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Abstract
Let F be an ordered topological vector space (over $\mathbb{R}$) whose positive cone F_+ is weakly closed, and let $E \subseteq F$ be a subspace. We prove that the set of positive continuous linear functionals on E that can be extended (positively and continuously) to F is weak-$*$ dense in the topological dual wedge E'_+. Furthermore, we show that this result cannot be generalized to arbitrary positive operators, even in finite-dimensional spaces.

Extension theorems for positive operators have been studied in great detail, and can be found in many textbooks on ordered vector spaces (e.g. [11,§V.5], [1,§1.4–1.5]). Many classical extension theorems provide sufficient conditions for a given operator to have a positive extension. These include a classical extension theorem of Kantorovich (see [1,Theorem 1.30]), and a theorem of Lotz on positive operators from a Banach sublattice of a Banach lattice to an AL-space (see [6,Proposition 3.2], [10,Theorem II.8.9]).

In general, not every positive operator can be extended, and the problem already arises when looking at functionals. In 1957, Mirkil proved that, for a finite-dimensional ordered vector space F whose positive cone F_+ is closed, all positive linear functionals on all subspaces of F can be extended if and only if F_+ is polyhedral (see [7,Corollary 1], [4,Theorem 4.13]). That same year, Bauer and Namioka independently found a necessary and sufficient condition for a positive continuous linear functional $E \rightarrow$

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$\mathbb{R}$ defined on a subspace E of a topological vector space F to have a positive and continuous extension to F (see [3,Theorem 1], [8,Theorem 4.4], [11,Theorem V.5.4]).

In this note, we look at things from a different perspective. Instead of determining whether or not a positive operator can be extended, we try to *approximate* it by extendable positive operators. We prove that the extendable positive linear functionals are weak-∗ dense in the dual wedge. Furthermore, we show that this result cannot be extended to arbitrary positive operators, even if the spaces are finite-dimensional.

Although the proofs in this note are relatively simple, the results appear to be unknown in the ordered vector spaces community. In particular, the author has not been able to locate an earlier proof (or statement) of Theorem 1 in the literature.

We follow notation and terminology from [1].

For positive functionals, we have the following positive result.

Theorem 1 *Let F be a preordered topological vector space, and let $E \subseteq F$ be a subspace. If the topological dual F' separates points on F, and if the positive wedge F_+ is weakly closed, then the set of all positive continuous linear functionals on E that can be extended positively and continuously to F is weak-∗ dense in E'_+.*

We make two remarks about the statement of Theorem 1. First, if F is locally convex, then we may replace the requirement that F_+ is weakly closed by the requirement that F_+ is closed. These two requirements are now equivalent, because F_+ is convex.

Second, if F_+ is weakly closed and if $F_+ \cap -F_+ = \{0\}$, then $\{0\}$ is also weakly closed, so the weak topology is Hausdorff. Hence, if F_+ is a cone, then the requirement that F' separates points on F is automatically met. Thus, the statement from the abstract is recovered.

Proof of Theorem 1 Define $\mathcal{R} := \{\varphi|_E : \varphi \in F'_+\} \subseteq E'_+$, and note that $\mathcal{R}$ is the wedge of positive continuous linear functionals $E \to \mathbb{R}$ that can be extended positively and continuously to F. Since F_+ is weakly closed, we have

$$F_+ = \left\{x \in F : \langle x, \varphi \rangle \geq 0 \text{ for all } \varphi \in F'_+\right\}.$$

It follows that

$$E_+ := F_+ \cap E = \left\{x \in E : \langle x, \varphi \rangle \geq 0 \text{ for all } \varphi \in F'_+\right\}.$$

This shows that E_+ is the predual wedge of $\mathcal{R}$. Hence, by the bipolar theorem, E'_+ is the weak-∗ closure of $\mathcal{R}$. ☐

A little more can be said if the subspace $E \subseteq F$ is finite-dimensional, as the following corollaries show.

Corollary 2 *Let F be a preordered topological vector space whose positive wedge F_+ is weakly closed, let $E \subseteq F$ be a finite-dimensional subspace, and let λ denote the Lebesgue (i.e. Haar) measure on E^*. Then λ-almost all positive linear functionals $E \to \mathbb{R}$ can be extended to a positive continuous linear functional $F \to \mathbb{R}$.*

Proof The set $\mathcal{R} \subseteq E_+^*$ of extendable positive linear functionals is a convex set, and it follows from Theorem 1 that $\overline{\mathcal{R}} = E_+^*$. The result follows since $E_+^* \setminus \mathcal{R} = \overline{\mathcal{R}} \setminus \mathcal{R} \subseteq \partial \mathcal{R}$, and the boundary of a convex set in Euclidean space has Lebesgue measure zero (see e.g. [5]). $\square$

Corollary 3 *Let F be a preordered topological vector space whose positive wedge F_+ is weakly closed, and let $E \subseteq F$ be a finite-dimensional subspace. Then every strictly positive linear functional on E can be extended to a positive continuous linear functional on F.*

If, in addition, F is finite-dimensional and $F_+ \cap -F_+ = \{0\}$, then the extension can be chosen to be strictly positive as well.

Proof If $E_+ \cap -E_+ \neq \{0\}$, then E does not have strictly positive linear functionals, so the statement is vacuously true. Assume henceforth that $E_+ \cap -E_+ = \{0\}$. Then E_+ is a closed cone, so the dual wedge E_+^* is generating. Therefore E_+^* has non-empty interior, and the strictly positive linear functionals on E are precisely the interior points of E_+^* (see [1,Lemma 2.17]).

By Theorem 1, the convex set $\mathcal{R}$ of extendable positive linear functionals is dense in E_+^*. It follows from finite-dimensional convex geometry that $\mathcal{R}$ and E_+^* have the same interior points (see [9,Theorem 6.3]). Hence, for every strictly positive linear functional $\varphi : E \to \mathbb{R}$, we have $\varphi \in \mathrm{int}(E_+^*) = \mathrm{int}(\mathcal{R}) \subseteq \mathcal{R}$, which shows that φ is extendable.

Assume now that F is finite-dimensional and that F_+ is a cone. Then F_+^* also has non-empty interior, and its interior points are precisely the strictly positive linear functionals on F. Now we use the fact that a linear map T between finite-dimensional spaces preserves the relative interior of every convex set M; that is, $\mathrm{relint}(T[M]) = T[\mathrm{relint}(M)]$ (see [9,Theorem 6.6]). Since $\mathcal{R}$ is the image of F_+^* under the canonical map $F^* \to F^*/E^\perp \cong E^*$, and since $\mathcal{R}$ and F_+^* are full-dimensional, it follows that every interior point of $\mathcal{R}$ is the image of an interior point of F_+^*. Equivalently, every strictly positive linear functional $E \to \mathbb{R}$ can be extended to a strictly positive linear functional $F \to \mathbb{R}$. $\square$

One might ask if Theorem 1 remains true if the positive linear functionals are replaced by positive operators. Unfortunately, this is not the case, even if the spaces are finite-dimensional, as we will now demonstrate.

A wedge $\mathcal{K} \subseteq E$ is a *simplex cone* (or *Yudin cone*) if there is an algebraic basis $\mathcal{B}$ of E such that $\mathcal{K}$ is the wedge generated by $\mathcal{B}$.

If E and G are vector spaces, and if $\varphi \in E^*$ and $z \in G$, then we write $z \otimes \varphi$ for the linear map $E \to G$, $x \mapsto \langle x, \varphi \rangle z$.

For our counterexample, we will consider the following situation.

Situation 4 Let E be a finite-dimensional vector space, and let $E_+ \subseteq E$ be a generating polyhedral cone which is *not* a simplex cone. Let $\varphi_1, \ldots, \varphi_m \in E_+^*$ be representatives of the extremal rays of E_+^*. Then $E_+ = \bigcap_{i=1}^m \{x \in E : \langle x, \varphi_i \rangle \geq 0\}$, and every positive linear functional is a positive combination of $\varphi_1, \ldots, \varphi_m$. Additionally, let $F := \mathbb{R}^m$ with the standard cone $F_+ := \mathbb{R}_{\geq 0}^m$, so that the map $T : E \to F$, $x \mapsto (\varphi_1(x), \ldots, \varphi_m(x))$ is bipositive. We will identify E with a subspace of F via this map.

Proposition 5 *In Situation 4, the positive linear maps $E \to E$ that can be extended to a positive linear map $F \to E$ are precisely the maps of the form $\sum_{i=1}^{k} x_i \otimes \psi_i$ with $x_1, \ldots, x_k \in E_+$ and $\psi_1, \ldots, \psi_k \in E_+^*$.*

Proof Let $e_1, \ldots, e_m$ denote the standard basis of $F = \mathbb{R}^m$. If $T : E \to E$ is a positive linear map that can be extended to a positive linear map $S : F \to E$, then we have

$$T(x) = S(\varphi_1(x), \ldots, \varphi_m(x)) = S(e_1)\varphi_1(x) + \cdots + S(e_m)\varphi_m(x),$$

so T can be written as $T = \sum_{j=1}^{m} S(e_j) \otimes \varphi_j$.

Conversely, suppose that $T = \sum_{i=1}^{k} x_i \otimes \psi_i$ with $x_1, \ldots, x_k \in E_+$ and $\psi_1, \ldots, \psi_k \in E_+^*$. Every ψ_i can be written as a positive combination of the $\varphi_1, \ldots, \varphi_m$, so after rearranging the terms we may write $T = \sum_{j=1}^{m} y_j \otimes \varphi_j$, where the y_j are positive combinations of the x_i. In particular, $y_1, \ldots, y_m \in E_+$. Therefore the map $S : F \to E$, $e_j \mapsto y_j$ is a positive extension of T. □

The following theorem of Barker and Loewy tells us that approximation by operators of the form described in Proposition 5 is not always possible.

Theorem 6 [Barker–Loewy, [2,Proposition 3.1]] *Let E be a finite-dimensional ordered vector space whose positive cone E_+ is closed and generating. Then the identity $\mathrm{id}_E : E \to E$ can be written as the limit of a sequence of operators of the form $\sum_{i=1}^{k} x_i \otimes \psi_i$ with $x_1, \ldots, x_k \in E_+$ and $\psi_1, \ldots, \psi_k \in E_+^*$ if and only if E_+ is a simplex cone.*

If E and F are as in Situation 4, then it follows from Proposition 5 and Theorem 6 that the identity $\mathrm{id}_E : E \to E$ cannot be approximated by positive operators that can be extended to positive operators $F \to E$.

Remark 7 Although Theorem 6 is exactly what Barker and Loewy proved and exactly what we used, we should point out that Tam later showed that the cone of operators of the form $\sum_{i=1}^{k} x_i \otimes \psi_i$ with $x_1, \ldots, x_k \in E_+$ and $\psi_1, \ldots, \psi_k \in E_+^*$ is already closed (see [12,Theorem 1]), and used this to give a simpler proof of Theorem 6 (see [12,Theorem 4]). As a result, another equivalent condition in Theorem 6 is that id_E itself can be written in this positive tensor form, rather than being a limit of operators of this form.

In Situation 4, an easier way to see that the cone generated by $\{x \otimes \psi : x \in E_+, \ \psi \in E_+^*\}$ is closed is to note that it is finitely generated. In any case, we conclude that the positive operators $E \to E$ which can be extended to a positive operator $F \to E$ form a closed cone which is strictly contained in the (closed) cone of all positive operators $E \to E$.

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