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Some less boring ones

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Vibrations of a circular drum: Some less boring ones

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1 Introduction

The standard Wave Equation in 2 dimensions is defined on a Cartesian grid as:

$$c^2\frac{d^2u}{dx^2} + c^2\frac{d^2u}{dy^2} = \frac{d^2u}{dt^2} \tag{1}$$

In virtually every textbook about Partial Differential Equations there is a section about the circular drum, which requires reformulating the Wave equation using Polar Coordinates:

$$\frac{c^2}{r^2}\frac{d^2u}{d\phi^2} + c^2\frac{d^2u}{dr^2} + \frac{c^2}{r}\frac{du}{dr} = \frac{d^2u}{dt^2} \tag{2}$$

After using Separation of Variables with separation constants λ and μ this leads to 3 ordinary equations:

$$\frac{d^2T}{dt^2} + \lambda^2c^2T = 0 \tag{3}$$

$$\frac{d^2P}{d\phi^2} + \mu^2P = 0 \tag{4}$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (r^2\lambda^2 - \mu^2)R = 0 \tag{5}$$

with solutions:

$$T(t) = C_1\sin(c\lambda t) + C_2\cos(c\lambda t) \tag{6}$$

$$P(\phi) = C_3\sin(\mu\phi) + C_4\cos(\mu\phi) \tag{7}$$

$$R(r) = C_5\mathrm{BesselY}(\mu,\lambda r) + C_6\mathrm{BesselJ}(\mu,\lambda r) \tag{8}$$

2 Standard problem with only a fixed outside boundary

This is by far the most commonly used example, and is discussed in most handbooks about Partial Differential Equations as well as on many websites, including e.g. Wikipedia (see ([7]).

If we restrict ourselves to single eigen modes for a circular drum with a fixed outside boundary (so a Dirichlet boundary) at $r = a$ some extra restrictions are required, enabling us to simplify the 3 functions from the introduction (section 1):

- We can simplify $T(t)$ to only contain the cosine term.
- $P(\phi)$ must be 2π periodic, so μ must be integer. This gives us the first wave number that we'll call 'm'.
- $R(r)$ can not have a singularity at the center (r=0) for the BesselY component, so $C_5 = 0$
- The BesselJ component should have zero amplitude at the boundary $r = a$, so $\mathrm{BesselJ}(\mu, \lambda a)$ should be 0.
- This requires λ/a to be an arbitrary zero for any BesselJ function with integer order m (or μ), or in other words: we need to scale the zeros of the BesselJ function with a factor a.
- The second wave number, which we'll call 'n' is simply a counter indicating which zero of the BesselJ function has to be scaled to be exactly on the boundary $r = a$.
- It is also enough to only use the cosine term for $P(\phi)$, but although not essential we subtract $\pi/4$ as to have the same orientation as when using both the sine and cosine.
- All other constants can be simply set to be either 0 or 1, as we don't care about the exact amplitude of a single mode.

This leads to the following:

$$T(t) = \cos\left(c\lambda_{m,n}t\right) \tag{9}$$
$$P(\phi) = \cos\left(m\phi - \pi/4\right) \tag{10}$$
$$R(r) = \mathrm{BesselJ}(m, \lambda_{m,n}r) \tag{11}$$

where the (scaled) zeros $\lambda_{m,n}$ of the BesselJ function have to be searched for in a numerical way, as there is no symbolic expression for roots of Bessel functions.
After that we can simply plot and/or animate the solutions: see (fig. 1).
The duration of a single period for this vibration is of course given by: $\dfrac{2\pi}{c\lambda_{m,n}}$

3 Some less boring ones: overview

As already mentioned above almost every textbook stops at the previous example, which can be seen as e.g. a base drum without any hole, such as is used in many classic rock bands.
However, there are also a lot more interesting, albeit also slightly more complicated examples that I would like to discuss below:

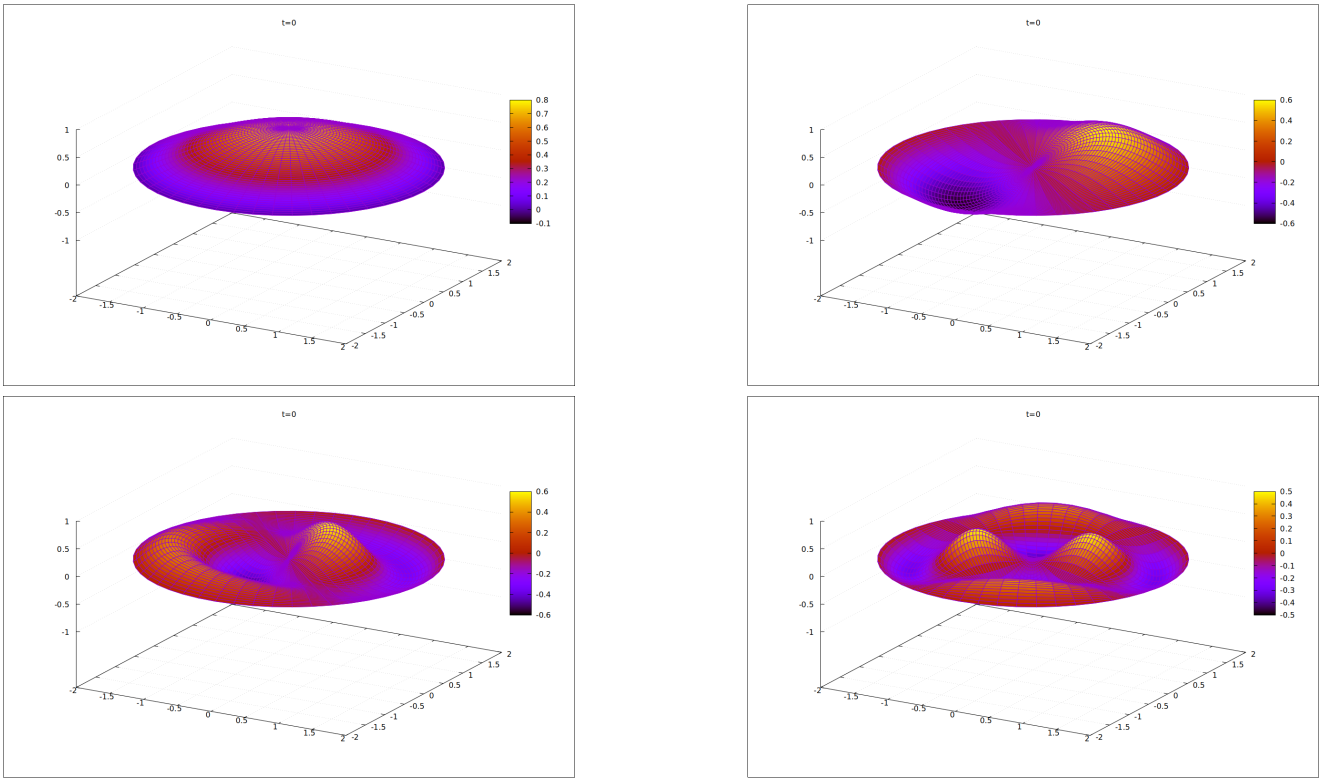

Figure 1: Modes 01, 11, 12 and 22 for a circular drum with only a fixed outside boundary

- Circular drum as in section (2), but now with damping, which also slightly changes the eigen modes.
- Circular drum with both a fixed inside boundary at $r = b$ and an outside boundary at $r = a$. This is often called an annular or ringshaped drum. E.g. a loudspeaker can be modelled as such a membrane.
- Circular drum with both a free moving inside boundary at $r = b$ and a fixed outside boundary at $r = a$. A nice example of this type is the common base drum with a centered hole, as is also often used in classic rock bands.
- Circular drum with a (small) fixed inside boundary at $r = b$, but with a free moving outside boundary at $r = a$. An example is again found in the rhythm section of rockbands as this type could be a simple model of a cymbal.
- Circular drum which is fixed at the center with $r = 0$, but with a free moving outside boundary at $r = a$. This could also be used as a model for a cymbal, but has some restrictions that are not found in the previous item with a small ring inside.

4 Solution with damping

If we add damping to the problem found in section (2) we find a new system of equations that is slightly more realistic if compared to membranes in the real world.

$$\frac{c^2}{r^2}\frac{d^2u}{d\phi^2} + c^2\frac{d^2u}{dr^2} + \frac{c^2}{r}\frac{du}{dr} = \frac{d^2u}{dt^2} + \gamma\frac{du}{dt} \tag{12}$$

After using Separation of Variables with separation constants λ and μ this leads again to 3 ordinary equations:

$$\frac{d^2T}{dt^2} + \gamma\frac{dT}{dt} + \lambda^2c^2T = 0 \tag{13}$$

$$\frac{d^2P}{d\phi^2} + \mu^2P = 0 \tag{14}$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (r^2\lambda^2 - \mu^2)R = 0 \tag{15}$$

with solutions:

$$T(t) = e^{\dfrac{-t\gamma}{2}}\left(C_1 \sin\frac{\sqrt{4c^2\lambda^2-\gamma^2}t}{2} + C_2\cos\frac{\sqrt{4c^2\lambda^2-\gamma^2}t}{2}\right) \tag{16}$$

$$P(\phi) = C_3\sin(\mu\phi) + C_4\cos(\mu\phi) \tag{17}$$

$$R(r) = C_5\text{BesselY}(\mu,\lambda r) + C_6\text{BesselJ}(\mu,\lambda r) \tag{18}$$

Following the same reasoning as before we can simplify these expressions, where the only difference is now the more complicated expression for $T(t)$:

$$T(t) = e^{\dfrac{-t\gamma}{2}}\left(\cos(\frac{\sqrt{4c^2\lambda^2-\gamma^2}t}{2})\right) \tag{19}$$

This also leads to a different duration for a single period of the damped motion: $\dfrac{4\pi}{\sqrt{4c^2\lambda^2-\gamma^2}}$

As the first frames (at $t=0$) of the modes are identical to the ones in section (2) and animations are a little difficult in an article like this, I would like to leave it to this and continue with the next, more interesting types of membranes.

5 Inside and outside boundary fixed (ring or hole in center)

The so called "annular drum" has both a fixed (Dirichlet) inner boundary and a fixed (Dirichlet) outer boundary. One example is e.g. a simple model for a loudspeaker.
This second boundary condition drastically changes some of the restrictions as put up in section (2):

- We can simplify $T(t)$ to only contain the cosine term (unchanged).
- $P(\phi)$ must be 2π periodic, so μ must be integer. This gives us the first wave number that we'll call 'm' (unchanged).

- The singularity for $R(r)$ at the center for the BesselY component is not important anymore now, so the solution now also contains BesselY functions!
- The solution should still have zero amplitude at the outer boundary $r = a$, which now leads to:

$$C_5 \mathrm{BesselY}(\mu, \lambda a) + C_6 \mathrm{BesselJ}(\mu, \lambda a) = 0 \tag{20}$$

- The solution should also have zero amplitude at the inner boundary $r = b$, which leads to:

$$C_5 \mathrm{BesselY}(\mu, \lambda b) + C_6 \mathrm{BesselJ}(\mu, \lambda b) = 0 \tag{21}$$

- Combining these 2 conditions leads to a fixed ratio between C_5 and C_6:

$$\frac{C_5}{C_6} = -\frac{\mathrm{BesselY}(\mu, \lambda b)}{\mathrm{BesselJ}(\mu, \lambda b)} = -\frac{\mathrm{BesselY}(\mu, \lambda a)}{\mathrm{BesselJ}(\mu, \lambda a)} \tag{22}$$

- Computing the values of λ is not so easy as in section (2) but from equation (22) we can find an expression that allows us to compute the values of λ:

$$Z(\lambda, m) = \mathrm{BesselY}(\mu, \lambda b)\mathrm{BesselJ}(\mu, \lambda a) - \mathrm{BesselY}(\mu, \lambda a)\mathrm{BesselJ}(\mu, \lambda b) = 0 \tag{23}$$

- The second wave number 'n' is again a counter, but now indicating the 1st, 2nd etc. zeros of the $Z(\lambda, m)$ function from above.
- All other constants can be simply set to be either 0 or 1, as we don't care about the exact amplitude of a single mode (unchanged).

This now leads to the following 3 functions, where the integration constants are replaced using either the condition for $r = a$ or $r = b$:

$$T(t) = \cos(c\lambda_{m,n} t) \tag{24}$$
$$P(\phi) = \cos(m\phi - \pi/4) \tag{25}$$
$$R(r) = \mathrm{BesselY}(\mu, \lambda_{m,n} b)\mathrm{BesselJ}(m, \lambda_{m,n} r) - \mathrm{BesselJ}(\mu, \lambda_{m,n} b)\mathrm{BesselY}(m, \lambda_{m,n} r) \tag{26}$$

The zeros $\lambda_{m,n}$ of the $Z(\lambda, m)$ function again have to be searched for in a numerical way e.g. using Regula Falsi or Bisection.

After that we can simply plot and/or animate the solutions: see fig. (2).
Note that the period hasn't changed if compared with section (2), as long as it is expressed in terms of $\lambda_{m,n}$.

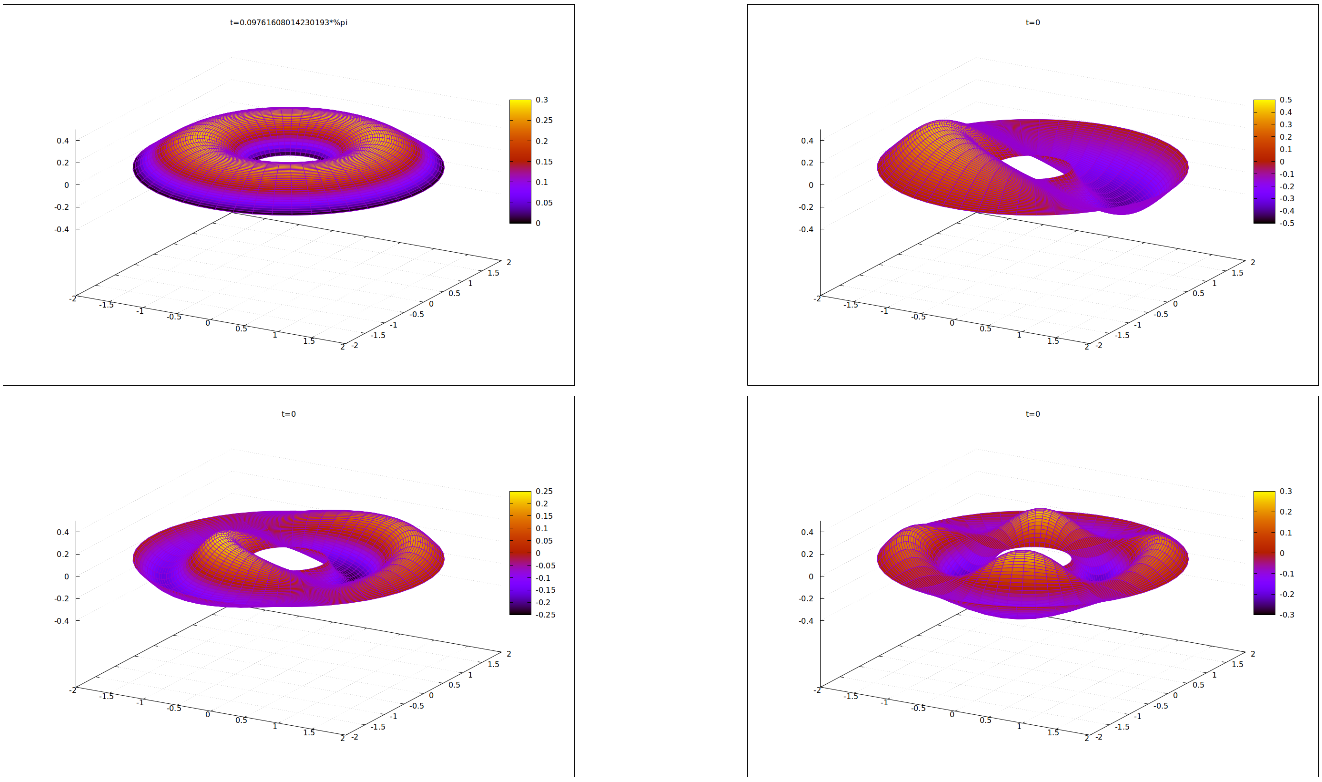

Figure 2: Modes 01, 11, 12 and 22 for a circlar drum with both inside and outside boundary fixed

6 Free inside and fixed outside boundary (hole in center)

Another nice membrane is the one with a hole in the center, where the inner boundary now has to obey a Neumann condition (spatial derivative must be zero at the inner boundary). This is the favorite shape of the base drum for many rock bands since the 70s and later.

Although most rock musicians have no clue about the differences between Bessel J and Y functions (maybe with the exception of Dr. Brian May from Queen ;-)), they are definitely able to hear the difference: these drums are said to have a slightly sharper beat than the ones from the first section.

The restrictions for this type of membrane are largely the same as in section (5):

- We can simplify $T(t)$ to only contain the cosine term (unchanged).
- $P(\phi)$ must be 2π periodic, so μ must be integer. This gives us the first wave number that we'll call 'm' (unchanged).
- The singularity for $R(r)$ at the center for the BesselY component is not important anymore now, so the solution now also contains BesselY functions (unchanged).
- The solution should still have zero amplitude at the outer boundary $r = a$, which leads to (unchanged):

$$C_5 \text{BesselY}(\mu, \lambda a) + C_6 \text{BesselJ}(\mu, \lambda a) = 0 \tag{27}$$

- However, the solution should now have a horizontal derivative at the inner boundary $r = b$, which leads to:
$$\frac{d(C_5 \text{BesselY}(\mu, \lambda b) + C_6 \text{BesselJ}(\mu, \lambda b))}{db} = 0 \tag{28}$$

  or:
$$\begin{aligned} &C_5(\text{BesselY}(\mu - 1, \lambda b) - \text{BesselY}(\mu + 1, \lambda b)\lambda + \\ &C_6(\text{BesselJ}(\mu - 1, \lambda b) - \text{BesselJ}(\mu + 1, \lambda b)\lambda = 0 \end{aligned} \tag{29}$$

- This again leads to an expression for a fixed ratio between C_5 and C_6:
$$\frac{C_5}{C_6} = -\frac{\text{BesselY}(\mu, \lambda a)}{\text{BesselJ}(\mu, \lambda a)} = -\frac{\text{BesselY}(\mu + 1, \lambda b) - \text{BesselY}(\mu - 1, \lambda b)}{\text{BesselJ}(\mu + 1, \lambda b) - \text{BesselJ}(\mu - 1, \lambda b)} \tag{30}$$

- Computing the values of λ is similar to the method in section (5) as from the 2 conditions above we can find an expression that allows us to compute the values of λ:
$$\begin{aligned} Z(\lambda, m) = \; &\text{BesselJ}(\mu, \lambda a)\dot{(}\text{BesselY}(\mu + 1, \lambda b) - \text{BesselY}(\mu - 1, \lambda b)) - \\ &\text{BesselY}(\mu, \lambda a)\dot{(}\text{BesselJ}(\mu + 1, \lambda b) - \text{BesselJ}(\mu - 1, \lambda b)) = 0 \end{aligned} \tag{31}$$

- The second wave number 'n' is again a counter, indicating the sequence of zeros of the (modified) $Z(\lambda, m)$ function from above.
- All other constants can be simply set to be either 0 or 1, as we don't care about the exact amplitude of a single mode (unchanged).

This now leads to the following:

$$T(t) = \cos(c\lambda_{m,n} t) \tag{32}$$
$$P(\phi) = \cos(m\phi - \pi/4) \tag{33}$$
$$R(r) = \text{BesselY}(\mu, \lambda_{m,n} a)\text{BesselJ}(m, \lambda_{m,n} r) - \text{BesselJ}(\mu, \lambda_{m,n} a)\text{BesselY}(m, \lambda_{m,n} r) \tag{34}$$

where the zeros $\lambda_{m,n}$ of the $Z(\lambda, m)$ function again have to be searched for in a numerical way.

After that we can simply plot and/or animate the solutions: see fig. (3).
Note that the period hasn't changed as long it is expressed in terms of $\lambda_{m,n}$.

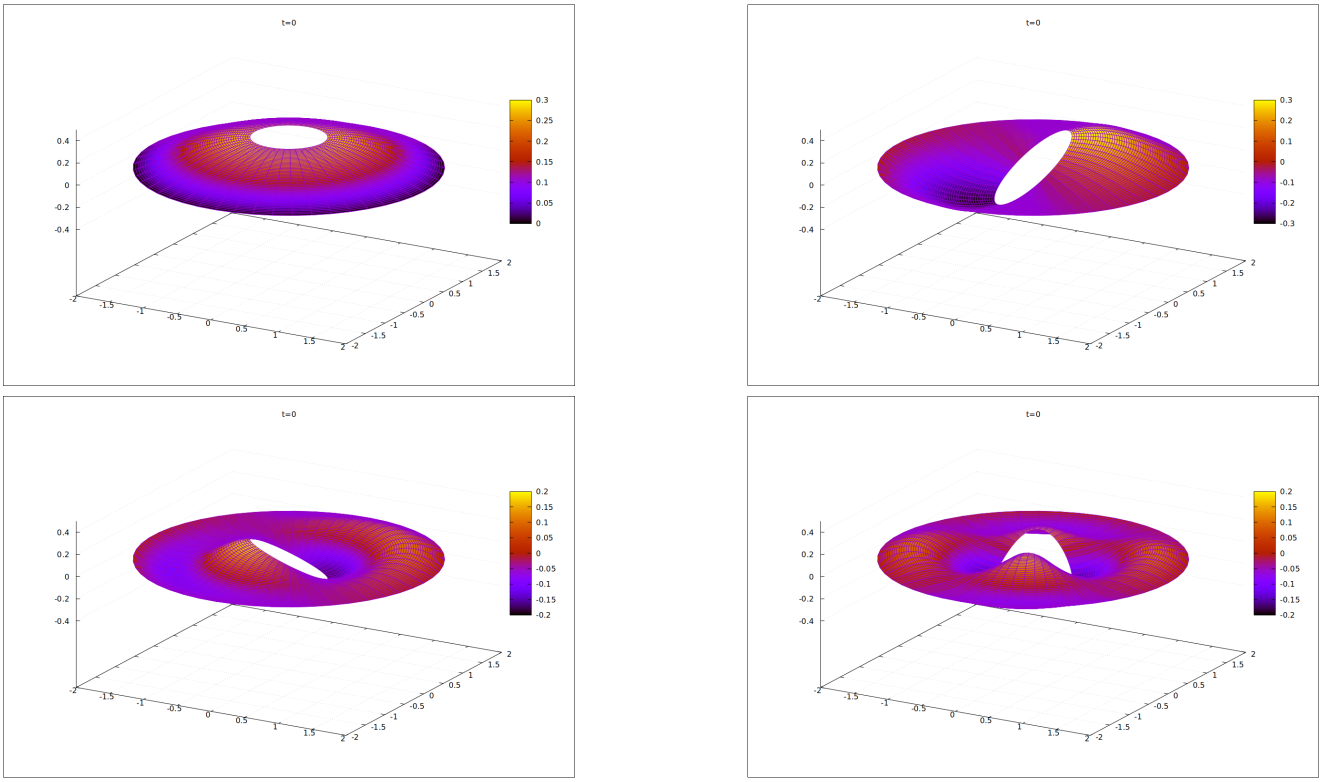

Figure 3: Modes 01, 11, 12 and 22 for a circular drum with free inside and fixed outside boundary

7 Fixed inside and free outside boundary

A membrane with a fixed inner Dirichlet boundary and a free Neumann outer boundary - although it looks totally different - is mathematically almost the same as the membrane discussed in section (6): it is enough to simply exchange the inner and outer radius (resp. 'b' and 'a') to find the solution to this problem.

This membrane could be seen as a simplified model of a cymbal, although to also take the stresses into account a more complete analysis should use a 4th order model. However, this model - albeit simple - is a nice start to see what motion is possible in such a cymbal and if you compare some high speed camera recordings of a cymbal being struck you can clearly see similar vibrations.

Let's check the restrictions for this type of membrane:

- We can simplify $T(t)$ to only contain the cosine term (unchanged).
- $P(\phi)$ must be 2π periodic, so μ must be integer. This gives us the first wave number that we'll call 'm' (unchanged).
- The singularity for $R(r)$ at the center for the BesselY component is again not important, so the solution now also contains BesselY functions (unchanged).
- The solution should have zero amplitude at the inner boundary $r = b$, which leads to (unchanged if compared to section (5)):

$$C_5 \text{BesselY}(\mu, \lambda b) + C_6 \text{BesselJ}(\mu, \lambda b) = 0 \tag{35}$$

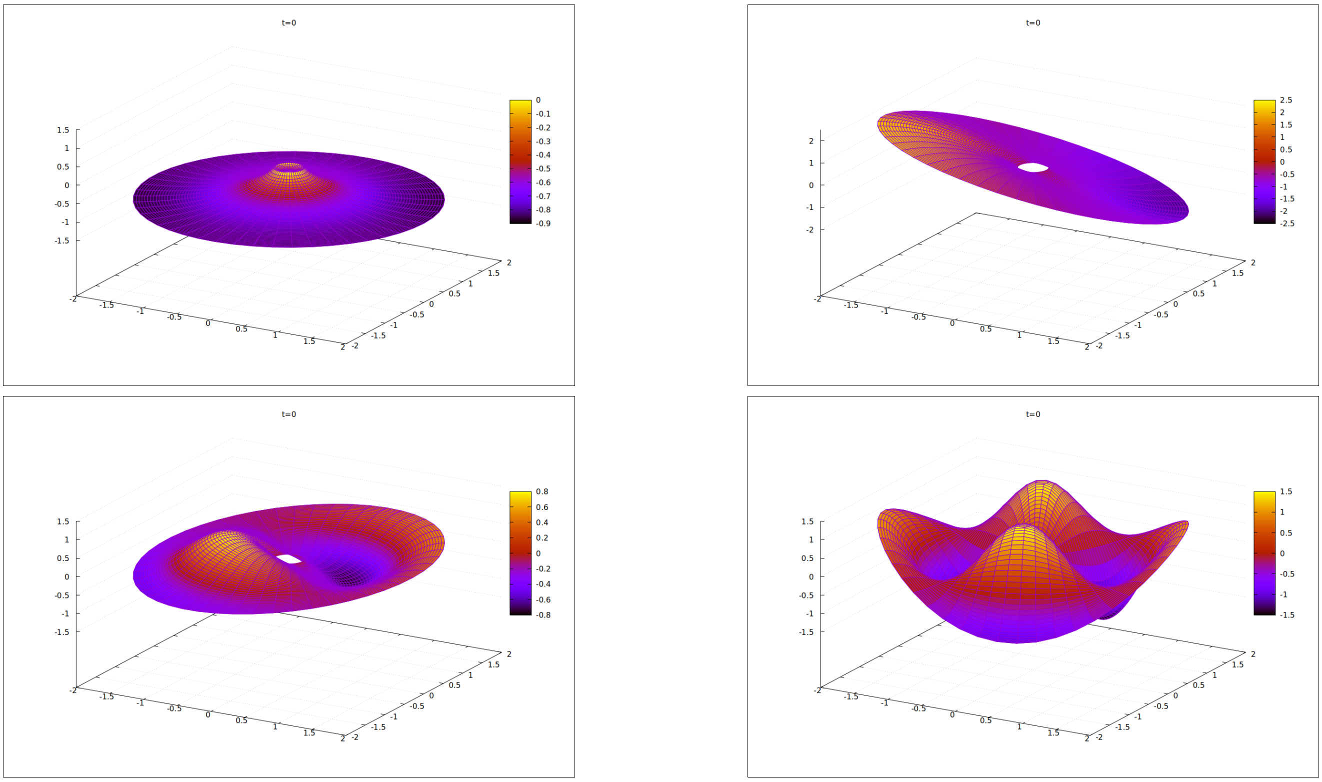

Figure 4: Modes 01, 11, 12 and 22 for a circular drum with fixed inside and free outside boundary

- However, the solution should now have a horizontal derivative at the outer boundary $r = a$, which leads to:

$$\frac{d(C_5\text{BesselY}(\mu, \lambda a) + C_6\text{BesselJ}(\mu, \lambda b))}{da} = 0 \tag{36}$$

or:

$$\begin{aligned} &C_5(\text{BesselY}(\mu-1, \lambda a) - \text{BesselY}(\mu+1, \lambda a)\lambda + \\ &C_6(\text{BesselJ}(\mu-1, \lambda a) - \text{BesselJ}(\mu+1, \lambda a)\lambda = 0 \end{aligned} \tag{37}$$

- This again leads to an expression for a fixed ratio between C_5 and C_6:

$$\frac{C_5}{C_6} = -\frac{\text{BesselY}(\mu, \lambda b)}{\text{BesselJ}(\mu, \lambda b)} = -\frac{\text{BesselY}(\mu+1, \lambda a) - \text{BesselY}(\mu-1, \lambda a)}{\text{BesselJ}(\mu+1, \lambda a) - \text{BesselJ}(\mu-1, \lambda a)} \tag{38}$$

- Computing the values of λ is completely similar to the method in section (6):

$$\begin{aligned} Z(\lambda, m) = \;& \text{BesselJ}(\mu, \lambda b)(\text{BesselY}(\mu+1, \lambda a) - \text{BesselY}(\mu-1, \lambda a)) - \\ & \text{BesselY}(\mu, \lambda b)(\text{BesselJ}(\mu+1, \lambda a) - \text{BesselJ}(\mu-1, \lambda a)) = 0 \end{aligned} \tag{39}$$

- The second wave number 'n' is again a counter, indicating the sequence of zeros of the $Z(\lambda, m)$ function from above (unchanged).

- All other constants can be simply set to be either 0 or 1, as we don't care about the exact amplitude of a single mode (unchanged).

This now leads to the following:

$$
\begin{aligned}
T(t) &= \cos(c\lambda_{m,n}t) & (40)\\
P(\phi) &= \cos(m\phi - \pi/4) & (41)\\
R(r) &= \text{BesselY}(\mu, \lambda_{m,n}b)\text{BesselJ}(m, \lambda_{m,n}r) - \text{BesselJ}(\mu, \lambda_{m,n}b)\text{BesselY}(m, \lambda_{m,n}r) & (42)
\end{aligned}
$$

The rest (including the period) is the same as for the previous membranes and leads to the plots shown in fig. (4).

8 Fixed center and free outside boundary

This is a more simplified version of the membrane with free (Neumann) boundary on the outside, as discussed in section (7). It can also be used as a (simplified) model of a cymbal and requires less effort than the one in (7).
Here the system has again an outside boundary at $r = a$ that can move freely. However, now we do not have a fixed inner boundary at $r = b$ but only a fixed center at $r = 0$.
The latter implies that no BesselY() functions are allowed anymore in the solution as they are always $\pm\infty$ at the center. Furthermore it also implies that the order of the BesselJ functions should be larger than 0 as BesselJ(0) is also non zero at the center.
For this reason we only have wave numbers 11 and up for this type of membrane.
The mathematical restrictions for this type of membrane have some resemblance to the simple membrane from (section 1):

- We can simplify $T(t)$ to only contain the cosine term (unchanged).
- $P(\phi)$ must be 2π periodic, so μ must be integer. This gives us the first wave number that we'll call 'm' (unchanged).
- $R(r)$ can not have a singularity at the center (r=0) which requires dropping the BesselY component, so $C_5 = 0$ (unchanged).
- The BesselJ component should now have a horizontal derivative at the boundary $r = a$, so:

$$C_6(\text{BesselJ}(\mu + 1, \lambda a) - \text{BesselJ}(\mu - 1, \lambda a))\lambda = 0. \tag{43}$$

- Also, because BesselJ(0) is NOT equal to 0 at the center the order 'm' must be at least 1 or higher for this problem.
- Computing the values of λ can be done as before (numerically) from equation (43):

$$Z(\lambda, m) = \text{BesselJ}(\mu + 1, \lambda a) - \text{BesselJ}(\mu - 1, \lambda a) = 0 \tag{44}$$

- The second wave number is again a counter for the zeros of the function $Z(\lambda, m)$ from above (unchanged).
- All constants can be simply set to be either 0 or 1, as we don't care about the exact amplitude of a single mode (unchanged).

The rest is the same as in section (2):

$$\begin{aligned} T(t) &= \cos\left(c\lambda_{m,n}t\right) & (45)\\ P(\phi) &= \cos\left(m\phi - \pi/4\right) & (46)\\ R(r) &= \mathrm{BesselJ}(m, \lambda_{m,n}r) & (47) \end{aligned}$$

After that we can simply plot and/or animate the solutions as before: see (fig. 5).

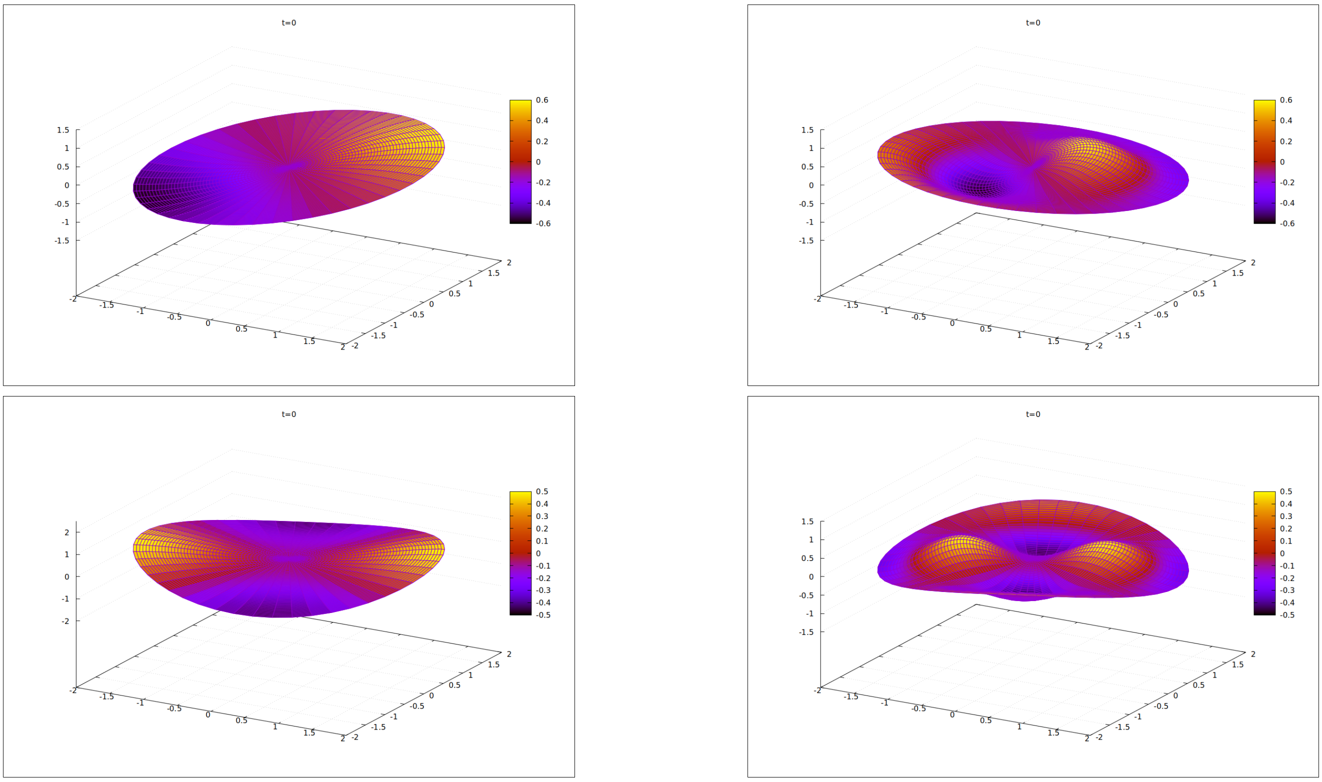

Figure 5: Modes 11, 12, 21 and 22 for a circular drum with a fixed center at $r = 0$ and free outside boundary

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