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STABILITY ANALYSIS OF THE BOUNDARY VALUE PROBLEM MODELLING A TWO-LAYER OCEAN

Kateryna Marynets

Delft Institute of Applied Mathematics
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft University of Technology
Mekelweg 4, 2628 CD Delft, The Netherlands

Abstract. We study boundedness of solutions to a linear boundary value problem (BVP) modelling a two-layer ocean with a uniform eddy viscosity in the lower layer and variable eddy viscosity in the upper layer. We analyse bounds of solutions to the given problem on the examples of different eddy viscosity profiles in the case of their parameter dependence.

1. **Introduction.** The motion of the near-surface water mass away from the wind direction is due to the *Ekman effect* (see discussion in [12]). It is caused by a balance between the Coriolis and frictional forces that are generated by the wind stress (for more details see [1, 3, 13, 15, 18, 20]).

The main characteristics of the Ekman flow (there is also a counterpart in the atmospheric boundary layer – see the discussion in the papers [6, 7, 15] – that we do not consider in the present paper) include:

- In the Northern Hemisphere one observes deflection of the surface current to the right of the dominating wind direction. In the Southern Hemisphere the flow is deflected to the left.
- The current speed decreases with depth, whereas the rotation of the water mass farther away from the wind direction forms a spiral (see Fig. 1).
- The net transport of water is at right angles to the wind direction.

In the Equatorial region, where the Coriolis force vanishes, the direction of the wind-drift current coincides with the wind direction (see discussions in [1, 5, 8, 9]).

There are already some results known in study of the Ekman flow. They are mainly focused on re-visiting of the Ekman's classic problem in terms of the predicted surface current deflection angle. The initial assumption was made by *Ekman* with respect to a constant vertical viscosity that led to the value of the deflection angle being equal to 45° (see discussion in [12]) . However, this did not correspond to the observations and thus, the model was adjusted to be more realistic.

One of the first attempts to align computational results with the field data was made by *Madsen* in [14]. He assumed a linear increase of the vertical eddy viscosity with vertical distance, from being equal to 0 at the free surface. Under this setting

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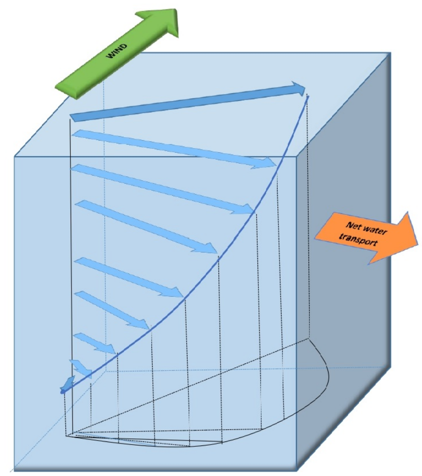

FIGURE 1. Ekman flow and the net water transport in the Northern Hemisphere

the author found that the angle between the surface drift current and the wind stress is of the order 10°, which was significantly different from the value predicted by Ekman.

For other classical results regarding the angle of deflection, we also refer to the discussions in the papers [2, 3].

Recently in [11] *Dritschel, Paldor* and *Constantin* have studied a piecewise-constant eddy viscosity profile (see Fig. 2) for a two-layer ocean of the form

$$K = \begin{cases} 1, & \text{for } z \in [-h, 0] \\ l^2, & \text{for } z \in (-\infty, -h). \end{cases}$$

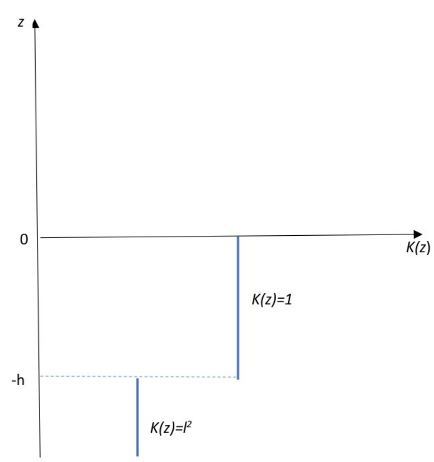

FIGURE 2. Piecewise-constant eddy viscosity profile for a two-layer ocean

This allowed them to construct a solution of the studied problem and to answer the question of dependence between the deflection angle and the eddy viscosity profile.

This idea was extended by *Roberti* in [16, 17]. In particular, in [16] he studied a three-layer flow with a piecewise-constant density (see Fig. 3)

$$K = \begin{cases} 1, & \text{for } z \in [-h_1, 0], \\ l_1^2, & \text{for } z \in [-h_2, -h_1), \\ l_2^2, & \text{for } z \in (-\infty, -h_2), \end{cases}$$

for $l_1, l_2 \geq 0$.

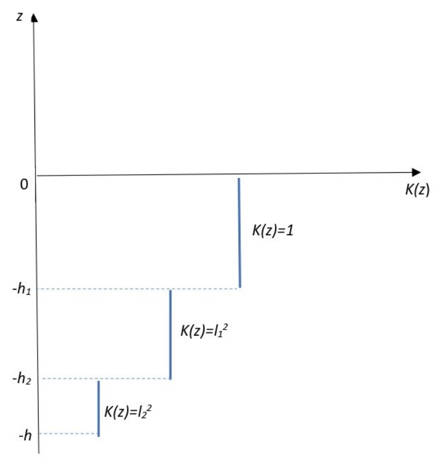

FIGURE 3. Piecewise-constant eddy viscosity profile for a three-layer ocean

Finally, in [4] *Constantin*, *Dritschel* and *Paldor* suggested to look into a depth dependent vertical density profile, where the eddy viscosity between the free surface and a certain depth $-h$ is a positive function of z and below the depth $-h$ it is equal to the eddy viscosity of the sea water (see Fig. 4).

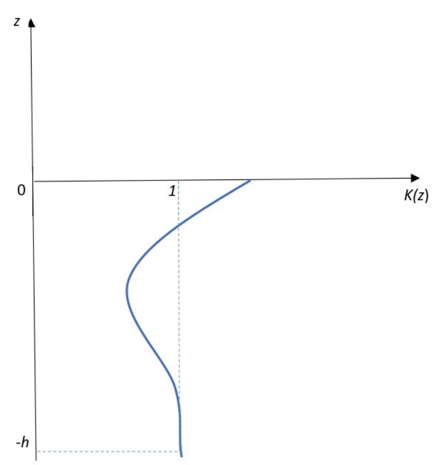

FIGURE 4. Piecewise-constant generalized eddy viscosity profile for a two-layer ocean

In other words,

$$K(z) = \begin{cases} l(z), & \text{for } z \in [-h, 0] \\ 1, & \text{for } z \in (-\infty, -h), \end{cases} \tag{1.1}$$

where $l(z) > 0$, for $z \in [-h, 0]$.

In this paper we investigate stability of solutions of the mathematical model of a two-layer ocean away from the Equator, where the density profile is given by 1.1.

The outline of the paper is the following. In *Section 2* we give a mathematical model of a two-layer ocean in terms of physical and non-dimensional variables. *Section 3* contains a detailed and robust analysis based on a logarithmic matrix norm. And last but not least, *Section 4* is devoted to analysis of three different parameter-dependent eddy viscosity profiles and computation of bounds of solutions to the corresponding problems.

2. **Mathematical model of a two-layer ocean.** According to [11] the horizontal momentum equation for a steady flow of a vertically homogeneous ocean of infinite depth under the f-plane approximation is given by

$$if\mathbf{U} = \frac{1}{\rho}\frac{\partial \boldsymbol{\tau}}{\partial Z} - \frac{1}{\rho}\nabla\mathbf{P} + \underline{\textit{higher order terms}}. \tag{2.1}$$

Here

- $\mathbf{U}(Z) = U(Z) + iV(Z)$ – complex horizontal velocity in the (X, Y) plane;
- Z – depth below the mean surface $Z = 0$;
- f – Coriolis parameter;
- ρ – water density;
- $\nabla\mathbf{P} = \frac{\partial P}{\partial X} + i\frac{\partial P}{\partial Y}$ – horizontal pressure gradient;
- $\boldsymbol{\tau}(Z) = \tau_x + i\tau_y$ – shear stress caused by molecular and turbulent processes.

The *higher order terms* in the equation 2.1 stand for interactions between variables and are presumed to be small.

Decomposition of the horizontal velocity into geostrophic (meaning that the force balance is between the Coriolis force and horizontal pressure gradient forces) and Ekman components

$$\mathbf{U} = \mathbf{U}_g + \mathbf{U}_e$$

results in separation of the correspondent flows, with the linear equation

$$if\mathbf{U}_e = \frac{1}{\rho}\frac{\partial \boldsymbol{\tau}}{\partial Z} \tag{2.2}$$

that governs dynamics of the wind-driven flow. Using the relation

$$\boldsymbol{\tau} = \rho\nu\frac{\partial \mathbf{U}_e}{\partial Z}$$

between the stress vector $\boldsymbol{\tau}$ within the fluid and the shear profile, with $\nu(Z)$ being a turbulent eddy viscosity coefficient, the Ekman's equations 2.2 for wind-driven ocean currents can be written in the form [11]:

$$if\mathbf{U}_e = \frac{\partial}{\partial Z}\left(\nu\frac{\partial \mathbf{U}_e}{\partial Z}\right). \tag{2.3}$$

Solution of the differential equation 2.3 is constrained by two boundary conditions:

$$\boldsymbol{\tau}_0 = \rho\nu\frac{\partial \mathbf{U}_e}{\partial Z} \quad \text{on} \quad Z = 0 \tag{2.4}$$

meaning that at the surface the shear stress balances the wind stress, and

$$\mathbf{U}_e \to 0 \quad \text{as} \quad Z \to -\infty \tag{2.5}$$

is due to vanishing of the wind-driven current with depth. Condition 2.5 is motivated by keeping the total kinetic energy finite.

If τ_0 denotes the magnitude of the surface wind stress, we can non-dimentionalize the boundary value problem (BVP) 2.3-2.5 by introducing the dimensionless variables:

- $K = \frac{f\rho\nu}{\tau_0}$ – eddy viscosity;
- $\mathbf{u} = \frac{\mathbf{U}_e}{\sqrt{2\tau_0/\rho}}$ – velocity;
- $z = \frac{Zf}{\sqrt{2\boldsymbol{\tau}_0/\rho}}$.

Then the non-dimensional linear governing equation for steady wind-driven ocean currents in mid-latitudes is

$$[K(z)\psi'(z)]' - 2i\psi(z) = 0, \text{ for } z < 0, \tag{2.6}$$

coupled with the corresponding boundary conditions

$$\psi'(0) = 1 \text{ on } z = 0, \tag{2.7}$$

$$\psi \to 0 \text{ as } z \to -\infty, \tag{2.8}$$

with $\psi = \mathbf{u}K(0)$ (see [4]).
Here

- the complex vector
$$\psi(z) = u(z) + iv(z) \tag{2.9}$$
represents the horizontal velocity field and
- $K(z)$ is the vertical depth-dependent nondimensional eddy viscosity.

Since the turbulence is practically confined to a near-surface ocean layer it is reasonable to assume, that below a certain depth h the eddy viscosity is equal to the eddy viscosity of sea water, normalized such that

$$K(z) = 1, \text{for } z < -h \tag{2.10}$$

with $K(z) > 0$ and not constant for $z \in (-h, 0]$ unconstrained, other than by a continuous dependence on z (see discussion in [4]).

3. **Boundedness of solutions.** Let us split the study of solutions to the BVP 2.6–2.8 into 2 problems. First, we look at the solutions to 2.6 for $z \in (-\infty, -h)$ and thus for a constant viscosity profile $K(z) = 1$. Secondly, we analyse the same equation 2.6, but in the case when $z \in [-h, 0]$ and the function $K = K(z)$ being already depth dependent. Additionally, we require an "agreement" condition for the eddy viscosity that reads $K(-h) = 1$.

Assume that $z \in (-\infty, -h)$ and the eddy viscosity satisfies condition 2.10. Then the differential equation 2.6 simplifies to the form of a linear autonomous second order ODE and reads as:

$$\psi''(z) - 2i\psi(z) = 0, \text{ for } z \in (-\infty, -h). \tag{3.1}$$

Additionally, function $\psi(z)$ satisfies the asymptotic condition 2.8.

In [4] the authors have obtained an explicit solution to 3.1 given by

$$\psi(z) = C_1 e^{(1+i)z} + C_2 e^{-(1+i)z}. \tag{3.2}$$

The asymptotic condition at $-\infty$ leads to the solution

$$\psi(z) = C_1 e^{(1+i)z}, \tag{3.3}$$

dependent on the arbitrary complex constant $C_1 < \infty$. Thus, for $z \in (-\infty, -h)$ the function $\psi(z)$ defined by 3.3 stays bounded.

Now let us analyse a solution to the differential equation 2.6 on the interval $[-h, 0]$ by rewriting it in the form:

$$(K(u+iv)')' - 2i(u+iv) = 0, \text{ for } z < 0.$$

Splitting up the real and imaginary parts yields to a system of second order differential equations

$$\begin{cases} (Ku')' = -2v, \\ (Kv')' = 2u, \end{cases} \tag{3.4}$$

subjected to the following initial conditions:

$$u'(0) = 1, \ v'(0) = 0.$$

Using the change of variables

$$\begin{aligned} u(z) &= x_1(z), \\ u'(z) &= x_2(z), \\ v(z) &= x_3(z), \\ v'(z) &= x_4(z) \end{aligned}$$

and recalling that $K(z)$ does not vanish for all $z \in (-\infty, 0]$, we reduce the order of the differential equations in 3.4 and thus, obtain a system of 4 first-order differential equations:

$$\begin{cases} x_1' = x_2, \\ x_2' = -\frac{K'}{K}x_2 - \frac{2}{K}x_3, \\ x_3' = x_4, \\ x_4' = \frac{2}{K}x_1 - \frac{K'}{K}x_4. \end{cases} \tag{3.5}$$

Note, that the system 3.5 represents a non-autonomous linear dynamical system that can be written in a matrix-vector form

$$x' = A(z)x, \tag{3.6}$$

where $x = (x_1, \ldots, x_4)^T$, $x' = (x_1', \ldots, x_4')^T$ and

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{K'}{K}(z) & -\frac{2}{K}(z) & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2}{K}(z) & 0 & 0 & -\frac{K'}{K}(z) \end{pmatrix}. \tag{3.7}$$

To proceed with the analysis of the system 3.6 we associate with the matrix-function $A(z)$ a *logarithmic matrix norm*:

$$\mu(A) = \lim_{\theta \to 0^+} \frac{|I + \theta A| - 1}{\theta}, \tag{3.8}$$

where I is the identity matrix and $\theta \in \mathbb{R}^+$ [10]. Function $\mu(\cdot)$ has the following properties [19]:

(i) $|\mu(A)| \leq ||A||$;
(ii) $\mu(A+B) \leq \mu(A) + \mu(B)$;
(iii)$Re(\lambda) \leq \mu(A)$;
(iv) $||e^{Ax}|| \leq e^{\mu(A)x}$.

Remark 1. *Even though in the literature function $\mu(\cdot)$ is sometimes referred to as a logarithmic norm, it is not an actual norm, since it can attain a negative sign.*

The result of Theorem 3 in [10] ensures, that if $y(t)$ is a solution to the system

$$y' = B(t)y, \tag{3.9}$$

then for all $t \geq t_0$ the inequality holds:

$$||y(t)|| \leq ||y(t_0)|| \exp\left(\int_{t_0}^{t} \mu[B(s)]ds\right). \tag{3.10}$$

Remark 2. *Note, that the estimate 3.10 holds for any logarithmic matrix norm $\mu(\cdot)$ as long as it is defined according to 3.8. For possible values of function $\mu(\cdot)$ we refer the reader to* [10].

To apply this result to the differential system 3.6 let us transform it into the form 3.9 using the substitution:

$$t = -z, \quad y(t) = x(-t) = x(z). \tag{3.11}$$

Then $y'(t) = -x'(z)$ and $B(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{K'}{K}(-t) & -\frac{2}{K}(-t) & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2}{K}(-t) & 0 & 0 & \frac{K'}{K}(-t) \end{pmatrix}$.

Thus, we obtain the system:

$$-y'(t) = B(t)y(t)$$

or in other words,

$$y'(t) = \tilde{B}(t)y(t), \tag{3.12}$$

where $t \in [0, h]$,

$$\tilde{B}(t) = -B(t) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -\frac{K'}{K}(-t) & \frac{2}{K}(-t) & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{2}{K}(-t) & 0 & 0 & -\frac{K'}{K}(-t) \end{pmatrix}. \tag{3.13}$$

To proceed with the analysis, let us fix the value of the logarithmic matrix norm to the one being given by

$$\mu[A] = \sup_k \left(\Re a_{kk} + \sum_{i, i\neq k} |a_{ik}|\right), \tag{3.14}$$

and calculate it for the matrix $\tilde{B}(t)$ in the differential system 3.12:

$$\mu[\tilde{B}(t)] = \max_t \left\{1, \frac{1}{K(-t)}[2 - K'(-t)]\right\}. \tag{3.15}$$

To investigate the logarithmic matrix norm $\mu[\tilde{B}(t)]$ in 3.15 we consider two cases:
I. Assume, that

$$\frac{1}{K(-t)}[2 - K'(-t)] \leq 1.$$

Then

$$\mu[\tilde{B}(t)] = 1$$

and

$$\int_0^t \mu[\tilde{B}(t)]ds = t \leq h < \infty.$$

II. Let

$$\frac{1}{K(-t)}[2 - K'(-t))] > 1,$$

then

$$\mu[\tilde{B}(t)] = \frac{1}{K(-t)}[2 - K'(-t)]$$

and

$$\begin{aligned}\int_0^t \mu[\tilde{B}(s)]ds &= \int_0^{-t} \frac{1}{K(s)}[2 - K'(s)]ds \\ &= -\int_{-t}^0 \frac{2}{K(s)}ds + \ln\left(\frac{K(-t)}{K(0)}\right) := K_h < \infty,\end{aligned}$$

$\forall t \in [0, h]$ and K being a continuous function defined on a bounded interval.

Thus, in view of 3.10 we conclude that the solution to the system 3.12 is also bounded on the interval $[0, h]$. Since the systems 3.12 and 3.6 are equivalent under substitution 3.11, the solution $x(z)$ to 3.6 is also bounded for $z \in [-h, 0]$.

4. Examples.

In this section we analyse linear and nonlinear parameter-dependent eddy viscosity profiles and compute the upper bounds of the norm of solutions with respect to the bifurcations of the logarithmic matrix norm $\mu[A(z)]$ for variable values of the unknown parameters.

Aligned with the analysis provided in the *Section 3* of the paper, we will only look at the upper ocean layer (up to the depth $-h$), since beneath this depth it was already shown that the solution is bounded.

4.1. Linear viscosity profile.

Assume, that the eddy viscosity profile is given by a linear function:

$$K(z, \lambda) = \lambda + \frac{\lambda - 1}{h}z, z \in [-h, 0], \tag{4.1}$$

where $\lambda > 0$ is some scalar parameter.

Note, that the value of parameter λ determines, whether the eddy viscosity is increasing ($0 < \lambda < 1$) or decreasing ($\lambda > 1$) with the ocean depth.

For a parameter-dependent eddy viscosity the logarithmic matrix norm $\mu[\cdot]$, defined by 3.14, 3.15, will also depend on a parameter λ. We will denote it by $\mu_\lambda[\cdot]$.

We want to investigate, if and how the ocean depth depends on the parameter λ and what is the bound of the norm of the solution to the differential system 3.6, 3.7.

Using the given eddy viscosity profile 4.1, it is easy to check that the matrix $\tilde{B}(t)$, defined via 3.13 for the associated system 3.12, is given by

$$\tilde{B}(t) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -\frac{\lambda-1}{\lambda h-(\lambda-1)t} & \frac{2h}{\lambda h-(\lambda-1)t} & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{2h}{\lambda h-(\lambda-1)t} & 0 & 0 & -\frac{\lambda-1}{\lambda h-(\lambda-1)t} \end{pmatrix},$$

where $t = -z \in [0, h]$ is a new independent variable.

For this particular matrix $\tilde{B}(t)$ the logarithmic matrix norm $\mu_h[\tilde{B}(t)]$, calculated by 3.14, 3.15, is

$$\mu_\lambda[\tilde{B}(t)] = \max_t \left\{1, \frac{2h-(\lambda-1)}{\lambda h-(\lambda-1)t}\right\} = \frac{2h-(\lambda-1)}{\lambda h-(\lambda-1)t} \geq 1, \tag{4.2}$$

for all $t \in [0,h]$ and $\lambda > 0$.

To evaluate the last inequality in 4.2 we consider two cases:

1. *Increasing eddy viscosity profile*: $0 < \lambda < 1$.

   In this setting and under condition 4.2 simple calculations show that $\lambda - 1 < 0 < h$ (thus, the depth of the ocean h in independent of the values of parameter λ) and

$$\begin{aligned}\int_0^t \mu_\lambda[\tilde{B}(s)]ds &= \int_0^t \frac{2h-(\lambda-1)}{\lambda h-(\lambda-1)s}ds = -\frac{2h-(\lambda-1)}{\lambda-1}\ln\left|\frac{\lambda h-(\lambda-1)t}{\lambda h}\right| \\ &= \underbrace{\left(\frac{2h}{\lambda-1}-1\right)}_{\leq -1}\ln\left(\frac{\lambda h}{\lambda h-(\lambda-1)t}\right) \leq \ln\left(\frac{\lambda h-(\lambda-1)t}{\lambda h}\right).\end{aligned} \tag{4.3}$$

   Then by substituting 4.3 into the estimate 3.10 we conclude:

$$\begin{aligned}||x(z)|| =&||y(t)|| \leq ||y(0)|| \underbrace{\frac{\lambda h-(\lambda-1)t}{\lambda h}}_{\leq \frac{1}{\lambda}} \\ \leq& \frac{1}{\lambda}||y(0)|| = \frac{1}{\lambda}||x(0)|| < +\infty, \forall z \in [-h,0],\end{aligned}$$

   where $t \in [0,h], \lambda \in (0,1)$.

2. *Decreasing eddy viscosity profile*: $\lambda > 1$.

It is easy to check that

$$\begin{aligned}\int_0^t \mu_\lambda[\tilde{B}(s)]ds &= \left(\frac{2h}{\lambda-1}-1\right)\ln\left(\frac{\lambda h}{\lambda h-(\lambda-1)t}\right) \\ &= \ln\left[\left(\frac{\lambda h}{\lambda h-(\lambda-1)t}\right)^{\frac{2h}{\lambda-1}-1}\right]\end{aligned}$$

(since $\frac{2h}{\lambda-1} - 1 \geq 1$ for $h \geq \lambda - 1$, and $(\frac{\lambda h}{\lambda h-(\lambda-1)t})^{\frac{2h}{\lambda-1}-1} \geq 1$) and thus,

$$||x(z)|| \leq \underbrace{\left(\frac{\lambda h}{\lambda h-(\lambda-1)t}\right)}_{\leq\lambda}^{\frac{2h}{\lambda-1}-1}||x(0)|| \leq \lambda^{\frac{2h}{\lambda-1}-1}||x(0)||.$$

Thus, for the increasing eddy viscosity profile dependence between the upper bound of solution and values of parameter λ is reciprocal. In the case of a decreasing viscosity profile this dependence is proportional to the power $\frac{2h}{\lambda-1} - 1$ of this parameter.

In Fig. 5 we depict the increasing and decreasing eddy viscosity profiles at a fixed depth $-h$ for two particular values of parameter λ.

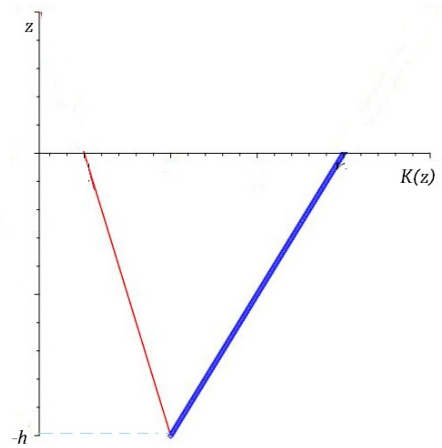

FIGURE 5. Linear eddy viscosity profiles for a two-layer ocean at a fixed ocean depth: red line – the increasing regime and blue points – the decreasing regime

4.2. **Polynomial viscosity profile.** Let us now study the qualitative behavior of solutions of the BVP 2.6-2.8 in the case of a polynomial eddy viscosity. In particular, let

$$K(z,a) = [a(z+h)+1]^2, z \in [-h,0].$$

Computations show that for $t=-z \in [0,h]$ matrix $\tilde{B}(t)$, defined by 3.13, is given by

$$\tilde{B}(t) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -\frac{2a}{a(h-t)+1} & \frac{2}{[a(h-t)+1]^2} & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{2}{[a(h-t)+1]^2} & 0 & 0 & -\frac{2a}{a(h-t)+1} \end{pmatrix},$$

and the associated to it logarithmic matrix norm 3.14, 3.15 is the following:

$$\mu_a[\tilde{B}(t)] = \frac{2-2a[a(h-t)+1]}{[a(h-t)+1]^2} \geq 1. \tag{4.4}$$

For this estimate to hold we require

$$h + \frac{1+a-\sqrt{a^2+2}}{a} \leq t \leq h + \frac{1+a+\sqrt{a^2+2}}{a},$$

which for all $t \in [0,h]$ leads to the bounds of the arbitrary parameter a being

$$0 < a < \frac{1}{2}.$$

In Fig. 6 we graph the eddy viscosity profile at a fixed once depth $-h$ for some admisible value of the parameter a.

Additionally, it is easy to show that

$$\begin{aligned} \int_0^t \mu_a[\tilde{B}(s)]ds &= \frac{2t}{(a(h-t)+1)(ah+1)} + 2\ln\left|1 - \frac{at}{ah+1}\right| \\ &\leq \frac{2t}{(a(h-t)+1)(ah+1)} \leq \frac{2}{a}, \end{aligned} \tag{4.5}$$

where $t \in [0,h]$ and $0 < a < \frac{1}{2}$.

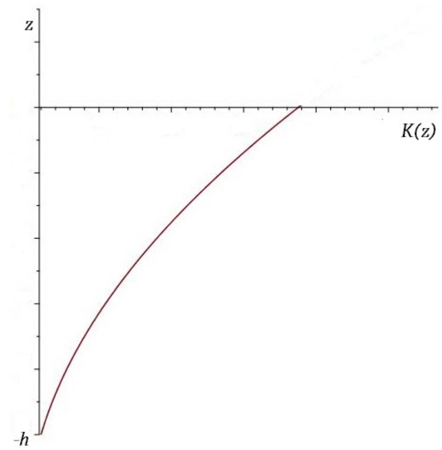

FIGURE 6. The polynominal eddy viscosity profile at a fixed ocean depth $-h$ for a fixed value of parameter a

Using 4.5, we obtain the upper bound of the norm $||x(z)||$ given by an inequality:

$$||x(z)|| = ||y(t)|| \leq ||y(0)||e^{2/a} = ||x(0)||e^{2/a} < +\infty,$$

$\forall z \in [-h, 0],$ and $0 < a < \frac{1}{2}$.

Thus, the bound of the solution shows an exponential dependence on the parameter a.

4.3. **Exponential eddy viscosity profile.** Consider the exponential eddy viscosity profile in the form

$$K(z, \nu) = e^{\nu(z+h)}, z \in [-h, 0],$$

where $\nu < +\infty$ is some scalar parameter.

The suggested in 3.14, 3.15 logarithmic matrix norm of the matrix

$$\tilde{B}(t) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -\nu & 2e^{-\nu(h-t)} & 0 \\ 0 & 0 & 0 & -1 \\ -2e^{-\nu(h-t)} & 0 & 0 & -\nu \end{pmatrix},$$

is defined by

$$\mu_\nu[\tilde{B}(t)] = 2e^{-\nu(h-t)} - \nu \geq 1, \tag{4.6}$$

where $t = -z \in [0, h]$.

Inequality 4.6 leads to the estimate of the scalar parameter ν:

$$0 < \nu \leq 1$$

and of the integral of the logarithmic matrix norm itself:

$$\int_0^t \mu_\nu[\tilde{B}(s)]ds = \frac{2}{\nu}e^{-\nu h}(e^{\nu t} - 1) - \nu t \leq \frac{2}{\nu}.$$

From the calculations above follows that the norm of the solution $x(z)$ to the differential system 3.6, 3.7 has the following upper bound:

$$||x(z)|| = ||y(t)|| \leq ||y(0)||e^{2/\nu} = ||x(0)||e^{2/\nu} < +\infty,$$

where $z \in [-h, 0]$, and $0 < \nu \leq 1$, expressing the fact of the exponential dependence on the parameter ν, present in the eddy viscosity profile.

The viscosity profile in the studied setting is depicted in Fig. 7.

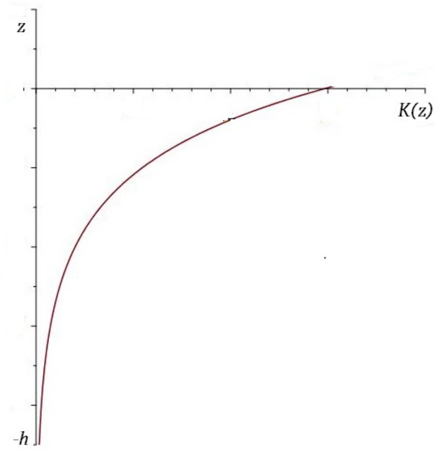

FIGURE 7. The exponential eddy viscosity profile at a fixed ocean depth $-h$ for a fixed value of parameter ν

5. **Conclusions.** To summarise, in this paper we have analysed solutions of the mathematical model of a two-layer ocean in the case of a piecewise constant eddy viscosity. Using the logarithmic matrix norm $\mu[\cdot]$ it was proved that all solutions of the given BVP 2.6–2.8 are bounded on the entire interval $(-\infty, 0]$. In the last section we have made an additional analysis of solutions to the given BVP for linear and nonlinear eddy viscosity profiles in the case of their parameter dependence.

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E-mail address: K.Marynets@tudelft.nl