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On solving wave equations on fixed bounded intervals involving Robin boundary conditions with time-dependent coefficients

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A B S T R A C T

In this paper, it is shown how characteristic coordinates, or equivalently how the well-known formula of d'Alembert, can be used to solve initial-boundary value problems for wave equations on fixed, bounded intervals involving Robin type of boundary conditions with time-dependent coefficients. A Robin boundary condition is a condition that specifies a linear combination of the dependent variable and its first order space-derivative on a boundary of the interval. Analytical methods, such as the method of separation of variables (SOV) or the Laplace transform method, are not applicable to those types of problems. The obtained analytical results by applying the proposed method, are in complete agreement with those obtained by using the numerical, finite difference method. For problems with time-independent coefficients in the Robin boundary condition(s), the results of the proposed method also completely agree with those as for instance obtained by the method of separation of variables, or by the finite difference method.

1. Introduction

The study of one-dimensional wave equations goes back to the middle of the 18th century when d'Alembert solved in Refs. [1–3], an initial value problem on an infinite interval (that is, on $-\infty < x < \infty$) by using characteristic coordinates. The formula for the solution of this problem is nowadays well-known, bears the name of d'Alembert, and can be found in all elementary books on partial differential equations.

This classical formula of d'Alembert can also be used to solve an initial value problem for a wave equation on a semi-infinite interval (that is, for instance on $0 < x < \infty$). For a Dirichlet type of boundary condition at $x = 0$ (that is, a condition for which the dependent variable is specified at $x = 0$), or for a Neumann type of boundary condition at $x = 0$ (that is, a condition for which the first order space derivative of the dependent variable is specified at $x = 0$), it is also well-known that the functions in the classical formula of d'Alembert should be extended as odd, or as even functions in x, respectively. A Robin boundary condition at $x = 0$ is a condition that specifies a linear combination of the dependent variable and its first order x-derivative on the boundary $x = 0$. How the functions should be extended for a Robin type of boundary condition (with

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constant coefficients) at $x = 0$, is less well-known, but it was already discovered at the end of the 19th century by Bryan in Ref. [4], and was also described in the Russian literature around the 1960s in Ref. [5]. Recently, in Ref. [6] this extension procedure for problems with a Robin type of boundary condition (with constant coefficients) at $x = 0$ was further formalized. Even more recently, in Refs. [7,8] the extension procedures on semi-infinite intervals for problems with a mass-spring-damper boundary condition at $x = 0$, were presented for a string equation and for an axially moving string equation, respectively.

On a bounded interval (that is, for instance on $0 < x < L < \infty$) the classical formula of d'Alembert can also be used to solve an initial value problem for a wave equation. In the literature only the cases where one has Dirichlet and/or Neumann boundary conditions, are solved by using the formula of d'Alembert, and leads to odd and/or even periodic extensions of the functions in the formula of d'Alembert. For other boundary conditions the formula of d'Alembert is not used, most likely, because it is not (well) known how to extend the functions in the formula of d'Alembert for other boundary conditions than those of Dirichlet type or of Neumann type. The reader is also referred to the papers [9,10] in which characteristic coordinates are used to solve problems for axially moving strings with Dirichlet and/or Neumann boundary conditions at the endpoints of the string. Usually the method of separation of variables (SOV), or the (equivalent) Laplace transform method is used to solve initial value problem for a wave equation on a bounded interval for various types of boundary conditions with constant coefficients. However, when a Robin boundary condition with a time-dependent coefficient is involved in the problem, then the aforementioned methods are not applicable. In this paper it will be shown how characteristic coordinates or equivalently, how the classical formula of d'Alembert can be used to solve an initial value problem for a wave equation on a bounded, fixed interval with at one endpoint a Dirichlet type of boundary condition, and at the other end a Robin type of boundary condition with a time-dependent coefficient. The Robin boundary condition with a time-dependent coefficient is an interesting one to study from the applicational (and from the mathematical) point of view. When one considers the transversal vibrations of a string which at one end is attached to a spring for which the stiffness properties change in time (due to fatigue, temperature change, and so on), then a Robin type of boundary condition is obtained with a time-varying coefficient. But also in the study of longitudinal vibrations of axially moving strings with time-varying lengths (as simple models for vibrations of elevator or mining cables), one obtains, after some transformations as a first order approximation of the problem, a wave equation for which at one end a Robin type of boundary condition with a time-varying coefficient has to be satisfied. The reader is referred to the papers [11–18] for further information on initial-boundary value problems for axially moving continua. Also in other fields of application the Robin boundary condition plays an important role and is sometimes called an impedance boundary condition in electromagnetic problems or a convective boundary condition in heat transfer problems.

The objective of this paper is to show how characteristic coordinates can be used to solve analytically an initial-boundary value problem for a wave equation on a bounded, fixed interval involving Robin types of boundary conditions with time-dependent coefficients. For these types of problems, no analytical solutions are yet available, and are presented in this paper (to the authors' knowledge) for the first time in the literature. This paper is organized as follows. In section 2 of this paper, the problem is formulated and it is shown shortly why the method of separation of variables cannot be used to these wave problems on a fixed, bounded interval involving at one endpoint a Robin type of boundary condition with a time-dependent coefficient and at the other endpoint a Dirichlet type of boundary condition. In section 3, the problem as formulated in section 2 will be solved by using the formula of d'Alembert. For some simple examples, the solutions constructed by the method as presented in section 3 will be compared in section 4 with numerical approximations, and (when applicable, that is, for time-independent coefficients in the Robin boundary condition) with solutions found by the method of SOV. Finally, in section 5 some conclusions will be drawn, and some remarks will be made about future research.

2. Statement of the problem

The governing equation of the transversal vibration of a string as shown in Fig. 1 can be derived by using Hamilton's principle (see, for instance [12,17])

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} - P \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad 0 < x < L, \;\; t \geq 0, \tag{1}$$

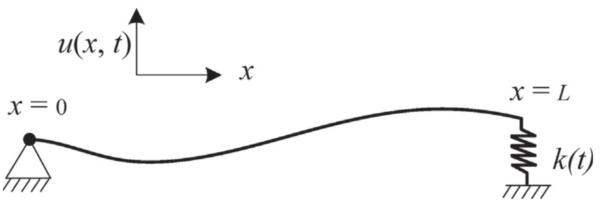

Fig. 1. The transverse vibrating string with a time-varying spring-stiffness support at $x = L$.

where ρ is the cable mass per unit length, P is the axial tension which is assumed to be constant, L is the distance between the supports, and $u(x,t)$ is the lateral displacement of a cable particle at position x at time t. The boundary conditions are given by

$$\begin{cases} u(0,t)=0 \\ Pu_x(L,t)+k(t)u(L,t)=0 \end{cases},\quad t\geq 0, \tag{2}$$

where $k(t)$ is the time-varying stiffness of the spring attached to the string at $x=L$. The boundary condition at $x=0$ is a Dirichlet type of boundary condition, and the boundary condition at $x=L$ is a Robin type of boundary condition with a time-varying coefficient $k(t)$.

Based on the Buckingham theorem, the following dimensionless quantities can be obtained to transform the governing equation (1) and the boundary conditions (2) to a non-dimensional form

$$\bar{x}=\frac{x}{L},\ \ \bar{u}=\frac{u}{L},\ \ \bar{t}=\frac{t}{L}\sqrt{\frac{P}{\rho}},\ \ \bar{k}=\frac{kL}{P}, \tag{3}$$

and hence, Eqs. (1) and (2) can be expressed as

$$u_{tt}-u_{xx}=0,\quad 0<x<1,\ \ t\geq 0, \tag{4}$$

$$\begin{cases} u(0,t)=0 \\ u_x(1,t)+k(t)u(1,t)=0 \end{cases},\quad t\geq 0. \tag{5}$$

where the overbar notations are omitted for convenience. The initial conditions for the string are assumed to be

$$\begin{cases} u(x,0)=f(x) \\ u_t(x,0)=g(x) \end{cases},\quad 0\leq x\leq 1, \tag{6}$$

When the method of separation of variables is used to find a nontrivial solution of Eqs. (4)–(5), it is assumed that there exist solutions in the form

$$u(x,t)=X(x)T(t). \tag{7}$$

Substituting (7) into the second boundary condition in Eq. (5) one obtains

$$\left.\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right|_{x=1}+k(t)X(1)=0. \tag{8}$$

This implies that $k(t)$ is time-independent, which contradicts to the fact that $k(t)$ is time-dependent. Thus, the method of separation of variables is not applicable to the problem, that is, the problem under consideration does not admit solutions in the form (7).

3. The analytical solution based on d'Alembert's method

According to the method of d'Alembert, the general solution to Eq. (4) and Eq. (6) is given by

$$u(x,t)=\frac{1}{2}[f(x-t)+f(x+t)]+\frac{1}{2}\int_{x-t}^{x+t}g(s)\mathrm{d}s. \tag{9}$$

It should be noted that the functions f and g are defined only when their arguments are in between 0 and 1, because $f(x)$ and $g(x)$ are only defined on the interval $[0,1]$. To obtain f and g on the domain outside $[0,1]$, the boundary conditions should be used. By substituting Eq. (9) into the boundary conditions (5), one obtains

$$f(t)+f(-t)+\int_{-t}^{t}g(s)\mathrm{d}s=0, \tag{10}$$

$$f'(1+t)+f'(1-t)+g(1+t)-g(1-t)+k(t)\left[f(1+t)+f(1-t)+\int_{1-t}^{1+t}g(s)\mathrm{d}s\right]=0, \tag{11}$$

where f and g are independent functions, because these functions represent the initial displacement and the initial velocity, respectively (and these physical quantities can be chosen independently). Once f and g have been determined completely outside the domain [0, 1], it follows from (9) that the solution of the initial-boundary value problem (4)–(6) has been constructed for all $t \geq 0$ and $0 \leq x \leq 1$. In the next two subsections it will be shown how f and g can be determined completely outside the domain [0, 1].

3.1. Extension of the function f

For $g \equiv 0$, Eqs. (10) and (11) become

$$f(t) + f(-t) = 0, \quad 0 \leq t \leq 1, \tag{12}$$

$$f'(1+t) + k(t)f(1+t) = -[f'(1-t) + k(t)f(1-t)], \quad 0 \leq 1-t \leq 1. \tag{13}$$

Eq. (12) implies that

$$f(t) = -f(-t), -1 \leq t \leq 0, \tag{14}$$

which defines f on the interval $[-1, 0]$. So, $f(x)$ is now defined on the interval$[-1, 1]$.

From Eq. (13), it now follows that the functions in the right hand side are defined and known for $0 \leq t \leq 2$. Then by solving this equation for the still unknown function $f(1+t)$ for $0 \leq t \leq 2$, one can determine f on the interval [1,3]. Let$y(t) = f(t+1)$, then Eq. (13) can be transformed into

$$y'(t) + k(t)y(t) = -[f'(1-t) + k(t)f(1-t)]. \tag{15}$$

Multiply both sides of (15) by the integrating factor $e^{\int_0^t k(s)ds}$, and then Eq. (15) can be rewritten in

$$\frac{d\left(e^{\int_0^t k(s)ds} y(t)\right)}{dt} = -e^{\int_0^t k(s)ds}[f'(1-t) + k(t)f(1-t)]. \tag{16}$$

Integrating (16) with respect to t from t is 0 to t, yields

$$e^{\int_0^t k(s)ds} y(t) - y(0) = \int_0^t -e^{\int_0^\tau k(s)ds}[f'(1-\tau) + k(\tau)f(1-\tau)]d\tau. \tag{17}$$

From (17) and from $y(t) = f(t+1)$ for $0 \leq t \leq 2$ it follows that

$$f(t+1) = e^{-\int_0^t k(s)ds} f(1) + e^{-\int_0^t k(s)ds} \int_0^t -e^{\int_0^\tau k(s)ds}[f'(1-\tau) + k(\tau)f(1-\tau)]d\tau. \tag{18}$$

And so, the function f is defined on the interval [1,3]. By Eq. (14) f is now defined on [-1, 3].

Let $f_{[i,j]}(x)$ denote the expression of f on the interval $[i, j]$. By using the boundary condition at $x = 0$, and by using Eq. (18), the expression for f on the interval $[-3, -1]$ can then be derived and is given by

$$f_{[-3,-1]}(t) = -f_{[1,3]}(-t), \quad -3 \leq t \leq -1. \tag{19}$$

Again, by using Eq. (19) and the boundary condition at $x = 1$, we can find the expression for f on the interval [3,5]. By repeating this extension procedure over and over again, the expression for $f(t)$ can then be found for all t with $-\infty \leq t \leq \infty$.

3.2. Extension of the function g

Let $f\equiv 0$, Eqs (10) and (11) become

$$\int_{-t}^{t} g(s)\mathrm{d}s = 0, \tag{20}$$

$$g(1+t) - g(1-t) + k(t)\int_{1-t}^{1+t} g(s)\mathrm{d}s = 0. \tag{21}$$

It readily follows from (20) that $g(t)$ should be extended as an odd function with respect to its argument at zero, that is, $g(-t) = -g(t)$. From Eq. (21) it follows that

$$g(1+t) + k(t)\int_{1}^{1+t} g(s)\mathrm{d}s = g(1-t) + k(t)\int_{1}^{1-t} g(s)\mathrm{d}s. \tag{22}$$

Obviously, the right-hand side is defined for $0 \le t \le 2$. By putting $y(t) = g(t+1)$, we again obtain an ODE for $y(t)$ (which is similar to Eq. (15)). Following the same extension procedure as for the function $f(t)$, the function $g(t)$ can then be found for all t with $-\infty \le t \le \infty$.

3.3. Wave reflections

As for the wave equation given by Eq. (4), the wave travelling speed is 1, which implies that the vibration information at the point $x = x_i$ (with $0 \le x_i \le 1$) will travel into two directions with speed 1, and at $t = 2$ the information will be back to the position x_i, as shown in Fig. 2. Thus, if we treat the status of the string at $t = 2$ as a new initial condition, we can then copy the extension steps as presented for the time-interval [0, 2] for the next time interval of length 2, that is, for $2 \le t \le 4$.

From the solution obtained by the d'Alembert method, the vibration data between $[x - t, x + t]$ is needed to derive the response of the particle at position x and at time t. Fig. 3 shows the domain of dependence, from which we can see that the information from $[-2, 3]$ is needed to determine the response of the whole string at t is 2. Then, by treating the state at $t = 2$ as a new initial condition and by using the same extension procedures (as presented for t from $t = 0$ to 2), the information that is needed to calculate the solution of the equation up to time $t = 4$ can be obtained.

By dividing the time domain into finite intervals of length 2, and by letting $f_{k,[i,j]}$ and $g_{k,[i,j]}$ be the "initial" functions defined on the space domain $[i, j]$ and for time $t = 2(k-1)$ with $k = 1, 2, 3, \cdots$, the initial condition extension relations are shown in Fig. 4.

For example, the responses of the string at time is $t = 5$ can be expressed as follows

$$\begin{aligned} u(x,5) &= \frac{1}{2}\left[f_{3,[-1,0]}(x-1) + f_{3,[1,2]}(x+1)\right] \\ &+\frac{1}{2}\left[\int_{x-1}^{0} g_{3,[-1,0]}(s)\mathrm{d}s + \int_{0}^{1} g_{3,[0,1]}(s)\mathrm{d}s + \int_{1}^{x+1} g_{3,[1,2]}(s)\mathrm{d}s\right]. \end{aligned} \tag{23}$$

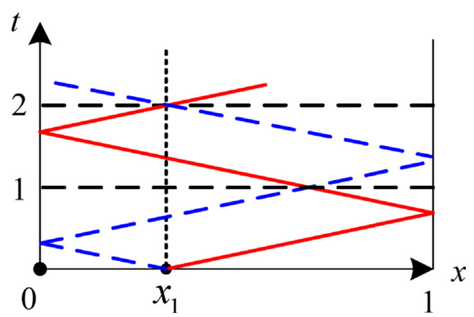

Fig. 2. Wave propagation and reflections.

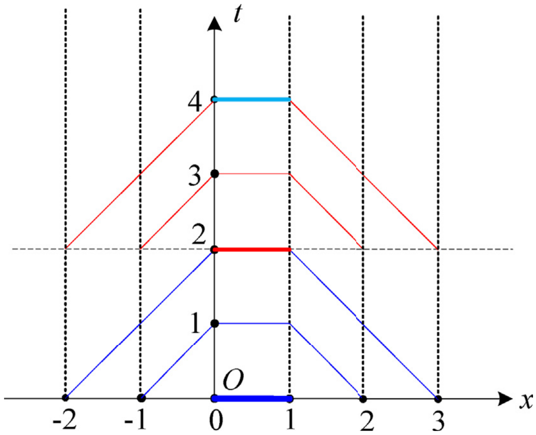

Fig. 3. Domain of dependence.

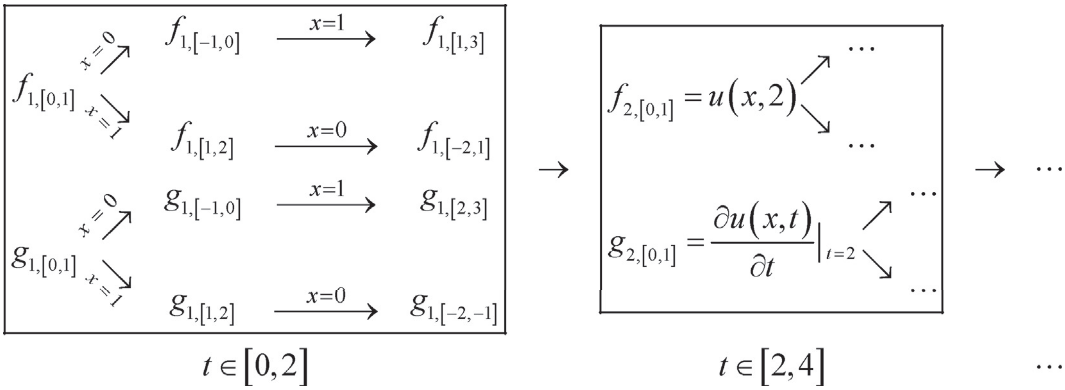

Fig. 4. Initial condition extension relations.

3.4. Numerical approximation

Here we introduce a uniform mesh Δx, a constant discretization time Δt, and a rectangular mesh consisting of points (x_i, t_j) with

$$x_i = i\Delta x, \quad t_j = j\Delta t, \tag{24}$$

where $i = 1, 2, 3, \cdots, N, j = 1, 2, \cdots$, with $N\Delta x = 1$. Following the finite difference method and by the Taylor series expansion, the second order space and time derivatives can be approximated by

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{x-1}, t_j)}{(\Delta x)^2} + \ddots\left((\Delta x)^2\right), \tag{25}$$

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) \approx \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j+1})}{(\Delta t)^2} + \ddots\left((\Delta t)^2\right). \tag{26}$$

Substituting the finite difference formulae into Eq. (4), and rearranging the terms, we end up with the linear iterative system

$$u_{i,j+1} = \sigma^2 u_{i+1,j} + 2\left(1 - \sigma^2\right)u_{i,j} + \sigma^2 u_{i-1,j} - u_{i,j-1}, \;\; i = 2, 3, \cdots, n-1, j = 1, 2, \cdots, \tag{27}$$

where $u_{i,j} = u(x_i, t_j)$, $\sigma = \Delta t/\Delta x$. From the boundary condition (5) it follows that

$$u_{0,j} = 0, \tag{28}$$

$$u_{n,j} = \frac{u_{n-1,j}}{1 + k(t_j)\Delta x}. \tag{29}$$

Let

$$\mathbf{u}^{(j)} = \begin{bmatrix} u_{1,j} & u_{2,j} & \cdots & u_{n-2,j} & u_{n-1,j} \end{bmatrix}^{\mathrm{T}}, \tag{30}$$

$$\mathbf{B} = \begin{bmatrix} 2\left(1-\sigma^2\right) & \sigma^2 & & & & \\ \sigma^2 & 2\left(1-\sigma^2\right) & \sigma^2 & & & \\ & \sigma^2 & \ddots & \ddots & & \\ & & \ddots & \ddots & & \sigma^2 \\ & & & & \sigma^2 & 2\left(1-\sigma^2\right) + \dfrac{\sigma^2}{1 + k(t_j)\Delta x} \end{bmatrix}, \tag{31}$$

then the iteration process can be rewritten in the following matrix form

$$\mathbf{u}^{(j+1)} = \mathbf{B}\mathbf{u}^{(j)} - \mathbf{u}^{(j-1)}, \tag{32}$$

where initial values imply:

$$u_{i,0} = f_i = f(x_i), \quad u_{i,1} = \frac{1}{2}\sigma^2 f_{i+1} + \left(1-\sigma^2\right) f_i + \frac{1}{2}\sigma^2 f_{i-1} + \Delta t g_i. \tag{33}$$

4. Examples

4.1. Case I: k(t) is constant

In this section, we first consider $k(t)$ as a constant by letting $k(t) = 1/2$, then the method of SOV can be applied here, and the solution can be found as

$$u(x,t) = \sum_{i=1}^{\infty} \left(A_i sin(\lambda_i x)\cos(\lambda_i t) + B_i sin(\lambda_i x) sin(\lambda_i t)\right), \tag{34}$$

where λ_i satisfies the transcendental equation $\lambda_i = -k\tan(\lambda_i)$. By substituting Eq. (34) into the initial conditions and by using the well-known orthogonality properties of the eigenfunctions, the coefficients A_i and B_i can be determined, yielding

$$A_i = \frac{\int_0^1 f(x)\sin(\lambda_i x)\mathrm{d}x}{\int_0^1 \sin^2(\lambda_i x)\mathrm{d}x}, \tag{35}$$

$$B_i = \frac{\int_0^1 g(x)\sin(\lambda_i x)\mathrm{d}x}{\lambda_i \int_0^1 \sin^2(\lambda_i x)\mathrm{d}x}. \tag{36}$$

The initial conditions are assumed to be (as an example)

$$\begin{cases} f(x) = \sin^2(1.7155x) \\ g(x) = 0 \end{cases}. \tag{37}$$

By using the first 10 terms in the Fourier series (as obtained by the method of SOV), the string shapes comparison at t is 2, 3, 4 and 5 using the method as presented in this paper, the method of SOV, and the finite difference method are given in Fig. 5. In this figure it can be seen that the methods are all in good agreement.

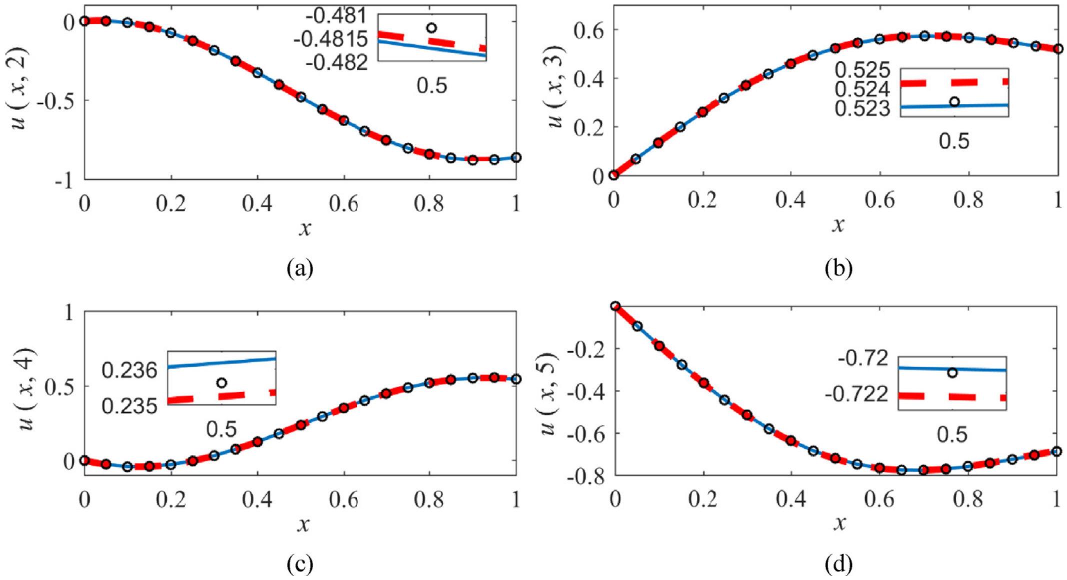

Fig. 5. String shape comparison when $k(t) = 1/2$, the proposed method (solid line), the finite difference method (dashed line) and the method of separation of variables (marked with o): (a) $u(x, 2)$, (b) $u(x, 3)$, (c) $u(x, 4)$ and (d) $u(x, 4)$.

4.2. 4.2case II: $k(t) = 1/(t + 2)$

Let $k(t)=1/(t+2)$. In this case, the method of SOV cannot be applied. For the same initial conditions (37), as in the previous case, the string wave shape comparisons between the proposed method and the finite difference method are shown in Fig. 6.

In Figs. 5 and 6, it can be seen that the proposed method agrees well with the method of SOV (when $k(t)$ is constant) and the finite difference method (in both cases). For the finite difference method, according to Eq. (29), its accuracy is $\because(\Delta x)$. And for the method of SOV, since only the first 10 terms in the Fourier series are considered, its result is also an approximation to the exact solution. In contrast, the proposed method provides the exact solution, and the minor differences between the proposed method and the two other methods can be found in the zoomed figures in Figs. 5 and 6.

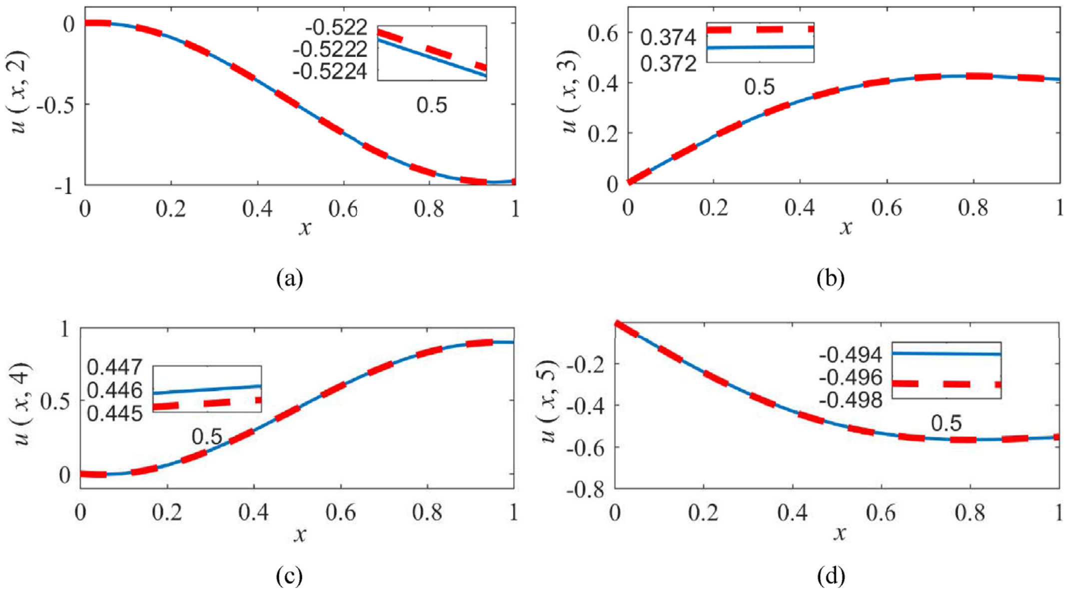

Fig. 6. String shapes comparisons when $k(t)=1/(t+2)$, the proposed method (solid line), the finite difference method (dashed line): (a) $u(x, 2)$, (b) $u(x, 3)$, (c) $u(x, 4)$ and (d) $u(x, 4)$.

5. Conclusions

In this paper, an analytical method is presented to solve wave equations on fixed, bounded intervals involving Robin type of boundary conditions with time-dependent coefficients. Based on the d'Alembert formula and on the boundary conditions, the initial conditions are extended on the whole x-domain. Taking into account the wave travelling speed and the total reflection time, the time domain is divided into smaller intervals of fixed length, so that the initial conditions extension procedure for each interval coincides with the previous ones. In this way one can obtain in a rather straightforward way an analytical expression for the solution on the time-interval $[0, 2n]$ with $n = 1, 2, 3, \ldots, N$ and N not too large. Of course one will encounter computational issues for large N. The proposed method is consistent compared with the method of SOV and the finite difference method. The presented method can also be applied to other initial-boundary value problems for PDEs like the heat equation. The proposed method also provides a way to test the accuracy of the analytical/numerical approximations. As mentioned in the introduction of this paper, the proposed method can be applied to solve the first order problem (obtained by a formal perturbation expansion) for the longitudinal vibrations of moving cables with similar boundary conditions. To construct a more accurate approximation, the next order problem, which involves terms that act as the external forces, should be solved. How to solve these nonhomogeneous problems is an interesting subject for future research, and includes the study of resonances in the problem.

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Appendix A. Supplementary data

Supplementary data related to this article can be found at https://doi.org/10.1016/j.jsv.2018.03.009.

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