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## On the monotonicity of tail probabilities

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# ON THE MONOTONICITY OF TAIL PROBABILITIES* 

BY

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#### Abstract

Let $S$ and $X$ be independent random variables, assuming values in the set of non-negative integers, and suppose further that both $\mathbb{E}(S)$ and $\mathbb{E}(X)$ are integers satisfying $\mathbb{E}(S) \geqslant \mathbb{E}(X)$. We establish a sufficient condition for the tail probability $\mathbb{P}(S \geqslant \mathbb{E}(S))$ to be larger than the tail $\mathbb{P}(S+X \geqslant \mathbb{E}(S+X))$, when the mean of $S$ is equal to the mode.


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## 1. MAIN RESULT

We are interested in the comparison between the tails $\mathbb{P}(S \geqslant \mathbb{E}(S))$ and $\mathbb{P}(S+X \geqslant$ $\mathbb{E}(S+X)$ ), where $S$ and $X$ are independent random variables. In everyday language, suppose an enterprise $S$ is successful if the result exceeds the mean; would it be beneficial to include one more enterprise $X$ ? In many applications, $S$ is a sum of independent random variables and $X$ adds one more to the sum. By the central limit theorem, $\mathbb{P}(S \geqslant \mathbb{E}(S))$ converges to $1 / 2$. Therefore, if $\mathbb{P}(S \geqslant \mathbb{E}(S))>1 / 2$ (the enterprise is favorably skewed), one would expect that adding one more term to the sum would lower this probability.

All random variables under consideration take values in $\mathbb{N} \cup\{0\}$. We establish an inequality that applies to random variables that satisfy certain "skewness" conditions. Throughout the text, given a positive integer $n$, we denote the set $\{1, \ldots, n\}$ by $[n]$.

[^0]Definition 1.1 (Right-skewness). Assume that $S$ is unimodal with mode $s$. Then we say that $S$ is right-skewed if

$$
\mathbb{P}(S=s-i) \leqslant \mathbb{P}(S=s+i-1) \quad \text { for all } i \in[s]
$$

In our definition, we allow that the mode is not unique. It is possible that $\mathbb{P}(S=s-1)=\mathbb{P}(S=s)$ and that is why we put the $\leqslant$ sign. If the inequality is strict, then the inequality in our main result is also strict.

Definition 1.2 (Left-loadedness). Let $X$ be a random variable such that $m:=\mathbb{E}(X)$ is an integer. For $i \in[m]$, set $\alpha_{i}:=\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant m+i)$. Then we say that $X$ is left-loaded if either of the following two conditions holds true:
$\left(L_{1}\right)$ : The sequence $\left\{\alpha_{i}\right\}_{i=1}^{m}$ changes sign once from positive to negative, i.e., there exists $\ell \in[m]$ such that $\alpha_{i} \geqslant 0$ for $i \leqslant \ell$, and $\alpha_{i} \leqslant 0$ for $i>\ell$.
$\left(L_{2}\right): \sum_{i=1}^{k} \alpha_{i} \geqslant 0$ for all $k \in[m]$.
A random variable can be both right-skewed and left-loaded. For instance, if $\mathbb{E}(S)=1$ then it is not hard to prove that $S$ is left-loaded. If such an $S$ is unimodal, such as the binomial distribution $\operatorname{Bin}(n, 1 / n)$, then it is also right-skewed. Another example is a geometric random variable with parameter $1 / n$. Our main result reads as follows.

THEOREM 1.1. Let $s \geqslant m$ be two positive integers. Suppose that $S$ and $X$ are independent random variables, assuming values in the set of non-negative integers, that satisfy the following conditions:

- $S$ is right-skewed with mode s.
- X is left-loaded with mean $m$.

Then $\mathbb{P}(S \geqslant s) \geqslant \mathbb{P}(S+X \geqslant s+m)$.
Note that we have replaced the mean of $S$ by its mode. If $S$ is binomial or Poisson with integer mean, then the mean is equal to the mode. We will show that Poisson random variables with integer mean are both right-skewed and left-loaded, and that binomial random variables are right-skewed if $p \leqslant 1 / 2$. We conjecture that a binomial random variable is left-loaded if it has integer mean and $p \leqslant 1 / 2$. This seems to be hard to prove and is related to an old inequality of Simmons [6].

Our inequality is well-established for standard random variables. Let Poi( $\lambda$ ) denote a Poisson random variable of mean $\lambda$. Teicher [7] showed that

$$
\begin{equation*}
\mathbb{P}(\operatorname{Poi}(k) \geqslant k) \geqslant \mathbb{P}(\operatorname{Poi}(k+1) \geqslant k+1) \quad \text { for all } k \geqslant 1, \tag{1.1}
\end{equation*}
$$

which follows from our result if we take $S \sim \operatorname{Poi}(k)$ and $X \sim \operatorname{Poi}(1)$. Let $\operatorname{Bin}(m, p)$ denote a binomial random variable of parameters $m$ and $p \in(0,1)$.

Chaundy and Bullard [1] showed that for every fixed positive integer $n \geqslant 1$ and probability $p=1 / n$,

$$
\begin{equation*}
\mathbb{P}(\operatorname{Bin}(n k, p) \geqslant k) \geqslant \mathbb{P}(\operatorname{Bin}(n(k+1), p) \geqslant k+1) \quad \text { for all } k \geqslant 1 \tag{1.2}
\end{equation*}
$$

This follows from our result if we take $S \sim \operatorname{Bin}(n k, p)$ and $X \sim \operatorname{Bin}(n, p)$ for $p=1 / n$. We remark that both inequalities 1.1 and (1.2) concern the monotonicity of tail probabilities of the form $\mathbb{P}\left(S_{k} \geqslant \mathbb{E}\left(S_{k}\right)\right)$, where $S_{k}$ is a sum of $k$ independent random variables of mean 1 . These results have been extended to the case of integer means (see [3, Theorem 2.1] and [4, Theorem 2.3]), and several of those extensions can be deduced from our main result. However, Theorem 1.1 provides a bit more, since it allows one to convolute different distributions. For example, it follows from the results in Section 3 that Theorem 1.1 implies that $\mathbb{P}(S \geqslant s) \geqslant \mathbb{P}(S+X \geqslant \mathbb{E}(S+X))$ for $S \sim \operatorname{Bin}(n, s / n)$ with $n \geqslant 2 s$, and $X \sim \operatorname{Poi}(m)$ with $s \geqslant m$, a result which may be seen as a "mixture" of 1.1) and (1.2).

The tail probability $\mathbb{P}(S \geqslant \mathbb{E}(S))$ has been extensively studied for Poisson random variables, motivated by a conjecture by Ramanujan that was eventually settled by Flajolet. This research is ongoing and results continue to be sharpened and extended; see [2] for recent progress and further references. It is not possible to deduce such refined results for parametrized families from our inequality, which puts relatively weak constraints on the distributions of $S$ and $X$.

## 2. PROOF OF MAIN RESULT

We begin with an observation.
Lemma 2.1. Let $X$ be a random variable, assuming non-negative integer values, such that $m:=\mathbb{E}(X)$ is an integer. Then

$$
\sum_{i=1}^{m}(\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant m+i))=\sum_{i \geqslant m+1} \mathbb{P}(X \geqslant m+i)
$$

In particular, $\sum_{i=1}^{m}(\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant m+i)) \geqslant 0$.
Proof. Notice that

$$
m=\sum_{i=1}^{m} \mathbb{P}(X \geqslant i)+\sum_{i=m+1}^{2 m} \mathbb{P}(X \geqslant i)+\sum_{i \geqslant 2 m+1} \mathbb{P}(X \geqslant i)
$$

which, upon transferring the first two sums on the right to the other side, is equivalent to

$$
\sum_{i=1}^{m}(\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant m+i))=\sum_{i \geqslant m+1} \mathbb{P}(X \geqslant m+i)
$$

We now prove our main result, which applies to random variables that are skewed to the right. One would expect that there exists a corresponding result for variables that are skewed to the left. However, our proof does not easily transfer to this case. One problem is that the inequality $\sum_{i=1}^{m}(\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant$ $m+i)) \geqslant 0$ holds for all random variables. It does not change sign if we skew the random variable to the left.

Proof of Theorem 1.1. If we condition on $S$ we have

$$
\begin{aligned}
\mathbb{P}(S+X \geqslant s+m) & =\sum_{i \geqslant 0} \mathbb{P}(X \geqslant s+m-i) \cdot \mathbb{P}(S=i) \\
& =\mathbb{P}(S \geqslant s+m)+\sum_{i=0}^{s+m-1} \mathbb{P}(X \geqslant s+m-i) \cdot \mathbb{P}(S=i)
\end{aligned}
$$

Hence $\mathbb{P}(S+X \geqslant s+m) \leqslant \mathbb{P}(S \geqslant s)$ is equivalent to

$$
\sum_{i=0}^{s+m-1} \mathbb{P}(X \geqslant s+m-i) \cdot \mathbb{P}(S=i) \leqslant \sum_{i=s}^{s+m-1} \mathbb{P}(S=i)
$$

which can be rearranged as

$$
\sum_{i=0}^{s-1} \mathbb{P}(S=i) \cdot \mathbb{P}(X \geqslant s+m-i) \leqslant \sum_{i=s}^{s+m-1} \mathbb{P}(S=i) \cdot \mathbb{P}(X \leqslant s+m-i-1)
$$

This is equivalent to

$$
\begin{equation*}
\sum_{i=1}^{s} \mathbb{P}(S=s-i) \cdot \mathbb{P}(X \geqslant m+i) \leqslant \sum_{i=1}^{m} \mathbb{P}(S=s+i-1) \cdot \mathbb{P}(X \leqslant m-i) \tag{2.1}
\end{equation*}
$$

Let $L$ and $R$ denote the left-hand side and the right-hand side of (2.1). Since $S$ is unimodal with mode $s \geqslant m$, we can estimate $L$ as follows:

$$
\begin{aligned}
L \leqslant & \sum_{i=1}^{m} \mathbb{P}(S=s-i) \cdot \mathbb{P}(X \geqslant m+i) \\
& +\mathbb{P}(S=s-m-1) \cdot \sum_{i=m+1}^{s} \mathbb{P}(X \geqslant m+i) \\
= & \ell_{1}+\ell_{2}
\end{aligned}
$$

with the convention that $\ell_{2}$ is equal to 0 when $s=m$. Now, since $S$ is right-skewed, we have

$$
\begin{equation*}
\ell_{1} \leqslant \sum_{i=1}^{m} \mathbb{P}(S=s+i-1) \cdot \mathbb{P}(X \geqslant m+i)=: R_{1} \tag{2.2}
\end{equation*}
$$

Using again the right-skewness of $S$ and Lemma 2.1, we have

$$
\begin{equation*}
\ell_{2} \leqslant \mathbb{P}(S=s+m) \cdot\left(\sum_{i=1}^{m}(\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant m+i))\right)=: R_{2} . \tag{2.3}
\end{equation*}
$$

It follows from (2.1)-2.3) that it is enough to show that $R_{1}+R_{2} \leqslant R$, or equivalently
$\sum_{i=1}^{m}(\mathbb{P}(S=s+i-1)-\mathbb{P}(S=s+m)) \cdot(\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant m+i)) \geqslant 0$.
For each $i \in[m]$, let $\Delta_{i}:=\mathbb{P}(S=s+i-1)-\mathbb{P}(S=s+m)$ as well as $\alpha_{i}:=\mathbb{P}(X \leqslant m-i)-\mathbb{P}(X \geqslant m+i)$, and note that (2.4) is equivalent to

$$
\begin{equation*}
\sum_{i=1}^{m} \Delta_{i} \cdot \alpha_{i} \geqslant 0 \tag{2.5}
\end{equation*}
$$

The unimodality of $S$ implies that $\Delta_{1} \geqslant \cdots \geqslant \Delta_{m} \geqslant 0$. We distinguish two cases.
Suppose first that $X$ satisfies condition $\left(L_{1}\right)$. Let $\ell \in[m]$ be such that $\alpha_{i} \geqslant 0$ for $i \leqslant \ell$, and $\alpha_{i} \leqslant 0$ for $i>\ell$. Then, since $\left\{\Delta_{i}\right\}_{i \in[m]}$ is non-increasing, it follows that

$$
\sum_{i=1}^{m} \Delta_{i} \cdot \alpha_{i} \geqslant \Delta_{\ell} \sum_{i=1}^{\ell} \alpha_{i}+\Delta_{\ell} \sum_{i=\ell+1}^{m} \alpha_{i}=\Delta_{\ell} \sum_{i \in[m]} \alpha_{i} \geqslant 0
$$

where the last estimate follows from the second statement in Lemma 2.1. Hence we obtain (2.5) and the result follows.

Now assume that $X$ satisfies condition $\left(L_{2}\right)$. Set $\Sigma_{i}:=\sum_{j=1}^{i} \alpha_{j}$ for $i \in[m]$, and notice that $\Sigma_{i} \geqslant 0$ by assumption. Using summation by parts, we have

$$
\sum_{i=1}^{m} \Delta_{i} \cdot \alpha_{i}=\Delta_{m} \cdot \Sigma_{m}+\sum_{i=1}^{m-1}\left(\Delta_{i}-\Delta_{i+1}\right) \cdot \Sigma_{i} \geqslant 0
$$

Hence, we obtain (2.5) and the result follows.

## 3. SKEWNESS OF RANDOM VARIABLES

The standard examples of non-negative random variables that take values in $\mathbb{N} \cup\{0\}$ are Poisson, binomial, or negative binomial. We examine their "skewness" properties.

Lemma 3.1. Fix a positive integer $s$, and let $S \sim \operatorname{Poi}(s)$. Then $S$ is rightskewed.

Proof. Since $s$ is a positive integer it follows that the mode of $S$ is equal to $s$. For $i \in[s]$, let $\beta_{i}=\frac{\mathbb{P}(S=s-i)}{\mathbb{P}(S=s+i-1)}$. Since the mode of $S$ is equal to $s$, it follows that
$\beta_{1} \leqslant 1$. Next, note that $\beta_{i} \geqslant \beta_{i+1}$ is equivalent to $s^{2} \geqslant s^{2}-i^{2}$, which is clearly correct for each $i \in[s]$. Hence, the sequence $\left\{\beta_{i}\right\}_{i=1}^{s}$ is non-increasing, and the fact that $\beta_{1} \leqslant 1$ finishes the proof.

Lemma 3.2. Fix a positive integer $s$, and let $S \sim \operatorname{Bin}(n, p)$ for some $n \geqslant 2 s$ with $p=s / n$. Then $S$ is right-skewed.

Proof. The proof is similar to the proof of Lemma 3.1. Let $\beta_{i}=\frac{\mathbb{P}(S=s-i)}{\mathbb{P}(S=s+i-1)}$ for $i \in[s]$. Since $S$ is unimodal with mode $s$, we have $\beta_{1} \leqslant 1$. Furthermore, $\beta_{i} \geqslant \beta_{i+1}$ is equivalent to

$$
\begin{equation*}
s^{2} \cdot\left((n-s+1)^{2}-i^{2}\right) \geqslant(n-s)^{2} \cdot\left(s^{2}-i^{2}\right) \tag{3.1}
\end{equation*}
$$

Now observe that (3.1) holds true when $s^{2} \cdot\left((n-s)^{2}-i^{2}\right) \geqslant(n-s)^{2} \cdot\left(s^{2}-i^{2}\right)$ and the latter is equivalent to $n-s \geqslant s$, which is true by assumption. Hence (3.1) holds true and we conclude that the sequence $\left\{\beta_{i}\right\}_{i \in[s]}$ is non-decreasing. The result follows.

We denote the negative binomial distribution by $N B(r, p)$ where $r \in \mathbb{N}$ is the number of failures and $p \in(0,1)$ is the probability of success. If $S \sim N B(r, p)$ then $\mathbb{P}(S=k)=\binom{k+r-1}{r-1} p^{k} q^{r}$ with $q=1-p$ the probability of failure. If $q=1 / n$, the negative binomial has mean $r(n-1)$ and mode $(r-1)(n-1)$.

Lemma 3.3. Let $S \sim N B(r, p)$ with $p=1-1 / n$ for some integer $n>1$. Then $S$ is right-skewed.

Proof. Let $a_{k}=\mathbb{P}(S=k)$. Then

$$
\frac{a_{k+1}}{a_{k}}=\frac{(k+r) p}{k+1}
$$

is $\leqslant 1$ if and only if $k+1 \geqslant p(r-1) / q$. In particular, $S$ is unimodal with mode $\lfloor p(r-1) / q\rfloor$, which is equal to $s=(n-1)(r-1)$ for our choice of $p$. To prove that $S$ is right-skewed, it suffices to show that $\frac{a_{s-i-1}}{a_{s-i}} \leqslant \frac{a_{s+i}}{a_{s+i-1}}$, in other words,

$$
\frac{s-i}{(s+r-1-i) p} \leqslant \frac{(s+r-1+i) p}{s+i}
$$

For our choice of $p$, this is equivalent to

$$
\frac{s-i}{s-i p} \leqslant \frac{s+i p}{s+i}
$$

which obviously holds true.
We have thus established the right-skewness of standard non-negative discrete random variables for certain parameters. Left-loadedness is more difficult to verify. We will prove that a Poisson random variable with integer mean is left-loaded.

Simmons [6] proved that a binomial random variable $X$ with integer mean $m$ satisfies $\mathbb{P}(X \leqslant m-1)>\mathbb{P}(X \geqslant m+1)$ if $n>2 m$. This has been generalized to other distributions by Perrin and Redside [5, Proposition 3.3].

Lemma 3.4. Let $X$ be a random variable with integer mean $m$. Then

$$
\mathbb{P}(X \leqslant m-1)>\mathbb{P}(X \geqslant m+1)
$$

if $X$ is Poisson.
Lemma 3.5. Fix a positive integer $m \geqslant 3$, and let $X \sim \operatorname{Poi}(m)$. Then

$$
\mathbb{P}(X \geqslant 2 m)>\mathbb{P}(X=0)
$$

Proof. It is enough to show that $\mathbb{P}(X=2 m)>\mathbb{P}(X=0)$, or equivalently that $m^{2 m}>(2 m)!$. This holds if $m=3$ and we proceed by induction:

$$
\begin{aligned}
(m+1)^{2(m+1)} & =\left(\frac{m+1}{m}\right)^{2 m} \cdot(m+1)^{2} \cdot m^{2 m} \\
& >4(m+1)^{2} \cdot(2 m)!>(2(m+1))!
\end{aligned}
$$

A sequence $\left\{a_{i}\right\}_{i=1}^{m}$ of real numbers is said to be $U$-shaped if there exists $\ell \in[m]$ such that $a_{1} \geqslant \cdots \geqslant a_{\ell}$ and $a_{\ell} \leqslant \cdots \leqslant a_{m}$.

Lemma 3.6. Let $m \geqslant 3$ be an integer, and let $X \sim \operatorname{Poi}(m)$. Then $X$ is leftloaded.

Proof. We show that $X$ satisfies condition $\left(L_{1}\right)$. Recall that $\alpha_{i}=\mathbb{P}(X \leqslant$ $m-i)-\mathbb{P}(X \geqslant m+i)$. We have to show that $\left\{\alpha_{i}\right\}_{i=1}^{m}$ changes sign once. Lemma 3.4 implies that $\alpha_{1}>0$ and Lemma 3.5 implies that $\alpha_{m} \leqslant 0$, and it suffices to show that the sequence $\left\{\alpha_{i}\right\}_{i=1}^{m}$ is U -shaped. Since for every $i \in[m-1]$ we have

$$
\alpha_{i+1}=\alpha_{i}-\mathbb{P}(X=m-i)+\mathbb{P}(X=m+i)
$$

it is enough to show that the sequence $\left\{b_{i}\right\}_{i=1}^{m}$, where $b_{i}:=\mathbb{P}(X=m-i)-$ $\mathbb{P}(X=m+i)$, changes sign once. To this end, for $i \in[m]$, let

$$
\beta_{i}=\frac{\mathbb{P}(X=m+i)}{\mathbb{P}(X=m-i)}
$$

Then $\beta_{i} \geqslant \beta_{i+1}$ is equivalent to $i^{2}+i \leqslant m$. Since the sequence $\left\{i^{2}+i\right\}_{i=1}^{m}$ is increasing, it follows that the sequence $\left\{\beta_{i}\right\}_{i=1}^{m}$ is U -shaped. Now note that $\beta_{1}<1$, and the proof of Lemma 3.5 implies that $\beta_{m} \geqslant 1$. Since $\left\{\beta_{i}\right\}_{i=1}^{m}$ is U-shaped, there exists a unique $k \in[m]$ such that $\beta_{i}<1$ for $i \leqslant k$, and $\beta_{i} \geqslant 1$ for $i \geqslant k+1$, which in turn yields $b_{i}>0$ for $i \leqslant k$, and $b_{i} \leqslant 0$ for $i \geqslant k+1$. In other words, the sequence $\left\{b_{i}\right\}_{i=1}^{m}$ changes sign once, as desired.

Lemma 3.7. Let $X \sim \operatorname{Poi}(m)$ for a natural number $m$. Then $X$ is left-loaded.

Proof. We need to verify the remaining two cases of $m=1$ and $m=2$. If $m=1$, then the second statement in Lemma 2.1 implies that $X$ satisfies condition $\left(L_{2}\right)$. If $m=2$, then Lemma 3.4 and the second statement in Lemma 2.1 imply that $X$ satisfies condition $\left(L_{2}\right)$. If $m \geqslant 3$ then Lemma 3.6 implies that $X$ satisfies condition $\left(L_{1}\right)$. The result follows.

In a similar way, one can show that a $\operatorname{Bin}(n, m / n)$ random variable is leftloaded for a certain range of parameters. More precisely, it satisfies condition $\left(L_{2}\right)$ when $m \in\{1,2\}$, and condition $\left(L_{1}\right)$ when $4 \leqslant m \leqslant n / 3$, but numerical experiments suggest that it is left-loaded for $m \leqslant n / 2$ (see the conjecture below). The same appears to be true for a negative binomial distribution with parameter $p=1-1 / n$.

## 4. CONCLUDING REMARKS

We expect that a binomial random variable is left-loaded if $p \leqslant 1 / 2$. More specifically, we conjecture the following.

Conjecture 4.1. Fix positive integers $n, m$ such that $n \geqslant 2 m$, and let $X \sim$ $\operatorname{Bin}(n, m / n)$. Then $X$ is left-loaded.

Condition $\left(L_{2}\right)$ says that $\sum_{i=1}^{k} \alpha_{i} \geqslant 0$ for all $1 \leqslant k \leqslant m$. Note that our conjecture extends Simmons' inequality (see [6] and [5]).

We have established the right-skewness of random variables for a limited set of parameter values. It is likely that this parameter range can be considerably extended.

The main restriction on our result is that $\mathbb{E}(X)$ is an integer. This is used in Lemma 2.1, which is just a rearrangement of terms. To extend our result to $X$ with non-integer mean, one needs to find a way around this lemma.

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