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DOI
10.5802/crmath.342

Publication date
2022

Document Version
Final published version

Published in
Comptes Rendus Mathematique

Citation (APA)
van Velthoven, J. T. (2022). Completeness of coherent state subsystems for nilpotent Lie groups. *Comptes Rendus Mathematique*, *360*, 799-808. https://doi.org/10.5802/crmath.342

INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

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Volume 360 (2022), p. 799-808

<https://doi.org/10.5802/crmath.342>

Comptes Rendus
Mathématique
2022, Vol. 360, p. 799-808
https://doi.org/10.5802/crmath.342

Harmonic analysis, Representation theory / *Analyse harmonique, Théorie des représentations*

Completeness of coherent state subsystems for nilpotent Lie groups

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Abstract. Let G be a nilpotent Lie group and let π be a coherent state representation of G. The interplay between the cyclicity of the restriction $\pi|_\Gamma$ to a lattice $\Gamma \leq G$ and the completeness of subsystems of coherent states based on a homogeneous G-space is considered. In particular, it is shown that necessary density conditions for Perelomov's completeness problem can be obtained via density conditions for the cyclicity of $\pi|_\Gamma$.

2020 Mathematics Subject Classification. 22E27, 42C30, 42C40, 81R30.

Funding. the author gratefully acknowledges support from the Research Foundation - Flanders (FWO) Odysseus 1 grant G.0H94.18N and the Austrian Science Fund (FWF) project J-4445.

Manuscript received 25 October 2021, accepted 31 January 2022.

1. Introduction

Let G be a connected unimodular Lie group and let $(\pi, \mathscr{H}_\pi)$ be an irreducible unitary representation of G. For a unit vector $\eta \in \mathscr{H}_\pi$, consider its orbit under the action π on $\mathscr{H}_\pi$,

$$\pi(G)\eta = \left\{\pi(g)\eta : g \in G\right\}. \tag{1}$$

As π is irreducible, $\pi(G)\eta$ is complete in $\mathscr{H}_\pi$. Two elements $\pi(g_1)\eta$ and $\pi(g_2)\eta$ differ from one another up to a phase factor, i.e. determine the same state or ray, only if $\pi(g_2^{-1}g_1)\eta \in \mathbb{C}\eta$.

Let $H \leq G$ be a closed subgroup that stabilises the state defined by $\eta \in \mathscr{H}_\pi$, i.e.

$$\pi(h)\eta = \chi(h)\eta, \quad h \in H, \tag{2}$$

where $\chi : H \to \mathbb{T}$ is a unitary character of H. Denote by $X = G/H$ the associated homogeneous G-space and let $\sigma : X \to G$ be a cross-section for the canonical projection $p : G \to X$. Then the system of coherent vectors

$$\{\eta_x\}_{x\in X} = \{\pi(\sigma(x))\eta\}_{x\in X}, \tag{3}$$

determine a π-*system of coherent states based on* X, in the sense of [24, 29].

It will be assumed that $X = G/H$ is unimodular, i.e. X admits a G-invariant positive Radon measure μ_X, and that η is *admissible*, that is,

$$\int_X |\langle \eta, \eta_x \rangle|^2 \, \mathrm{d}\mu_X(x) < \infty. \tag{4}$$

Then there exists an admissibility constant $d_{\pi,\eta} > 0$ such that

$$\int_X |\langle f, \eta_x \rangle|^2 \, \mathrm{d}\mu_X(x) = d_{\pi,\eta}^{-1} \|f\|_{\mathscr{H}_\pi}^2, \quad \text{for all} \quad f \in \mathscr{H}_\pi. \tag{5}$$

The identity (5) implies, in particular, that the system (3) is overcomplete, i.e. the system $\{\eta_x\}_{x\in X}$ contains proper subsystems which are complete in $\mathscr{H}_\pi$.

For an irreducible representation $(\pi, \mathscr{H}_\pi)$ of G that is square-integrable modulo the center $Z = Z(G)$ (resp. the kernel $K = \ker(\pi)$), any vector $\eta \in \mathscr{H}_\pi$ satisfies (2) and (4) for $H = Z$ (resp. $H = K$). Another common choice [12, 22, 26, 29] for the index space $X = G/H$ is a symplectic G-space or a homogeneous Kähler manifold that arises as a phase space in geometric quantization [34]. Subgroups $H \leq G$ defining such a phase space do not need to satisfy (2) for all $\eta \in \mathscr{H}_\pi$ and might not be contained in the isotropy group of a chosen η.

In [24, 26], a particular focus is on coherent states for which the stabilising subgroup $H \leq G$ is assumed to be maximal with the property (2), that is, $H = G_{[\eta]}$, where

$$G_{[\eta]} := \left\{ g \in G : \pi(g)\eta = e^{i\phi(g)}\eta \right\} \tag{6}$$

is the stabiliser of η for the G-action in the projective Hilbert space $\mathrm{P}(\mathscr{H}_\pi)$. The associated coherent states are so-called *Perelomov-type coherent states*; see Section 4.

Perelomov's completeness problem [24, 26] concerns the completeness of subsystems arising from discrete subgroups $\Gamma \leq G$ for which the volume of $\Gamma\backslash X$ is finite. More explicitly, subsystems parametrised by an orbit $\Gamma' := \Gamma \cdot o$ of the base point $o := eH \in X$,

$$\{\eta_{\gamma'}\}_{\gamma' \in \Gamma'} = \{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma'}. \tag{7}$$

Criteria for the completeness of subsystems (7) involving the volume of the coset space $\Gamma\backslash X$ and the admissibility constant $d_{\pi,\eta} > 0$ were posed as a problem in [24, p. 226] and [26, p. 44]. Note that if $H = G_{[\eta]}$, then $X = G/G_{[\eta]}$ depends on η, and so does the volume of $\Gamma\backslash G/G_{[\eta]}$.

The classical example of coherent states arises from the Heisenberg group $G = \mathbb{H}^1$ and the Schrödinger representation $(\pi, L^2(\mathbb{R}))$ of $\mathbb{H}^1$. For any $\eta \in L^2(\mathbb{R}) \setminus \{0\}$, the stabiliser $G_{[\eta]}$ defined in (6) coincides with the centre $Z(\mathbb{H}^1)$ of $\mathbb{H}^1$, and $X = G/G_{[\eta]} \cong \mathbb{R}^2$. Therefore, the coherent state system (3) is parametrised by the classical phase space $\mathbb{R}^2$ and the subsystem (7) associated to $\Gamma \subset \mathbb{H}^1$ is parametrised by a lattice $\Gamma' \subset \mathbb{R}^2$. If the square-integrable representation (mod Z) π is treated as a projective representation ρ of $G/G_{[\eta]} \cong \mathbb{R}^2$, then the coherent vectors (3) and the subsystem (7) arise as orbits of $\mathbb{R}^2$ and Γ', respectively. In particular, a subsystem $\{\pi(\sigma(\gamma'))\eta\}_{\gamma'\in\Gamma'}$ is complete in $L^2(\mathbb{R})$ if, and only if, η is a cyclic vector for $\rho|_{\Gamma'}$, i.e. the linear span of $\rho(\Gamma')\eta$ is dense in $L^2(\mathbb{R})$. This shows that Perelomov's completeness problem for the Heisenberg group is equivalent to determining whether a vector is cyclic for the restriction $\rho|_{\Gamma'}$. If η is the Gaussian, the cyclicity of η has been completely characterised in [2, 23] (see also [21]) in terms of the co-volume or density of the lattice. The necessity of these density conditions have been shown to hold for arbitrary vectors and in arbitrary dimensions [28], but a density condition alone is not sufficient for describing the cyclicity of the Gaussian in higher-dimensions [7, 27]. The criteria [2,23,28] coincide with the density conditions characterising the cyclicity of the restricted projective representations as obainted in, e.g. [3, 30].

In other settings than the Heisenberg group, the stabilisers $G_{[\eta]}$ defined in (6) do not need to be normal subgroups and could depend crucially on the vector $\eta \in \mathscr{H}_\pi \setminus \{0\}$. For example, this occurs for the holomorphic discrete series π of $G = \mathrm{PSL}(2,\mathbb{R})$, where $G_{[\eta]} = \mathrm{PSO}(2)$ for a class of rotation-invariant vectors η. Hence, the coherent vectors (3) do not arise as orbits of a (projective) representation of $G/G_{[\eta]}$ and the subsystems (7) are not parametrised by an associated discrete subgroup. Perelomov's problem for the highest weight vector has been studied for this setting in [9, 10, 25], and the criteria for the cyclicity of $\pi|_{\Gamma}$ are quite different from the completeness of coherent state subsystems; see [31, Section 9.1] for an overview.

Of particular interest are representations and vectors that support a system of coherent states based on an index manifold $X = G/H$ with additional properties, such as a symplectic [16, 17] or complex structure [13, 18]. For nilpotent Lie groups, another common choice (cf. [26, Section 10]) is the manifold X to be the corresponding coadjoint orbit $\mathscr{O}_\pi$ of the representation π, which forms the classical phase space, like in the special case of the Heisenberg group.

The purpose of this note is to combine characterisations of coherent state representations [13, 16, 18] and criteria for the cyclicity of restricted representations [3, 31] to obtain necessary density conditions for (variants of) Perelomov's completeness problem on nilpotent Lie groups.

The first result on the completeness of subsystems concerns π-systems of coherent states based on the coadjoint orbit $\mathscr{O}_\pi$. (cf. Section 2 for the precise definitions.)

Theorem 1. *Let G be a connected, simply connected nilpotent Lie group and let $\Gamma \leq G$ be a discrete, co-compact subgroup. Suppose $(\pi, \mathscr{H}_\pi)$ is an irreducible representation of G that admits an admissible vector $\eta \in \mathscr{H}_\pi \setminus \{0\}$ defining a π-system of coherent states based on a homogeneous G-space $X = G/H \cong \mathscr{O}_\pi$, with admissibility constant $d_{\pi,\eta} > 0$. Then*

(i) $H = \{g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathscr{H}_\pi}\}$;

(ii) *If $\{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma \cdot o}$ is complete in $\mathscr{H}_\pi$, then* $\operatorname{covol}(p(\Gamma)) d_{\pi,\eta} \leq 1$.

(The value $\operatorname{covol}(p(\Gamma)) d_{\pi,\eta}$ *is independent of the normalisation of G-invariant measure on X.)*

Theorem 1 considers π-systems of coherent states parametrised by the canonical phase space $\mathscr{O}_\pi$ (cf. [26, Section 10]), and provides a necessary condition for the completeness of associated subsystems. The representations satisfying the hypothesis of Theorem 1 are called *coherent state representations* in [16], and are characterised as those being an irreducible representation whose associated coadjoint orbit is a linear variety. The considered representations are therefore essentially square-integrable, like in the special case of the Heisenberg group.

The second result concerns π-systems of coherent states associated to vectors yielding a symplectic projective orbit (cf. Section 4 for the precise definitions.)

Theorem 2. *Let G be a connected, simply connected nilpotent Lie group and let $\Gamma \leq G$ be a discrete, co-compact subgroup. Suppose $(\pi, \mathscr{H}_\pi)$ is an irreducible representation of G that admits an admissible vector $\eta \in \mathscr{H}_\pi \setminus \{0\}$ yielding a symplectic orbit and defines a π-system of coherent states based on $X = G/G_{[\eta]}$, with admissibility constant $d_{\pi,\eta} > 0$. Then*

(i) $G_{[\eta]} = \{g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathscr{H}_\pi}\}$;

(ii) *If $\{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma \cdot o}$ is complete in $\mathscr{H}_\pi$, then* $\operatorname{covol}(p(\Gamma)) d_{\pi,\eta} \leq 1$.

In contrast to Theorem 1, the index manifold $X = G/G_{[\eta]}$ in Theorem 2 is selected via the maximal subgroup (6) stabilising the state determined by $\eta \in \mathscr{H}_\pi \setminus \{0\}$. The vectors $\eta \in \mathscr{H}_\pi$ yielding a symplectic orbit play a distinguished role in geometric quantization [12, 22]. Theorem 2 applies, in particular, to smooth vectors of a square-integrable representation (see Proposition 10) and to so-called *highest weight vectors* (see Remark 12).

The proofs of Theorem 1 and Theorem 2 are relatively simple and short, but they hinge on a combination of several non-trivial statements on coherent state representations [13, 16, 18] and density conditions for restricted discrete series [3, 31]. More explicitly, exploiting results of [13, 16, 18], it will be shown that the completeness of coherent state subsystems is equivalent to the admissible vector being a cyclic vector for a restricted *projective* representation; the necessary density conditions then being a direct consequence of [31].

Notation

For a complex vector space $\mathcal{H}$, the notation $\mathrm{P}(\mathcal{H})$ will be used for its projective space, i.e. the space of all one-dimensional subspaces. The subspace or ray generated by $\eta \in \mathcal{H} \setminus \{0\}$ will be denoted by $[\eta] := \mathbb{C}\eta$. Henceforth, unless stated otherwise, G is a connected, simply connected nilpotent Lie group with exponential map $\exp : \mathfrak{g} \to G$. Haar measure on G is denoted by μ_G. If $\Lambda \leq G$ is a discrete subgroup, then the co-volume is defined as $\operatorname{covol}(\Lambda) := \mu_{G/\Lambda}(G/\Lambda)$, where $\mu_{G/\Lambda}$ denotes G-invariant Radon measure on G/Λ.

2. Coherent state representations of nilpotent Lie groups

This section provides preliminaries on irreducible representations of nilpotent Lie groups and associated coherent states. References for these topics are the books [6] and [1, 26].

2.1. *Coadjoint orbits*

Let $\mathfrak{g}^*$ denote the dual vector space of $\mathfrak{g}$. The coadjoint representation $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ is defined by $\mathrm{Ad}^*(g)\ell = \ell \circ \mathrm{Ad}(g)^{-1}$ for $g \in G$ and $\ell \in \mathfrak{g}^*$. The stabiliser of $\ell \in \mathfrak{g}^*$ is the connected closed subgroup $G(\ell) = \{g \in G : \mathrm{Ad}^*(g)\ell = \ell\}$, its Lie algebra is the annihilator subalgebra $\mathfrak{g}(\ell) = \{X \in \mathfrak{g} : \ell([Y,X]) = 0,\ \forall\, Y \in \mathfrak{g}\}$.

For $\ell \in \mathfrak{g}^*$, its *coadjoint orbit* is denoted by $\mathcal{O}_\ell := \mathrm{Ad}^*(G)\ell$ and endowed with the relative topology from $\mathfrak{g}^*$. The orbit $\mathcal{O}_\ell$ is homeomorphic to $G/G(\ell)$; in notation: $\mathcal{O}_\ell \cong G/G(\ell)$.

2.2. *Irreducible representations*

A Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is *subordinated* to $\ell \in \mathfrak{g}^*$ if $\ell(X) = 0$ for every $X \in [\mathfrak{p},\mathfrak{p}]$. If $\mathfrak{p}$ is subordinate to ℓ, then the map $\chi_\ell : \exp(\mathfrak{p}) \to \mathbb{T}$, $\chi_\ell\big(\exp(X)\big) = e^{2\pi i \ell(X)}$ defines a unitary character of $P = \exp(\mathfrak{p})$. The associated induced representation of G is denoted by $\pi_\ell = \pi(\ell,\mathfrak{p}) = \mathrm{ind}_P^G\big(\chi_\ell\big)$.

For every π in the unitary dual $\widehat{G}$ of G, there exists $\ell \in \mathfrak{g}^*$ and a subalgebra $\mathfrak{p} \subset \mathfrak{g}$, subordinate to ℓ, such that π is unitarily equivalent to $\pi_\ell = \pi(\ell,\mathfrak{p})$. A representation $\pi_\ell = \pi(\ell,\mathfrak{p})$, with $\mathfrak{p}$ subordinate to $\ell \in \mathfrak{g}^*$, is irreducible if, and only if, $\mathfrak{p}$ is a maximal subalgebra subordinated to $\ell \in \mathfrak{g}^*$ satisfying $\dim(\mathfrak{p}) = \dim(\mathfrak{g}) - \dim(\mathcal{O}_\ell)/2$, a so-called *(real) polarisation.*

Two irreducible induced representations $\mathrm{ind}_{\exp(\mathfrak{p})}^G(\chi_\ell)$ and $\mathrm{ind}_{\exp(\mathfrak{p}')}^G(\chi_{\ell'})$ are unitarily equivalent if and only if the linear functionals $\ell,\ell' \in \mathfrak{g}^*$ belong to the same coadjoint orbit. The orbit associated to the equivalence class $\pi \in \widehat{G}$ will also be denoted by $\mathcal{O}_\pi$.

2.3. *Moment set*

Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G. Denote by $\mathcal{H}_\pi^\infty$ the space of smooth vectors for π, i.e. the space of $\eta \in \mathcal{H}_\pi$ for which $g \mapsto \pi(g)\eta$ is smooth.

The derived representation $\mathrm{d}\pi : \mathfrak{g} \to L(\mathcal{H}_\pi^\infty)$ is defined by

$$\mathrm{d}\pi(X)\eta = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(\exp(tX))\eta, \quad X \in \mathfrak{g},\ \eta \in \mathcal{H}_\pi^\infty. \tag{8}$$

It can be extended complex linearly to a representation of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$.

The *moment map* of π is the mapping $J_\pi : \mathcal{H}_\pi^\infty \to \mathfrak{g}^*$ defined by

$$J_\pi(\eta)(X) = \frac{1}{i}\frac{\langle \mathrm{d}\pi(X)\eta, \eta\rangle}{\langle \eta,\eta\rangle}, \quad X \in \mathfrak{g},\ \eta \in \mathcal{H}_\pi^\infty. \tag{9}$$

Note that the right-hand side of (9) only depends on the ray $[\eta]$ generated by $\eta \in \mathcal{H}_\pi^\infty \setminus \{0\}$.

The moment map J_π is equivariant with respect to the canonical G-actions on $\mathscr{H}_\pi^\infty$ and $\mathfrak{g}^*$, i.e. $J_\pi(\pi(g)\eta)(X) = (\mathrm{Ad}(g)^* J_\pi(\eta))(X)$ for $g \in G$, $X \in \mathfrak{g}$ and $\eta \in \mathscr{H}_\pi^\infty$. In particular, $J_\pi(G \cdot \eta)$ is the coadjoint orbit $\mathscr{O}_{J_\pi(\eta)}$ of $J_\pi(\eta) \in \mathfrak{g}^*$.

The *moment set* I_π of π is the closure $I_\pi := \overline{J_\pi(\mathscr{H}_\pi^\infty)}$ in $\mathfrak{g}^*$. Its relation to the coadjoint $\mathscr{O}_\pi$ of $\pi \in \widehat{G}$ is

$$I_\pi = \overline{\mathrm{conv}}(\mathscr{O}_\pi), \tag{10}$$

where $\overline{\mathrm{conv}}$ denotes the closed convex hull; see [33, Theorem 4.2].

2.4. *Coherent state representations*

Henceforth, it is assumed that $(\pi, \mathscr{H}_\pi)$ is non-trivial. Let $\eta \in \mathscr{H}_\pi$ be a unit vector and let $H \leq G$ be a closed subgroup such that there exists a unitary character $\chi : H \to \mathbb{T}$ satisfying

$$\pi(h)\eta = \chi(h)\eta, \quad h \in H. \tag{11}$$

Denote $X := G/H$ and let μ_X be G-invariant Radon measure on X, which is unique up to scalar multiplication. Fix a Borel cross-section $\sigma : X \to G$ for the quotient map $p : G \to X$. The vector η is called *admissible* if

$$\int_X |\langle \eta, \pi(\sigma(x))\eta\rangle|^2 \,\mathrm{d}\mu_X(x) < \infty. \tag{12}$$

A pair (η, χ) satisfying (11) and (12) is said to define a *π-system of coherent states* based on $X = G/H$. The condition (12) is independent of the particular choice of section σ.

For a π-system of coherent states, there exists an *admissibility constant* $d_{\pi,\eta} > 0$ such that, for all $f \in \mathscr{H}_\pi$,

$$\int_X |\langle f, \pi(\sigma(x))\eta\rangle|^2 \,\mathrm{d}\mu_X(x) = d_{\pi,\eta}^{-1} \|f\|_{\mathscr{H}_\pi}^2. \tag{13}$$

For further properties on square-integrability modulo a subgroup, see, e.g. [17, 19].

An irreducible representation $(\pi, \mathscr{H}_\pi)$ is called a *coherent state representation* if it admits a π-system of coherent states based on connected, simply connected homogeneous G-space X.[1]

3. Completeness of coherent state subsystems

This section considers the relation between subsystems of coherent states parametrised by a simply connected G-space and lattice orbits of an associated projective representation.

3.1. *Projective kernel*

The *kernel* and *projective kernel* of a unitary representation $(\pi, \mathscr{H}_\pi)$ of G are defined by

$$\ker(\pi) = \{g \in G : \pi(g) = I_{\mathscr{H}_\pi}\} \quad \text{and} \quad \mathrm{pker}(\pi) = \{g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathscr{H}_\pi}\},$$

respectively. If $(\pi, \mathscr{H}_\pi)$ is non-trivial and irreducible, then $\mathrm{pker}(\pi) \leq G$ is a connected, closed normal subgroup, and there exists $\chi_\pi : \mathrm{pker}(\pi) \to \mathbb{T}$ such that $\pi(g) = \chi_\pi(g) I_{\mathscr{H}_\pi}$ for $g \in \mathrm{pker}(\pi)$.

The following observation plays a key role in the sequel. Its proof hinges on [16, Lemma 3.5], which characterises coherent state representations π in terms of their coadjoint orbit $\mathscr{O}_\pi$.

Proposition 3. *Let $H \leq G$ be a connected subgroup. Suppose π admits a π-system of coherent states based on G/H. Then $H = \mathrm{pker}(\pi)$. In particular, $H \leq G$ is normal.*

[1] The definition of a coherent state representation used here is the same as in [16, 17, 19], but differs from the definition in [13, 14, 18], where the square-integrability assumption (12) is not part of the definition.

Proof. If π admits a pair (η, χ) satisfying (11) and (12), then π is unitarily equivalent to a subrepresentation of the induced representation $\operatorname{ind}_H^G \chi$, see, e.g. [16, Proposition 1.2]. Since $H \le G$ is assumed to be connected, it follows by [16, Lemma 3.5] that $H = G(\ell)$ for any $\ell \in \mathcal{O}_\pi$. By [4, Theorem 2.1], the projective kernel of an arbitrary irreducible representation π of G is given by $\operatorname{pker}(\pi) = \bigcap_{\ell \in \mathcal{O}_\pi} G(\ell)$. Therefore, $\operatorname{pker}(\pi) = \bigcap_{\ell \in \mathcal{O}_\pi} G(\ell) = H$. □

The conclusion of Proposition 3 may fail for disconnected subgroups $H \le G$ whenever π has a discrete kernel:

Remark 4. Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G.

(a) If π is square-integrable modulo $K = \ker(\pi)$, then $\pi|_K$ satisfies (11) for the trivial character $\chi \equiv 1$ and any vector $\eta \in \mathcal{H}_\pi$ defines a π-system of coherent states based on G/K.

(b) If π is square-integrable modulo $Z = Z(G)$, then $\pi|_Z$ satisfies (11) for the central character $\chi \in \widehat{Z}$ and any vector $\eta \in \mathcal{H}_\pi$ defines a π-system of coherent states based on G/Z. Moreover, $\operatorname{pker}(\pi) = Z(G)$ by [6, Corollary 4.5.4].

3.2. *Necessary density conditions*

A *uniform subgroup* $\Gamma \le G$ is a discrete subgroup such that $\Gamma\backslash G$ is compact. For a nilpotent Lie group G, the uniformity of a discrete subgroup $\Gamma \le G$ is equivalent to Γ being a lattice, i.e. having finite co-volume; see [6, Corollary 5.4.6].

The following result provides a criterium for cyclicity of restricted (projective) representations in terms of the lattice co-volume or density (cf. [31, Theorem 7.4]).

Theorem 5 ([31]). *Let $(\pi, \mathcal{H}_\pi)$ be an irreducible, square-integrable projective unitary representation of a unimodular group G, with formal dimension $d_\pi > 0$. Let $\Gamma \le G$ be a lattice. If there exists $\eta \in \mathcal{H}_\pi$ such that $\pi(\Gamma)\eta$ is complete in $\mathcal{H}_\pi$, then* $\operatorname{covol}(\Gamma) d_\pi \le 1$.

For a genuine representation π of G that is square-integrable modulo the centre $Z(G)$, a version of Theorem 5 can also be deduced from [3, Theorem 5]; see also [3, Theorem 3] for a converse in the setting of nilpotent Lie groups. However, in order to treat a representation π that is merely square-integrable modulo $\ker(\pi)$ (equivalently, $\operatorname{pker}(\pi)$), the projective version of Theorem 5 is particularly convenient for the purposes of the present note.

The following completeness result for coherent state subsystems can simply be obtained by combining Proposition 3 and Theorem 5.

Theorem 6. *Let $H \le G$ be a connected subgroup. Suppose $(\pi, \mathcal{H}_\pi)$ is an irreducible representation that admits an admissible vector $\eta \in \mathcal{H}_\pi$ defining a π-system of coherent states based on $X = G/H$, with admissibility constant $d_{\pi,\eta} > 0$. Then*

(i) $H = \operatorname{pker}(\pi)$*;*

(ii) *If $\Gamma \le G$ is uniform and $\{\pi(\sigma(\gamma'))\eta\}_{\gamma' \in \Gamma \cdot o}$ is complete, then* $\operatorname{covol}(p(\Gamma)) d_{\pi,\eta} \le 1$.

Proof. By Proposition 3, the admissibility of π implies that $H = \operatorname{pker}(\pi) \le G$ is normal. Hence, the induced mapping $\pi' : G/H \to \mathcal{U}(\mathcal{H}_\pi),\ x \mapsto \pi(\sigma(x))$ forms an irreducible projective representation of G/H. Since the measure μ_X is Haar measure on $X = G/H$, it follows that π' is square-integrable on G/H by the admissibility condition (12). In particular, the constant $d_{\pi,\eta} > 0$ in (13) coincides with the (unique) formal dimension $d_{\pi'} > 0$ of the projective representation $(\pi', \mathcal{H}_\pi)$ normalised according to the G-invariant measure μ_X.

Suppose $\Gamma \le G$ is a uniform subgroup. As in the proof of Proposition 3, the admissibility of π implies that $\operatorname{pker}(\pi) = G(\ell)$ for any $\ell \in \mathcal{O}_\pi$. A combination of [6, Proposition 5.2.6] and [6, Theorem 5.1.11] therefore yields that $\Gamma \cap H$ is a uniform subgroup of $H = \operatorname{pker}(\pi)$. Hence, the image $p(\Gamma)$ is a uniform subgroup of G/H by [6, Lemma 5.1.4 (a)].

In combination, applying Theorem 5 to $(\pi', \mathcal{H}_\pi)$ and $p(\Gamma) \le G/H$ yields the result. □

Remark 7. The constant $d_{\pi,\eta} > 0$ coincides with the formal dimension $d_{\pi'} > 0$ of the projective representation $(\pi', \mathscr{H}_\pi)$ of $X = G/\operatorname{pker}(\pi)$. In particular, the product $\operatorname{covol}(p(\Gamma))d_{\pi'}$ is independent of the choice of G-invariant measure μ_X: if $\mu'_X = c \cdot \mu_X$ for $c > 0$, then $\operatorname{covol}'(p(\Gamma)) = c \cdot \operatorname{covol}(p(\Gamma))$ and $d'_{\pi'} = d_{\pi'}/c$.

Theorem 1 follows directly from Proposition 3 and Theorem 6:

Proof of Theorem 1. By assumption, there exists an admissible $\eta \in \mathscr{H}_\pi$ and associated character $\chi : H \to \mathbb{T}$ defining a π-system of coherent states based on $G/H \cong \mathscr{O}_\pi$. Since $\mathscr{O}_\pi$ is simply connected, it follows that $H \subset G$ is connected, see, e.g. [11, Proposition 1.94]. The conclusions are therefore a direct consequence of Proposition 3 and Theorem 6. □

4. Perelomov-type coherent states

Let $(\pi, \mathscr{H}_\pi)$ be an irreducible representation of G. Then π yields an action of G on the projective spaces $\mathrm{P}(\mathscr{H}_\pi)$ and $\mathrm{P}(\mathscr{H}_\pi^\infty)$ by $g \cdot [\eta] = [\pi(g)\eta]$.

A system of *Perelomov-type coherent states* is a G-orbit in $\mathrm{P}(\mathscr{H}_\pi)$,

$$G \cdot [\eta] = \big\{[\pi(g)\eta] \; : \; g \in G\big\}.$$

Let $G_{[\eta]}$ be the isotropy group of $\eta \in \mathscr{H}_\pi \setminus \{0\}$ in the projective space $\mathrm{P}(\mathscr{H}_\pi)$,

$$G_{[\eta]} := \big\{g \in G \; : \; \pi(g)\eta \in \mathbb{C}\eta\big\}. \tag{14}$$

Denote by $X = G/G_{[\eta]}$ the associated homogeneous space and let $\sigma : X \to G$ be a Borel section for the quotient map $p : G \to X$. Then a Perelomov-type coherent state system is determined by the system of vectors,

$$\{\eta_x\}_{x \in X} = \{\pi(\sigma(x))\eta\}_{x \in X}.$$

See [24, Section 2] and [26, Chapter 2] for the basic properties of Perelomov-type states.

Let $\chi_\eta : G_{[\eta]} \to \mathbb{T}$ be the unitary character of $G_{[\eta]}$ such that $\pi(g)\eta = \chi_\eta(g)\eta$ for all $g \in G_{[\eta]}$. Note that $G_{[\eta]}$ is the maximal subgroup satisfying the property (11) for a chosen η.

The following sections consider Perelomov-type coherent states of vectors $\eta \in \mathscr{H}_\pi^\infty \setminus \{0\}$ with the property that $G/G_{[\eta]}$ has a symplectic or complex structure. Such systems are of particular interest for geometric quantization, see [22] and [26, Section 16].

4.1. *Symplectic projective orbits*

Following [12, 13], an orbit $G \cdot [\eta] = \{[\pi(g)\eta] : g \in G\}$ is called *symplectic* if $[\eta] \in \mathrm{P}(\mathscr{H}_\pi^\infty)$ and $G \cdot [\eta]$ is a symplectic submanifold of $\mathrm{P}(\mathscr{H}_\pi)$.

The following simple characterisation of symplectic orbits will be used below, see, e.g. [8, Theorem 26.8] or [5, Proposition 2.1] for proofs.

Lemma 8 ([8]). *Let $[\eta] \in \mathrm{P}(\mathscr{H}_\pi^\infty)$ and let $J_\pi : \mathrm{P}(\mathscr{H}_\pi^\infty) \to \mathfrak{g}^*$ be the momentum map of π. The orbit $G \cdot [\eta]$ is symplectic if, and only if, the stabiliser $G_{[\eta]}$ is an open subgroup of $G(J_\pi(\eta))$.*

For the purposes of this note, the significance of a symplectic orbit is that its stabiliser subgroups coincides with the projective kernel, and hence does not depend on the chosen vector. This is demonstrated by the following proposition.

Proposition 9. *Suppose $\eta \in \mathscr{H}_\pi^\infty \setminus \{0\}$ is such that $G \cdot [\eta]$ is symplectic. Then $G_{[\eta]}$ is connected. In particular, if η is an admissible vector defining a π-system of coherent states based on $G/G_{[\eta]}$, then $G_{[\eta]} = \operatorname{pker}(\pi)$.*

Proof. If $G\cdot[\eta]$ is symplectic, then $G\cdot[\eta]$ forms a Hamiltonian G-space, with momentum map $J_\pi : G\cdot[\eta] \to \mathfrak{g}^*$ given as in (9), see, e.g. [13, Section 2.5]. Set $\ell := J_\pi([\eta])$. Then, by Lemma 8, the stabiliser $G_{[\eta]}$ is an open subgroup of $G(\ell)$. Since $G(\ell)$ is connected (cf. Section 2.1), it follows that $G_{[\eta]} = G(\ell)$ is connected. The last assertion follows from Proposition 3. □

The following provides a partial converse to Proposition 9.

Proposition 10. *Suppose $(\pi,\mathscr{H}_\pi)$ is square-integrable modulo* $\mathrm{pker}(\pi)$. *Then, for any* $[\eta] \in \mathrm{P}(\mathscr{H}_\pi^\infty)$, *the orbit $G\cdot[\eta]$ is symplectic and* $G_{[\eta]} = \mathrm{pker}(\pi)$.

Proof. Let $\eta \in \mathscr{H}_\pi^\infty \setminus \{0\}$ be fixed. The inclusion $\mathrm{pker}(\pi) \subseteq G_{[\eta]}$ is immediate. Conversely, if $g \in G_{[\eta]}$, then

$$J_\pi([\pi(g)\eta]) = \frac{1}{i}\frac{\langle \pi(g)\eta, \mathrm{d}\pi(X)\pi(g)\eta\rangle}{\langle \pi(g)\eta,\pi(g)\eta\rangle} = \frac{1}{i}\frac{\langle \eta, \mathrm{d}\pi(X)\eta\rangle}{\langle \eta,\eta\rangle} = J_\pi([\eta]), \quad X \in \mathfrak{g},$$

so that by the G-equivariance of J_π it follows that $\mathrm{Ad}^*(g)J_\pi([\eta]) = J_\pi([\eta])$. This means that $g \in G(J_\pi([\eta]))$, and it remains to show that $G(J_\pi([\eta])) \subseteq \mathrm{pker}(\pi)$.

Since $\pi \in \widehat{G}$ is square-integrable modulo $\mathrm{pker}(\pi)$, it is also square-integrable modulo $\ker(\pi)$, see, e.g., [4, Corollary 2.1]. It follows therefore by [6, Theorem 4.5.2] and [6, Theorem 3.2.3] that $\mathscr{O}_\pi$ is a linear variety of the form $\mathscr{O}_\pi = \ell + \mathfrak{k}^\perp$ for $\ell \in \mathscr{O}_\pi$, with $\mathfrak{k}$ being the Lie algebra of $\mathrm{pker}(\pi)$. In addition, [6, Theorem 3.2.3] yields that $\mathfrak{g}(\ell) = \mathfrak{k}$ for $\ell \in \mathscr{O}_\pi$, so that $G(\ell) = \mathrm{pker}(\pi)$ for $\ell \in \mathscr{O}_\pi$. By [33, Theorem 4.2] (see also Equation (10)) it follows, in particular, that

$$J_\pi([\eta]) \in J_\pi(\mathrm{P}(\mathscr{H}_\pi^\infty)) \subseteq I_\pi = \overline{\mathrm{conv}}(\mathscr{O}_\pi) = \mathscr{O}_\pi,$$

where $I_\pi := \overline{J_\pi(\mathscr{H}_\pi^\infty)}$ denotes the moment set of π. Therefore, $G(J_\pi([\eta])) = \mathrm{pker}(\pi)$.

Lastly, since $G_{[\eta]} = \mathrm{pker}(\pi) = G(J_\pi([\eta]))$ by the arguments above, the orbit $G\cdot[\eta]$ is symplectic by Lemma 8. □

Proof of Theorem 2. If $G\cdot[\eta]$ is symplectic, then $G_{[\eta]}$ is connected by Proposition 9. Therefore, if η determines a π-system of coherent states based on $G/G_{[\eta]}$, the conclusions of Theorem 2 follow directly from Theorem 6. □

4.2. *Highest weight vectors*

In [13,18], an orbit $G\cdot[\eta] = \{[\pi(g)\eta] : g \in G\}$ is called *complex* if $[\eta] \in \mathrm{P}(\mathscr{H}_\pi^\infty)$ and $G\cdot[\eta]$ is a complex submanifold of $\mathrm{P}(\mathscr{H}_\pi)$.

The following lemma characterises complex orbits in terms of a (complex) stabiliser; cf. [13, Proposition 2.8] and [20, Lemma XV.2.3].

Lemma 11 ([13]). *Let* $\mathfrak{s} = (\mathfrak{g})_{\mathbb{C}}$. *For* $[\eta] \in \mathrm{P}(\mathscr{H}_\pi^\infty)$, *let* $\mathfrak{s}_{[\eta]} = \{X \in \mathfrak{s} : \mathrm{d}\pi(X)\eta \in \mathbb{C}\cdot\eta\}$.

The following assertions are equivalent:

(i) *The orbit $G\cdot[\eta]$ is complex;*
(ii) $\mathfrak{s}_{[\eta]} + \overline{\mathfrak{s}_{[\eta]}} = \mathfrak{s}$.

A stabiliser $\mathfrak{s}_{[\eta]}$ satisfying part (ii) of Lemma 11 is called *maximal* in [26, Section 2.4], where it is part of a principle for selecting coherent states that minimise the uncertainty principle. Such vectors and associated orbits play an important role in Berezin's quantization, see [26, Section 16]. In addition, vectors of this type are intimately related to highest weight modules and representations (cf. [18, 20]) and are also referred to as *highest weight vectors.*

Remark 12. By [13, Proposition 2.8], any complex orbit is automatically symplectic in the sense of Section 4.1. Theorem 2 applies therefore to highest weight vectors.

Remark 13. The significance of a complex orbit $G\cdot[\eta]$ is that the quotient manifold $G/G_{[\eta]}$ admits a complex structure (cf. [20, Section XV.2]). In turn, for certain (classes of) representations admitting highest weight vectors, the representation space may be realised as a space of holomorphic functions (see [26, Section 2.4] and [32]); in particular, see [15, Section 5] for complex orbits for the Heisenberg group. For nilpotent Lie groups, the existence of complex orbits appears to be restrictive, i.e. [14, Theorem 1] asserts that the only irreducible representations with a discrete kernel admitting complex orbits are those of Heisenberg groups. In contrast, symplectic orbits do exist for all groups admitting square-integrable representations by Proposition 10.

Acknowledgements

Thanks are due to Bas Janssens for helpful discussions and to the anonymous referee for providing helpful comments and suggestions.

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