^4\pi^4$, there will always be one nontrivial stable solution (see Fig. 4a and b). As ω_2 gets larger and larger, another unstable equilibrium appears (see Fig. 4c). This equilibrium splits up into one stable point and one saddle point (see Fig. 4d and e). As $\omega_2=\frac{1}{2}C+\frac{1}{2}\frac{3^3\alpha}{2^5\beta^2L^{12}\pi^{12}}$, the saddle and the first equilibrium point coincide and disappear (see Fig. 4g). This mechanism makes sense by considering the cubic polynomial in $\hat{R}$ in Eq (97). For $(C-2\omega_2)>-\frac{3^3\alpha}{2^5\beta^2L^{12}\pi^{12}}$ we have at most three nontrivial equilibria, that is, two stable equilibria and one unstable saddle type equilibrium. While if $(C-2\omega_2)>-\frac{3^3\alpha}{2^5\beta^2L^{12}\pi^{12}}$, there won't be any nontrivial equilibrium.

Physically, this means that when the nonlinear elastic force is present and a relatively small damping is present, then there will be at most two nontrivial stable solutions. Here, the initial conditions will determine to which stable solution the solution will converge. Next, we will look at the interesting case when $\beta=0$.

4.3.3 Case $\beta=0$

When $\beta=0$, system (90) will become

$$\begin{aligned}\frac{dR_{L,0}}{dt_2}&=\frac{1}{L^3\pi^3}\sin(\psi_{L,0}),\\\frac{d\psi_{L,0}}{dt_2}&=(C-2\omega_2)+\frac{3\alpha L^2\pi^2}{16}R_{L,0}^2+\frac{1}{L^3\pi^3}\frac{\cos(\psi_{L,0})}{R_{L,0}}.\end{aligned}\tag{99}$$

with $C=-\frac{V_0^4}{2L^6\pi^6}-\frac{3V_0^2S(L)}{L^2\pi^2}+\frac{8\alpha V_0^4}{L^8\pi^8}+\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\frac{4\alpha V_0^4}{n^8\pi^8}$.

For system (99) we can also find a first integral. By rewriting system (99) as

$$\begin{aligned}\frac{d\psi_{L,0}}{dR_{L,0}}&=\Big((C-2\omega_2)+\frac{3\alpha L^2\pi^2}{16}R_{L,0}^2\\&\quad+\frac{\cos(\psi_{L,0})}{L^3\pi^3R_{L,0}}\Big)\frac{L^3\pi^3}{\sin(\psi)}\\\frac{d\cos(\psi_{L,0})}{dR_{L,0}}&=-L^3\pi^3(C-2\omega_2)\\&\quad-\frac{3\alpha L^5\pi^5}{16}R_{L,0}^2-\frac{\cos(\psi_{L,0})}{R_{L,0}}\\\frac{d\cos(\psi_{L,0})}{dR_{L,0}}+\frac{\cos(\psi_{L,0})}{R_{L,0}}&=-L^3\pi^3(C-2\omega_2)-\frac{3\alpha L^5\pi^5}{16}R_{L,0}^2.\end{aligned}$$

and by solving the last linear equation, we obtain as first integral of system (99)

$$F(R_{L,0},\psi_{L,0})=F(\mathbf{0})+R_{L,0}\cos(\psi_{L,0})$$

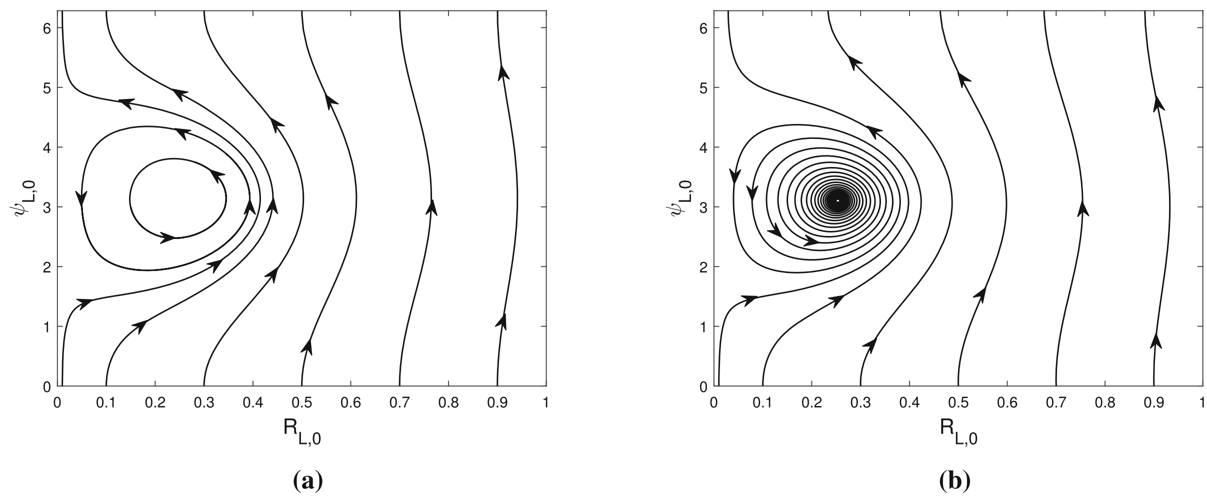


Fig. 3 The phase portraits of system (90) for $\alpha > 0$ and for various β. Here, we see the change in stability of the nontrivial equilibrium from Lyapunov stable to asymptotically stable when damping β is changed from 0 to positive values. When the nonlinear elastic force is present, all solutions will be stable (either asymptotically stable when damping is present or else Lyapunov stable when no damping is present)

$$+\frac{L^3\pi^3(C-2\omega_2)}{2}R_{L,0}^2 +\frac{3\alpha L^5\pi^5 R_{L,0}^4}{64} = \text{constant}. \tag{100}$$

Using the Taylor expansion for this first integral in a neighborhood of the equilibrium point (R^*, ψ^*) we get

$$\begin{aligned} F(R_{L,0}, \psi_{L,0}) &= F(R^*, \psi^*) \\ &+ \left[L^3\pi^3(C-2\omega_2) + \frac{9\alpha L^5\pi^5}{16}(R^*)^2\right](R_{L,0}-R^*)^2 \\ &- R^*\cos(\psi^*)(\psi_{L,0}-\psi^*)^2 \\ &- \sin(\psi^*)(R_{L,0}-R^*)(\psi_{L,0}-\psi^*) + HOT, \end{aligned} \tag{101}$$

where HOT stands for higher order terms. By using the first integral (101) and Morse's theorem it follows that the equilibrium points are center points and/or saddle points.

The phase portraits for the case $\beta = 0$ and $\alpha = 0$ can be seen in Fig. 2, that correspond with $2\omega_2 = C$ (Fig. 2c), $2\omega_2 > C$ (Fig. 2d–e), and $2\omega_2 < C$ (Fig. 2a–b). While the phase portraits when $\beta = 0$ and $\alpha > 0$, can be found in Fig. 5 and in Fig. 3a. Figure 3a is for $C = 2\omega_2$. For the case $C \neq 2\omega_2$, there are three essentially different phase portraits which correspond to negative, zero, and positive values of the discriminant of the cubic Eq. (97). These three different phase portraits are given in Fig. 5. Here we see that in the absence of the structural damping, the solutions are always Lyapunov stable when the elastic force is present. In all of these superharmonic cases, we see that there usually exist two unstable saddle equilibria on the $\psi_{L,0}$-axis, that is, when $R_{L,0} = 0$.

4.4 The subharmonic case (case $\omega_0 = 2M^2\pi^2$ for a fixed $M \in \mathbb{N}$)

In this subsection we will consider the subharmonic case $\omega = 2M^2\pi^2 + \mathcal{O}(\varepsilon)$ (that is, $\omega_0 = 2M^2\pi^2$) for a fixed $M \in \mathbb{N}$. By substituting $v_{k,0}$ and $v_{k,1}$ into the $\mathcal{O}(\varepsilon^2)$ Eq. (81), we obtain

$$\begin{aligned} \mathcal{L}v_{k,2} &= \cos(k^2\pi^2 t_0)\Bigg(-2k^2\pi^2\frac{\partial B_{k,1}}{\partial t_1} + \frac{V_0^4}{k^4\pi^4}A_{k,0} - 2k^2\pi^2\frac{\partial B_{k,0}}{\partial t_2} \\ &\quad + 2V_0^2 A_{k,1} + 6V_0^2 S(k)A_{k,0} - \beta k^6\pi^6 B_{k,0} \\ &\quad + 2\delta_{k,M}V_0\big(A_{k,0}\sin(\omega_1 t_1 + \omega_2 t_2) \\ &\quad + B_{k,0}\cos(\omega_1 t_1 + \omega_2 t_2)\big) \\ &\quad - \frac{\alpha k^2\pi^4}{2}A_{k,0}\Bigg[\frac{8k^2V_0^4H^2(k)}{k^8\pi^8} + \frac{k^2}{4}(A_{k,0}^2 + B_{k,0}^2) \\ &\quad + \sum_{n=1}^{\infty}\frac{n^2}{2}(A_{n,0}^2 + B_{n,0}^2) + \sum_{n=1}^{\infty}\frac{4n^2V_0^4H^2(n)}{n^8\pi^8}\Bigg]\Bigg), \\ &\quad + \sin(k^2\pi^2 t_0)\Bigg(2k^2\pi^2\frac{\partial A_{k,1}}{\partial t_1} + \frac{V_0^4}{k^4\pi^4}B_{k,0} + 2k^2\pi^2\frac{\partial A_{k,0}}{\partial t_2} \\ &\quad + 2V_0^2 B_{k,1} + 6V_0^2 S(k)B_{k,0} + \beta k^6\pi^6 A_{k,0} \\ &\quad + 2\delta_{k,M}V_0\big(A_{k,0}\cos(\omega_1 t_1 + \omega_2 t_2) \end{aligned}$$

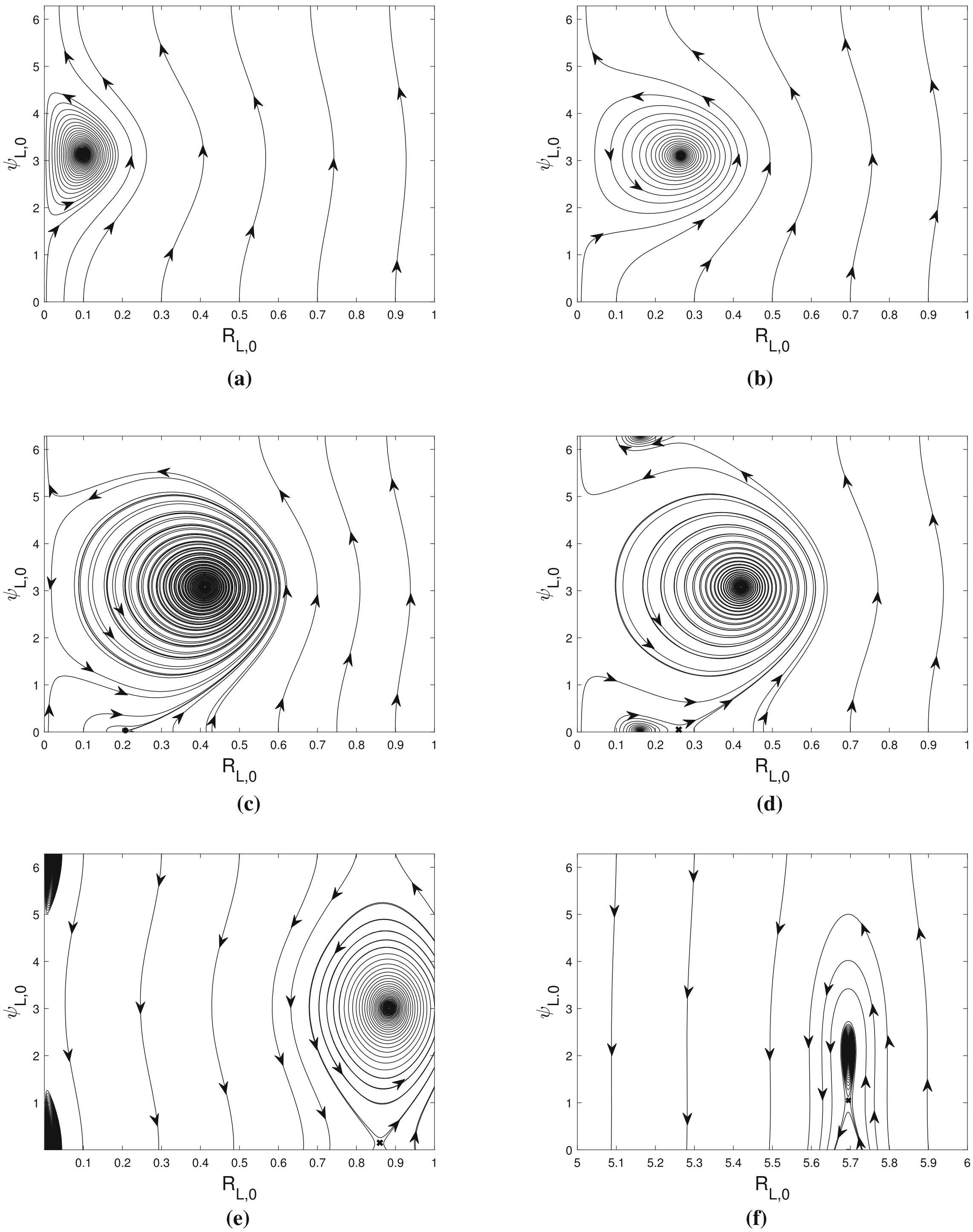
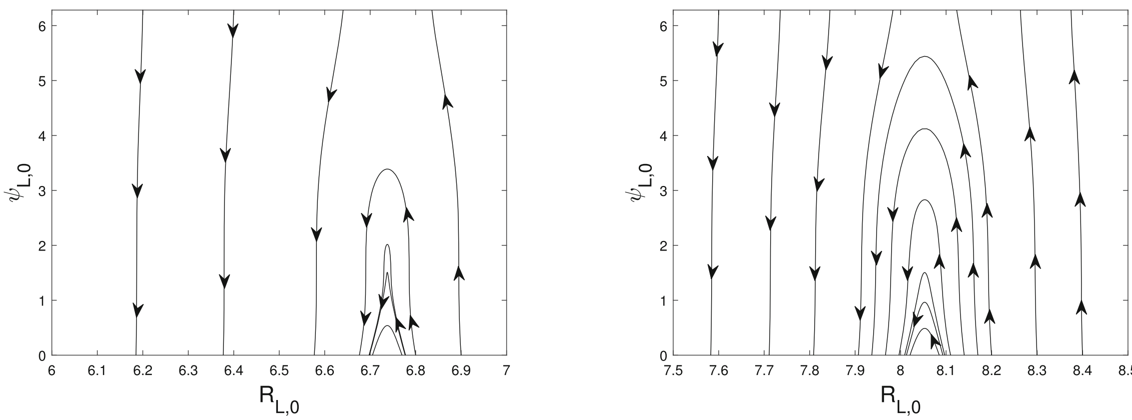
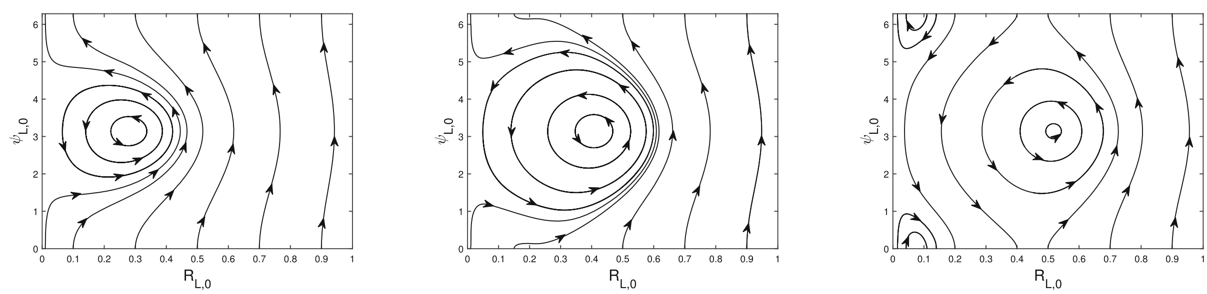


Fig. 4 The phase portrait of system (90) when $\alpha > 0$, for β relatively small, and for various values of ω_2. As ω_2 gets larger and larger, a nontrivial equilibrium point occurs (Figure **c**)) which for larger ω_2 bifurcates in two nontrivial equilibria with different stability properties (Fig. **d**). In Fig. **e**-**h**, the first equilibrium point and the saddle equilibrium coincide and disappear for larger values of ω_2



(g) (h)

Fig. 4 continued



(a) (b) (c)

Fig. 5 The phase portrait of system (90) for $\beta = 0$ and $\alpha > 0$. Figure **a**, **b**, and **c** are phase portraits for negative, zero, and positive values of the discriminant of the cubic Eq. (97), respectively. When the elastic force is present but the structural damping is not present, we see that the nontrivial equilibria will always be Lyapunov stable

$$
\begin{aligned}
&-B_{k,0}\sin(\omega_1 t_1+\omega_2 t_2)\big)\\
&-\tfrac{\alpha k^2\pi^4}{2}B_{k,0}\Big[\tfrac{8k^2V_0^4H^2(k)}{k^8\pi^8}+\tfrac{k^2}{4}(A_{k,0}^2+B_{k,0}^2)\\
&+\sum_{n=1}^{\infty}\tfrac{n^2}{2}(A_{n,0}^2+B_{n,0}^2)+\sum_{n=1}^{\infty}\tfrac{4n^2V_0^4H^2(n)}{n^8\pi^8}\Big]\Big)\\
&+NST, \qquad (102)
\end{aligned}
$$

where

$$
\delta_{k,M}=\begin{cases}0, & \text{for } k\neq M,\\ 1, & \text{for } k=M.\end{cases}
$$

and

$$
A_{k,0}(t_1,t_2)=C_{k,0}(t_2)\cos\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right)+D_{k,0}(t_2)\sin\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right),
$$

$$
B_{k,0}(t_1,t_2)=C_{k,0}(t_2)\sin\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right)-D_{k,0}(t_2)\cos\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right).
$$

To avoid secular terms in $v_{k,2}$, it follows from (102) that $A_{k,1}$ and $B_{k,1}$ have to satisfy

$$
\begin{aligned}
\tfrac{\partial A_{k,1}}{\partial t_1}+\tfrac{V_0^2}{k^2\pi^2}B_{k,1}=&-\tfrac{V_0^4}{2k^6\pi^6}B_{k,0}-\tfrac{\partial A_{k,0}}{\partial t_2}-\tfrac{3V_0^2S(k)}{k^2\pi^2}B_{k,0}\\
&-\tfrac{\beta k^4\pi^4}{2}A_{k,0}+\tfrac{\alpha\pi^2}{4}B_{k,0}X(k)\\
&+\delta_{k,M}\tfrac{V_0}{k^2\pi^2}\big(-A_{k,0}\cos(\omega_1 t_1+\omega_2 t_2)\\
&+B_{k,0}\sin(\omega_1 t_1+\omega_2 t_2)\big),\\
\tfrac{\partial B_{k,1}}{\partial t_1}-\tfrac{V_0^2}{k^2\pi^2}A_{k,1}=&\tfrac{V_0^4}{2k^6\pi^6}A_{k,0}-\tfrac{\partial B_{k,0}}{\partial t_2}+\tfrac{3V_0^2S(k)}{k^2\pi^2}A_{k,0}\\
&-\tfrac{\beta k^4\pi^4}{2}B_{k,0}-\tfrac{\alpha\pi^2}{4}A_{k,0}X(k)\\
&+\delta_{k,M}\tfrac{V_0}{k^2\pi^2}\big(A_{k,0}\sin(\omega_1 t_1+\omega_2 t_2)\\
&+B_{k,0}\cos(\omega_1 t_1+\omega_2 t_2)\big),
\end{aligned}
$$

where $X(k)$ is again given by (84). Combining these two equations, we obtain

$$
\tfrac{\partial^2 A_{k,1}}{\partial t_1^2}+\tfrac{V_0^4}{k^4\pi^4}A_{k,1}=\tfrac{V_0^2}{k^2\pi^2}\cos\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right)\Big[-2\tfrac{dD_{k0}}{dt_2}-\beta k^4\pi^4 D_{k,0}
$$

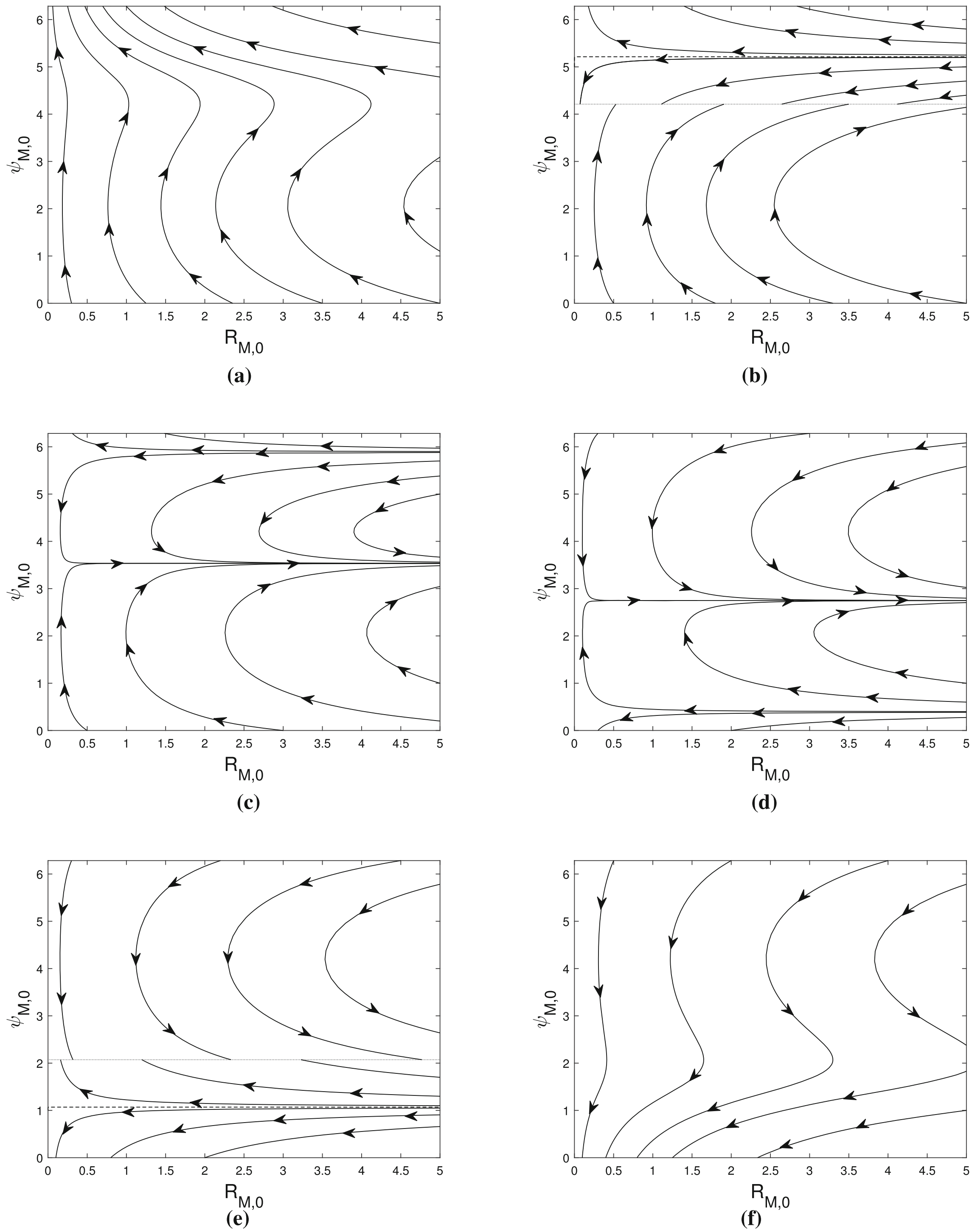


Fig. 6 The phase portrait of system (107) when $\alpha = 0$, β is relatively small, and for various values of ω_2. In Fig. **a**, there are no nontrivial equilibria. When $\omega_2 = D - \sqrt{\Delta}$ $\left(\Delta = \frac{4V_0^2}{M^4\pi^4} - \beta^2 M^8 \pi^8\right)$, a manifold of equilibria (the dotted line in Fig. **b**) and the separatrix between the orbits (the dashed line) occur. As ω_2 gets larger and larger, the phase portrait change from Fig. **c** to **d**. Here we have unbounded solutions due to the range of resonance frequencies. In Fig. **e**, that is, when $\omega_2 = D + \sqrt{\Delta}$ we have a similar phase portrait as in Fig. **b**. When ω_2 gets larger, we have similar phase portraits as in Fig. **a** with reversed trajectories. All solutions are bounded except when ω_2 is in the range of the resonance frequency (that is, in Fig. **c** and **d**)

$$-2\Big(\tfrac{V_0^4}{2k^6\pi^6}+\tfrac{3V_0^2S(k)}{k^2\pi^2}-\tfrac{\alpha\pi^2X(k)}{4}\Big)C_{k,0}\Big]$$
$$+\tfrac{V_0^2}{k^2\pi^2}\sin\left(\tfrac{V_0^2}{k^2\pi^2}t_1\right)\Big[2\tfrac{dC_{k,0}}{dt_2}+\beta k^4\pi^4C_{k,0}$$
$$-2\Big(\tfrac{V_0^4}{2k^6\pi^6}+\tfrac{3V_0^2S(k)}{k^2\pi^2}-\tfrac{\alpha\pi^2X(k)}{4}\Big)D_{k,0}\Big]$$
$$+\tfrac{\omega_1V_0\delta_{k,M}}{k^2\pi^2}\Big[C_{k,0}\sin\left(\left(\omega_1+\tfrac{V_0^2}{k^2\pi^2}\right)t_1+\omega_2t_2\right)$$
$$-D_{k,0}\cos\left(\left(\omega_1+\tfrac{V_0^2}{k^2\pi^2}\right)t_1+\omega_2t_2\right)\Big]. \tag{103}$$

For all modes $k\neq M$ secular terms in $A_{k,1}$ and $B_{k,1}$ can be avoided when $C_{k,0}$ and $D_{k,0}$ satisfy (as follows from (103)) the same equations as given by (85). Hence, we will have the same stable equilibria, and $C_{k,0}$ and $D_{k,0}$ will tend to zero for $t_2\to\infty$.

Case $k=M$

Now, we will begin the discussion for the case when the excited mode is $k=M$. Here, we have three subcases to consider

1. $\omega_1=0$,
2. $\omega_1=-\frac{2V_0^2}{M^2\pi^2}$,
3. $\omega_1\neq 0$ and $\omega_1\neq-\frac{2V_0^2}{M^2\pi^2}$.

When $\omega_1=0$, to avoid secular terms in $A_{M,1}$ and $B_{M,1}$, $C_{M,0}$ and $D_{M,0}$ have to satisfy

$$\frac{dC_{M,0}}{dt_2}=-\frac{\beta M^4\pi^4}{2}C_{M,0}+\Big(\frac{V_0^4}{2M^6\pi^6}+\frac{3V_0^2S(M)}{M^2\pi^2}-\frac{\alpha\pi^2X(M)}{4}\Big)D_{M,0}, \tag{104}$$

$$\frac{dD_{M,0}}{dt_2}=-\Big(\frac{V_0^4}{2M^6\pi^6}+\frac{3V_0^2S(M)}{M^2\pi^2}-\frac{\alpha\pi^2X(M)}{4}\Big)C_{M,0}-\frac{\beta M^4\pi^4}{2}D_{M,0}. \tag{105}$$

By combining the two equations in (104), and by introducing $R_{M,0}^2=C_{M,0}^2+D_{M,0}^2$, we obtain again equation (86), that is,

$$\frac{dR_{M,0}^2}{dt_2}=-\beta M^4\pi^4R_{M,0}^2. \tag{106}$$

Hence, we have the same stable equilibria, and $C_{M,0}$ and $D_{M,0}$ tend to zero for $t_2\to\infty$. For the subcase when $\omega_1\neq 0$ and $\omega_1\neq-\frac{2V_0^2}{M^2\pi^2}$, by eliminating the secular terms in (103), we again end up with equation (106), and the same stable equilibria for $C_{M,0}$ and $D_{M,0}$ are obtained.

For the subcase $\omega_1=-\frac{2V_0^2}{M^2\pi^2}$, the functions $C_{M,0}$ and $D_{M,0}$ have to satisfy

$$\frac{dC_{M,0}}{dt_2}=-\left(\frac{\beta M^4\pi^4}{2}+\frac{V_0}{M^2\pi^2}\cos(\omega_2t_2)\right)C_{M,0}+\left(\frac{V_0^4}{2M^6\pi^6}+\frac{3V_0^2S(M)}{M^2\pi^2}-\frac{\alpha\pi^2X(M)}{4}-\frac{V_0}{M^2\pi^2}\sin(\omega_2t_2)\right)D_{M,0},$$

$$\frac{dD_{M,0}}{dt_2}=-\left(\frac{V_0^4}{2M^6\pi^6}+\frac{3V_0^2S(M)}{M^2\pi^2}-\frac{\alpha\pi^2X(M)}{4}+\frac{V_0}{M^2\pi^2}\sin(\omega_2t_2)\right)C_{M,0}-\left(\frac{\beta M^4\pi^4}{2}-\frac{V_0}{M^2\pi^2}\cos(\omega_2t_2)\right)C_{M,0}.$$

By introducing polar coordinates

$$C_{M,0}(t_2)=R_{M,0}(t_2)\cos(\phi_{M,0}(t_2)),$$
$$D_{M,0}(t_2)=R_{M,0}(t_2)\sin(\phi_{M,0}(t_2)),$$

the system for $C_{M,0}$ and $D_{M,0}$ becomes:

$$\frac{dR_{M,0}}{dt_2}=-\Big(\frac{V_0}{M^2\pi^2}\cos(2\phi_{M,0}(t_2)-\omega_2t_2)+\frac{\beta M^4\pi^4}{2}\Big)R_{M,0},$$
$$\frac{d\phi_{M,0}}{dt_2}=\frac{V_0}{M^2\pi^2}\sin(2\phi_{M,0}(t_2)-\omega_2t_2)-\frac{V_0^4}{2M^6\pi^6}-\frac{3V_0^2S(M)}{M^2\pi^2}+\frac{\alpha\pi^2X(M)}{4}.$$

By putting $\psi_{M,0}(t_2)=2\phi_{M,0}(t_2)-\omega_2t_2$, an autonomous system for $R_{M,0}$ and $\psi_{M,0}$ is obtained:

$$\frac{dR_{M,0}}{dt_2}=-\left(\frac{V_0}{M^2\pi^2}\cos(\psi_{M,0})+\frac{\beta M^4\pi^4}{2}\right)R_{M,0},$$
$$\frac{d\psi_{M,0}}{dt_2}=\frac{2V_0}{M^2\pi^2}\sin(\psi_{M,0})-\frac{V_0^4}{M^6\pi^6}-\frac{6V_0^2S(M)}{M^2\pi^2}+\frac{\alpha\pi^2X(M)}{2}-\omega_2.$$

Assuming that no initial energy is present in all modes $k\neq M$, we can simplify $X(M)$, such that we obtain the following autonomous system

$$\begin{aligned}\frac{dR_{M,0}}{dt_2}&=-\left(\frac{V_0}{M^2\pi^2}\cos(\psi_{M,0})+\frac{\beta M^4\pi^4}{2}\right)R_{M,0},\\ \frac{d\psi_{M,0}}{dt_2}&=(D-\omega_2)+\frac{3\alpha\pi^2}{8}M^2R_{M,0}^2+\frac{2V_0}{M^2\pi^2}\sin(\psi_{M,0}),\end{aligned} \tag{107}$$

where

$$D=-\frac{V_0}{M^6\pi^6}-\frac{6V_0^2S(M)}{M^2\pi^2}+\frac{16\alpha V_0^4}{M^8\pi^8}+\sum_{n=1}^{\infty}\frac{8\alpha V_0^4}{n^8\pi^8}.$$

The jacobian of the vector field (107) is given by

$$\begin{pmatrix}-\left(\frac{V_0}{M^2\pi^2}\cos(\psi_{M,0})+\frac{\beta M^4\pi^4}{2}\right) & \frac{V_0}{M^2\pi^2}\sin(\psi_{M,0})\\ \frac{3\alpha M^2\pi^2}{4}R_{M,0} & \frac{2V_0}{M^2\pi^2}\cos(\psi_{M,0})\end{pmatrix},$$

where the corresponding eigenvalues of the jacobian matrix satisfy

$$\lambda^2-\left(\frac{V_0}{M^2\pi^2}\cos(\psi_{M,0})-\frac{\beta M^4\pi^4}{2}\right)\lambda-\cos(\psi_{M,0})\left(\frac{2V_0^2}{M^4\pi^4}\cos(\psi_{M,0})+\beta V_0M^2\pi^2\right)-\frac{3\alpha V_0}{4}R_{M,0}\sin(\psi_{M,0})=0. \tag{108}$$

The first group of equilibria of system (107) satisfy $R_{M,0}=0$ and $\sin(\psi_{M,0})=\frac{(\omega_2-D)M^2\pi^2}{2V_0}$. For $0\leq\psi_{M,0}<2\pi$ and depending on the value of $\frac{(\omega_2-D)M^2\pi^2}{2V_0}$, there can be at most two equilibrium points on that part of the $\psi_{M,0}$-axis,

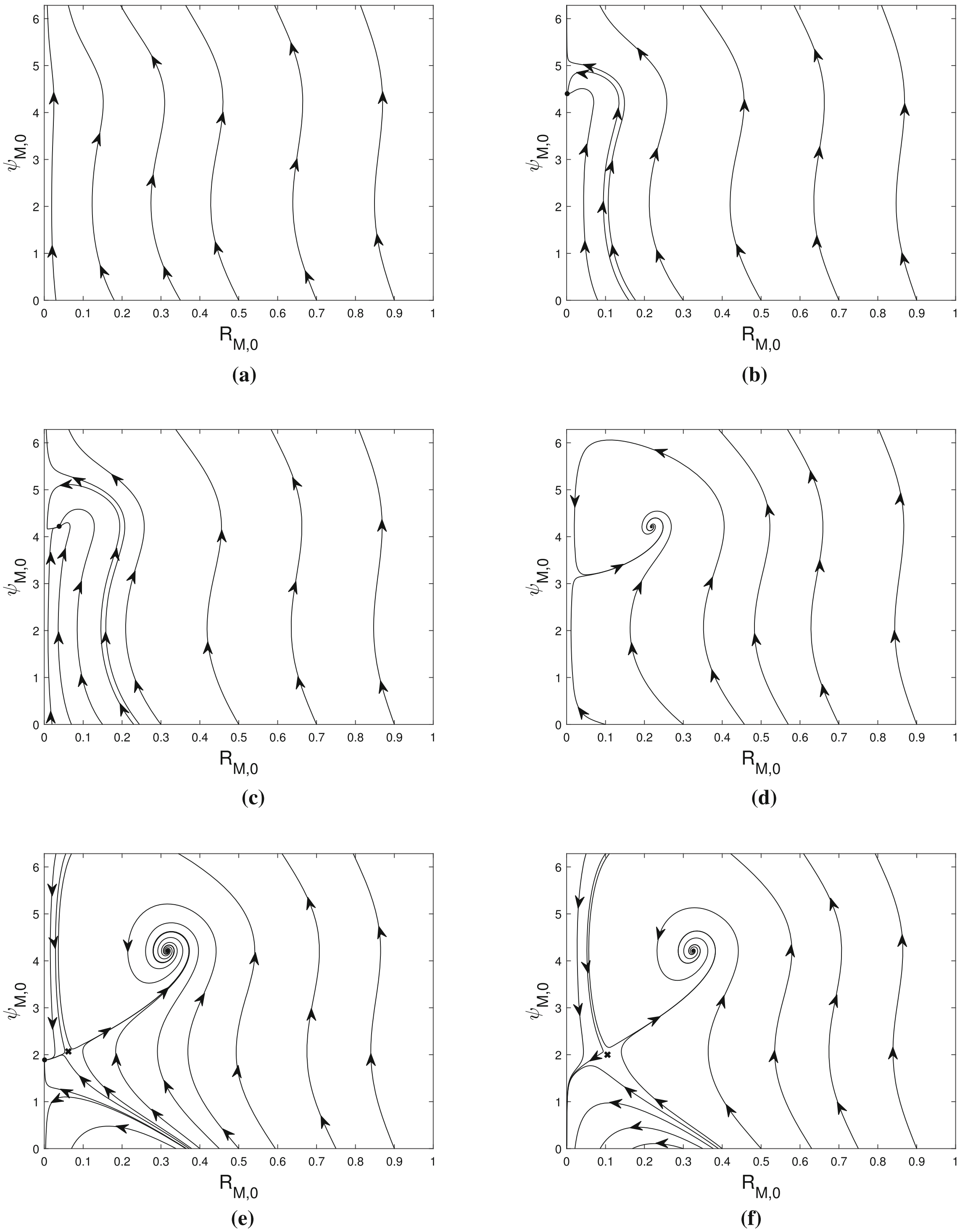


Fig. 7 The phase portrait of system (90) when $\alpha > 0$, β is relatively small, and for various values of ω_2. In Figure (a) there are no nontrivial equilibria. As ω_2 gets larger, one equilibrium of the first group of equilibria occurs in Figure (b). In Figure (c) and (d), there are 2 equilibria from the first group and one equilibrium from the second group of equilibria. While in Figure (e), we have two equilibria from both groups of equilibria. The first group of equilibria disappear as ω_2 gets larger as we see in Figure (f). From Figure (a) to (f), we see a transition of phase portrait as ω_2 becomes larger and larger. Here, all solutions are bounded

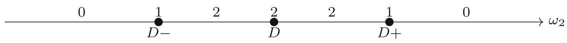


Fig. 8 The number of equilibria for the first group of equilibrium points which satisfy system (107), where $D+ = D + \frac{2V_0}{M^2\pi^2}$ and $D- = D - \frac{2V_0}{M^2\pi^2}$. The number of equilibria is given above the ω_2-line. Here, we see the change in number of equilibria of the first group of equilibria when ω_2 is varied

see Figure 7b–e. When $\frac{(\omega_2 - D)M^2\pi^2}{2V_0} = 0$ (corresponding to $\sin(\psi_{M,0} = 0)$), that is, $\omega_2 = D$, there are two equilibria, that is, one unstable equilibrium in $(0, 0)$, and one equilibrium in $(0, \pi)$, which is stable when $\beta > \frac{2V_0}{M^6\pi^6}$ and unstable when $0 \le \beta < \frac{2V_0}{M^6\pi^6}$. Also if $0 < \left|\frac{(D-\omega_2)M^2\pi^2}{2V_0}\right| < 1$, that is, $D - \frac{2V_0}{M^2\pi^2} < \omega_2 < D + \frac{2V_0}{M^2\pi^2}$, but $\omega_2 \ne D$, then there exist two equilibria. One of them is unstable and the other one is stable if $0 \le \beta < \frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4}{4V_0^2}(D-\omega)^2}$ (small) and unstable if $\beta > \frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4}{4V_0^2}(D-\omega_2)^2}$ (large). When $\omega_2 = D \pm \frac{2V_0}{M^2\pi^2}$, we have either an equilibrium in $(0, \frac{\pi}{2})$ or in $(0, \frac{3\pi}{2})$ which both have negative and zero eigenvalues. And, finally, no equilibria if $\omega_2 < D - \frac{2V_0}{M^2\pi^2}$ or $\omega_2 > D + \frac{2V_0}{M^2\pi^2}$ (Fig. 7a and f). In Fig. 8 we present a diagram for the number of equilibria of this first group of equilibria.

For the second group of equilibria, we have to satisfy

$$\cos(\psi_{M,0}) = -\frac{\beta M^6\pi^6}{2V_0}, \tag{109}$$

$$0 = (D - \omega_2) + \frac{3\alpha\pi^2M^2}{8}R_{M,0}^2 + \frac{2V_0}{M^2\pi^2}\sin(\psi_{M,0}). \tag{110}$$

From Eq. (109), it follows that there are no $\psi_{M,0}$ if $\beta > \frac{2V_0}{M^6\pi^6}$ (β large), one $\psi_{M,0}$ if $\beta = \frac{2V_0}{M^6\pi^6}$, and two values for $\psi_{M,0}$ if $\beta < \frac{2V_0}{M^6\pi^6}$ (β small). If we combine (109) and (110), then $R_{M,0}$ has to satisfy:

$$\left[(D - \omega_2) + \frac{3\alpha\pi^2M^2}{8}R_{M,0}^2\right]^2 = \frac{4V_0^2}{M^4\pi^4} - \beta^2M^8\pi^8. \tag{111}$$

Case $\boldsymbol{\alpha = 0}$ *(no elastic forces)*

If α is taken to be zero, then there are infinitely many equilibria if $\omega_2 = D \pm \sqrt{\frac{4V_0^2}{M^4\pi^4} - \beta^2M^8\pi^8}$. We will look at two subcases here, that is when $\beta = 0$ and when $\beta > 0$. For the subcase $\beta = 0$, based on the analysis as presented in section 3 of this paper (see case 1.3), we summarize the results in Fig. 9. For the special condition $\omega_2 = D \pm \frac{2V_0}{M^2\pi^2}$, we have unbounded solutions.

For the subcase $\beta > 0$, we have bounded solutions when $\omega_2 \le D - \frac{2V_0}{M^2\pi^2}$ or $\omega_2 \ge D + \frac{2V_0}{M^2\pi^2}$. When $D - \frac{2V_0}{M^2\pi^2} < \omega_2 < D + \frac{2V_0}{M^2\pi^2}$ and $0 < \beta < \frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4}{4V_0^2}(D-\omega_2)^2}$ (small), the solutions will be unbounded, but become bounded

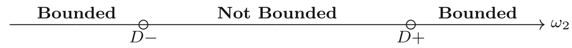


Fig. 9 Diagram of the boundedness of solutions of system (107) when $\alpha = 0$, and $\beta < \frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4}{4V_0^2}(D-\omega_2)^2}$, where $D+ = D + \frac{2V_0}{M^2\pi^2}$ and $D- = D - \frac{2V_0}{M^2\pi^2}$. In the absence of the nonlinear elastic force and relatively small damping, the solution can become unbounded for a certain range of ω_2

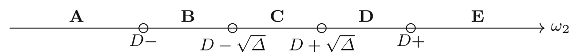


Fig. 10 Diagram for the total number of equilibria of system (107) when $\alpha = 0$ and β is relatively small, where $D+ = D + \frac{2V_0}{M^2\pi^2}$, $D- = D - \frac{2V_0}{M^2\pi^2}$, and $\Delta = \frac{4V_0^2}{M^4\pi^4} - \beta^2M^8\pi^8$. In domain A, there are no equilibria of system (107). While when $D- \le \omega_2 \le D+$, that is, in domain B, C, and D, the first group of equilibria exists. The stable equilibrium of the second group of equilibria exists when $\omega_2 > D - \sqrt{\Delta}$, that is, in domain C, D, and E. When $\omega_2 > D + \sqrt{\Delta}$, that is, in the domain D and E, an unstable saddle equilibrium exists. Unlike in the case when $\alpha = 0$, the solutions are always bounded in the domains B, C, and D. Thus, when the nonlinear elastic force is present and the structural damping is relatively small, the solution can become unbounded for a certain range of ω_2. While if the damping is quite large, all solutions will be bounded. The $D-$ and $D+$ mark the range when there are at most two equilibria from the first group. To the right of $D - \sqrt{\Delta}$ the existence of a stable equilibrium from the second group of equilibria is guaranteed, while to the right of $D + \sqrt{\Delta}$ the existence of an unstable saddle from the second group of equilibria is guaranteed

if $\beta > \frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4}{4V_0^2}(D-\omega_2)^2}$ (large). For the special case $D - \frac{2V_0}{M^2\pi^2} < \omega_2 < D + \frac{2V_0}{M^2\pi^2}$ and β equals $\frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4}{4V_0^2}(D-\omega_2)^2}$, the solutions will always be bounded. The phase portrait for this special case, where there are infinitely many equilibria, can be seen in the middle phase portrait of Fig. 6b and e (the dotted line represents the manifold of infinitely many equilibria). We present the summary of the boundedness of solutions in Fig. 10.

Case $\boldsymbol{\alpha > 0}$

Simplifying (111), $R_{M,0}$ should satisfy

$$R_{M,0}^2 = \frac{8}{3\alpha\pi^2M^2}\left(\omega_2 - D \pm \sqrt{\Delta}\right),$$

where $\Delta = \frac{4V_0^2}{M^4\pi^4} - \beta^2M^8\pi^8$. In this case we will have two non-negative $R_{M,0}$ if $\beta < \frac{2V_0}{M^6\pi^6}$ and $\omega_2 \ge D + \sqrt{\Delta}$, and one $R_{M,0}$ if $\beta = \frac{2V_0}{M^6\pi^6}$ and $\omega_2 \ge D + \sqrt{\Delta}$. While if $\beta > \frac{2V_0}{M^6\pi^6}$ or $\omega_2 < D + \sqrt{\Delta}$, no $R_{M,0}$ can be found. By combining these $R_{M,0}$ conditions and $\psi_{M,0}$ conditions, we have two equilibria when $\beta < \frac{2V_0}{M^6\pi^6}$ and $\omega_2 \ge D + \sqrt{\Delta}$, where one is stable and the other is unstable (7(e) and (f)). We can also have one equilibrium if $\beta = \frac{2V_0}{M^6\pi^6}$ and $\omega_2 \geq D + \sqrt{\Delta}$ (one zero and one negative eigenvalue) or a stable equilibrium point when $D - \sqrt{\Delta} < \omega_2 < D + \sqrt{\Delta}$ and $\beta \leq \frac{2V_0}{M^6\pi^6}$. If $\beta > \frac{2V_0}{M^6\pi^6}$ (large) or $\omega_2 < D - \sqrt{\Delta}$, then there are no equilibrium points. Here, we will give a complete description on all the equilibria when the structural damping is relatively small, that is, $\beta < \frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4}{4V_0^2}(D-\omega_2)^2}$. The first group of equilibria exists if $D - \frac{2V_0}{M^2\pi^2} \leq \omega_2 \leq D + \frac{2V_0}{M^2\pi^2}$, that is the domains B, C, D in Fig. 9. The stable equilibrium from the second group of equilibria appears when $\omega_2 \geq D - \sqrt{\frac{4V_0}{M^4\pi^4} - \beta^2 M^8\pi^8}$ (the domain C, D, and E in Fig. 9), while the unstable saddle equilibrium of the second group appears when $\omega_2 \geq D + \sqrt{\frac{4V_0}{M^4\pi^4} - \beta^2 M^8\pi^8}$ (the domain D and E in Fig. 9).

When the structural damping is quite large, all solutions will be stabilized regardless the existence of elastic forces. In the absence of the elastic force and when the structural damping is relatively small, we will have a range of resonance frequencies when $D - \frac{2V_0}{M^2\pi^2} < \omega_2 < D + \frac{2V_0}{M^2\pi^2}$, which are the same resonance frequencies as in Case 1.3. Thus, when the elastic force is present, this constant D will serve as a correction to the resonance frequency in the subharmonic Case 1.3. But it will turn out that in this range of frequencies, the solution is still bounded when the elastic force is present.

When the nonlinear elastic force is present, we can have unbounded solutions if the following two requirements are met, that is,

1. The first group of equilibria has two unstable saddles, and the conditions are given by:

   $D - \frac{2V_0}{M^2\pi^2} < \omega_2 < D + \frac{2V_0}{M^2\pi^2}$

   and

   $0 \leq \beta < \frac{2V_0}{M^6\pi^6}\sqrt{1 - \frac{M^4\pi^4(D-\omega_2)^2}{4V_0^2}}$.

2. The second group of equilibria does not exist, and the condition is given by

   (a) $\beta > \frac{2V_0}{M^6\pi^6}$, or
   (b) $0 \leq \beta < \frac{2V_0}{M^6\pi^6}$ and $\omega_2 < D - \sqrt{\Delta}$

Case 1 and 2(a) are impossible to occur.
We will discuss the possibility of the case 1 and 2(b). Because $D - \omega_2 > \sqrt{\Delta}$, then we have

$$D - \omega_2 > 0 \text{ and } (D-\omega_2)^2 + \beta^2 M^8\pi^8 - \frac{4V_0^2}{M^4\pi^4} > 0.$$

While from the existence condition of the second group of equilibria, we have

$$\left((D-\omega_2) + \frac{3\alpha M^2\pi^2}{8}R_{M,0}\right)^2 + \beta^2 M^8\pi^8 = \frac{4V_0^2}{M^4\pi^4}.$$

This means

$$(D-\omega_2)^2 + \beta^2 M^8\pi^8 - \frac{4V_0^2}{M^4\pi^4} = -\frac{3}{4}\alpha M^2\pi^2(D-\omega_2)R_{M,0} - \left(\frac{3\alpha M^2\pi^2}{8}R_{M,0}\right)^2 < 0.$$

This also cannot happen. Thus, we do not have unbounded solutions when the nonlinear elastic force is present.

So far, in the analysis we assumed that only energy was present in mode $k = M$. Now we will consider the case when initial energy is present in mode $k = M$ and in mode $k = 2M$. The interaction between two modes, that is,

($k = M$ and $k = 2M$)

We will assume that $\omega_0 = 2M^2\pi^2$ for some fixed $M \in \mathbb{N}$, and $\omega_1 = -\frac{2V_0^2}{M^2\pi^2}$. So, we actually assume that the actuation frequency is $\mathcal{O}(\varepsilon^2)$ close to a subharmonic frequency of the beam. From (103)-(107) it follows that $R_{M,0}$, $R_{2M,0}$, $\psi_{M,0}$, and $\psi_{2M,0}$ have to satisfy

$$\begin{aligned}
\frac{dR_{M,0}}{dt_2} &= -\left(\frac{V_0}{M^2\pi^2}\cos(\psi_{M,0}) + \frac{\beta M^4\pi^4}{2}\right)R_{M,0},\\
\frac{dR_{2M,0}}{dt_2} &= -8\beta M^4\pi^4 R_{2M,0},\\
\frac{d\psi_{M,0}}{dt_2} &= \frac{2V_0}{M^2\pi^2}\sin(\psi_{M,0}) - \frac{V_0^4}{M^6\pi^6} - \frac{6V_0^2 S(M)}{M^2\pi^2} + \frac{\alpha\pi^2 X(M)}{2} - \omega_2,\\
\frac{d\phi_{2M,0}}{dt_2} &= -\left(\frac{V_0^4}{2^7 M^6\pi^6} + \frac{3V_0^2 S(2M)}{2^2 M^2\pi^2} - \frac{\alpha\pi^2 X(2M)}{4}\right).
\end{aligned} \tag{112}$$

Using the same arguments as before, it follows directly that the $R_{2M,0}$ equation can be solved, that is, $R_{2M,0} = R_0 e^{-8\beta M^4\pi^4 t_2}$, where R_0 is an initial value for $R_{2M,0}$, and by simplifying (112) further, we obtain:

$$\begin{aligned}
\frac{dR_{M,0}}{dt_2} &= -\left(\frac{V_0}{M^2\pi^2}\cos(\psi_{M,0}) + \frac{\beta M^4\pi^4}{2}\right)R_{M,0},\\
\frac{d\psi_{M,0}}{dt_2} &= (D-\omega_2) + \frac{2V_0}{M^2\pi^2}\sin(\psi_{M,0})\\
&\quad + \frac{3}{8}\alpha M^2\pi^2 R_{M,0}^2 + \alpha M^2\pi^2 R_0^2 e^{-16\beta M^4\pi^4 t_2}.
\end{aligned} \tag{113}$$

Here, the phase portraits (see Fig. 11) are qualitatively similar to the ones when only the excited mode $k = M$ is present. If we look carefully, system (113) is similar to system (107) with one additional term $\alpha M^2\pi^2 R_0^2 e^{-16\beta M^4\pi^4 t_2}$, which is decreasing exponentially to zero.

5 Conclusions and remarks

In this paper, the oscillations of a simply supported microbeam which is actuated by a DC and AC electric load have been studied. In the first part of the paper, we looked at the influence of the electrostatic force without damping and nonlinear elastic force. Here we found accurate approximations of the exact solution, including the solution to the first super- and subharmonic resonance cases on time-scales of order $1/\varepsilon$ for various frequencies of the electrostatic force. We also found accurate approximations of the natural frequencies and the super- and subharmonic resonance frequencies of the actuated microbeam

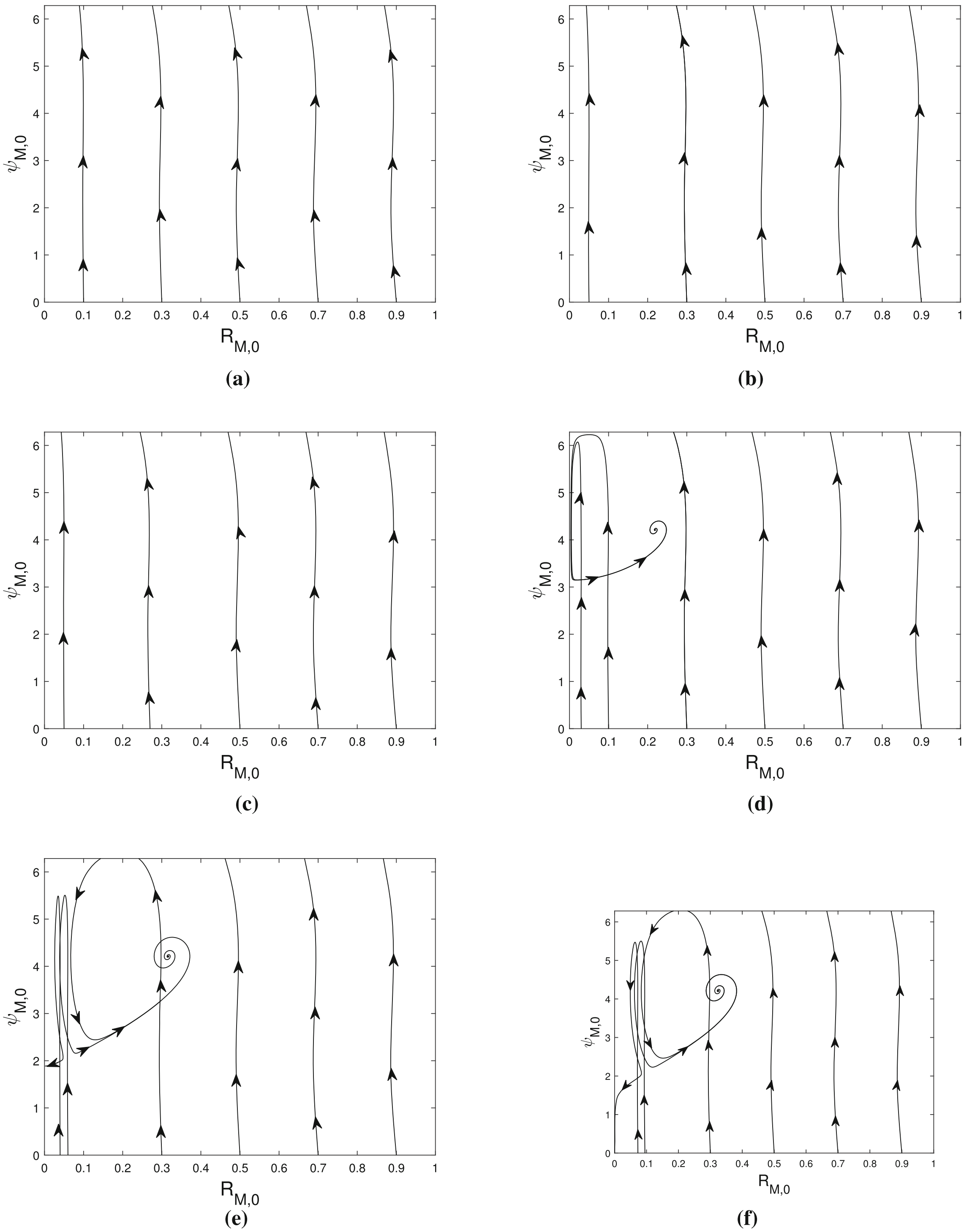


Fig. 11 The phase portrait of system (113) when $\alpha > 0$, β is relatively small, and for various values of ω_2. These phase portraits are given for the same parameter values as in Figure 7. Figure 11a to c are qualitatively the same phase portraits, unlike the ones in Fig. 7a to c

up to order ε^3. It is interesting to see that the subharmonic resonance frequency occurs for values in a certain interval (compared to a single value which we usually encounter for resonance frequencies).

In the second part of the paper, we considered similar models including viscous damping of order ε and models including structural damping and nonlinear elastic forces of order ε^2. Two cases have been considered in detail. In the first case, viscous damping is present but no elastic forces are assumed to be present. We saw that, although the frequency of the electrostatic force is set order ε^2 close to the eigenfrequency of the actuated microbeam, the order ε damping already stabilizes the solution. The second case to be considered was when the viscous damping is changed to a relatively smaller structural damping with additional elastic forces. We studied two special subcases, the superharmonic and subharmonic cases. For the superharmonic and subharmonic cases, we found that the solutions are always bounded if damping is quite large. In the superharmonic case, when the nonlinear elastic force and the damping are not present, we can have unbounded solutions for certain values of ω_2, which coincide with the resonance frequencies. For all other cases, the solution is always bounded. For the subharmonic case, we found that when the nonlinear elastic force is not present and a relatively small damping is present, we can have unbounded solutions for certain frequencies as in the Case 1.3. When the nonlinear elastic force is present, all solutions are bounded also.

The analysis in this paper shows that for this actuated beam problem with simply supported end conditions, truncation is allowed. But, one still has to consider the sub- and superharmonic case to understand fully that this is allowed.

For future work, different assumptions on how small or how large the model parameters are, will lead to different models. Moreover, other boundary conditions can be applied and a tensile axial force can be included in the model. Extending the model to two-dimensional cases (for instance, rectangular or circular domains) are also other options to proceed, and to apply the presented approach given in this paper. Those changes will surely add more complexity to the model analysis.

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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