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On a multiple timescales perturbation approach for a stefan problem with a time-dependent heat flux at the boundary

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Abstract In this paper, a classical Stefan problem is studied. It is assumed that a small, time-dependent heat influx is present at the boundary, and that the initial values are small. By using a multiple timescales perturbation approach, it is shown analytically (most likely for the first time in the literature) how the moving interface and its stability are influenced by the time-dependent heat influx at the boundary and by the initial conditions. Accurate approximations of the solution of the problem are constructed, which are valid on long timescales. The constructed approximations turn out to agree very well with solutions of problems for which similarity solutions are available (in numerical form).

Keywords Multiple timescales · Stefan problem · Time-dependent heat flux

Mathematics Subject Classification 35K55 · 35R37 · 35C20 · 80A22 · 35B20

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1 Introduction

For more than a century, Stefan or moving boundary problems have been studied. Originally, Stefan problems describe the phase transitions of materials, but nowadays, it has many applications ranging from problems such as solidification of a liquid [23], formation of a mushy area between a solid and a liquid phase [21], three-phase transition models for gas formation [1], heat convection in the liquid phase [20,22], diffusion in glassy polymers [12,13], the dissolution of particles in multi-component alloys [25], and to many, many other applications. The readers are referred to the classical books [5,7] for a more comprehensive review of the existing variations in application of Stefan problems.

The classical Stefan problems consist of parabolic partial differential equations with boundary conditions given at a fixed position and at a moving position, and with some specified conditions at an initial time. The existence and uniqueness of a solution for such Stefan problem have been proved in many mathematical papers (see for instance [2,27]). On the other hand, exact analytical solutions are rarely available. And those available are usually constructed by using the similarity method, which implies that only very special Stefan problems are solved.

Most Stefan problems are solved approximately by using numerical methods (see for instance [4,10,12,13,16,18,19,24,25]). A smaller part of the Stefan prob-

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lems is approximately solved by using straightforward perturbation expansion (see for instance [3,6,9,12,13,15,17,26,28]). To expand the solution of the problem, a small parameter is usually introduced in the aforementioned papers, and this choice of the small parameter usually leads to a problem for an elliptic partial differential equation. For such problems, arbitrary initial conditions and arbitrary time-dependent boundary conditions (given at the fixed position) cannot be included in the approximation. In fact, one constructs a stationary solution for a Stefan problem with very special initial conditions and boundary conditions (at the fixed position).

In this paper, a classical Stefan problem similar to the ones that have been studied in [3] and in [17] is considered and is formulated in Sect. 2 of this paper. In [3,17], the authors used the moving boundary variable as the time-like variable and assumed that the Stefan number is small. In this way, the problem was reformulated in a problem for a perturbed elliptic equation, which does not permit arbitrary initial conditions. In fact, the approach as is introduced in [3,17] starts off from the stationary solution. This also implies that the heat influx at the boundary cannot be an arbitrary, time-dependent function. In this paper, we use a different approach. First, using a transformation in the spatial variable, we fix the moving boundary. Furthermore, by rescaling the time variable by using a transformation which depends on the moving boundary variable, we rewrite the system as a weakly nonlinear diffusion equation subject to nonlinear boundary conditions and subject to initial conditions on a fixed spatial domain. Thus, our approach does not assume a small Stephan number.

In contrast to [3,17] where the authors used a straighforward, naive perturbation approach, a multiple timescales perturbation method is used and applied in Sect. 3 of this paper such that arbitrary initial conditions and an arbitrary time-dependent heat inflow at the boundary can be taken into account. Accurate approximations of the solution of the Stefan problem will be constructed, and it will be shown how the initial values and the boundary heat inflow influence the solution. The readers are referred to the classical and standard books [8,11,14] on perturbation methods, which all explain and describe clearly how the multiple timescales perturbation method can be used and applied to all kinds of problems described by differential equations. In Sect. 4 of this paper, the obtained approximations are compared to numerical approximations of solutions which are obtained by using the similarity method. In contrast with the approximation constructed in [3,17] which diverges after a relatively short time, our approximations agrees very well with the one constructed using similarity method, on long timescales. Finally, in Sect. 5 of this paper, we draw some conclusions.

Some notations. Throughout this paper, we will be using the notation: ∂_η for $\frac{\partial}{\partial \eta}$ and ∂_η^2 for $\frac{\partial^2}{\partial \eta^2}$. Furthermore, when the function depends only on one variable, we will be using $'$ to denote the first-order derivative with respect to the variable.

2 Formulation of the problem

Consider a semi-infinite sheet of ice which is melting. We assume that the heat transfer is one dimensional. This implies that the sheet of melting ice can be modeled on the interval: $[0, \infty)$. Using θ as the time variable, let $S(\theta) \in (0, \infty)$ be the location of the boundary between the solid and the liquid phase. In this paper, we restrict our attention to the liquid phase of the melting ice. We denote the temperature of the material at the location $X \in [0, S(\theta))$ and the time θ by $T(X, \theta)$. Then, the dynamics of this temperature profile is governed by the heat equation (or diffusion equation):

$$\rho c \partial_\theta T(X, \theta) = K \partial_X^2 T(X, \theta), \ 0 < X < S(\theta), \ \theta > 0, \tag{1a}$$

where ρ represents the density of the material, while c and K represent the heat capacity and the heat conductivity of the water. At the interface of the phases, that is, at $X = S(\theta)$, the heat exchange is governed by the so-called *Stefan condition*:

$$-K \partial_X T(S(\theta), \theta) = \rho L S'(\theta), \tag{1b}$$

$$T(S(\theta), \theta) = T_m, \tag{1c}$$

where L and T_m are the latent heat and melting temperature of the material, respectively. For the other boundary condition at $X = 0$, we consider a heat inflow described by the Neumann boundary condition

$$\partial_X T(0, \theta) = -\varepsilon \bar{H}(\theta), \tag{1d}$$

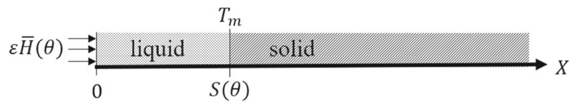

Fig. 1 Melting process of a semi-infinite ice sheet

where $0 < \varepsilon \ll 1$ is a small positive parameter, and $\bar{H}$ is a given positive function of θ. Lastly, the initial condition is given by

$$T(X,0) = T_m + \varepsilon \bar{F}(X), \quad 0 < X < S(0) = b, \quad \text{(1e)}$$

where b is a positive constant. In Fig. 1, the melting ice process is displayed graphically. The system of equations (1a)-(1e) defines a classical and well-known Stefan problem.

To reduce the number of parameters in the aforementioned Stefan problem, the following rescalings and redefinitions of functions are introduced:

$$\tau = \frac{K}{c\rho}\theta, \quad U(X,\tau) = \frac{c}{L}\left(T(X,\theta) - T_m\right),$$
$$F(X) = \frac{c}{L}(\bar{F}(X) - T_m), \quad \text{and} \quad H(\tau) = \frac{c}{L}\bar{H}(\theta).$$

Then, problem (1a)-(1e) can be reformulated in the following nondimensional form:

$$\partial_\tau U(X,\tau) = \partial_{XX} U(X,\tau), \quad 0 < X < S(\tau), \quad \tau > 0, \quad \text{(2a)}$$
$$-\partial_X U(S(\tau),\tau) = S'(\tau), \quad \tau \geq 0, \quad \text{(2b)}$$
$$U(S(\tau),\tau) = 0, \quad \tau \geq 0, \quad \text{(2c)}$$
$$\partial_X U(0,\tau) = -\varepsilon H(\tau) \quad \tau \geq 0, \quad \text{(2d)}$$
$$U(X,0) = \varepsilon F(X), \quad 0 < X < S(0) = b, \quad \text{(2e)}$$

where F is positive-definite function.

Traditionally, one defines a transformation in space to fix the moving boundary. As a consequence, one ends up with a diffusion equation with time-dependent coefficients. In this paper, we will also rescale time again to eliminate the presence of the moving variable $S(\tau)$ from the problem to be studied.

Let

$$t(\tau) = \int_0^\tau S^{-2}(\eta)d\eta, \quad x = \frac{X}{s(t)}, \quad v(x,t) = U(X,\tau),$$
$$s(t) = S(\tau), \quad \text{and} \quad h(t) = H(\tau). \quad \text{(3)}$$

Substituting the aforementioned transformations (3) into the problem (2a)-(2e) for $U(X,\tau)$, one obtains the following problem for $v(x,t)$:

$$-\partial_x v(x,t)\frac{x}{s(t)}s'(t) + \partial_t v(x,t) = \partial_{xx} v(x,t), \quad \text{(4a)}$$
$$\partial_x v(0,t) = -\varepsilon s(t)h(t), \quad \text{(4b)}$$
$$\partial_x v(1,t) = -\frac{1}{s(t)}s'(t), \quad \text{(4c)}$$
$$v(1,t) = 0, \quad \text{(4d)}$$
$$v(x,0) = \varepsilon f(x), \quad \text{(4e)}$$

where $f(x) = F(X)$. The dependence on $s(t)$ in (4a) can be removed by substituting (4c) into (4a), yielding

$$\partial_t v(x,t) = \partial_{xx} v(x,t) - x\partial_x v(x,t)\partial_x v(1,t).$$

Furthermore, from (4b), it follows that

$$s(t) = -\frac{\partial_x v(0,t)}{\varepsilon h(t)}, \quad \text{and}$$
$$s'(t) = -\frac{1}{\varepsilon}\left[\frac{\partial_{xt} v(0,t)h(t) - \partial_x v(0,t)h'(t)}{h^2(t)}\right].$$

Substituting these expressions into (4c), one obtains

$$-h(t)\partial_x v(0,t)\partial_x v(1,t) = \partial_{xt} v(0,t)h(t) - h'(t)\partial_x v(0,t).$$

Since small initial conditions are considered, the following rescaling is introduced: $v = \varepsilon u$, and problem (4a)-(4e) becomes

$$\partial_t u(x,t) = \partial_{xx} u(x,t) - \varepsilon x \partial_x u(x,t)\, \partial_x u(1,t),$$
$$0 < x < 1, \quad t \geq 0, \quad \text{(5a)}$$
$$u(1,t) = 0, \quad t \geq 0, \quad \text{(5b)}$$
$$-\varepsilon\partial_x u(1,t)\partial_x u(0,t) = \partial_{xt} u(0,t) - \frac{h'(t)}{h(t)}\partial_x u(0,t), \quad t \geq 0, \quad \text{(5c)}$$
$$u(x,0) = f(x), \quad 0 < x < 1. \quad \text{(5d)}$$

Problem (5a)-(5d) is a weakly nonlinear diffusion equation subject to nonlinear boundary conditions and initial conditions on a fixed spatial domain. In the next section, we will use the two timescales perturbation method to approximately solve this problem.

3 The two timescales perturbation method

When straightforward perturbation expansion for the solution $u(x,t)$ of problem (5a)-(5d) is used, then secular (that is, unbounded) terms in t will occur in the approximations for $u(x,t)$. To avoid these secular terms and to obtain approximations which are valid on long timescales, a two timescales perturbation method will be used to (approximately) solve problem (5a)-(5d).

It is assumed that the solution depends on $t_0 = t$ and $t_1 = \varepsilon t$, and that $u(x,t)$ can be expanded in

$$u(x,t) = u_0(x,t_0,t_1) + \varepsilon u_1(x,t_0,t_1) + \mathcal{O}(\varepsilon^2).$$

The time derivative operator is then $\partial_t = \partial_{t_0} + \varepsilon \partial_{t_1}$. By using the two timescales t_0 and t_1, by using the perturbation expansion for $u(x,t)$, and by taking apart terms of equal orders in ε, one obtains a family of initial-boundary value problems for $u_0, u_1, u_2, \ldots$: (see also [8,11,14] for a description of the method)

$$\mathcal{O}(1): \ \partial_{t_0} u_0(x,t_0,t_1) = \partial_{xx} u_0(x,t_0,t_1), \tag{6a}$$

$$\partial_{xt_0} u_0(0,t_0,t_1) = \frac{h'(t)}{h(t)} \partial_x u_0(0,t_0,t_1), \tag{6b}$$

$$u_0(1,t_0,t_1) = 0, \tag{6c}$$

$$u_0(x,0,0) = f(x), \tag{6d}$$

$$\mathcal{O}(\varepsilon): \ \partial_{t_0} u_1(x,t_0,t_1) + \partial_{t_1} u_0(x,t_0,t_1)$$
$$= \partial_{xx} u_1(x,t_0,t_1) \tag{6e}$$
$$- x \partial_x u_0(x,t_0,t_1) \partial_x u_0(1,t_0,t_1), \tag{6f}$$

$$- \partial_x u_0(1,t_0,t_1) \partial_x u_0(0,t_0,t_1) = \partial_{xt_0} u_1(0,t_0,t_1)$$
$$+ \partial_{xt_1} u_0(0,t_0,t_1) - \frac{h'(t)}{h(t)} \partial_x u_1(0,t_0,t_1), \tag{6g}$$

$$u_1(1,t_0,t_1) = 0, \tag{6h}$$

$$u_1(x,0,0) = 0, \tag{6i}$$

$$\mathcal{O}(\varepsilon^2): \ \partial_{t_0} u_2(x,t_0,t_1) + \partial_{t_1} u_1(x,t_0,t_1)$$
$$= \partial_{xx} u_2(x,t_0,t_1)$$
$$- x[\partial_x u_0(x,t_0,t_1) \partial_x u_1(1,t_0,t_1)$$
$$+ \partial_x u_1(x,t_0,t_1) \partial_x u_0(1,t_0,t_1)], \tag{6j}$$
$$- [\partial_x u_1(1,t_0,t_1) \partial_x u_0(0,t_0,t_1)$$
$$+ \partial_x u_0(1,t_0,t_1) \partial_x u_1(0,t_0,t_1)]$$
$$= \partial_{xt_0} u_2(0,t_0,t_1) + \partial_{xt_1} u_1(0,t_0,t_1)$$
$$- \frac{h'(t)}{h(t)} \partial_x u_2(0,t_0,t_1), \tag{6k}$$

$$u_2(1,t_0,t_1) = 0, \tag{6l}$$

$$u_2(x,0,0) = 0, \ldots \tag{6m}$$

In the next subsections, the $\mathcal{O}(1)$-problem, $\mathcal{O}(\varepsilon)$-problem, and $\mathcal{O}(\varepsilon^2)$-problem will be studied.

3.1 Solving the $\mathcal{O}(1)$-problem

Before solving the $\mathcal{O}(1)$-problem (6a)-(6d), we first simplify the boundary condition at $x = 0$ by integrating (6b) with respect to t_0, yielding

$$\partial_x u_0(0,t_0,t_1) = k_0(t_1) h(t_0),$$

where $k_0(t_1)$ is an arbitrary function depending on t_1. Moreover, we derive from (4b) that $\partial_x u_0(0,t) = -s(t)h(t)$. Since $h(0) > 0$, this implies that $k_0(0) = -s(0) = -b$. Using the transformation $\tilde{u}_0(x,t_0,t_1) = u_0(x,t_0,t_1) + k_0(t_1)h(t_0)(1-x)$, we obtain an initial-boundary value problem with homogeneous boundary conditions for $\tilde{u}_0$:

$$\partial_{t_0} \tilde{u}_0(x,t_0,t_1) = \partial_{xx} \tilde{u}_0(x,t_0,t_1) + H_0(x,t_0,t_1),$$
$$\partial_x \tilde{u}_0(0,t_0,t_1) = \tilde{u}_0(1,t_0,t_1) = 0,$$
$$\tilde{u}_0(x,0,0) = f(x) - bh(0)(1-x),$$

where $H_0(x,t_0,t_1) = k_0(t_1)h'(t_0)(1-x)$. The problem for $\tilde{u}(x,t_0,t_1)$ can readily be solved, and so $u(x,t_0,t_1)$ is given by

$$u_0(x,t_0,t_1) = \sum_{n=1}^{\infty} [u_{0n}(t_0,t_1)\phi_n(x)] - k_0(t_1)h(t_0)(1-x),$$

where $\phi_n(x) = \cos(\omega_n x)$, $\omega_n = \left(n - \frac{1}{2}\right)\pi$, and

$$u_{0n}(t_0,t_1) = e^{-\omega_n^2 t_0} \times \left[C_{0n}(t_1) - \frac{2k_0(t_1)}{\omega_n^2} \int_0^{t_0} e^{\omega_n^2 \eta} h'(\eta) d\eta \right].$$

Note that, this solution still contains some unknown functions in t_1, i.e., $k_0(t_1)$ and $C_{0n}(t_1)$. These functions will be chosen as such secular terms in u_1 can be avoided. To determine $u_0(x,t_0,t_1)$ completely, we have to solve the $\mathcal{O}(\varepsilon)$ problem (6f)-(6i).

3.2 Solving the $\mathcal{O}(\varepsilon)$-problem

Consider the $\mathcal{O}(\varepsilon)$ problem (6f)-(6i). Let us simplify (6g) to

$$\partial_x u_1(0,t_0,t_1) = h(t_0)\left[\int_0^{t_0} \frac{g_1(\eta,t_1)}{h(\eta)} d\eta + k_1(t_1)\right], \tag{7}$$

where $g_1(t_0,t_1) = -\partial_{xt_1} u_0(0,t_0,t_1) - \partial_x u_0(1,t_0,t_1)$ $\partial_x u_0(0,t_0,t_1)$, and $k_1(t_1)$ is an arbitrary function. We define $G_1(t_0,t_1) = \partial_x u_1(0,t_0,t_1)$. As in the previous subsection, we transform $\tilde{u}_1(x,t_0,t_1) = u_1(x,t_0,t_1) + G_1(t_0,t_1)(1-x)$ to obtain homogeneous boundary conditions for $\tilde{u}_1$, and to obtain for $\tilde{u}_1$:

$$\begin{aligned}
&\partial_{t_0}\tilde{u}_1(x,t_0,t_1) = \partial_{xx}\tilde{u}_1(x,t_0,t_1) + H_1(x,t_0,t_1),\\
&\partial_x \tilde{u}_1(0,t_0,t_1) = \tilde{u}_1(1,t_0,t_1) = 0,\\
&\quad \tilde{u}_1(x,0,0) = k_1(0)h(0)(1-x),
\end{aligned}$$

where

$$\begin{aligned}
H_1(x,t_0,t_1) &= \partial_{t_0}G_1(t_0,t_1)(1-x) - \partial_{t_1}u_0(x,t_0,t_1)\\
&\quad -x[\partial_x u_0(x,t_0,t_1)][\partial_x u_0(1,t_0,t_1)].
\end{aligned}$$

The problem for $\tilde{u}_1$ can simply be solved, and so we obtain

$$\begin{aligned}
u_1(x,t_0,t_1) = &\sum_{n=1}^{\infty}[u_{1n}(t_0,t_1)\phi_n(x)]\\
&- G_1(t_0,t_1)(1-x),
\end{aligned}$$

where

$$\begin{aligned}
u_{1n}(t_0,t_1) &= C_{1n}(t_1)e^{-\omega_n^2 t_0}\\
&\quad + \int_0^{t_0} e^{\omega_n^2(\eta - t_0)} H_{1n}(\eta,t_1)d\eta, \quad \text{and}\\
H_{1n}(t_0,t_1) &= 2\int_0^1 H_1(x,t_0,t_1)\cos(\omega_n x)\,dx.
\end{aligned}$$

Secular terms that occur in this solution have to be removed to avoid unbounded solution. In doing so, we can derive explicit expressions for the unknown functions $k_0(t_1)$ and $C_{0n}(t_1)$. The $\mathcal{O}(\varepsilon)$ solution itself also has other still unknown functions $k_1(t_1)$ and $C_{1n}(t_1)$ which can be used to avoid secular terms in u_2.

To compute u_0 completely, the heat inflow function $h(t)$ has to be specified. In Sect. 3.4 and 3.5, we will explicitly determine $k_0(t_1)$ and $C_{0n}(t_1)$ in $u_0(x,t_0,t_1)$ when the heat inflow is constant at the boundary $x=0$, and when the heat inflow is time-periodic, respectively.

3.3 Solving the $\mathcal{O}(\varepsilon^2)$-problem

Let us now consider the problem (6j)-(6m). We need to compute the solution of $\mathcal{O}(\varepsilon^2)$ problem to obtain later the $\mathcal{O}(\varepsilon)$ part of the solution for the moving interface $s(t)$. Observe first that

$$\begin{aligned}
&\partial_x u_1(x,t_0,t_1) = \sum_{n=1}^{\infty} u_{1n}(t_0,t_1)\phi_n'(x) + G_1(t_0,t_1) \Rightarrow\\
&\partial_{xt_1}u_1(0,t_0,t_1) = \partial_{t_1}G_1(t_0,t_1).
\end{aligned}$$

Now, defining

$$\begin{aligned}
g_2(t_0,t_1) = &-\partial_{xt_1}u_1(0,t_0,t_1)\\
&- [\partial_x u_1(1,t_0,t_1)\partial_x u_0(0,t_0,t_1)\\
&+ \partial_x u_0(1,t_0,t_1)\partial_x u_1(0,t_0,t_1)]\\
&= -\partial_{t_1}G_1(t_0,t_1)\\
&- \Bigg[\sum_{n=1}^{\infty}\phi_n'(1)\,(u_{1n}(t_0,t_1)G_0(t_0,t_1)\\
&+ u_{0n}(t_0,t_1)G_1(t_0,t_1))\\
&+ 2G_0(t_0,t_1)G_1(t_0,t_1)\Bigg],
\end{aligned}$$

equation (6k) can be simplified to

$$u_{2x}(0,t_0,t_1) = h(t_0)\left[\int_0^{t_0}\frac{g_2(\eta,t_1)}{h(\eta)}d\eta + k_2(t_1)\right]. \tag{8}$$

We set $G_2(t_0,t_1) = \partial_x u_2(0,t_0,t_1)$. Because now we have a similar problem as for the $\mathcal{O}(\varepsilon)$ problem, we can proceed in the same way and obtain

$$\begin{aligned}
u_2(x,t_0,t_1) = &\sum_{n=1}^{\infty}[u_{2n}(t_0,t_1)\phi_n(x)]\\
&- G_2(t_0,t_1)(1-x),
\end{aligned}$$

where

$$\begin{aligned}
u_{2n}(t_0,t_1) &= C_{2n}(t_1)e^{-\omega_n^2 t_0}\\
&\quad + \int_0^{t_0} e^{\omega_n^2(\eta-t_0)}H_{2n}(\eta,t_1)d\eta,\\
H_{2n}(t_0,t_1) &= 2\int_0^1 H_2(x,t_0,t_1)\cos(\omega_n x)\,dx,
\end{aligned}$$

and where

$$H_2(x,t_0,t_1) = \partial_{t_0} G_2(1-x) - x\big[\partial_x u_0(x,t_0,t_1)\partial_x u_1(1,t_0,t_1) + \partial_x u_1(x,t_0,t_1)\partial_x u_0(1,t_0,t_1)\big].$$

3.4 The case with constant heat inflow at the boundary $x = 0$

Let us consider the simplest situation where the heat flux at the boundary $x = 0$ is constant. An exact solution for this problem is not available, but by using the similarity method, an ordinary differential equation, which describes the moving interface, can be obtained. The numerical approximation of the solution of this ordinary differential equation will be compared in Sect. 4.3 of this paper with the approximation we will now be constructing by using the multiple timescales perturbation method.

Suppose $h(t) = a$, where a is a positive constant. Then, the $\mathcal{O}(1)$ solution u_0 is

$$u_0(x,t_0,t_1) = \sum_{n=1}^{\infty} C_{0n}(t_1)e^{-\omega_n^2 t_0}\cos(\omega_n x) - ak_0(t_1)(1-x).$$

For the $\mathcal{O}(\varepsilon)$ problem, the boundary condition (7) becomes

$$\partial_x u_1(0,t_0,t_1) = ak_1(t_1) - t_0\left(a^2k_0^2(t_1) + ak_0'(t_1)\right) + ak_0(t_1)\sum_{n=1}^{\infty}\left[\frac{\phi_n'(1)}{\omega_n^2}u_{0n}(t_0,t_1)\right].$$

To avoid unbounded solutions in t_0, we have to take $ak_0^2(t_1)+k_0'(t_1) = 0$ with initial condition $k_0(0) = -b$. That gives us

$$k_0(t_1) = \frac{b}{bat_1 - 1}. \tag{9}$$

Next, we compute

$$\int_0^{t_0} e^{\omega_n^2\eta}H_{1n}(\eta,t_1)d\eta = -\left[C_{0n}'(t_1) - \frac{3ak_0(t_1)C_{0n}(t_1)}{2}\right]t_0 + n.s.t,$$

where $n.s.t$ stands for non-secular terms. So, to avoid unbounded solutions in t_0, it follows that $C_{0n}'(t_1) - \frac{1}{2}3ak_0(t_1)C_{0n}(t_1) = 0$. Solving this equation for $C_{0n}(t_1)$, and by using the initial conditions (6d), it follows that

$$C_{0n}(t_1) = \left(f_n - \frac{2ab}{\omega_n^2}\right)(1-abt_1)^{\frac{3}{2}},$$

where f_n is the n-th Fourier series coefficient of $f(x)$. Thus, the solution $v(x,t)$ is

$$v(x,t) = \varepsilon\left[\sum_{n=1}^{\infty}\left(f_n - \frac{2ab}{\omega_n^2}\right)\frac{(1-ab\varepsilon t)^{\frac{3}{2}}}{e^{\omega_n^2 t}}\cos(\omega_n x) + \frac{ab(1-x)}{1-ab\varepsilon t}\right] + \mathcal{O}(\varepsilon^2). \tag{10}$$

Solution (10) contains a singularity at $ab\varepsilon t = 1$. However, this is an artificial singularity due to the time scaling in (3). Transforming back to the time variable τ will remove this singularity. From (10), it can be seen that the influence of the initial temperature distribution (given by the Fourier coefficients f_n) on the solution $v(x,t)$ decreases exponentially in time. This implies that the moving interface is stable in time. The moving interface will be discussed further in Sects. 3.6 and in 4.

3.5 The case with periodic heat flux at the boundary $x = 0$

Let us consider a more general case: the case where the incoming heat at $x = 0$ is T-periodic and positive definite. We assume that T is $\mathcal{O}(1)$. We can expand $h(t)$ in its Fourier series $a+\sum_{n=1}^{\infty}(A_n\sin(\kappa_n t)+B_n\cos(\kappa_n t))$, where $\kappa_n = \frac{2n\pi}{T}$ for some constants a, A_n, and B_n. The solution for u_{0n} then becomes

$$u_{0n}(t_0,t_1) = Q_n(t_1)e^{-\omega_n^2 t_0} - R_n(t_0,t_1),$$

where

$$Q_n(t_1) = C_{0n}(t_1) + \sum_{m=1}^{\infty}\frac{2\kappa_m k_0(t_1)}{\omega_n^2(\omega_n^4+\kappa_m^2)}(A_m\omega_n^2 + B_m\kappa_m), \quad \text{and} \tag{11}$$

$$R_n(t_0,t_1) = \sum_{m=1}^{\infty}\frac{2\kappa_m k_0(t_1)}{\omega_n^2(\omega_n^4+\kappa_m^2)}$$

$$\left[(A_m\omega_n^2 + B\kappa_m)\cos(\kappa_m t_0) + (A_m\kappa_m - B_m\omega_n^2)\sin(\kappa_m t_0)\right]. \tag{12}$$

Computing u_{1x} gives us

$$u_{1x}(0, t_0, t_1) = -h(t_0)\Bigg[t_0\left(k_0'(t_1) + ak_0^2(t_1)\right) - \sum_{n=1}^{\infty}\frac{k_0^2(t_1)}{\kappa_n}(B_n\sin(\kappa_n t_0) - A_n\cos(\kappa_n t_0)) + k_0(t_1)\sum_{n=1}^{\infty}\left(\phi_n'(1)\int_0^{t_0}u_{n0}(\eta, t_1)d\eta\right) - k_1(t_1)\Bigg].$$

We recognize the first term inside the square bracket as a secular term. As a consequence, $k_0(t_1)$ has to satisfy (9). Computing further, the solution leads to similar results as in the case of the constant heat flux, i.e.,

$$\int_0^{t_0} e^{\omega_n^2\eta}H_{1n}(\eta, t_1)d\eta = -\left[Q_n'(t_1) - \frac{3ak_0(t_1)Q_n(t_1)}{2}\right]t_0 + n.s.t..$$

Removing the secular term gives us $Q_n(t_1) = Q_n(0)(1-abt_1)^{\frac{3}{2}}$, from which we can derive

$$C_{0n}(t_1) = \left[f_n - \frac{2b}{\omega_n^2}\left(a + \sum_{m=1}^{\infty}\left(B_m + \kappa_m\frac{A_m\omega_n^2 + B_m\kappa_m}{(\omega_n^4 + \kappa_m^2)}\right)\right)\right](1-abt_1)^{\frac{3}{2}} - \sum_{m=1}^{\infty}\frac{2\kappa_m k_0(t_1)(A_m\omega_n^2 + B_m\kappa_m)}{\omega_n^2(\omega_n^4 + \kappa_m^2)}.$$

Thus, the solution $v(x, t)$ can be written in a similar form as the solution for the constant case:

$$v(x, t) = \varepsilon\left[\sum_{n=1}^{\infty}\left(Q_n(\varepsilon t)e^{-\omega_n^2 t_0} - R_n(t, \varepsilon t)\right)\cos(\omega_n x) + \frac{ab(1-x)}{1-ab\varepsilon t}\right] + \mathcal{O}(\varepsilon^2), \tag{13}$$

where Q and R are given by (11) and (12), respectively.

As an example, we can assume that the heat flux at $x = 0$ is alternating on and off and is described by a square wave $h(t) = 2a\sum_{n=1}^{\infty}(-1)^n H(t - nT)$, where H is the Heaviside function, and T is the period of the wave. In this case, the Fourier series of the flux $h(t)$ has the form

$$h(t) = a + \frac{4a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{2\pi(2n-1)t}{T}\right).$$

We can directly use the general solution (13) by setting

$$\kappa_n = \omega_n, \quad A_n = \frac{2a}{\omega_n}, \quad \text{and} \quad B_n = 0.$$

Thus, $Q_n(t_1)$ and $R_n(t_1)$ become

$$Q_n(t_1) = \left[f_n - \frac{2ab}{\omega_n^2}\left(1 + 2\sum_{m=1}^{\infty}\frac{\omega_m\omega_n^2}{(\omega_n^4 + \omega_m^2)}\right)\right](1-abt_1)^{\frac{3}{2}},$$

$$R_n(t_0, t_1) = \sum_{m=1}^{\infty}\frac{4ak_0(t_1)}{\omega_n^2(\omega_n^4 + \omega_m^2)}\left[\omega_n^2\cos(\omega_m t_0) + \omega_m\sin(\omega_m t_0)\right].$$

Observe that the periodic heat flux leads to the same function for $k_0(t_1)$ as in the previous Subsection. From (13), it can be seen that $v(x, t)$ and so the moving interface $s(t)$ are mainly determined by the overall average of the periodic heat flux at the boundary $x = 0$, that is, by the constant a. The non-constant periodic parts in $h(t)$ only lead to relatively small, periodic fluctuations (with average zero over a period T) in the moving interface position $s(t)$.

3.6 The formula for the moving boundary

We still need to compute the moving interface $s(t)$ from the solution of the temperature profile $v(x, t)$. From (4b), we can directly see that

$$s(t) = -\frac{\partial_x v(0, t)}{\varepsilon h(t)} = k_0(\varepsilon t) + \mathcal{O}(\varepsilon).$$

In this Subsection, we restrict ourselves to the case with constant heat flux at $x = 0$ (see Sect. 3.4). We write first that

$$s(t_0, t_1) = -\frac{u_x(0, t_0, t_1)}{a} = -\frac{1}{a}(u_{0x}(0, t_0, t_1) + \varepsilon u_{1x}(0, t_0, t_1) + \mathcal{O}(\varepsilon^2)).$$

If we only take the $\mathcal{O}(1)$ part, then the solution will be
$s(t) = \dfrac{b}{1 - ab\varepsilon t} + \mathcal{O}(\varepsilon)$,
which in its original variables S and τ become

$$S(\tau) = b + \varepsilon a\tau + \mathcal{O}(\varepsilon^2). \tag{14}$$

This result is valid at least up to $\tau = \mathcal{O}(1)$. To obtain better approximations on longer timescales, we need to include higher order terms. In this case, we have

$$s(t_0, t_1) = \frac{b}{1 - abt_1} + \varepsilon R(t_0, t_1) + \mathcal{O}(\varepsilon^2), \tag{15}$$

where

$$R(t_0, t_1) = -\frac{u_{1x}(0, t_0, t_1)}{a} = -k_0(t_1) \sum_{n=1}^{\infty} \left[\frac{\phi_n'(1) C_{0n}(t_1)}{\omega_n^2} e^{-\omega_n^2 t_0} \right] - k_1(t_1).$$

We need to know $k_1(t_1)$ explicitly. To obtain $k_1(t_1)$, we have to compute the solution $u_2(x, t_0, t_1)$ of the $\mathcal{O}(\varepsilon^2)$ problem, where we have boundary condition (8). To find k_1, we only need to identify the secular terms in $u_{2x}(0, t_0, t_1)$. If we look at the formula of g_2, we see that secular terms can be avoided when

$$-ak_1'(t_1) - 2a^2 k_1(t_1) k_0(t_1) = 0,$$

and so,

$$k_1(t_1) = \frac{k_1(0)}{(1 - abt_1)^2}.$$

Since $\partial_x u_1(0, 0) = 0$, we find that

$$k_1(0) = -b \sum_{n=1}^{\infty} \frac{(-1)^n C_{0n}(0)}{\omega_n}.$$

Thus,

$$k_1(t_1) = -\frac{b}{(1 - abt_1)^2} \sum_{n=1}^{\infty} \frac{(-1)^n C_{0n}(0)}{\omega_n}.$$

So, we can compute R in (15) explicitly, yielding

$$R(t_0, t_1) = b(1 - abt_1)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{\omega_n} \left(f_n - \frac{2ab}{\omega_n^2} \right) \left[e^{-\omega_n^2 t_0} - (1 - abt_1)^{-\frac{5}{2}} \right].$$

Writing in full, we have

$$s(t) = \frac{b}{1 - ab\varepsilon t} \left[1 + \varepsilon (1 - ab\varepsilon t)^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{\omega_n} \left(f_n - \frac{2ab}{\omega_n^2} \right) \left[e^{-\omega_n^2 t} - (1 - ab\varepsilon t)^{-\frac{5}{2}} \right] \right] + \mathcal{O}(\varepsilon^2).$$

Transforming this solution back to the original variables S and τ is difficult, as it involves infinite series. In this case, we will compute $S(\tau)$ implicitly. To do so, we compute first $s(t)$ for some values of t up to the N-th term. Then, for each t, we compute τ by using the inverse transformation

$$\tau(t) = \int_0^t s^2(\eta) d\eta. \tag{16}$$

For each τ obtained, we map it to the respective $s(t)$ to obtain the value of $S(\tau) = s(t(\tau))$. This procedure can be done numerically in Sect. 4.3 where we compare it with a similarity solution. We will also compare it with results in the literature [3,17], where a straightforward perturbation approach was used.

4 Numerical comparisons

4.1 Previous work

In [3,17], an initial boundary value problem similar to (2a)-(2e) is considered. In both papers straightforward, naive perturbation expansions are used and it is well known that the obtained results are usually not accurate on long timescales (see [8,11,14]). We use those results for comparison with ours. The approximations as obtained in [3,17] for the moving boundary variable are in the following form with adjusted notation.

$$S(\tau) = \frac{\sqrt{7} \tan\left(\frac{\sqrt{7}}{2} \gamma^2 \tau - \arctan\left(\frac{1}{\sqrt{7}} \right) \right) + 1}{4\gamma} + C, \tag{17}$$

where γ is the Stefan number times the constant heat flux at the boundary, and C is a constant corresponding to the initial condition. In our case, $C = b$, the Stefan number equals to one, and the heat flux is the small number εa.

4.2 An equation for the similarity solution

An explicit analytical solution for the problem, we are considering, is not available. However, we can use the similarity method to obtain a first-order, nonlinear ordinary differential equation which describes the analytical solution. To obtain an approximation of the solution of this differential equation, we have to integrate the equation by using numerical methods. For our problem, the similarity transformations are

$$U(X,\tau) = y(z)\sqrt{\tau}, \quad z = \frac{X}{\sqrt{\tau}}.$$

Subsituting these transformations into the problem gives us the following problem:

$$y'' + \frac{z}{2}y' - \frac{y}{2} = 0, \tag{18a}$$

$$y'(0) = -\varepsilon a, \tag{18b}$$

$$y'\left(S(\tau)\tau^{-\frac{1}{2}}\right) = -S'(\tau), \tag{18c}$$

$$y\left(S(\tau)\tau^{-\frac{1}{2}}\right) = 0. \tag{18d}$$

The differential equation (18a) for y can readily be solved, yielding

$$y(z) = c_2 z - c_1\left[e^{-\frac{z^2}{4}} + \sqrt{\pi}\frac{z}{2}\text{erf}\left(\frac{z}{2}\right)\right],$$

and by using the boundary conditions (18b)-(18d), we finally obtain

$$S'(\tau) = \frac{2\varepsilon a e^{-\frac{S^2}{4\tau}}\sqrt{\tau}}{2e^{-\frac{S^2}{4\tau}}\sqrt{\tau} + S\sqrt{\pi}\text{erf}\left(\frac{S}{2\sqrt{\tau}}\right)}. \tag{19}$$

The differential equation (19) cannot be solved explicitly. In the next section, we will use a numerical integration method (i.e., the Adaptive Runge–Kutta method) to obtain numerical approximations of S as function of τ.

4.3 Numerical results

In this subsection, a constant heat flux at the boundary $x = 0$ is considered. We will compare and plot in Fig. 2 to Fig. 5 the approximations as obtained in [3,17], the numerical approximations as obtained from the similarity approach (see Sect. 4.2 of this paper), and the approximations as obtained using the two timescales perturbation approach (see Sect. 3.4. Different sets of values for the parameters a, b, and ε are chosen. In Fig. 2, the choice for the parameters is $a = b = 1$, and $\varepsilon = 0.1$. For small times τ, the three approximations are close to each other. For larger times τ, the approximation obtained by using a straightforward perturbation method blows up (due to a singularity in the approximation (17), i.e., the tan-function becomes large), where as the other two approximations stay close to each other. The small difference between the "similarity" approximation and the "two timescales perturbation" approximation is most likely due to the fact that higher-order correction terms (in ε) in the last approximation are neglected. In Fig. 3 to Fig. 5, the choices for the parameters are $\{a = 1, b = 0.3, \varepsilon = 0.1\}$, $\{a = 2, b = 0.3, \varepsilon = 0.1\}$, and $\{a = 1, b = 0.3, \varepsilon = 0.05\}$, respectively. Also, in Fig. 3 to 5, similar behavior as in Fig. 2 can be observed, and it can be concluded that the (exact) similarity solution and the approximation as obtained by the two timescales perturbation method stay close to each other for long times τ.

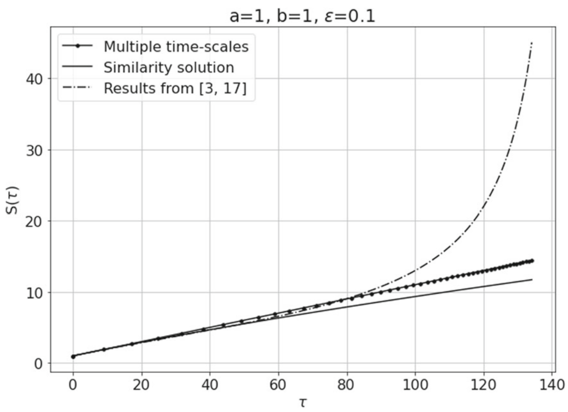

Fig. 2 Approximations of moving boundary profiles $S(\tau)$ by using different methods, and $a = 1$, $b = 1$, $\varepsilon = 0.1$

5 Conclusion

In this paper, it is shown how the multiple timescales perturbation method can be applied to approximate the solution of a classical Stefan problem with a time-dependent heat flux at the boundary. How the solution

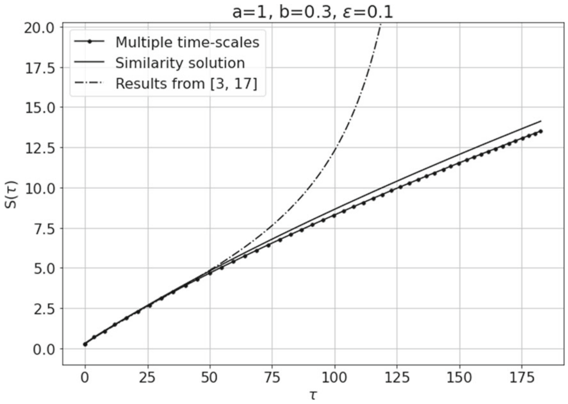

Fig. 3 Approximations of moving boundary profiles $S(\tau)$ by using different methods, and $a = 1$, $b = 0.3$, $\varepsilon = 0.1$

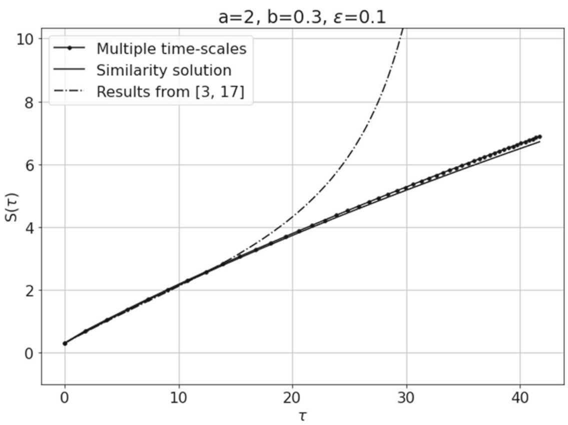

Fig. 4 Approximations of moving boundary profiles $S(\tau)$ by using different methods, and $a = 2$, $b = 0.3$, $\varepsilon = 0.1$

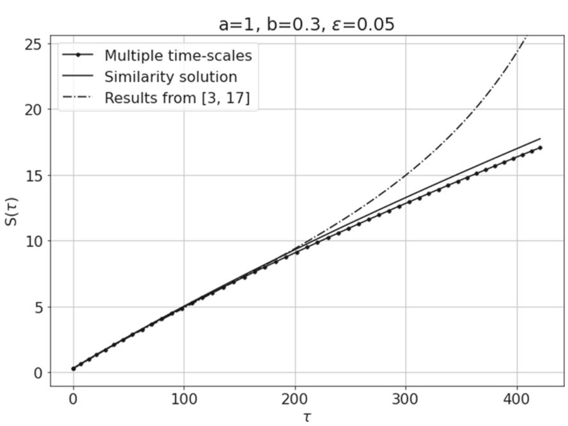

Fig. 5 Approximations of moving boundary profiles $S(\tau)$ by using different methods, and $a = 1$, $b = 0.3$, $\varepsilon = 0.05$

is influenced by this heat flux and by the initial conditions is shown explicitly. It is also shown that the results as obtained by the multiple timescales perturbation method agree well with those for which "exact" solutions in numerical form are available. The applicability of the multiple timescales perturbation method to this Stefan problem opens possibilities for future research to more complicated moving boundary problems.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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