

**Dynamics of a weakly nonlinear string on an elastic foundation with a partly prescribed discrete spectrum**

Abramian, Andrei K.; Vakulenko, Sergei A.; van Horssen, Wim T.

**DOI**

[10.1007/s11071-022-08142-7](https://doi.org/10.1007/s11071-022-08142-7)

**Publication date**

2022

**Document Version**

Final published version

**Published in**

Nonlinear Dynamics

**Citation (APA)**

Abramian, A. K., Vakulenko, S. A., & van Horssen, W. T. (2022). Dynamics of a weakly nonlinear string on an elastic foundation with a partly prescribed discrete spectrum. *Nonlinear Dynamics*, 111(6), 5221-5235. <https://doi.org/10.1007/s11071-022-08142-7>

**Important note**

To cite this publication, please use the final published version (if applicable). Please check the document version above.

**Copyright**

Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

**Takedown policy**

Please contact us and provide details if you believe this document breaches copyrights. We will remove access to the work immediately and investigate your claim.

***Green Open Access added to TU Delft Institutional Repository***

***'You share, we take care!' - Taverne project***

**<https://www.openaccess.nl/en/you-share-we-take-care>**

Otherwise as indicated in the copyright section: the publisher is the copyright holder of this work and the author uses the Dutch legislation to make this work public.



# Dynamics of a weakly nonlinear string on an elastic foundation with a partly prescribed discrete spectrum

Andrei K. Abramian · Sergei A. Vakulenko · Wim T. van Horssen

Received: 9 July 2022 / Accepted: 26 November 2022  
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

**Abstract** In this paper the dynamics of a weakly nonlinear elastic string on a Winkler elastic foundation is studied. The foundation may be spatially heterogeneous. At one end of the string a mass-spring system is attached, and the other end of the string is fixed. The string is assumed to be long, and the lower part of the spectrum of the string is prescribed. It is shown that localized modes exist and that the dynamics of the string for large times is determined by these localized modes. The frequencies of these localized modes can be controlled by special choices for the spatial heterogeneities in the elastic foundation. Analytical and numerical results are presented to illustrate the findings.

**Keywords** Nonlinear string · Inhomogeneous Winkler foundation · Prescribed spectrum

## 1 Introduction

In many engineering applications it is useful to construct a structure with a given (lower part of the) spec-

trum for the natural eigenfrequencies. One example is the problem to determine the structure parameters (for instance, the mass and stiffness parameters) such that undesirable resonances can be avoided (see [1]). Another example is determining the mass and stiffness parameters in a numerical model for a mechanical system such that the first natural eigenfrequencies coincide with experimental data for these frequencies (see [2–6]). Similar examples can be mentioned in the field of identification of structural damage from frequency data (see [7–10]), and in the field of determination of loads acting on elastic beams or plates during vibrations or impact of objects on surfaces of the structure [11, 12]. All problems mentioned above are inverse problems in vibration theory, and in this paper, we consider one of these problems for a weakly nonlinear elastic string on a Winkler foundation. Particular types of flexible structures, like tall suspension bridges, or iced overhead transmission lines, can be subjected to oscillations due to various causes. Simple models which describe these oscillations are given in the form of nonlinear second-order partial differential equations, as can be seen for example in [13, 14]. This motivate us to investigate the spectrum for strings with large lengths. In this paper it will be shown that the spectrum includes modes corresponding to localized string excitations, and non-localized modes. The localized oscillations of infinite-length strings on the weakly nonlinear foundation were considered in [15, 16]. The last mentioned studies for the case of a weak cubic nonlinear elas-

---

A. K. Abramian (✉) · S. A. Vakulenko  
Institute for Problems in Mechanical Engineering, Saint Petersburg, Russia  
e-mail: andabr33@yahoo.co.uk

W. T. van Horssen  
Delft Institute of Applied Mathematics, Faculty EEMCS, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands  
e-mail: W.T.vanHorssen@tudelft.nl

tic foundation demonstrate that introducing a nonlinearity destroys trapped modes. The resonant oscillations in linear infinite-length strings were considered in [17]. The localization phenomena for spatially inhomogeneous infinite-length strings were considered in [18, 19]. The effect of a structure symmetry on the formation of high-frequency trapped modes was studied in [20]. It was shown that some natural frequencies of the finite structures are equal to discrete frequencies of a similar infinite structures.

A resonance analysis in our paper will show that the large time behaviour of the weakly nonlinear and long string is mainly determined by the localized modes, because the resonances where the non-localized modes are involved are much weaker than the resonances between localized modes. By using classical ideas from Schrödinger operator theory we show that the frequencies of the localized modes are controllable by varying the Winkler foundation. Note that the problem on the control of the first  $N$  eigenvalues was considered in [21]. In that paper it is shown, by using a Darboux transformation [22], how to construct longitudinally vibrating rods having prescribed values for the first  $N$  natural frequencies, under a given set of boundary conditions. The rods and their normal modes can be constructed explicitly by analytical expressions. However, these formulas become complicated for a large number  $N$  of discrete modes. The author of [23] suggested an analytical, exact procedure for the reconstruction of simply supported vibrating beams having given values for the first  $N$  natural frequencies. The author mentioned that “the results hold for beams in which the product between the bending stiffness and the linear mass density is constant, and the analysis is based on the fact that this class of beams is spectrally equivalent to a family of strings fixed at the ends”. Further, the author of [23] uses recent results on the exact construction of second-order Sturm–Liouville operators with prescribed natural frequencies. Another approach is proposed recently in [24] for Schrödinger operators, where singular potentials are used to control a part of the discrete spectrum of a rod. In this paper, we consider a more complicated situation; namely, we consider boundary conditions, which are not very well studied. The boundary condition at the right string edge is the standard zero Dirichlet condition while at the left edge we have a boundary condition corresponding to an oscillator coupled to the string. We use an asymptotic approach for such strings on a Winkler foundation,

which gives a more transparent procedure to control the string’s discrete spectrum. This approach allows us to precisely control part of the string spectrum. Furthermore, we show that if the string is under a weak nonlinear perturbation, then these controlled modes determine the dynamics of the weakly nonlinear string for large times. It should be noted, that our approach can be considered as a generalization of the method suggested in [24]. The paper is organized as follows. First we state an a priori estimate to prove existence of solutions. After that we consider a linear operator which determines the linearized problem. Then we show how one can define the type of Winkler foundation to control the discrete spectrum of the linearized operator. In the last section we qualitatively investigate possible resonances induced by a weak nonlinearity, and show that only resonances between localized modes are essential. On the basis of the obtained results we conclude that it is possible to control the large time dynamics of weakly nonlinear and long strings via localized mode frequencies.

## 2 Statement of the problem

The equation describing the dynamics of a weakly nonlinear elastic string on a Winkler foundation is given by [25]:

$$u_{tt} - c^2 u_{xx} + B(x)u = \epsilon f(u), \quad (1)$$

where  $u(x, t)$  is the string transverse displacement,  $x \in [0, L]$  is the longitudinal coordinate,  $t > 0$  is the time,  $c$  is the wave speed of the transverse waves in the string,  $\epsilon > 0$  is a small parameter,  $f(u)$  is a smooth function, which defines nonlinear forces acting on the string, and a bounded function  $B(x)$  defines the coefficient of the elastic Winkler foundation, which depends on  $x$ . Initial conditions have the form

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad (2)$$

where  $u_0, u_1$  are smooth functions. We consider a problem for a string, which has a length  $L$  with an oscillator connected to the left string edge, and a fixed end condition at the right string edge. Then, the following boundary conditions hold:

$$Mu_{tt} + Ku = b_0 u_x \quad x = 0, \quad t > 0, \quad b_0 > 0, \quad (3)$$

where  $b_0u_x$  is the component of force exerted by the string on the mass of the oscillator for small string displacements,  $K > 0$  is the spring rigidity,  $M > 0$  is the mass of the oscillator (see [26]), and

$$u(L, t) = 0. \tag{4}$$

In order to prove the well-posedness of the problem and to perform the spectral analysis, we need to introduce some assumptions on  $B(x)$ . These assumptions follow the standard ideas in quantum mechanics and in spectral theory for Schrödinger operators [27, 28]. For infinite strings we suppose that the derivative of  $B$  is a sufficiently fast decreasing function, for example,  $|B'(x)| < \text{Const}(1 + |x|)^{-s}$ ,  $s > 2$ . Then the limit  $B(+\infty) = \lim_{x \rightarrow +\infty} B(x)$  exist, and we can present that function as  $B = B(x_0) + W(x, x_0)$ , where  $W(x, x_0) = \int_{x_0}^x B'(y)dy$  and  $x_0 = \infty$ .

Further we suppose that  $B(+\infty) > 0$ , and denote this constant by  $a^2$  with  $a > 0$ . For strings of large lengths  $L$ , i.e.  $L \gg 2c\pi\sqrt{M/K}$  (which means that the string length is much larger than the distance from the string edge to the point where a wave with speed  $c$  will arrive for the time equal to the period of the oscillator) we use the same definition for  $B(x)$ , but now with  $x_0 = L$ .

### 3 The well-posedness of the problem

We are looking for solutions  $u(x, t)$  of the initial boundary value problem (IBVP) defined by (1)–(4), which are bounded in  $L^2[0, L]$ . Under certain conditions on  $f$ , existence of such solutions follows from an a priori estimate, which gives a limit from above for the string energy. We use the following standard notation

$$(u, v) = \int_0^L u(x)v(x)dx, \quad \|u\| = (u, u)^{1/2}.$$

Let us introduce the function

$$\Phi(u) = \int_0^u f(s)ds.$$

Multiplying the left and the right hand sides of (1) by  $u_t$ , and by integrating by parts, one obtains

$$\frac{d(E_s + E_{osc})[u(\cdot, t)]}{dt} = 0, \tag{5}$$

where

$$E_s[u] = \frac{1}{2}(\|u_t\|^2 + c^2\|u_x\|^2 + \int_0^L (a^2 + W)u^2dx) - \epsilon \int_0^L \Phi(u)dx$$

is the string energy, consisting of the kinetic energy, the potential energy, contribution of the elastic foundation, and a nonlinear contribution, and

$$E_{osc} = \frac{c^2(Mu_t(0)^2 + Ku(0)^2)}{2b_0}$$

is the oscillator energy. Equation (5) implies that

$$E[u(x, t)] = \text{Const}, \text{ where } E = E_s + E_{osc}. \tag{6}$$

Assume that

$$\int_0^L \Phi(u(x))dx \leq C_0\|u\|^2 \tag{7}$$

for a positive constant  $C_0$  and all continuous functions  $u(x)$ .

**Lemma I** *Assume that condition (7) holds and moreover*

$$a^2c^2 > \|W\|_1^2, \tag{8}$$

where

$$\|W\|_1 = \int_0^L |W(x, x_0)|dx.$$

Then for sufficiently small  $\epsilon > 0$  one has

$$\|u_t(\cdot, t)\|, \|u_x(\cdot, t)\|, \|u(\cdot, t)\| < \bar{C} \quad \forall t > 0, \tag{9}$$

where a positive constant  $\bar{C}$  depends on the norms  $\|u_0\|, \|u_1\|$  of the initial data.

Lemma I gives us a priori estimates of the  $L_2$ -norms of  $u$  and  $u_t$ , which show that solutions of (1) exist for all times  $t$  if, the initial data have bounded  $L_2$ -norms, that is,

$$\|u_0\|, \|u_1\| < c_0.$$

*Proof* Equation (6) implies

$$\begin{aligned}
 & 2E_{osc} + \|u_t\|^2 + c^2\|u_x\|^2 + \int_0^L (a^2 + W(x, x_0))u^2 dx \\
 & = 2\epsilon \int_0^L \Phi(u) dx + C,
 \end{aligned} \tag{10}$$

where the constant  $C$  is equal to the energy value at the initial time  $t = 0$ :  $C = E[u(x, 0)]$ . The oscillator energy  $E_{osc}$  is nonnegative; thus, from (7) it follows that (10) implies

$$\begin{aligned}
 & \|u_t\|^2 + c^2\|u_x\|^2 + a^2\|u\|^2 \\
 & - \sup_{x \in (0, L)} |u(x, t)|^2 \int_0^L |W(x, x_0)| dx \\
 & \leq 2\epsilon C_0 \|u\|^2 + C.
 \end{aligned} \tag{11}$$

Note that

$$\frac{1}{2} |u(x, t)|^2 = \left| \int_x^L uu_s ds \right| \leq \|u\| \|u_x\|.$$

Let  $b > 0$  be a constant, then

$$2\|u\| \|u_x\| \leq b^2 \|u\|^2 + b^{-2} \|u_x\|^2.$$

We set  $b^2 = c^{-2} \|W\|_1 + \delta$ , where  $\delta > 0$  is small enough. We note that then  $c^2 - b^{-2} \|W\|_1 = C_1 > 0$  and  $a^2 - b^2 \|W\|_1 = C_2 > 0$ . By the last estimates and (11) we obtain

$$\|u_t\|^2 + C_1 \|u_x\|^2 + C_2 \|u\|^2 \leq 2\epsilon C_0 \|u\|^2 + C,$$

which for sufficiently small  $\epsilon > 0$  gives us the result of Lemma I. □

#### 4 Operators with prescribed discrete spectrum

Let us introduce a linear operator associated with our problem:

$$\mathbf{L}u = -c^2 D_x^2 u + (a^2 + W(x, x_0))u, \tag{12}$$

where  $D_x = d/dx$ . So, we are dealing with the Schrödinger operator. However, the boundary conditions are non-standard:

$$-\lambda Mu(0) + Ku(0) = b_0 u_x(0), \tag{13}$$

$$u(L) = 0 \tag{14}$$

and the spectral problem has the form

$$\mathbf{L}u = \lambda u, \tag{15}$$

where the function  $u(x)$  is defined on  $[0, L]$  and satisfies (13). We consider the following problem.

*Spectrum control problem (SCP)* Let  $\lambda_i \in (0, a^2)$ , be given different numbers for  $i = 1, \dots, N$ . Find a potential  $W$  such that the spectrum  $\text{Spec}(\mathbf{L})$  of the operator  $\mathbf{L}$  includes all  $\lambda_i$ , for  $i = 1, \dots, N$ , and the remaining spectrum lies in the domain

$$D_N = \left\{ \lambda \in \mathbb{R} : \lambda < \min_{1 \leq i \leq N} \lambda_i - \delta \right\}$$

where  $\delta > 0$ .

The meaning of such a statement for the problem is that the numbers  $\lambda_i$  for  $i = 1, \dots, N$  define the main frequencies of the free oscillations of the string.

The well-studied situation arises when  $x \in (-\infty, +\infty)$ . In this case we suppose that

$$W(x, x_0) = 0 \quad |x| > L_0, \tag{16}$$

i.e. the spatial heterogeneities are localized.

To describe an asymptotical approach to the **SCP** problem, we first consider the Schrödinger operator

$$\mathbf{H}\psi = -D_x^2 \psi + V(x)\psi, \tag{17}$$

and the spectral problem

$$\mathbf{H}\psi = \tilde{\lambda} \psi,$$

assuming that the potential  $V(x)$  is a smooth well-localized function (in certain cases it may be a piecewise constant function).

For the operator  $\mathbf{H}$  the SCP problem can be reformulated as

*Discrete spectrum control problem (DSCP)* Let  $\tilde{\lambda}_i < 0$ , be different numbers for  $i = 1, \dots, N$ . Find a fast in  $x$  decreasing potential  $V(x)$  such that the discrete spectrum  $\text{Spec}(\mathbf{H})$  of the operator  $\mathbf{H}$  consists of the eigenvalues  $\tilde{\lambda}_i$ .

This problem is well studied, and we state different approaches in the coming subsections. The potentials  $W, V$ , and the corresponding eigenvalues are related as

follows

$$a^2 + W = c^2V, \quad \lambda_i = \tilde{\lambda}_i + a^2. \tag{18}$$

#### 4.1 Spectrum structure of the Schrödinger operators **H**

To simplify the statement, we first consider the case of the Schrödinger operator for  $x \in (-\infty, +\infty)$  ( case **I**) and then we consider ( in Sect. 5) the Schrödinger operator on  $x \in (0, L)$  for large  $L \gg 2c\pi\sqrt{M/K}$  (case **II**)).

##### 4.1.1 General facts

In case **I** the spectrum consists of a discrete part and of a continuous part. All discrete eigenvalues  $\lambda_i, i = 1, 2, \dots, N$  have multiplicity one, and the corresponding eigenfunctions  $\psi_i$  have bounded norms  $\|\psi_i\|$ . The continuous spectrum lies in  $(0, +\infty)$ , and the eigenfunctions of the continuous spectrum are not localized (that is, do not belong to  $L_2(\mathbb{R})$ ). So, we have a discrete set of well-localized eigenfunctions, and a continuous set of densely located eigenvalues which corresponds to non-localized eigenfunctions. Under our assumptions on  $W$  the spectrum of the non-localized functions always lies below a part of the spectrum which corresponds to the localized eigenfunctions.

##### 4.1.2 Bargmann potentials and KdV

The problem DSCP can be solved analytically by using so-called Bargmann potentials, which are connected to multi-soliton solutions of the famous KdV equation:

$$V_\tau - 6V V_x + V_{xxx} = 0. \tag{19}$$

The Bargmann potentials  $V$  can be found as follows. For  $N = 1$  our **DSCP** problem on  $(-\infty, +\infty)$  can be solved by a one-soliton solution of (19). So, we obtain

$$V(x, x_0) = -2\kappa^2 \cosh^{-2}(\kappa(x - x_0)). \tag{20}$$

We can take any  $x_0$ , since  $x_0$  is a free parameter . For each  $x_0$  the operator **H** has a single eigenvalue in the discrete spectrum that equals  $\lambda = -\kappa^2$ . The corresponding eigenfunction has the form

$$\psi(x, \tau) = \sqrt{\kappa/2} \cosh^{-1}(\kappa(x - x_0)). \tag{21}$$

For  $\kappa \rightarrow +\infty$  we obtain that  $V/\kappa \rightarrow \delta(x - x_0)$ , where  $\delta$  is the Dirac delta-function that leads to a particular case of potentials as considered in [24]. The corresponding Winkler foundation  $W$  for the operator **L** has the form

$$W = -2c^2\kappa^2 \cosh^{-2}(\kappa(x - x_0)) - a^2. \tag{22}$$

Further,  $N$ -soliton solutions of (19) and the corresponding  $V = V_N$  can be found by the following algebraic algorithm.

Let  $\lambda_j = -\kappa_j^2$ . Let us take the square  $N \times N$  matrix **M** with entries depending on  $x$  and  $\tau$ :

$$M_{ij}(x, \tau) = \delta_{ij} + \frac{c_j^2(0)}{\kappa_i + \kappa_j} \exp(-(\kappa_j - \kappa_i)x - 8\kappa_j^3\tau),$$

where  $c_j(0)$  are arbitrary parameters, and where  $\delta_{ij}$  is the Kronecker delta symbol(that is,  $\delta_{ij} = 1$  for  $i = j$  and else 0). Then,

$$V_N(x, \tau) = 2D_x^2 \ln \det \mathbf{M}$$

gives us the needed potential for any choice of  $c_j(0)$  and  $\tau$ .

Note that when  $\tau \rightarrow +\infty$ , this potential transforms into a chain of solitons separated by large intervals. This means that for large  $t$  the potential  $V_n$  is a sum of well-separated and well-localized potentials. This fact inspires an asymptotic approach as stated in the coming subsections. Note that the formulas for the Bargmann potentials can be obtained by the Darboux transformation [22], and they are complicated for large  $n$  (under a general choice for the parameters  $\kappa_j, \tau, c_j$ ). This shows that in general the method of [21] leads to complicated formulas when  $N \gg 1$ . We thus need an asymptotical approach to have more transparent formulas and to handle more complicated boundary conditions such as (3).

##### 4.1.3 Asymptotic approach: potentials with many localized wells

Let  $V$  be a sum of well-localized and separated potential wells  $\tilde{V}_j$ :

$$V = V_N = \sum_{j=1}^N \tilde{V}_j(x - \bar{x}_j),$$

where each  $\bar{V}_j(x)$  is a well-localized function, for example, satisfying the inequality

$$|\bar{V}_j(x)| < C \exp(-c|x|).$$

It is clear that, for example, functions  $\bar{V}_j$  of the form (20) satisfy this inequality. Suppose that for each  $j$  the spectral problem (17) has a single eigenvalue  $\bar{\lambda}_j$  for the discrete spectrum. Note that according to the results of the previous section for functions  $\bar{V}_j$  defined by (20) this assumption holds; however, there are also possibly other variants; for example, we can take a rectangular potential well of an appropriate width and height (see [27]). Let us define the distance between the centres of the potential wells as

$$Dist(\bar{x}) = \min_{i \neq j} |x_i - x_j|.$$

It is well known that if the distance  $Dist(\bar{x})$  between the wells is large, we have  $N$  eigenvalues  $\lambda_j$  in the discrete spectrum of the operator  $\mathbf{H}$  with the potential  $V_N$ , which are exponentially close to  $\bar{\lambda}_j$ , that is,

$$\lambda_j = \bar{\lambda}_j + O(\exp(-c \text{Dist}(\bar{x}))), \tag{23}$$

where  $c > 0$  is a constant which is independent of  $L$ . This shows that, to control the  $N$  values of the discrete spectrum with an exponential accuracy, we can use the formula:

$$V_N = -2 \sum_{j=1}^N \kappa_j^2 \cosh^{-2}(\kappa_j(x - \bar{x}_j)), \quad \kappa_j = \sqrt{-\lambda_j} \tag{24}$$

under the condition that  $Dist(\bar{x}) \gg 1$ .

Note that we can also use other potentials instead of hyperbolic cosines, for example,

$$V_N^{\text{rec}} = \sum_{j=1}^N V_{\text{rec}}(x - \bar{x}_j, a_j, U_j), \tag{25}$$

where we can take rectangular potential wells

$$\begin{aligned} V_{\text{rec}}(x, a, U) &= 0, \quad (|x| > a), \\ V_{\text{rec}}(x, a, U) &= U, \quad (|x| < a). \end{aligned} \tag{26}$$

Each well depends on two parameters, the width  $a$ , and the height  $U$ . Let us introduce two important auxiliary

quantities,  $\gamma = 2a^{-1}U^{-1/2}$  and  $\xi = ka/2$ . For even eigenfunctions we have

$$\cos(\xi) = \pm\gamma\xi, \quad \tan \xi > 0,$$

and for odd eigenfunctions

$$\sin(\xi) = \pm\gamma\xi, \quad \tan \xi < 0$$

(see [27]). The eigenvalue is defined by the relation  $\lambda = k^2$ . The eigenfunction has the form

$$\begin{aligned} \psi &= C_1 \cos(kx), \quad (|x| > a), \\ \psi &= C_2 \cosh(\sqrt{k + U}x), \quad (|x| < a). \end{aligned} \tag{27}$$

The number of corresponding eigenvalues is determined by  $\gamma$ . For shallow wells, where  $\gamma > 0$  is large, we have a single even eigenfunction with an eigenvalue close to zero:  $\xi \approx \gamma^{-1} + O(\gamma^{-3})$ . In the opposite case, where  $U_0a^2$  is large and  $\gamma$  is small, we obtain a number of even and odd eigenfunctions, and this number is  $\text{Const}\gamma^{-1}$ . Note that for shallow wells  $V$  we always have a single localized eigenfunction with a small eigenvalue (not depending on the well form).

### 5 A string of a finite but large length

In Sect. 4.1 we considered infinitely long strings. In this section, we consider a string of large length  $L \gg 2c\pi\sqrt{M/K}$  subject to the boundary condition (3) and (4). The concept of localized modes makes sense for infinite strings where a mode  $\psi(x)$  is localized if  $\|\psi\|^2 < \infty$ . This concept is standard and it was developed in quantum mechanics [27,28]. For a finite length string we suggest the following interpretation. Intuitively, for large lengths  $L$  a normalized eigenfunction  $\psi(x)$  (such that  $\|\psi\| = 1$ ) is localized if  $\max |\psi(x)| = O(1)$ . For example, for harmonic modes  $\psi \propto \sin(kx)$  we have then  $\max |\psi(x)| = O(L^{-1/2})$  whereas for  $\psi \propto \exp(-b|x|)$  with  $b > 0$  one has  $\max |\psi(x)| = O(b^{-1/2})$ .

One can propose the following formal definition. Let us consider the family of strings for different lengths  $L$ ,  $L \in (1, +\infty)$ . Consider the corresponding family of normalized eigenfunctions  $\psi(x, L)$  which continuously depend on  $L$ . The eigenfunctions  $\psi(x, L)$  are called localized modes if  $\max |\psi(x, L)| = O(1)$  as  $L \rightarrow +\infty$ .



### 5.1 Scalar product and self-adjoint operator

The linear dynamic string equation under the boundary conditions (3) and (4) also leads to a self-adjoint spectral problem, if we modify the standard  $L_2$ -scalar product.

In order to see this, let us consider two solutions  $\exp(i\omega_j t)\psi_j(x)$ ,  $j = 1, 2$  for different frequencies  $\omega_j$ . Then, we have

$$-c^2 \frac{d^2 \psi_j}{dx^2} + W(x, x_0)\psi_j = -(\omega_j^2 + a^2)\psi_j.$$

We multiply both sides of this equation for  $j = 1$  with  $\psi_2$ , and similarly the equation for  $j = 2$  with  $\psi_1$ . Adding the equations, integrating the so-obtained equation from  $x = 0$  to  $x = L$ , and taking into account the conditions (3) and (4), we find that

$$(\omega_1^2 - \omega_2^2) \left( b\psi_1(0)\psi_2(0) + \int_0^L \psi_1\psi_2 dx \right) = 0,$$

where  $b = b_0/(c^2 M)$ . So, we observe that the eigenfunctions  $\psi_1, \psi_2$  are orthogonal if we introduce the following scalar product in an appropriate function space consisting of functions  $u$  with bounded norm  $\|u\|_b = (u, u)_b^{1/2}$ , where

$$(u, v)_b = \int_0^L u(x)v(x)dx + b^{-1}u(0)v(0).$$

The corresponding spectral problem is associated with a self-adjoint operator, and we can use the usual Galerkin procedure since the eigenfunctions are orthogonal. We can now use the results for case I (see Sect. 4.1) to achieve an asymptotical control of the discrete spectrum for  $L \gg 2c\pi\sqrt{M/K}$ .

### 5.2 Perturbation of the discrete spectrum by boundary effects for a finite length string

Assume  $L \gg 2c\pi\sqrt{M/K}$ . Let us consider a chain of localized wells located at the points  $\bar{x}_j$  such that  $0 < \bar{x}_1 < \dots < \bar{x}_N$ . We suppose as in the previous section that  $Dist(\bar{x}) \gg 2c\pi\sqrt{M/K}$ , and we introduce the additional conditions

$$\bar{x}_1 \gg 2c\pi\sqrt{M/K}, \quad L - \bar{x}_N \gg 2c\pi\sqrt{M/K}.$$

Let us introduce  $d = \min\{Dist(\bar{x}), \bar{x}_1, L - \bar{x}_N\}$ . Then, we can estimate the influence of the boundary conditions and the finite size of a string as follows. Let us for simplicity take  $N = 1$ , i.e. we are dealing with a single potential well. Then, we have a single eigenfunction  $\psi$ . Let us denote by  $\psi^\infty$  the localized eigenfunction for case I, i.e. for  $x \in (-\infty, +\infty)$ , and let  $\lambda^{(\infty)}$  be the corresponding eigenvalue. We denote by  $\tilde{\lambda}_1$  the perturbation of this eigenvalue, and thus  $\lambda_1 = \lambda^{(\infty)} + \tilde{\lambda}_1$  is the eigenvalue for our finite but long string. Then, we construct  $\psi$  with the help of the following representation:

$$\psi = \psi^{(\infty)} + \psi_b + \tilde{\psi}, \tag{28}$$

where  $\psi_b = C_1 \exp(-ax) + C_2 \exp(-a(L - x))$  and  $\tilde{\psi}$  is a correction satisfying the boundary conditions (3) and (4) such that

$$(\tilde{\psi}, \psi^{(\infty)})_b = 0. \tag{29}$$

We adjust the constants  $C_1, C_2$  from the condition that the function  $\phi = \psi^{(\infty)} + \psi_b$  satisfies the boundary conditions (3) and (4). Then, for those constants we obtain the estimate

$$|C_i| < c \max\{\exp(-a\bar{x}_1), \exp(-a(L - \bar{x}_1))\}, \quad i = 1, 2.$$

We apply the standard perturbation theory. It follows from (29) that the perturbation  $\tilde{\lambda}$  of the eigenvalue can be estimated as

$$|\tilde{\lambda}_1| < c_1 \max\{\exp(-a\bar{x}_1), \exp(-a(L - \bar{x}_1))\}. \tag{30}$$

In fact, let us substitute  $\psi$  in the main equation that defines  $\lambda_1$ . Then, we have

$$\begin{aligned} -c^2(\psi^\infty + \psi_b + \tilde{\psi})_{xx} + (a^2 + W)((\psi^\infty + \psi_b + \tilde{\psi})) \\ = (\lambda^{(\infty)} + \tilde{\lambda}_1)((\psi^\infty + \psi_b + \tilde{\psi})). \end{aligned} \tag{31}$$

Part of the terms cancel out, and we obtain

$$\begin{aligned} -c^2\tilde{\psi}_{xx} + (a^2 + W)\tilde{\psi} \\ = \tilde{\lambda}_1((\psi^\infty + \psi_b + \tilde{\psi}) - W\psi_b + \lambda^{(\infty)}\psi_b). \end{aligned} \tag{32}$$

We multiply both sides of this equation by  $\psi^\infty$  and we integrate over  $[0, L]$ . By taking into account (29), we obtain the estimate (30).

For the general case with  $N$  potential wells and  $N$  eigenvalues  $\lambda_j^\infty$  we can obtain

$$|\tilde{\lambda}_j| < c_j \max\{\exp(-a\bar{x}_1), \exp(-a(L - \bar{x}_N))\}. \tag{33}$$

So, if we neglect exponentially small corrections then the study on long strings ( case **II**) can be reduced to the well-studied case for infinitely long strings ( case **I**).

### 5.3 Special modes localized at $x = 0$

For the boundary condition (3) there is an eigenfunction  $\psi_0$  which is localized at  $x = 0$ . To find it, we set

$$\psi_0 = \exp(-\gamma x) \quad \lambda_0 = \omega^2 = -c^2\gamma^2 + a^2. \tag{34}$$

Then, we obtain

$$\gamma = \frac{-b_0 + (b_0^2 - 4(K - Ma^2)Mc^2)^{1/2}}{2Mc^2}, \tag{35}$$

and the function  $\psi_0$  is localized if  $\text{Re}\gamma > 0$ . This means that  $K < Ma^2$ . Finally, for long strings with  $L \gg 2c\pi\sqrt{M/K}$  under the condition  $K < Ma^2$ , we have  $N_{loc} \geq N + 1$  localized modes  $\Psi_n$  such that

$$|\Psi_n(x)| < C_n \exp(-\kappa_n|x - \bar{x}_n|), \tag{36}$$

for  $n = 0, 1, \dots, N$  and some  $\kappa_n > 0, C_n > 0, \bar{x}_n \in (0, L)$  for  $n \geq 1$  and  $\bar{x}_0 = 0$ . If the localized mode at  $x = 0$  is absent, then  $N_{loc} \geq N$ . Moreover, there are non-localized eigenfunctions, which for large  $|x|$  and  $L$  have the form

$$\psi_k = L^{-1/2}(a_k \cos(\omega_k x) + b_k \sin(\omega_k x)), \tag{37}$$

where  $a_k, b_k$  are coefficients, and for the frequencies  $\omega_k$  for large  $L$  we have the asymptotics

$$\omega_k^2 = (c^2(\pi k/L)^2 + a^2)(1 + O(L^{-1})). \tag{38}$$

The last formula can be obtained by the standard perturbation theory (see ‘‘Appendix’’). For large  $L$  the frequencies  $\omega_k$  densely fill the semi-axis  $(a^2, +\infty)$  and are located in intervals of length of order  $L^{-1}$ . We assume that all functions are normalized:  $\|\Psi_n\|_b = 1$  and  $\|\omega_k\|_b = 1$ . In the remaining part of the paper we will show that for weakly nonlinear, long strings the localized modes corresponding to the prescribed spectrum also determine the large time behaviour.

## 6 Dynamics of weakly nonlinear strings

### 6.1 Fourier decomposition

Our plan is as follows. Assuming that nonlinear effects are small, we want to construct asymptotic solutions for the string displacement, and we try to find asymptotic solutions for  $u(x, t)$  by the Galerkin decomposition of  $u(x, t)$ . We look for the string displacement  $u(x, t)$  in the form:

$$u(x, t) = \sum_{n=1}^N U_n(t)\Psi_n(x) + \sum_{k=0}^{+\infty} \hat{u}_k(t)\psi_k(x), \tag{39}$$

where  $U_n(t)$  and  $\hat{u}_k(t)$  are unknown coefficients,  $\Psi_n$  are localized eigenfunctions of the corresponding spectral problem, and  $\psi_k(x)$  are non-localized eigenfunctions, which correspond to a continuous spectrum in the limit for  $L \rightarrow +\infty$ . For the unknown coefficients  $U_n(t)$  and  $\hat{u}_k(t)$  one has the following system of differential equations :

$$\frac{d^2 U_n}{dt^2} + \Omega_n^2 U_n = \epsilon(f(u), \Psi_n)_b, \tag{40}$$

where  $n = 1, \dots, N$ , and

$$\frac{d^2 \hat{u}_k}{dt^2} + \omega_k^2 \hat{u}_k = \epsilon(f(u), \psi_k)_b, \tag{41}$$

where  $\Omega_n^2 = \lambda_n$  and  $\omega_k^2$  are frequencies defined by relation (38). For large  $L$  the frequencies  $\omega_k$  densely fill the semi-axis  $(a^2, +\infty)$  and are located in intervals of length of order  $L^{-1}$ . Equations (40) and (41) describe the dynamics of the infinite system of coupled oscillators. We can simplify the problem for small  $\epsilon > 0$  using the method as presented in the next subsection.

### 6.2 Asymptotic solutions for the amplitude coefficients

We suppose that  $f(u)$  is an analytic function such that

$$f(u) = f_2 u^2 + f_3 u^3 + \dots$$

where  $f_2, f_3, \dots$  are coefficients. Then, for small displacements, we have to take into account the term  $f_2 u^2$  by following the arguments as given in [29]. Such

type of force per unit length is the force acting on a string due to springs which have a different behaviour for compression and extension (see [29]). The asymptotic approach to study equations (40) and (41) is well known, see [28,30,31]. Let  $T = \epsilon t$  be a slow time. Furthermore, it should be observed that the right-hand sides of (40) and (41) are of order  $\epsilon$ . We seek solutions  $U_n$  of (40) in the form

$$U_n = U_{n,0}(t, T) + \epsilon U_{n,1}(t, T) + \dots, \tag{42}$$

where

$$U_{n,0} = A_n(T) \sin(\Omega_n t + \phi_n(T)),$$

and where the amplitude  $A_n(T)$  and the phase  $\phi_n(T)$  are unknown (slowly varying in time) functions. Similarly,

$$\hat{u}_k = \hat{u}_{k,0}(t, T) + \epsilon \hat{u}_{k,1}(t, T) + \dots, \tag{43}$$

where

$$\hat{u}_{k,0} = a_k(T) \sin(\omega_k t + \hat{\phi}_k(T))$$

and where the amplitude  $a_k(T)$  and the phase  $\hat{\phi}_k(T)$  are unknown (slowly varying in time) functions.

From (43) it follows that

$$\begin{aligned} \partial_t U_{n,0} &= (\Omega_n + \epsilon \phi_{nT}) A_n(T) \cos(\Omega_n t + \phi(T)) + \epsilon A_n(T)_T \sin(\Omega_n t + \phi_n(T)), \tag{44} \\ \partial_{tt} U_{n,0} &= -\Omega_n^2 A_n \sin(\Omega_n t + \phi_n(T)) \\ &\quad + 2\epsilon \Omega_n (A_n)_T \cos(\Omega_n t + \phi_n(T)) \\ &\quad - 2\epsilon \Omega_n A_n \phi_{nT} \sin(\Omega_n t + \phi_n(T)) + O(\epsilon^2). \tag{45} \end{aligned}$$

Analogous relations hold for  $\hat{u}_{k,0}$ .

By substituting these relations into (40), and by taking together terms of order  $\epsilon$ , one obtains the following equation for  $U_{n,1}$ :

$$(U_{n,1})_{tt} + \Omega_n^2 U_{n,1} = S_n(A, \phi, t, \epsilon), \tag{46}$$

where

$$\begin{aligned} S_n &= 2\Omega_n \left( -A_n)_T \cos(\Omega_n t + \phi_n(T)) \right. \\ &\quad \left. + A_n \phi_{nT} \sin(\Omega_n t + \phi_n(T)) \right) + R_n(A, \phi, t), \tag{47} \end{aligned}$$

and

$$R_n(A, \hat{a}, \phi, t) = R_n^{LL} + R_n^{LN} + R_n^{NN}, \tag{48}$$

and where  $R_n^{LL}, R_n^{LN}, R_n^{NN}$  are terms induced by an interaction between localized mode interactions, localized and non-localized ones, and between non-localized ones, respectively. Here  $A = (A_1, A_2, \dots)$  and  $\phi = (\phi_1, \phi_2, \dots)$  denote infinite sequences of the functions  $A_n(t)$  and  $\phi_n(t)$ , respectively. Below, taking into account that amplitudes  $A_n$  are small, we set  $f = f_2 u^2$  and neglect terms of higher order. Then we have

$$\begin{aligned} R_n^{LL} &= \sum_{m_1=1}^N \sum_{m_2=1}^N A_{m_1} A_{m_2} \cos(\Omega_{m_1} t + \phi_{m_1}) \\ &\quad \times \cos(\Omega_{m_2} t + \phi_{m_2}) (\Psi_{m_1}, \Psi_{m_2}, \Psi_n)_b, \tag{49} \end{aligned}$$

$$\begin{aligned} R_n^{LN} &= \sum_{m=1}^N \sum_{k=0}^{\infty} A_m a_k \cos(\Omega_m t + \phi_m) \cos(\omega_k t + \phi_k) \\ &\quad \times (\Psi_m, \psi_k, \Psi_n)_b, \tag{50} \end{aligned}$$

$$\begin{aligned} R_n^{NN} &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1} a_{k_2} \cos(\omega_{k_1} t + \phi_{k_1}) \cos(\omega_{k_2} t + \phi_{k_2}) \\ &\quad \times (\psi_{k_1}, \psi_{k_2}, \Psi_n)_b. \tag{51} \end{aligned}$$

Let  $\langle f \rangle_T$  denotes the average of a continuous uniformly bounded functions  $f$ :

$$\langle f \rangle_T = T^{-1} \int_0^T f(t) dt.$$

For large times  $t = O(\epsilon^{-1})$  Eq. (46) has a bounded solution in  $t$  if and only if

$$\langle S_n(A, \phi, t, \epsilon) \cos(\Omega_n t + \phi_n) \rangle_T = 0, \tag{52}$$

and

$$\langle S_n(A, \phi, t, \epsilon) \sin(\Omega_n t + \phi_n) \rangle_T = 0, \tag{53}$$

Finally, it follows from (52) and (53) that the system of equations for the amplitudes  $A$ , and for the phases  $\phi$  has the form:

$$\Omega_n \frac{dA_n}{dT} = \langle R_n(A, \phi, t) \cos(\Omega_n t + \phi_n) \rangle_T, \tag{54}$$

and

$$\Omega_n A_n \frac{d\phi_n}{dT} = -\langle R_n(A, \phi, t) \sin(\Omega_n t + \phi_n) \rangle_T, \quad (55)$$

and

$$\omega_k \frac{da_k}{dT} = \langle R_n(A, \phi, t) \cos(\omega_k t + \hat{\phi}_k) \rangle_T, \quad (56)$$

and

$$\omega_k \frac{d\phi_k}{dT} = -\langle R_n(A, \phi, t) \sin(\omega_k t + \hat{\phi}_k) \rangle_T. \quad (57)$$

We will investigate this system qualitatively in the next section.

### 6.3 Resonances

Relations (54), (56), (49), (50) and (51) show that in our system three types of resonances occur. We have

1. **LL** resonances between localized modes;
2. **LN** resonances between localized and non-localized modes;
3. **NN** resonances between non-localized modes.

**LL** resonances between modes  $m_1, m_2$  and  $n$  can only occur under the condition that

$$\tau_{m_1, m_2, n}^\pm = |\Omega_{m_1} \pm \Omega_{m_2}| - \Omega_{n, n}.$$

where

$$|\tau_{m_1, m_2, n}^\pm| = O(\epsilon), \quad (58)$$

in which  $\tau_{m_1, m_2, n}$  are detuning parameters.

We can avoid these resonances by adjusting the frequencies  $\Omega_m$ . Similarly, **LN** resonances between localized modes  $m, n$  and a non-localized mode  $k$  are possible when

$$|\tilde{\tau}_{m, n, k}^\pm| = O(\epsilon), \quad \tilde{\tau}_{m, n, k}^\pm = |\Omega_m \pm \omega_k| - \Omega_n, \quad (59)$$

where  $\tilde{\tau}_{m, n, k}^\pm$  are detuning parameters. It is impossible to avoid the **LN** or **NN** resonances because the frequencies  $\omega_k$  are located densely. However, the effects of these **NN** and **LN** resonances are small for large  $L$

and  $\epsilon \rightarrow 0$ . In fact, consider relation (50). We obtain the following typical sum over all non-localized modes

$$\sum_{k=1/L, 2/L, \dots} L^{-1} (\exp(i(\Omega_{mn} - \omega_k)\tau) - 1) ((\Omega_{mn} - \omega_k)\tau)^{-1}, \quad (60)$$

where  $\Omega_{mn} = \Omega_n \pm \Omega_m$ ,  $\omega_k = \sqrt{c^2 k^2 + a^2}$ , and  $\tau = O(\epsilon^{-1})$  is large. Note that the function  $(\exp(i(\Omega_{mn} - \omega_k)\tau) - 1) ((\Omega_{mn} - \omega_k)\tau)^{-1}$  is correctly defined even for  $\Omega_{mn} = \omega_k$  (this follows from the Taylor series for  $\exp(i(\Omega_{mn} - \omega_k)\tau)$ ). However, this function takes large values for large  $\tau$  and small  $|\Omega_{mn} - \omega_k|$ , and therefore we must be careful in estimating the sum (60). The factor  $L^{-1}$  occurs due to the property that amplitudes of non-localized modes are proportional to  $L^{-1/2}$  [see relation (37)]. The main idea is that in the limit  $L \rightarrow \infty$  the sum (60) is an approximation of the integral

$$I_{\text{reg}} = \tau^{-1} \int_0^{+\infty} (\exp(i(\Omega_{mn} - \omega_k)\tau) - 1) \times ((\Omega_{mn} - \omega_k))^{-1} dk,$$

which can be regularized (for example, in a principal value sense) and is therefore small as  $\epsilon \rightarrow 0$ . Note that in practical situations  $L$  is finite (but possibly large) and  $\epsilon \rightarrow 0$ . Then one can expect that the set of frequencies  $\Omega_n$  for which the sum (60) is not small for large  $L$  has a small measure  $\mu_{\epsilon, L}$ , which converges to 0 as  $\epsilon \rightarrow 0$ . These arguments can be confirmed by numerical simulations and by the following lemma. Note that for each localized mode we have at most 1 resonance with non-localized modes, and there is possibly a number of **NN** resonances.

To estimate the total contribution of the **LN** and **NN** resonances, we can use the following lemma. We set  $T = \epsilon^{-1}$  in the equations (54), (55), (56), (57).

**Lemma II** *Let  $\epsilon > 0$  be small and  $L \gg 2c\pi\sqrt{M/K}$ . Then the total contributions of the **LN** and **NN** resonances can be estimated by  $O((\epsilon(\ln \epsilon))L^{1/2})$ .*

*Remark* This lemma shows that the **LN** and **NN** resonance contributions are small if  $L/2c\pi\sqrt{M/K} \ll \epsilon^{-2+s}$ , where  $s \in (0, 1)$ . Thus for the practical situations, as described before, it can be concluded that the resonance contributions are small.

*Proof* We consider the **LN** resonances. Estimates for the **NN** resonances are more sophisticated, but they can

be done in a similar way. From (50) it follows that a typical sum that defines the resonance contribution has the form:

$$S_{\text{reso}} = L^{-1/2} \sum_{n=1}^{N_{\text{loc}}} g(k_n) \frac{1 - \exp((\Omega - \omega_{k_n})\tau)}{(\Omega - \omega_{k_n})\tau}. \quad (61)$$

where  $\Omega$  is a frequency,  $k_n = \sqrt{c^2 n^2 / L^2 + a^2}$ ,  $\tau = O(\epsilon^{-1})$  and  $g(k)$  is a smooth function depending on  $k$  such that

$$|g(k)| < C(1 + |k|)^{-1-s}, \quad s > 0. \quad (62)$$

The factor  $L^{-1/2}$  occurs as a result of the asymptotics (37) describing non-localized modes for large  $|x|$ . Let  $\rho$  be a small parameter, which will be adjusted later. We decompose the sum (61) into two sums,  $\tilde{S}_{\text{reso},\rho}$  and  $\bar{S}_{\text{reso},\rho}$ . The sum  $\tilde{S}_{\text{reso},\rho}$  is a sum over  $n$  such that

$$|\Omega - \omega_{k_n}| < \rho, \quad (63)$$

and in  $\bar{S}_{\text{reso},\rho}$  we are summing up over all integers  $n$  such that (63) does not hold. The sum  $\tilde{S}_{\text{reso},\rho}$  includes the  $O(\rho L)$  terms, which are bounded as  $\epsilon \rightarrow 0$ . Therefore, we have

$$|\tilde{S}_{\text{reso},\rho}| < \text{Const} L^{-1/2} \rho L < c \rho L^{1/2}. \quad (64)$$

Note that  $\bar{S}_{\text{reso},\rho}$  contains a logarithmic singularity; therefore, it can be estimated by (62) as follows:

$$|\bar{S}_{\text{reso},\rho}| < \text{Const} L^{-1/2} L |\ln \rho| \tau^{-1}. \quad (65)$$

So, by (64) and (66) we find that

$$S_{\text{reso}} < c L^{1/2} (\rho + |\ln \rho| \tau^{-1}) \quad (66)$$

where  $c > 0$  is uniform in  $\epsilon, \rho$ . We choose a  $\rho$ , which minimizes the right hand side of the last inequality. This leads to  $\rho = \tau^{-1} = O(\epsilon)$ . We substitute this value of  $\rho$  into the last estimate and obtain

$$S_{\text{reso}} < c_1 L^{1/2} \epsilon |\ln \epsilon|,$$

where  $c_1 > 0$  is uniform in  $L, \rho$ , and this proves the lemma. Note that the contributions of the NN resonances also are small under the conditions of our lemma.  $\square$

## 6.4 Interaction between localized and non-localized modes

In [15] for a semi-infinite string with a weak cubic nonlinearity (that is for  $f = -u^3$ ) and under boundary conditions similar to (3), an interesting effect is found: the amplitude  $A(t)$  of a localized (at  $x = 0$ ) mode slowly decreases on a time scale of  $O(\epsilon^{-2})$ . Note that such a string has a constant energy, and so this effect is only possible if there is a transfer of energy between the localized mode and the non-localized modes. Now we will estimate the impact on non-local mode activation by localized modes. We also consider a large string with  $L \gg 2c\pi\sqrt{M/K}$ , nonlinearity  $f = f_2 u^2$ , and apply here a simplified approach.

To simplify the statement, let us consider the case of a single localized mode  $\Psi(x)$ , which can be localized at  $x = 0$  or around another point of the string. Let us represent the string displacement  $u(x, t)$  by the Fourier decomposition:

$$u(x, t) = \hat{U}_{\text{loc}}(t) \Psi(x) + \sum_{k=1}^{\infty} \hat{u}_k(t) \psi_k(x),$$

where  $\hat{U}_{\text{loc}}, \hat{u}_k(t)$  are unknown Fourier coefficients depending on time. We assume that the amplitudes of the non-localized modes  $\hat{u}_k$  are small and that the amplitude of the localized mode  $\hat{U} = \hat{U}_{\text{loc}}$  has order 1. Then, by neglecting some small terms, we obtain the following system of equations for the unknown amplitudes:

$$\frac{d^2 \hat{U}}{dt^2} + \Omega^2 \hat{U} + c_0 \epsilon \hat{U}^2 = -c_0 \epsilon \sum_{k=1}^{\infty} b_k \hat{U} \hat{u}_k, \quad (67)$$

where the terms of order  $\epsilon \hat{u}_k^2$ , are neglected, and  $\Omega$  is the frequency of the localized mode, and

$$\frac{d^2 \hat{u}_k}{dt^2} + \omega_k^2 \hat{u}_k = -\epsilon b_k U^2, \quad (68)$$

where the terms of order  $\epsilon \hat{U}^2 (\hat{u}_k)$  are neglected and  $\omega_k$  is the frequency of the localized mode. The coefficients  $b_k$  are given by

$$b_k = \langle \Psi^2, \psi_k \rangle.$$

The initial conditions are  $u_k(0) = 0, v_k(0) = 0$  (where  $v_k(t) = du_k/dt$ ), in order to study the excitation of the non-local modes by the localized mode. The main idea to simplify the system (67)–(68) is based on using the asymptotic representation for  $\hat{U} \approx A(\tau) \sin(\Omega t + \phi(\tau))$ , where  $\tau = \epsilon t$ . By using this representation, we find an asymptotical solution of (68) up to terms of the order  $O(\epsilon)$ , where the first two contributions depend on  $A, A_\tau$ . Then we can reduce system (67)–(68) to a single equation for  $\hat{U}$ . We obtain

$$\hat{u}_k = \epsilon(\alpha_k + \beta_k \cos(2(\Omega t + \phi))) + \tilde{u}_k + O(\epsilon), \tag{69}$$

where

$$\alpha_k = \frac{b_k A^2}{4(\omega_k^2 - \Omega^2)}, \quad \beta_k = -\frac{A^2}{2(\omega_k^2 - 4\Omega^2)},$$

and

$$\tilde{u}_k = C_k \cos(\omega_k t), \quad C_k = -\alpha_k - \beta_k. \tag{70}$$

By substituting (69–70) into the right hand side of (67), and by using the averaging method (according to (54)) we obtain the following equation for  $A$ :

$$\Omega \frac{dA}{d\tau} = -f_2 \epsilon A (\sin(\Omega t + \phi) \cos(\Omega t + \phi) \hat{u}_k(t))_T. \tag{71}$$

Observe that the right hand side of Eq. (71) has the order  $\epsilon$  and that a **NL** resonance exists when for some  $k = k_*$  we have  $2\Omega \approx \omega_{k_*}$ . Let  $\mu$  be the corresponding detuning parameter defined by the relation:

$$\mu \epsilon = 2\Omega - \omega_{k_*}. \tag{72}$$

We assume that  $\mu = O(1)$  and that there is a single resonance  $k_*$  such that (72) holds (this is possible if  $L \ll \epsilon^{-1} 2c\pi \sqrt{M/K}$ ). Then, we obtain

$$\Omega \frac{dA}{d\tau} = -f_2 \epsilon b_{k_*}^2 A \cos(\mu\tau). \tag{73}$$

This equation shows that depending on the sign of the nonlinearity and on the detuning parameter value, we can have a decrease or increase of the amplitude  $A(\tau)$  for times  $t$  of order  $\epsilon^{-2}$ . Indeed, we have

$$A(\tau) = A(0) \exp\left(-\epsilon \mu^{-1} \Omega^{-1} f_2 b_{k_*}^2 \sin(\mu\tau)\right).$$

The effect of the increase ( or decrease) is bounded, at least for  $\mu = O(1)$ . For small  $\mu$  we have  $\mu^{-1} \sin(\mu\tau) \approx \tau$  and the resonance effect becomes stronger. Then, we can observe that the amplitude increase of  $A$  depends on the sign of the coefficient  $f_2$ . It is interesting to consider the case when  $(2c\pi)^{-1} L \sqrt{K/M} \gg \epsilon^{-1}$ . In this case the period of the oscillator is much smaller than the time during which a wave travels from one end to the other end. Wave vectors  $k$  are proportional to  $n/L$ , where  $n$  is an integer, and the distance between the frequencies is  $d\omega(k)/dn = c/L$ . Then, we have a resonance set of values for  $k$ , and the analysis for this case is more complicated. However, to simplify the problem, we observe that for the case  $\mu \ll 1$ , and for the nonlinearity  $f = f_2 u^2 + f_3 u^3$ , we obtain

$$\Omega \frac{dA}{d\tau} = \epsilon(-f_2 c_2 A + f_3 c_3 A^2), \tag{74}$$

where  $c_j > 0$  are constants. This result coincides with [15] for the case  $f_2 = 0, f_3 < 0$  where we observe a decreasing amplitude. For  $f_2 \neq 0, f_3 < 0$  and  $0 < A(0) < A_* = f_2 c_2 / f_3 c_3$  we can have an increasing amplitude within a time period. This growth, however, is bounded by the value  $A_*$  for large times since due to the cubic term the nonlinearity is saturated and the energy  $E[u]$  is bounded from below.

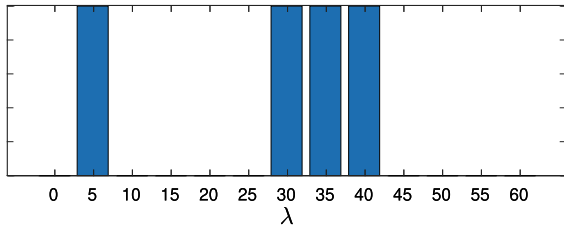
### 7 Concluding remarks

In some practical applications as mentioned in Introduction, we can encounter the following two cases:

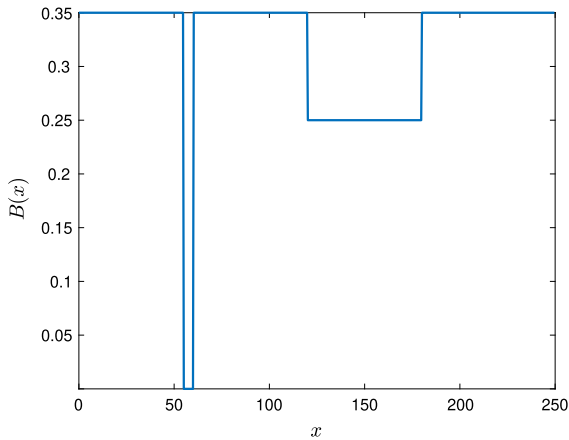
**A** A single eigenvalue of the structure is located beneath a dense group of other eigenvalues;

**B** eigenvalues are grouped in separate areas where they are densely located.

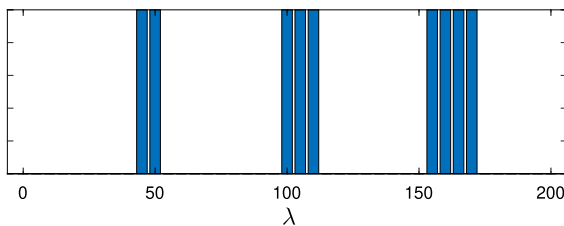
Let us briefly explain how to get a string with a prescribed spectrum. Consider case **A** (a typical example is shown in Fig. 1). Let the first minimal eigenvalue be  $a_0^2$  and let the remaining and localized spectrum be in the interval  $[a_1, b_1]$ . In this case we can use the fact mentioned earlier for a swallow well which contains exactly one eigenfunctions. We choose  $a < a_0$  but close to  $a_0$ , so  $a - a_0 \ll a_0$ . The plot of  $W(x)$  corresponds to two potential wells separated at a great



**Fig. 1** A typical spectrum for the case, where the single eigenvalue of the structure is located beneath a dense group of other eigenvalues, a few of the eigenvalues are located inside an interval, and a single one is located far away from this group at  $\lambda = 5$ . The domains of eigenvalue localization are shown by dark bars

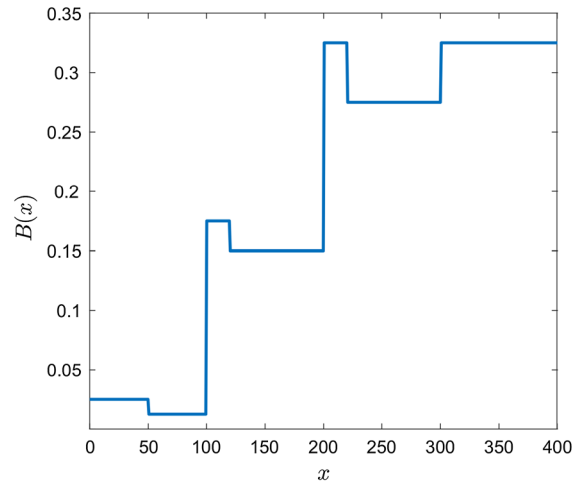


**Fig. 2** The form of the elastic foundation function  $a^2 + W(x, x_0)$  and location of potential wells, which gives the spectrum as shown in Fig. 1. The first potential well is deep and narrow and produces a single eigenvalue close to  $\lambda = 5$ . The second well has a larger width and it produces a few localized modes, and the corresponding eigenvalues are located in the interval  $[27, 42]$ . Wells should be separated by a long distance



**Fig. 3** A typical spectrum for the case, where the eigenvalues are grouped in separate areas where they are densely located. Here we have 3 groups of eigenvalues

distance, for example in the order of the string length. The first well is a swallow one and the second is a deep one (see Fig. 2). In case **B** we suppose that we

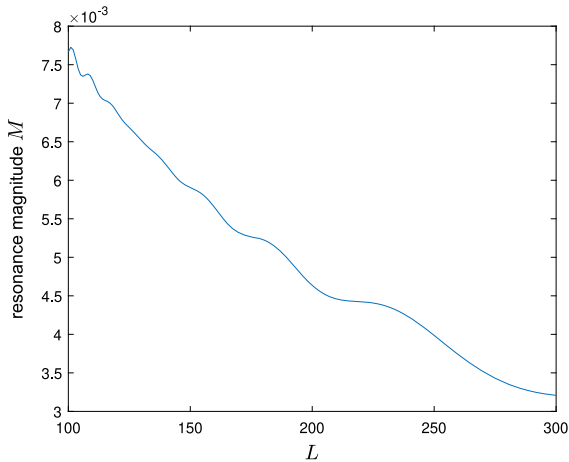


**Fig. 4** The form of the elastic foundation function  $a^2 + W(x, x_0)$  and the location of the potential wells which create the spectrum, as shown in Fig. 3

have  $m$  groups of eigenvalues with eigenvalue density  $d_\lambda$ . Then we proceed as follows. We take a potential  $V$  consisting of  $m$  potential wells  $V_j(x)$  separated by large distances. For each well we adjust its form, in particular in such a way that the well width  $a_j$  is proportional to  $d_\lambda^{-1/2}$ , and the well height is proportional to the number of eigenvalues in the corresponding group. In the general case the algorithm to create a string of large length  $L \gg 1$  with a prescribed discrete spectrum is as follows. We first solve the SCP problem for the operator  $\mathbf{H}$ . To this end, we define  $\kappa_j$  by  $\kappa = \sqrt{\lambda_j}$ ,  $j = 1, \dots, N$  and choose the points  $\bar{x}_j$  such that  $0 = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 \dots < \bar{x}_N < \bar{x}_{N+1} = L$ , where  $\bar{x}_{i+1} - \bar{x}_i > L/2N$ . Then we define  $V_N(x)$  by the relation (24) and  $W$  by  $W = c^2 V_N - a^2$ .

A typical spectrum for the case, where the eigenvalues are grouped in separate areas where they are densely located, is shown in Fig. 3. The form of the elastic foundation function  $a^2 + W(x)$  and the location of the potential wells which create the spectrum, as shown in Fig. 3, are presented in Fig. 4. The magnitude of resonances (defined by relation (61) between localized and non-localized modes as a function of the string length is shown in Fig. 5.

We consider the dynamics of strings on an elastic weakly nonlinear foundation, which have a prescribed spectrum. The foundation may also be spatially heterogeneous. The research has been conducted with boundary conditions which describe an oscillator coupled to



**Fig. 5** The magnitude of resonances [defined by relation (61)] between localized and non-localized modes as a function of the string length. The parameters are  $a^2 = 0.5$ , the minimal string length  $L_{\min} = 100$ , the maximal string length  $L_{\max} = 300$ , the speed  $c = 2$ , the time  $T = 500 = \epsilon^{-1}$ ,  $\Omega_n - \Omega_m = 0.63$ , and the number of modes is  $N = 20$

the string at one end and the other end is fixed. In this case the string can have localized modes. We show that these localized modes determine the dynamics of the long string for large times. We can control the frequencies of the localized modes for a special choice of the spatial heterogeneities in the elastic foundation. We use an asymptotic approach for such strings, which gives a more transparent procedure to control the string discrete spectrum. This approach allows us to precisely control a part of the spectrum. Furthermore, we show that if the string is under a weak nonlinear perturbation, then these controlled modes determine the dynamics of the weakly nonlinear string for large times.

**Author contributions** AKA contributed to the conceptualization, methodology and formal analysis. SAV was involved in the methodology and formal analysis. WTVH assisted in the conceptualization and validation.

**Funding** The work received no additional funding.

**Data availability** The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

## Appendix

In this “Appendix”, we derive formula (38). Let us consider first the case  $W(x, x_0) = 0$ . Then the solution of the spectral problem (15) has the form  $\psi_k = A \sin(kx) + B \cos(kx)$ , where the boundary conditions (13) imply that

$$A \sin kL + B \cos kL = 0, \tag{75}$$

$$(K - \lambda_k M)B = b_0 k A \tag{76}$$

and  $\lambda_k = a^2 - c^2 k^2$ . This system reduces to the following equation for the unknown wave number  $k$ :

$$\tan(kL) = \frac{b_0 k}{K + M c^2 k^2 - M a^2}.$$

To find an asymptotical solution  $k$  for this equation for large  $L$ , we introduce the variable  $z = kL$ . Then, one obtains

$$\tan z = \frac{b_0 z}{L(\mu + M c^2 (z/L)^2)}, \quad \mu = K - M a^2.$$

For large  $L$  one has  $z = n\pi + O(L^{-1})$ , where  $n$  are non-negative integers. This relation gives us (38) for  $W = 0$ . To estimate the effect of the localized perturbations to  $\lambda_k$  and  $\omega_k$ , we apply the standard perturbation theory. The perturbation  $\tilde{\lambda}$  of  $\lambda$  is

$$\tilde{\lambda} = \frac{\int_0^L W(x, x_0) \psi_k(x)^2 dx}{\int_0^L \psi_k(x)^2 dx}.$$

For large  $L$  the denominator of this fraction is of order  $L$ , while the numerator has the order 1 (because  $W$  is not zero on part of the interval). Therefore, we conclude that  $\tilde{\lambda} = O(L^{-1})$ , which finally proves (38).

For a localized mode at  $x = 0$  (induced by the oscillator) one can show, by an analogous estimate, that  $\tilde{\lambda} = O(\exp(-cL))$ , where  $c > 0$  is a constant.

## References

1. Mottershead, J.E., Ram, Y.M.: Inverse eigenvalue problems in vibration absorption: passive modification and active control. *Mech. Syst. Sig. Process.* **20**, 5–44 (2006)
2. Capecchi, D., Vestroni, F.: Identification of finite element models in structural dynamics. *Eng. Struct.* **15**, 21–30 (1993)



3. Friswell, M.I., Motterhead, J.E.: Finite Element Model Updating in Structural Dynamics. Kluwer Academic Publishers, Dordrecht (1995)
4. Capecchi, D., Vestroni, F.: Monitoring of structural systems by using frequency data. *Earthq. Eng. Struct. Dyn.* **28**, 447–461 (2000)
5. Dilena, M., Fedele Dell’Oste, M., Fernandez-Saez, J., Morassi, A., Zaera, A.: Recovering added mass in nanoresonator sensors from finite axial eigenfrequency data. *Mech. Syst. Sig. Process.* **130**, 122–151 (2019)
6. Dilena, M., Fedele Dell’Oste, M., Fernandez-Saez, J., Morassi, A., Zaera, A.: Hearing distributed mass in nanobeam resonators. *Int. J. Solid. Struct.* **193–194**, 568–592 (2020)
7. Vestroni, F., Capecchi, D.: Damage evaluation in cracked vibrating beams using experimental frequencies and finite element models. *J. Vib. Control* **2**, 69–86 (1996)
8. Dilena, M., Morassi, A.: A damage analysis of steel-concrete composite beams via dynamic methods. Part II: Analytical models and damage detection. *J. Vib. Control* **9**, 529–565 (2003)
9. Dilena, M., Morassi, A.: Reconstruction method for damage detection in beams based on natural frequency and antiresonant frequency measurements. *J. Eng. Mech.* **136**, 329–344 (2010)
10. Pau, A., Greco, A., Vestroni, F.: Numerical and experimental detection of concentrated damage in a parabolic arch by measured frequency variations. *J. Vib. Control* **17**(4), 605–614 (2011)
11. Kawano, A., Morassi, A.: Uniqueness in the determination of loads in multi-span beams and plates. *Eur. J. Appl. Math.* **30**(1), 176–195 (2019)
12. Kawano, A., Morassi, A., Zaera, R.: Dynamic identification of pretensile forces in a spider orb-web. *Mech. Syst. Sig. Process.* **169**, 108703 (2022)
13. Lazer, A.C., McKenna, P.J.: Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. *SIAM Rev.* **32**, 537–578 (1990)
14. van Horssen, W.T.: An asymptotic theory for a class of initial-boundary value problems for weakly nonlinear wave equations with an application to a model of the galloping oscillations of overhead transmission lines. *SIAM J. Appl. Math.* **48**, 1227–1243 (1988)
15. Kaplunov, J., Nolde, E.: An example of a quasi-trapped mode in a weakly non-linear elastic waveguide. *C. R. Méc.* **336**(7), 553–558 (2008)
16. Indeitsev, D.A., Osipova, E.V.: Localization of nonlinear waves in elastic bodies with inclusions. *Acoust. Phys.* **50**, 420–426 (2004)
17. Shishkina, E.V., Gavrilov, S.N., Mochalova, Y.A.: Passage through a resonance for a mechanical system, having time-varying parameters and possessing a single trapped mode. The principal term of the resonant solution. *J. Sound Vib.* **481**, 115422 (2020)
18. Luongo, A.: Mode localization in dynamics and buckling of linear imperfect continuous structures. *Nonlinear Dyn.* **25**, 133–156 (2001)
19. Indeitsev, D.A., Kuklin, T.S., Mochalova, Y.A.: Localization in a Bernoulli–Euler beam on an inhomogeneous elastic foundation. *Vestn. St. Petersburg. Univ. Math.* **48**(1), 41–48 (2015)
20. Abramian, A.K., Andreyev, V.L., Indejtchev, D.A.: Resonance oscillations of infinite and finite elastic structures with inclusions. *JASA* **95**, 3007 (1994). <https://doi.org/10.1121/1.408819>
21. Morassi, A.: Explicit construction of rods and beams with given natural frequencies. In: *Topics in Modal Analysis, Volume 7. Conference Proceedings of the Society for Experimental Mechanics Series*. Springer, New York (2014)
22. Matveev, V.B., Salle, M.A.: *Darboux Transformations and Solitons*. Springer, Berlin (1991)
23. Morassi, A.: Exact construction of beams with a finite number of given natural frequencies. *J. Vib. Control* **21**(3), 591–600 (2015)
24. Behrndt, J., Khrabustovsky, A.: Singular Schrödinger operators with prescribed spectral properties. *J. Funct. Anal.* **282**, 109252 (2022)
25. Younesian, D., Hosseinkhani, A., Askari, H., et al.: Elastic and viscoelastic foundations: a review on linear and nonlinear vibration modeling and applications. *Nonlinear Dyn.* **97**, 853–895 (2019)
26. Yurke, B.: Conservative model for the damped harmonic oscillator. *Am. J. Phys.* **52**, 1099 (1984)
27. Landau, L.D., Lifshitz, E.M.: *Quantum Mechanics: Non-Relativistic Theory*, vol. 3, 3rd edn. Pergamon Press, Oxford (1977)
28. Kato, T.: *Perturbation Theory for Linear Operators*. Springer, Berlin (1991)
29. Boertjens, G.J., van Horssen, W.T.: An asymptotic theory for a weakly nonlinear beam equation with a quadratic perturbation. *SIAM J. Appl. Math.* **60**(2), 602–632 (2000)
30. Bogoliubov, N., Mitropolsky, Y.A.: *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Gordon and Breach, New York (1961)
31. De Bruijn, N.: *Asymptotic Methods in Analysis*. North-Holland, Amsterdam (1961)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.