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Hong, Huifen; Baldi, Simone; Yu, Wenwu; Yu, Xinghuo

DOI
10.1109/TCYB.2021.3128051

Publication date
2022

Document Version
Final published version

Published in
IEEE Transactions on Cybernetics

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Huifen Hong, Simone Baldi, *Senior Member, IEEE*, Wenwu Yu, *Senior Member, IEEE*, and Xinghuo Yu, *Fellow, IEEE*

***Abstract*—This article investigates the distributed time-varying optimization problem for second-order multiagent systems (MASs) under limited interaction ranges. The goal is to seek the minimum of the sum of local time-varying cost functions (CFs), where each CF is only available to the corresponding agent. Limited communication range refers to the scenario where the agents have limited sensing and communication capabilities, that is, a pair of agents can communicate with each other only if their distance is within a certain range. To handle such a problem, a new continuous connectivity-preserving mechanism is presented to preserve the connectivity of the considered network. Then, two distributed optimization algorithms are presented to solve the optimization problem with time-varying CFs and time-invariant CFs, respectively. Theoretical analysis and two numerical examples are provided to verify the effectiveness of the methods.**

***Index Terms*—Connectivity-preserving mechanism, distributed optimization, finite-time consensus, second-order multiagent system (MAS), time-varying cost functions (CFs).**

Manuscript received 9 July 2021; revised 7 October 2021; accepted 6 November 2021. Date of publication 30 November 2021; date of current version 18 November 2022. This work was supported in part by the National Natural Science Foundation of China under Grant 62073076; in part by the Smart Grid Joint Fund of State Key Program of National Natural Science Foundation of China under Grant U2066203; in part by the Australian Research Council under Discovery Program under Grant DP200101197; in part by the Natural Science Foundation of Jiangsu Province under Grant BK20210591; in part by the University Science Research Project of Jiangsu Province under Grant 21KJB510044; in part by the Doctor of Entrepreneurship and Innovation in Jiangsu Province under Grant JSSCBS20210509; and in part by the Scientific Foundation of Nanjing University of Posts and Telecommunications under Grant NY220147. This article was recommended by Associate Editor Z.-G. Hou. *(Corresponding author: Wenwu Yu.)*

Huifen Hong is with the School of Mathematics, Southeast University, Nanjing 210096, China, and also with the College of Automation, College of Artificial Intelligence, Nanjing University of Posts and Telecommunications, Nanjing 210023, China (e-mail: honghuifen@seu.edu.cn).

Simone Baldi is with the School of Mathematics, Southeast University, Nanjing 210096, China, and also with the Delft Center for Systems and Control, Delft University of Technology, 2628 Delft, The Netherlands (e-mail: s.baldi@tudelft.nl).

Wenwu Yu is with the School of Mathematics, Frontiers Science Center for Mobile Information Communication and Security, Southeast University, Nanjing 210096, China, and also with the Department of Electrical Engineering, Nantong University, Nantong 226019, China (e-mail: wwyu@seu.edu.cn).

Xinghuo Yu is with the School of Engineering, RMIT University, Melbourne, VIC 3001, Australia (e-mail: x.yu@rmit.edu.au).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TCYB.2021.3128051.

Digital Object Identifier 10.1109/TCYB.2021.3128051

I. Introduction

One of the most popular research topics in multiagent systems (MASs) is the distributed consensus problem, that is, driving the states of a group of agents to a common value (also called the consensus value) by designing a distributed protocol [1]. In general, the consensus problem includes two cases: 1) leader–follower consensus [2] and 2) leaderless consensus [3]. Leader–follower consensus means that all followers can track the state of the leader, while leaderless consensus implies that a set of agents reaches an agreement and then follows a given trajectory formed by the dynamics of all agents. Leaderless consensus is a fundamental problem in MASs, and can be applied in many practical areas, such as distributed average tracking [4], consensus on the median value [5], and distributed optimization [6].

Distributed optimization aims to optimize the sum of local cost functions (CFs), where each local CF is only available to the corresponding agent: in this setting, the consensus value should track the optimal trajectory of the total CF. To solve this problem, a subgradient-based algorithm [7] and two gossip algorithms [8] were proposed, whereas as a distributed optimization problem with constraint was investigated in [9] by using the projection operator. A distributed randomized gradient-free optimization protocol of MASs with constraint sets over weight-unbalanced digraphs was proposed in [10].

Note that all the above results are discrete-time distributed optimization algorithms. Recently, the continuous-time distributed optimization problem has attracted increasing attention because of its potential applications in MAS coordination. Wang and Elia [11] presented a distributed algorithm by introducing a dynamic integrator. Based on the results in [11] and [12], the consensus-based optimization problem was investigated in [6] for weight-balanced digraphs and some new continuous-time coordination algorithms were presented in [13]. Yang *et al.* [14] studied the distributed optimization taking communication delays into consideration. Yue *et al.* [15] addressed the distributed optimization of a strictly convex summation-separable CF with possibly nonconvex local functions over strongly connected digraphs. The distributed constrained convex optimization problem of a sum of nonsmooth CFs was studied in [16]. A fixed-time consensus-based optimization algorithm was proposed in [17]. While [11]–[17] considered agents whose dynamics are modeled by single integrator (first-order dynamics), in many applications, such as attitude alignment of multiple unmanned aerial vehicles [18], the agents are modeled by double-integrator (second-order dynamics) or even more general linear dynamics: [19]–[24] investigated the distributed optimization problem for second-order and general linear dynamics, respectively.

In many practical settings, such as tracking an optimal solution as it changes over time, the CFs in optimization problems rely explicitly on time [25]–[27]. Recent works in this direction include [19], [28]–[30]. For example, a class of convex optimization problem with time-varying CFs was investigated in [28] in a centralized setting. Distributed quadratic optimization problem with a neighboring coupled time-varying CF was studied in [29]. A tracking control protocol was proposed in [30] for a class of nonlinear MASs to solve the time-varying distributed optimization problem. In the time-varying case, the convergence analysis is more complex.

In the aforementioned works, the considered communication graphs are all state-independent. However, in many practical applications, the sensing and communication range of each agent is limited and, thus, the corresponding communication graph is determined by the spatial distances. The main idea for addressing such limited communication ranges is to design a connectivity-preserving mechanism to preserve the connectivity property of the initial graph all the time. Existing results include asymptotic [31]–[33] and finite-time consensus [34]–[36] for MASs with a limited communication range. However, few works have discussed distributed optimization problem subject to limited communication ranges. In [37], such a problem was considered for first-order MASs with identical gradients of the local CFs. A drawback of the connectivity-preserving mechanisms proposed in most works is the discontinuities, that is, sudden jump of the consensus weights resulting from the variation of the edge set. This might result in poor performance.

In view of the above observations, there still exist several open problems for distributed time-varying optimization with second-order MASs under limited communication ranges, which motivate this work. First, new connectivity-preserving mechanisms are needed which avoid discontinuities in the consensus weights. Then, no distributed connectivity-preserving optimization algorithm has been presented for second-order MASs with time-varying CFs. Compared with the literature, the main contributions of the article are as follows.

1) Sudden jump of weights resulting from the newly formed edges can be efficiently avoided, as compared to [31]–[36], by designing a new connectivity-preserving mechanism.
2) Two distributed optimization algorithms are proposed that can make the agents reach finite-time consensus, which is more challenging than asymptotic consensus in [19].
3) The local CFs are assumed to be time-varying instead of time invariant as in [6], [11]–[14], and [23]. Note that the protocols in these works are not applicable to the time-varying CF case.
4) In the special case that the local CFs are time invariant, the proposed distributed optimization algorithm results in a simplified algorithm requiring less assumptions on the local CFs.

The structure of this article is as follows. In Section II, the problem statement is given. In Section III, a connectivity-preserving mechanism is designed and optimization control algorithms are proposed. Numerical simulations are presented in Section IV to corroborate the theoretical results and conclusions are drawn in Section V. Notations and background knowledge on graph theory and homogeneity theory used in this article can be found in the Appendix.

II. Problem Statement

Consider a MAS consisting of N agents labeled from 1 to N. Each agent evolves according to the following second-order dynamics:

$$\begin{aligned}\dot{x}_i(t) &= v_i(t)\\ \dot{v}_i(t) &= u_i(t), \quad i \in \mathcal{I}_N\end{aligned} \tag{1}$$

of which $x_i, v_i, u_i \in \mathbb{R}^n$, respectively, represent the position, the velocity, and the control input of agent i.

The second-order MAS aims to solve the following optimization problem in a distributed way:

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{N} f_i(x,t) \tag{2}$$

where $f_i(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is a time-varying local CF which is only known to agent i. $f(x,t) = \sum_{i=1}^{N} f_i(x,t)$ is assumed to be uniformly strongly convex for all $t \geq 0$, that is, $(\nabla f(x,t) - \nabla f(y,t))^{\mathrm{T}}(x-y) \geq C_f\|x-y\|^2$ for some $C_f > 0$ and $x, y \in \mathbb{R}^n$. Furthermore, each $f_i(x,t)$ is twice continuously differentiable with respect to x.

The following lemma is a standard result of convex optimization.

Lemma 1 [38]: For any differentiable convex function $f(x) : \mathbb{R}^n \to \mathbb{R}$, it holds: $f(x)$ is minimized $\iff$ $\nabla f(x) = 0$.

The optimization problem (2) is equivalent to

$$\min_{x_i\in\mathbb{R}^n} \sum_{i=1}^{N} f_i(x_i,t) \text{ subject to } x_i = x_j \quad \forall i,j \in \mathcal{I} \tag{3}$$

that is, the agents are required to reach some form of consensus. The following notion of consensus is considered in this work.

Definition 1: The second-order MAS (1) is said to achieve finite-time consensus, if there exists a finite time $T > 0$ such that $\lim_{t\to T}\|x_i(t) - x_j(t)\| = 0$, $\lim_{t\to T}\|v_i(t) - v_j(t)\| = 0$, and $x_i(t) = x_j(t)$, $v_i(t) = v_j(t)$ $\forall t \geq T$, $i,j \in \mathcal{I}$.

Consistent with literature [34]–[36], we consider the following state-dependent communication.

Assumption 1: At time t, any two agents i and j are said to be neighbors of each other if and only if $\|x_i(t) - x_j(t)\| < R$.

The goal of the article is to propose a distributed control protocol such that: 1) the initial edges can be preserved under the assumption that the initial graph is connected; 2) the second-order MAS (1) can achieve finite-time consensus; and 3) the states $x_i(t), i \in \mathcal{I}$ can track the optimal trajectory $x^*(t) = \arg\min_{x\in\mathbb{R}^n} f(x,t)$ asymptotically, and the error of $\sum_{j=1}^N f_j(x_j,t) - \sum_{j=1}^N f_j(x^*(t),t)$ can converge to zero asymptotically, that is, $\lim_{t\to\infty}\|x_i(t) - x^*(t)\| = 0, i \in \mathcal{I}$, and $\lim_{t\to\infty}(\sum_{j=1}^N f_j(x_j,t) - \sum_{j=1}^N f_j(x^*(t),t)) = 0$.

III. Main Results

In this section, a new connectivity-preserving mechanism is proposed. Then, two distributed optimization algorithms are given for time-varying and time-invariant optimization problems, respectively, which can maintain the connectivity, achieve finite-time consensus, and track the optimal trajectory asymptotically.

A. Connectivity-Preserving Mechanism Design

Network connectivity ensures information sharing among agents along the communication links. Hence, preserving the connectivity of the communication graph is critical for MASs to achieve the global coordinated objective.

To design the connectivity-preserving mechanism, we first introduce two classes of functions $c(s)$ and $\phi(s)$. Here, $c(s) : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows.

$c(s)$ is continuous on $\mathbb{R}^+$. $c(s) = 1$ when $s \in [0, \tau R)$ with $\tau \in (0, 1)$; $0 < c(s) \le 1$ when $s \in [\tau R, R)$; $c(s) = 0$ when $s \in [R, +\infty)$.

$\phi(s) : \mathbb{R}^+ \to \mathbb{R}^+$ is an artificial potential function that satisfies the following properties.

P1) $\phi(s)$ is continuous on $[0, R)$. $\phi(s) = 1$ when $s \in [0, \tau R)$ with $\tau \in (0, 1)$; $\phi(s) \ge 1$ when $s \in [\tau R, R)$; and $\phi(s) = 0$ when $s \in [R, +\infty)$.

P2) $\int_0^r \phi(s) s ds \to \infty$ when $r \to R^-$.

Furthermore, we let

$$a(\|x_i(t) - x_j(t)\|) = \begin{cases} 0, & \|x_i(0) - x_j(0)\| < R \text{ and } \|x_i(t) - x_j(t)\| \ge R \\ 1, & \|x_i(0) - x_j(0)\| < R \text{ and } \|x_i(t) - x_j(t)\| < R \\ c(\|x_i(t) - x_j(t)\|), & \|x_i(0) - x_j(0)\| \ge R \end{cases} \tag{4}$$

and

$$\varpi(\|x_i(t) - x_j(t)\|) = \begin{cases} \phi(\|x_i(t) - x_j(t)\|), & \|x_i(0) - x_j(0)\| < R \\ c(\|x_i(t) - x_j(t)\|), & \|x_i(0) - x_j(0)\| \ge R. \end{cases} \tag{5}$$

Remark 1: Here, we provide two examples for the above-mentioned function classes $c(s)$ and $\phi(s)$

$$c(s) = \begin{cases} 1, & 0 \le s < \tau R \\ \frac{R^2 - s^2}{(1-\tau^2)R^2}, & \tau R \le s < R \\ 0, & s \ge R \end{cases} \tag{6}$$

$$\phi(s) = \begin{cases} 1, & 0 \le s < \tau R \\ \frac{(1-\tau^2)R^2}{R^2 - s^2}, & \tau R \le s < R \\ 0, & s \ge R. \end{cases} \tag{7}$$

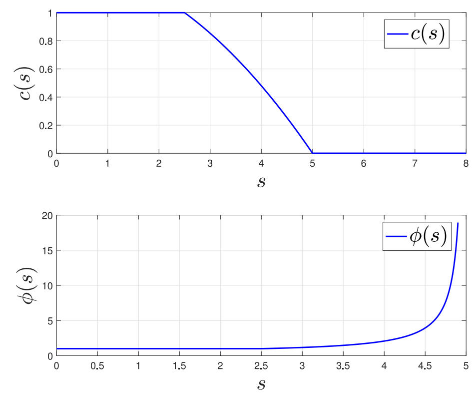


Fig. 1. Illustration of $c(s)$ and $\phi(s)$ with $R = 5$ and $\tau = 0.5$.

Fig. 1 shows the shape of the functions $c(s)$ and $\phi(s)$ with $R = 5$ and $\tau = 0.5$. Later, we will show that the properties of $c(s)$ can avoid sudden jump of weights resulting from the newly formed edges.

Remark 2: Note that $a(\|x_i(t) - x_j(t)\|) = 0$ and $\varpi(\|x_i(t) - x_j(t)\|) = 0$ if $\|x_i(t) - x_j(t)\| \ge R$, which reflects the limited interaction range of the agents. The aforementioned properties will be later used in the stability analysis to prove that if the edge $\epsilon_{ij}(0)$ exists, the potential function $\phi(s)$ can preserve this edge at all time instants. Specifically, the first property of $\phi(s)$ and $c(s)$ will allow $a(\|x_i(t) - x_j(t)\|)$ and $\varpi(\|x_i(t) - x_j(t)\|)$ to converge to a constant as consensus is achieved; the second property of $\phi(s)$ will be used to ensure that the initial edges are maintained as time evolves.

The function $c(s)$ is introduced to reduce the control effort, due to the property $c(s) \le \phi(s)$ for all $s \in \mathbb{R}^+$. Finally, it has to be remarked that continuity of $c(s)$ can avoid jumps of $a(\|x_i(t) - x_j(t)\|)$ and $\varpi(\|x_i(t) - x_j(t)\|)$ at t resulting from the edge (i, j) being included into or excluded from $\mathcal{E}(t)$, thus avoid sudden change of controllers.

B. Distributed Optimization Protocol Design

We propose the following distributed connectivity-preserving optimization protocol:

$$\begin{aligned} u_i(t) = &-k_1 \sum_{j=1}^N \varpi_{ij}(t)(x_i - x_j) - k_2 \sum_{j=1}^N a_{ij}(t)(v_i - v_j) \\ &- l_1 \sum_{j=1}^N a_{ij}(t)\text{sgn}(x_i - x_j) - l_2 \sum_{j=1}^N a_{ij}(t)\text{sgn}(v_i - v_j) \\ &+ \psi_i(x_i, v_i, t) \end{aligned} \tag{8}$$

where k_1, k_2, l_1, and l_2 are some feedback gains to be designed later

$$\begin{aligned} \psi_i(x_i, v_i, t) = &-\frac{d}{dt}\left(H_i^{-1}(x_i, t)\left(\nabla f_i(x_i, t) + \frac{\partial}{\partial t}\nabla f_i(x_i, t)\right)\right) \\ &- H_i(x_i, t)\nabla f_i(x_i, t) \end{aligned} \tag{9}$$

and $H_i(x_i, t)$ is the Hessian matrix of the differentiable function $f_i(x_i, t), i \in \mathcal{I}_N$, $a_{ij}(t) \triangleq a(\|x_i - x_j\|)$, and $\varpi_{ij}(t) \triangleq \varpi(\|x_i - x_j\|)$, in which $a(\|x_i - x_j\|)$ and $\varpi(\|x_i - x_j\|)$ are defined in (4) and (5), respectively.

Remark 3: The role of the first four terms in (8) is to make the agents achieve finite-time consensus. Furthermore, the weights $\varpi_{ij}(t)$ can preserve the initial edges due to property P2 of $\phi(s)$. The last term $\psi_i(x_i, v_i, t)$ in (8) is for optimization purpose to drive the position state of all agents to the optimal trajectory.

Assumption 2: It is assumed that ψ_i in (9) can be decomposed as $\psi_i(x_i, v_i, t) = g(x_i, v_i, t)+d_i(x_i, v_i, t)$, where $g(x, v, t)$ is Lipschitz continuous, that is, there exist two positive numbers ℓ_1 and ℓ_2 such that $\|g(x, v, t) - g(x', v', t)\| \le \ell_1\|x - x'\| + \ell_2\|v - v'\|$ for all $x, x', v, v' \in \mathbb{R}^n$, and $d_i(x, v, t)$ is error bounded, that is, $\|d_i(x, v, t) - d_j(x', v', t)\| \le \bar{d}$ with $\bar{d} \ge 0$ for $x, x', v, v' \in \mathbb{R}^n$ and $i, j \in \mathcal{I}$.

Assumption 3: It is assumed that $H_1(x, t) = \cdots = H_N(x, t) = H(x, t)$ and $H(x, t)$ is invertible.

Assumptions 2 and 3 are analogous to assumptions that are common in the literature. In particular, Assumption 2 is needed to handle nonsmooth artificial potential functions: analogous assumptions can be found in [17] and [39]. Assumption 3 imposes some homogeneity in the local functions: a more restrictive assumption can be found in [24] where all the gradients of the local CFs are taken to be identical. In Section III-D, we will show that in some special cases, Assumption 3 can be removed.

To derive the main results, some lemmas are introduced.

Lemma 2 [40]: $(\sum_{i=1}^N z_i^2)^{(1/2)} \le \sum_{i=1}^N |z_i| \le N^{(1/2)}(\sum_{i=1}^N z_i^2)^{(1/2)}$.

Lemma 3: Denote $\Lambda(t) = [\varpi_{ij}(t)]_{N\times N}, A(t) = [a_{ij}(t)]_{N\times N}$ and $A^2(t) = [a_{ij}^2(t)]_{N\times N}$ as three adjacency matrices associated to graps $\mathcal{G}_{\Lambda(t)}$, $\mathcal{G}_{A(t)}$, and $\mathcal{G}_{A^2(t)}$, respectively. $\mathcal{L}_{\Lambda(t)}, \mathcal{L}_{A(t)}$, and $\mathcal{L}_{A^2(t)}$ are their Laplacian matrices, respectively. If the initial graph $\mathcal{G}_{A(0)}$ is connected and the initial edges are always preserved, then the following inequalities hold:

$$\lambda_2(\mathcal{L}_{\Lambda(t)}) \ge \lambda_2(\mathcal{L}_0), \lambda_2(\mathcal{L}_{A(t)}) \ge \lambda_2(\mathcal{L}_0), \lambda_2(\mathcal{L}_{A^2(t)}) \ge \lambda_2(\mathcal{L}_0) \tag{10}$$

where $\mathcal{L}_0 = \mathcal{L}_{A(0)}$ is the Laplacian matrix of the graph $\mathcal{G}_{A(t)}$ at $t = 0$.

Proof: See Appendix C1. ■

C. Performance Analysis

Let $\bar{x} = (1/N)\sum_{j=1}^N x_j$ and $\bar{v} = (1/N)\sum_{j=1}^N v_j$. Define $e_{x_i} = x_i - \bar{x}$ and $e_{v_i} = v_i - \bar{v}, i \in \mathcal{I}$ as the consensus errors. It is not difficult to obtain that $e_{x_i} - e_{x_j} = x_i - x_j$ and $e_{v_i} - e_{v_j} = v_i - v_j$. Because the graph is undirected, one obtains $\sum_{j=1}^N u_j = \sum_{j=1}^N \psi_j(x_j, v_j, t)$. Then, the error system of (1) can be written as

$$\begin{aligned} \dot{e}_{x_i}(t) &= e_{v_i} \\ \dot{e}_{v_i}(t) &= \psi_i(x_i, v_i, t) - \frac{1}{N}\sum_{j=1}^N \psi_j(x_j, v_j, t) \\ &\quad - k_1\sum_{j=1}^N \varpi_{ij}(t)(e_{x_i} - e_{x_j}) - k_2\sum_{j=1}^N a_{ij}(t)(e_{v_i} - e_{v_j}) \\ &\quad - l_1\sum_{j=1}^N a_{ij}(t)\mathrm{sgn}(e_{x_i} - e_{x_j}) - l_2\sum_{j=1}^N a_{ij}(t)\mathrm{sgn}(e_{v_i} - e_{v_j}) \end{aligned} \tag{11}$$

where $\psi_i(x_i, v_i, t) - (1/N)\sum_{j=1}^N \psi_j(x_j, v_j, t) = g(x_i, v_i, t) - (1/N)\sum_{j=1}^N g(x_j, v_j, t) + d_i(x_i, v_i, t) - (1/N)\sum_{j=1}^N d_j(x_j, v_j, t)$ based on Assumption 2.

Denote $e_x = (e_{x_1}^{\mathrm{T}}, \ldots, e_{x_N}^{\mathrm{T}})^{\mathrm{T}}$ and $e_v = (e_{v_1}^{\mathrm{T}}, \ldots, e_{v_N}^{\mathrm{T}})^{\mathrm{T}}$. Choose a Lyapunov candidate function

$$\begin{aligned} V_1(t) &= k_1\sum_{i,j=1}^N \int_0^{\|e_{x_i}-e_{x_j}\|} \varpi(s)ds \\ &\quad + \frac{k_2}{2}\sum_{i,j=1}^N \int_0^{\|e_{x_i}-e_{x_j}\|} a(s)sds \\ &\quad + l_2\sum_{i,j=1}^N \int_0^{\|e_{x_i}-e_{x_j}\|} a(s)ds \\ V_2(t) &= e_x^{\mathrm{T}}e_x + e_x^{\mathrm{T}}e_v + e_v^{\mathrm{T}}e_v \\ V(t) &= V_1(t) + V_2(t). \end{aligned} \tag{12}$$

Remark 4: Noting that the signum function is utilized in the control protocol (8), system (11) is discontinuous. Thus, the solutions of (11) are to be understood in the Filippov sense. Actually, Filippov solutions always exist here since the right-hand side of (11) is measurable as well as locally essentially bounded [41]. On the other hand, the candidate Lyapunov function $V(t)$ is continuously differentiable, that is, the set-valued Lie derivative at any discontinuous point takes single value. Motivated by these observations, the differential inclusion notation will not be utilized in the proof, so as to avoid symbol redundancy: however, the following lemma and theorem would hold also in such discontinuous setting.

Lemma 4: Suppose the initial graph $\mathcal{G}_{A(0)}$ is connected and $V(0)$ is bounded. If the feedback gains satisfy the following conditions:

$$\begin{aligned} k_1 &> (\ell_1 + \iota(2 + 2\ell_1 + \ell_2))/\lambda_2(\mathcal{L}_0) \\ k_2 &> (\ell_2 + 0.5)/\lambda_2(\mathcal{L}_0) + (2 + 2\ell_1 + \ell_2)/(8\iota\lambda_2(\mathcal{L}_0)) \\ l_1 &> \left(\sqrt{2}\bar{d}N^{\frac{1}{2}}\right)/\lambda_2^{\frac{1}{2}}(\mathcal{L}_0) + l_2 \\ l_2 &> \left(\sqrt{2}\bar{d}N^{\frac{1}{2}}\right)/\lambda_2^{\frac{1}{2}}(\mathcal{L}_0) \end{aligned} \tag{13}$$

where ι is an arbitrary positive number, then the initial edges $\epsilon_{ij}(0) \in \mathcal{E}(0)$ will always be preserved under Assumptions 1–3.

Proof: See Appendix C2. ■

Remark 5: It is noted that the parameters ℓ_1, ℓ_2, and $\bar{d}$ in (13) are known, which are defined in Assumption 2. The constant ι is a given positive number. The global information N and $\lambda_2(\mathcal{L}_0)$ can be estimated by invoking the distributed parameter estimation method [42], [43]. Hence, the control gains k_1, k_2, and l_2 can be chosen according to condition (13). When l_2 is selected, l_1 can be given by (13).

Next, the main results will be presented.

Theorem 1: If the conditions in Lemma 4 are satisfied, then:

1) MAS (1) can achieve finite-time consensus under the control protocol (8);
2) $\lim_{t\to\infty}\|x_i(t) - x^*(t)\| = 0, i \in \mathcal{I}$ and $\lim_{t\to\infty}(\sum_{j=1}^N f_j(x_j, t) - \sum_{j=1}^N f_j(x^*(t), t)) = 0$ hold.

Proof:

1) From Lemma 4, we know that the initial edges are maintained all the time, hence both $a(s)$ and $\varpi(s)s$ are continuous based on their definitions. Thus, the Lyapunov function $V(t)$ is continuous differentiable. By invoking the Lyapunov stability theory of nonsmooth systems in [44] and (29), the error system (11) is globally uniformly asymptotically stable at its equilibrium point $e_{x_i} = e_{v_i} = 0, i \in \mathcal{I}$. Thus, condition 3) in Lemma 6 in Appendix B is satisfied. Denote $B_{\tau R} = \{(e_x^{\mathrm{T}}, e_v^{\mathrm{T}}) \in \mathbb{R}^{2nN} | \|e_{x_i} - e_{x_j}\| \leq \tau R, \|e_{v_i} - e_{v_j}\| \leq \tau R\ \forall i, j \in \mathcal{I}\}$. By noting the definitions of $a_{ij}(t)$ and $\varpi_{ij}(t)$, one obtains $a_{ij}(t) = \varpi_{ij}(t) = 1$ in the set $B_{\tau R}$. With e_x and e_v evolving within the ball $B_{\tau R}$, the error system (11) becomes

$$\begin{aligned}\dot{e}_{x_i}(t) &= e_{v_i}\\ \dot{e}_{v_i}(t) &= \psi_i(x_i, v_i, t) - \frac{1}{N}\sum_{j=1}^{N}\psi_j(x_j, v_j, t)\\ &\quad - k_1\sum_{j=1}^{N}(e_{x_i} - e_{x_j}) - k_2\sum_{j=1}^{N}(e_{v_i} - e_{v_j})\\ &\quad - l_1\sum_{j=1}^{N}\mathrm{sgn}(e_{x_i} - e_{x_j}) - l_2\sum_{j=1}^{N}\mathrm{sgn}(e_{v_i} - e_{v_j}).\end{aligned} \tag{14}$$

The nominal system of (14) is

$$\begin{aligned}\dot{e}_{x_i}(t) &= e_{v_i}\\ \dot{e}_{v_i}(t) &= -l_1\sum_{j=1}^{N}\mathrm{sgn}(e_{x_i} - e_{x_j}) - l_2\sum_{j=1}^{N}\mathrm{sgn}(e_{v_i} - e_{v_j}).\end{aligned} \tag{15}$$

It is not difficult to see system (15) with variables $(e_{x_1}^{\mathrm{T}}, \ldots, e_{x_N}^{\mathrm{T}}, e_{v_1}^{\mathrm{T}}, \ldots, e_{v_N}^{\mathrm{T}})^{\mathrm{T}}$ is locally homogeneous of degree $\sigma = -1 < 0$ with dilation $(\underbrace{2, \ldots, 2}_{nN}, \underbrace{1, \ldots, 1}_{nN})$. The perturbed terms in (14) is denoted as $\omega(t, e_x, e_v) = (\omega_1^{\mathrm{T}}, \ldots, \omega_{2N}^{\mathrm{T}})^{\mathrm{T}}$ with $\omega_i \in \mathbb{R}^n, i = 1, \ldots, 2N$. $\omega_i = 0$ for $i = 1, \ldots, N$. Because $\omega_{i+N} = \psi_i(x_i, v_i, t) - (1/N)\sum_{j=1}^{N}\psi_j(x_j, v_j, t) - k_1\sum_{j=1}^{N}(e_{x_i} - e_{x_j}) - k_2\sum_{j=1}^{N}(e_{v_i} - e_{v_j}), i \in \mathcal{I}$ are bounded in the ball $B_{\tau R}$, there exist $M_j \geq 0$ such that $\|\omega_j\| \leq M_j$ for $j = N+1, \ldots, 2N$. Thus, conditions 1) and 2) in Lemma 6 in Appendix B are satisfied. Based on Lemma 6 in Appendix B, the MAS (1) with control protocol (8) can achieve finite-time consensus.

2) From 1), one obtains that there is a finite time T, for all $t \geq T$, $x_1 = \cdots = x_N = \bar{x}$, $v_1 = \cdots = v_N = \bar{v}$. Hence, the dynamics of MAS (1) becomes

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{v}\\ \dot{\bar{v}}(t) &= \frac{1}{N}\sum_{j=1}^{N}\psi_j(\bar{x}, \bar{v}, t).\end{aligned} \tag{16}$$

Select the Lyapunov candidate function as

$$\begin{aligned}W(t) &= \frac{1}{2}\left(\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\right)^{\mathrm{T}}\left(\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\right)\\ &\quad + \frac{1}{2}\left(N\bar{v} + \sum_{j=1}^{N}w_j(\bar{x}, t)\right)^{\mathrm{T}}\left(N\bar{v} + \sum_{j=1}^{N}w_j(\bar{x}, t)\right)\end{aligned} \tag{17}$$

where $w_j(\bar{x}, t) = H_j^{-1}(\bar{x}, t)([\partial/\partial t]\nabla f_j(\bar{x}, t) + \nabla f_j(\bar{x}, t))$. The derivative of $W(t)$ is

$$\begin{aligned}\dot{W}(t) &= \left(\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\right)^{\mathrm{T}}\left(\sum_{j=1}^{N}H_j(\bar{x}, t)\bar{v} + \sum_{j=1}^{N}\frac{\partial}{\partial t}\nabla f_j(\bar{x}, t)\right)\\ &\quad - \left(N\bar{v} + \sum_{j=1}^{N}w_j(\bar{x}, t)\right)^{\mathrm{T}}\left(\sum_{j=1}^{N}H_j(\bar{x}, t)\nabla f_j(\bar{x}, t)\right)\\ &= \left(\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\right)^{\mathrm{T}}\left(NH(\bar{x}, t)\bar{v} + \sum_{j=1}^{N}\frac{\partial}{\partial t}\nabla f_j(\bar{x}, t)\right)\\ &\quad - \left(N\bar{v} + \sum_{j=1}^{N}w_j(\bar{x}, t)\right)^{\mathrm{T}}\left(H(\bar{x}, t)\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\right)\\ &= -\left(\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\right)^{\mathrm{T}}\left(\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\right)\end{aligned} \tag{18}$$

where Assumption 3 is utilized in the above equation. Thus, when $\sum_{j=1}^{N}\nabla f_j(\bar{x}, t) \neq 0$, one has $\dot{W}(t) < 0$. By $W(t) \geq 0$ and $\dot{W}(t) \leq 0$, one can deduce that $\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)$ is bounded and $0 \leq W(t) \leq W(0)$ for $t \geq 0$. By integrating both sides of (18), it derives that $\int_0^{\infty}\|\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\|^2 dt = -W(\infty) + W(0) \leq \infty$, which implies $\int_0^{\infty}\|\sum_{j=1}^{N}\nabla f_j(\bar{x}, t)\|^2 dt$ is bounded. By Barbalat's lemma in [45], one obtains $\lim_{t\to+\infty}\sum_{j=1}^{N}\nabla f_j(\bar{x}, t) = 0$. Moreover, by noting that $\sum_{j=1}^{N}f_j(x, t)$ is convex, the optimal trajectory $x^*(t)$ satisfies the first-order optimality condition $\sum_{j=1}^{N}\nabla f_j(x^*(t), t) = 0\ \forall t$. Hence, one obtains $\lim_{t\to+\infty}(\nabla f(\bar{x}, t) - \nabla f(x^*(t), t)) = \lim_{t\to+\infty}\sum_{j=1}^{N}(\nabla f_j(\bar{x}, t) - \nabla f_j(x^*(t), t)) = 0$. By the definition of uniformly strongly convexity and invoking the Cauchy–Swhartz inequality, one has $\|\bar{x} - x^*(t)\|^2 \leq (1/C_f)(\nabla f(\bar{x}, t) - \nabla f(x^*(t), t))^{\mathrm{T}}(\bar{x} - x^*(t)) \leq (1/C_f)\|\nabla f(\bar{x}, t) - \nabla f(x^*(t), t)\| \cdot \|\bar{x} - x^*(t)\|$. Hence, one obtains $0 \leq \|\bar{x} - x^*(t)\| \leq (1/C_f)\|\nabla f(\bar{x}, t) - \nabla f(x^*(t), t)\|$. Thus, $\lim_{t\to+\infty}\|\bar{x} - x^*(t)\| = 0$. On the other hand, the convexity of $f(\bar{x}, t)$ implies that $\forall t$

$$0 \leq f(\bar{x}, t) - f(x^*(t), t) \leq \nabla f(\bar{x}, t)^{\mathrm{T}}(\bar{x} - x^*(t)).$$

Hence, $0 \leq f(\bar{x}, t) - f(x^*(t), t) \leq \|\nabla f(\bar{x}, t)\| \cdot \|\bar{x} - x^*(t)\|$. From the above analysis, we know that $\|\nabla f(\bar{x}, t)\|$ is bounded for all $t \geq 0$. Thus, $\lim_{t\to+\infty}(f(\bar{x}, t) - f(x^*(t), t)) = 0$. This proof is completed. ■

Remark 6: If the trajectory that minimizes the CF is known to a subset of the followers, one can apply the leader-following

TABLE I
COMPARISON OF CONSENSUS-BASED DISTRIBUTED OPTIMIZATION

Methods	Dynamics	Convergence rate of consensus	Limited ranges
(8) in [17]	First-order	Finite time	No
(10) in [37] (13) in [46]	First-order	Finite time	Yes
(29) in [19]	Second-order	Infinite time	No
This work	Second-order	Finite time	Yes

approach to steer all the agents to the optimal trajectory of the total CF in a distributed way. However, in our work, agent i only knows its own local CF $f_i(x_i, t)$, which means that none of the agents has the knowledge of total CF $\sum_{i=1}^{N} f_i(x_i, t)$. Hence, the optimal trajectory of the total CF is unknown to all the agents. It should be emphasized that the optimal trajectory of $f_i(x_i, t)$ is generally different from the optimal trajectory of $\sum_{i=1}^{N} f_i(x_i, t)$. Such a property is what distinguishes the method proposed here from the leader-following approach.

Remark 7: Compared with the optimization algorithms in [17], [37], and [46], the control protocol (8) is designed for the more general case of second-order MASs and time-varying CFs. The algorithms in [19] and [47] were both proposed for second-order MASs. Compared with them, our proposed control protocol (8) can achieve finite-time consensus. Furthermore, connectivity-preserving is considered here. The detailed comparison between the proposed design and some related works is listed in Table I.

D. Special Case of Time-Invariant CFs

In this section, we consider a special case where the CFs are time invariant, that is

$$\min_{x_i \in \mathbb{R}^n} \sum_{i=1}^{N} f_i(x_i) \text{ subject to } x_i = x_j \quad \forall i, j \in \mathcal{I}.$$

In contrast with uniformly strongly convexity defined after (2), we can assume the milder condition that $f(x) = \sum_{i=1}^{N} f_i(x)$ is C-strongly convex function, which means that there exists a positive constant C such that $f(x) \geq f(y) + \nabla f(y)^{\mathrm{T}} (x - y) + (C/2)\|x - y\|^2$, for $x, y \in \mathbb{R}^n$. Furthermore, each $f_i(x)$ is assumed to be twice continuously differentiable with respect to x.

In this case, the control protocol for MAS (1) is designed as

$$\begin{aligned} u_i(t) = &-k_1 \sum_{j=1}^{N} \varpi_{ij}(t)\big(x_i - x_j\big) - k_2 \sum_{j=1}^{N} a_{ij}(t)\big(v_i - v_j\big) \\ &- l_1 \sum_{j=1}^{N} a_{ij}(t)\mathrm{sgn}\big(x_i - x_j\big) - l_2 \sum_{j=1}^{N} a_{ij}(t)\mathrm{sgn}\big(v_i - v_j\big) \\ &+ \psi_i(x_i, v_i) \end{aligned} \tag{19}$$

where k_1, k_2, l_1, l_2 are some feedback gains to be designed later, and

$$\psi_i(x_i, v_i) = -H_i(x_i)v_i - \nabla f_i(x_i). \tag{20}$$

Theorem 2: Assume that $\mathcal{G}_{A(0)}$ is connected and $V(0)$ is bounded. If the feedback gains satisfy the same conditions as (13), then the following statements hold under Assumptions 1 and 2.

1) The initial edges will always be preserved.
2) The MAS (1) with control protocol (19) can achieve finite-time consensus.
3) $\lim_{t\to+\infty} \|x_i(t) - x^*\| = 0, i \in \mathcal{I}$ holds, where x^* is the minimized point of $\sum_{j=1}^{N} f_j(x)$.

Proof: The proofs of 1) and 2) are similar with those in Lemma 4 and Theorem 1. We just need to prove 3). From 2), we know that there exists a finite time T, for all $t \geq T$, $x_1 = \cdots = x_N = \bar{x}, v_1 = \cdots = v_N = \bar{v}$. Hence, the dynamics of MAS (1) becomes

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{v} \\ \dot{\bar{v}}(t) &= -\frac{1}{N} \sum_{j=1}^{N} \big(H_j(\bar{x})\bar{v} + \nabla f_j(\bar{x})\big). \end{aligned} \tag{21}$$

Let $\tilde{x} = \bar{x} - x^*$ and $\tilde{v} = \bar{v}$. Transferring the equilibrium point of system (21) to the origin, one obtains

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{v} \\ \dot{\tilde{v}}(t) &= -\frac{1}{N} \sum_{j=1}^{N} \big(H_j(\tilde{x} + x^*)\tilde{v} + \nabla f_j\big(\tilde{x} + x^*\big)\big). \end{aligned} \tag{22}$$

Select the Lyapunov candidate function as $W_0(\tilde{x}, \tilde{v}) = N(\sum_{j=1}^{N} f_j(\tilde{x} + x^*) - \sum_{j=1}^{N} f_j(x^*)) + (1/2)\|N\tilde{v} + \sum_{j=1}^{N} \nabla f_j(\tilde{x} + x^*)\|^2$. Note that $W_0(\tilde{x}, \tilde{v})$ is continuously differentiable and $W_0(0, 0) = 0$. Because $\sum_{j=1}^{N} f_j(x)$ is C-strongly convex, $W_0(\tilde{x}, \tilde{v}) \geq (NC/2)\|\tilde{x}\|^2 + (1/2)\|N\tilde{v} + \sum_{j=1}^{N} \nabla f_j(\tilde{x} + x^*)\|^2$. Thus, W_0 is radially unbounded in regard to $\tilde{x}$ and $\tilde{v}$.

The time derivative of W_0 along trajectory (22) is

$$\begin{aligned} \dot{W}_0 = &N \sum_{j=1}^{N} \big(\nabla f_j\big(\tilde{x} + x^*\big)\big)^{\mathrm{T}} \tilde{v} \\ &- \left(N\tilde{v} + \sum_{j=1}^{N} \nabla f_j\big(\tilde{x} + x^*\big)\right)^{\mathrm{T}} \sum_{j=1}^{N} \nabla f_j\big(\tilde{x} + x^*\big) \\ = &-\left(\sum_{j=1}^{N} \nabla f_j\big(\tilde{x} + x^*\big)\right)^{\mathrm{T}} \left(\sum_{j=1}^{N} \nabla f_j\big(\tilde{x} + x^*\big)\right). \end{aligned} \tag{23}$$

Denote $M = \{(\tilde{x}, \tilde{v}) | \dot{W}_0 = 0\} = \{(\tilde{x}, \tilde{v}) | \sum_{j=1}^{N} \nabla f_j(\tilde{x} + x^*) = 0\}$. According to Lemma 1, one has that $\sum_{j=1}^{N} \nabla f_j(\tilde{x} + x^*) = 0$ means $\tilde{x} = 0$. Then, one obtains $\tilde{v} = 0$ from (22). Thus, $M = \{(\tilde{x}, \tilde{v}) | \tilde{x} = 0, \tilde{v} = 0\}$. By invoking LaSalle's invariance principle [48], it can be concluded the origin is globally asymptotically stable. Then, $\lim_{t\to\infty} \|\bar{x}(t) - x^*\| = 0$. The proof is completed. ■

Remark 8: Note that Assumption 3 is not required in this case, that is, each of $H_i(x)$ is not necessarily invertible and the assumption on identical Hessians is removed.

IV. NUMERICAL EXAMPLES

In this section, we give two examples to show the effectiveness of the two control protocols (8) and (19), respectively.

Consider MASs with 5, 10, and 100 agents in 2-D space, whose dynamics are described in (1). The initial

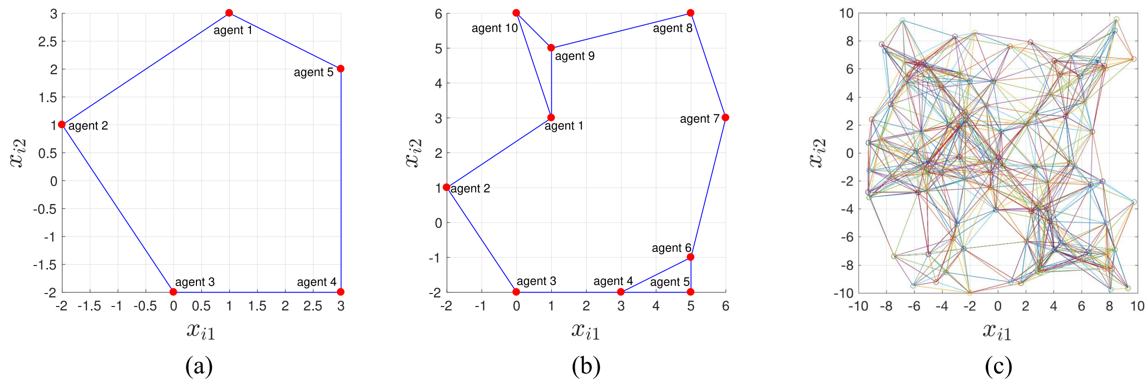


Fig. 2. Initial communication graphs $\mathcal{G}_{A(0)}$ with communication range $R = 5$. (a) MAS with five agents, where the initial position states are $x_1(0) = (1, 3)^{\mathrm{T}}$, $x_2(0) = (-2, 1)^{\mathrm{T}}, x_3(0) = (0, -2)^{\mathrm{T}}, x_4(0) = (3, -2)^{\mathrm{T}}$, and $x_5(0) = (3, 2)^{\mathrm{T}}$, resulting in $\lambda_2(\mathcal{L}_0) = 1.3820$; (b) MAS with ten agents, where the initial position states are $x_1(0) = (1, 3)^{\mathrm{T}}, x_2(0) = (-2, 1)^{\mathrm{T}}, x_3(0) = (0, -2)^{\mathrm{T}}, x_4(0) = (3, -2)^{\mathrm{T}}, x_5(0) = (5, -2)^{\mathrm{T}}, x_6(0) = (5, -1)^{\mathrm{T}}, x_7(0) = (6, 3)^{\mathrm{T}}$, $x_8(0) = (5, 6)^{\mathrm{T}}, x_9(0) = (1, 5)^{\mathrm{T}}$, and $x_{10}(0) = (0, 6)^{\mathrm{T}}$, resulting in $\lambda_2(\mathcal{L}_0) = 0.3820$; and (c) MAS with 100 agents where the initial position of each agent is randomly generated in the subspace $[-10, 10]^2 \subset \mathbb{R}^2$, resulting in $\lambda_2(\mathcal{L}_0) = 1.2009$.

TABLE II
DESIGN OF CONTROL GAINS IN THREE CASES WITH $\iota = 1$ IN EXAMPLE 1

	k_1	k_2	l_1	l_2
$N = 5$	> 4.3415	> 1.5376	> 14.5570	> 7.2785
$N = 10$	> 15.7068	> 5.5628	> 39.1570	> 19.5785
$N = 100$	> 4.9963	> 1.7695	> 69.8373	> 34.9186

communication graphs are depicted in Fig. 2. The functions $c(s)$ and $\phi(s)$ are chosen as (6) and (7) with $\tau = 0.5$. Denote $d_{ij} = \|x_i - x_j\|$ as the relative distance between agents i and j.

Example 1: In this example, we consider the time-varying CFs: $f_i(x_i, t) = (1/2)([1/2]x_{i1} - \cos(0.1(q(i) - 5)t + 0.1) - 1)^2 + (1/2)(x_{i2} - 2\sin(0.1(q(i) - 5)t) - [(q(i) - 5)/20])^2$, where $q(i) = \mathrm{mod}(i + 9, 10) + 1$, $i \in \mathcal{I}_N$ and $N = 5, 10, 100$. By calculation, it can be obtained that $\nabla f_i(x_i, t) = (0.25x_{i1} - 0.5\cos(0.1(q(i) - 5)t + 0.1) - 0.5, x_{i2} - 2\sin(0.1(q(i) - 5)t) - [(q(i) - 5)/20])$, while $H_i(x_i, t) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}, i \in \mathcal{I}_N$ which satisfies Assumption 3. From (9), one obtains

$$\psi_i(x_i, v_i, t) = g(x_i, v_i, t) + d_i(x_i, v_i, t)$$

where $g(x_i, v_i, t) = \begin{pmatrix} -\frac{1}{16}x_{i1} - v_{i1} + \frac{1}{8} \\ -x_{i2} - v_{i2} \end{pmatrix}$, and $d_i(x_i, v_i, t) = \begin{pmatrix} d_{i1}(x_i, v_i, t) \\ d_{i2}(x_i, v_i, t) \end{pmatrix}$ with $d_{i1}(x_i, v_i, t) = ([1/8] - 0.02(q(i) - 5)^2)\cos(0.1(q(i) - 5)t + 0.1) - 0.2(q(i) - 5)\sin(0.1(q(i) - 5)t + 0.1)$ and $d_{i2}(x_i, v_i, t) = [(q(i) - 5)/20] + (2 - 0.02(q(i) - 5)^2)\sin(0.1(q(i) - 5)t) + 0.2(q(i) - 5)\cos(0.1(q(i) - 5)t)$. It is easy to obtain that $g(x_i, v_i, t)$ is Lipschitz continuous with $\ell_1 = 1$ and $\ell_2 = 1$, and $d_i(x_i, v_i, t)$ is error bounded with $\bar{d} = 2.7058$. Hence, Assumption 2 is satisfied.

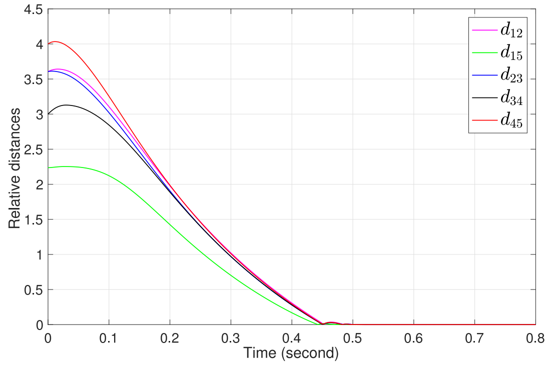
By setting $\sum_{j=1}^{N} \nabla f_j(x^*, t) = 0$, we obtain that $x^*(t) = (2 + 2/N)\sum_{j=1}^{N} \cos(0.1(q(j) - 5)t + 0.1), (1/N)\sum_{j=1}^{N}((q(j) - 5)/20 + 2\sin(0.1(q(j) - 5)t)))^{\mathrm{T}}$ is the optimal trajectory based on Lemma 1. Let $\iota = 1$. According to (13), the design of control gains in three cases is shown in Table II.

We choose $k_1 = 15.7068, k_2 = 5.5628, l_1 = 69.8373$, and $l_2 = 34.9186$ such that condition (13) is satisfied for three different cases. The profile of the distances corresponding to the initial adjacent agents in graph $\mathcal{G}_{A(0)}$ under control protocol (8) is illustrated in Fig. 3. One can observe that the initial edges are preserved since the corresponding distances are always less than the communication range $R = 5$.

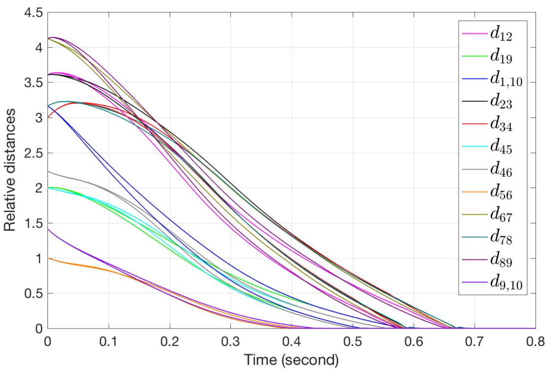
Fig. 4(a) shows the trajectories of control inputs (8). For comparison purposes, let us select the discontinuous function $c(s) = \begin{cases} 1, & 0 \le s < R \\ 0, & s \ge R \end{cases}$ instead of (6): note that the connectivity-preserving mechanism with this function leads to discontinuous weights like [34] and [36]. Fig. 4(b) shows the trajectories of control inputs (8) resulting from the discontinuous function. From this figure, it can be seen that there exist several nonsmooth points in the beginning, which are induced by the discontinuous weights. Fig. 4(a) validates that the proposed connectivity-preserving mechanism with continuous weights can avoid sudden jumps at the beginning of evolution.

Denote $(1/N)\sum_{i=1}^{N}\|x_i - \bar{x}\|$ and $(1/N)\sum_{i=1}^{N}\|x_i - x^*(t)\|$ as the average consensus error and the average optimization error, where $\bar{x} = (1/N)\sum_{i=1}^{N} x_i$. Fig. 5 shows the trajectories of the average consensus error (left) and the average optimization error (right) for different numbers of agents. In general, the larger the number of agents, the slower the convergence rate of consensus. On the other hand, the number of agents has minor influence on the convergence rate of optimization, which can be explained by the fact that the convergence rate of optimization just relies on the term $(1/N)\sum_{i=1}^{N}\psi_i$.

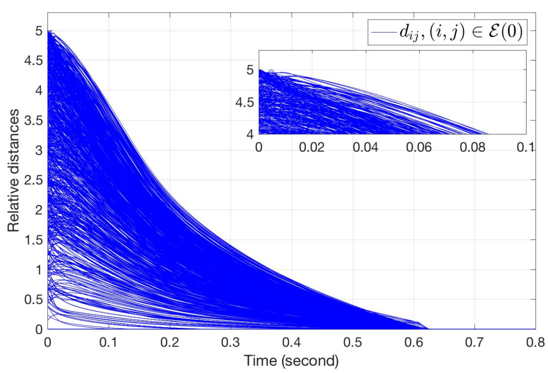
Example 2: In this example, we consider the time-invariant CFs: $f_i(x_i) = (1/2)(x_{i1} - x_{i2} - (p(i) + 5)/5)^2 + (1/2)x_{i2}^2$, where $p(i) = \mathrm{mod}(i + 4, 5) + 1, i \in \mathcal{I}_N$ and $N = 5, 10, 100$. By calculation, one can obtain that $\nabla f_i(x_i) = (x_{i1} - x_{i2} - (p(i) + 5)/5, -x_{i1} + 2x_{i2} + (p(i) + 5)/5)$ and $H_i(x_i) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, i \in \mathcal{I}_N$. From (20), one obtains $\psi_i(x_i, v_i) = g(x_i, v_i) + d_i(x_i, v_i)$, where $g(x_i, v_i) = \begin{pmatrix} -v_{i1} + v_{i2} - x_{i1} + x_{i2} \\ v_{i1} - 2v_{i2} + x_{i1} - 2x_{i2} \end{pmatrix}$, and $d_i(x_i, v_i) = \begin{pmatrix} \frac{p(i)+5}{5} \\ -\frac{p(i)+5}{5} \end{pmatrix}$. It is not difficult to check that $g(x_i, v_i)$ is Lipschitz continuous with $\ell_1 = \ell_2 = 2\sqrt{2}$, and $d_i(x_i, v_i)$ is error-bounded with



(a)



(b)



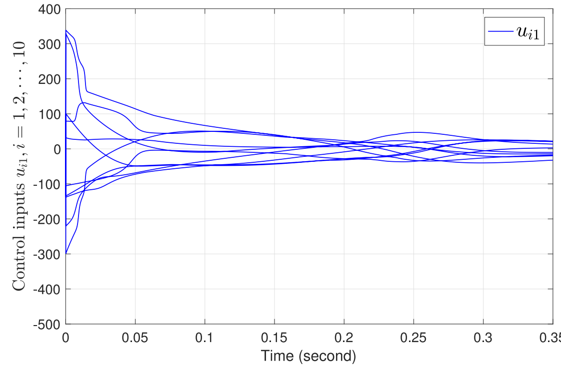
(c)

Fig. 3. Evolution of relative distances for adjacent agents in graph $\mathcal{G}(0)$ under control protocol (8) in Example 1 with $k_1 = 15.7068, k_2 = 5.5628$, $l_1 = 69.8373$, and $l_2 = 34.9186$ in three different cases: (a) MAS with five agents; (b) MAS with ten agents; and (c) MAS with 100 agents. From this figure, one can observe that the corresponding distances $d_{ij}, (i, j) \in \mathcal{E}(0)$ are always less than the communication range $R = 5$, which means that the initial edges can be well maintained.

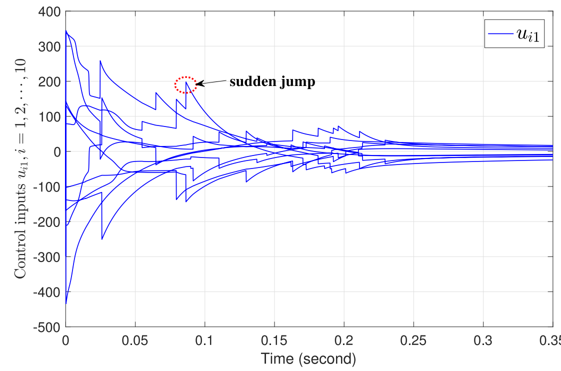
TABLE III
DESIGN OF CONTROL GAINS IN THREE CASES WITH $\iota = 1$ IN EXAMPLE 2

	k_1	k_2	l_1	l_2
$N = 5$	> 9.6337	> 3.3568	> 6.0868	> 3.0434
$N = 10$	> 34.8526	> 12.1442	> 16.3730	> 8.1865
$N = 100$	> 11.0864	> 3.8630	> 29.2017	> 14.6008

$\bar{d} = 1.1314$. By setting $\sum_{j=1}^{N} \nabla f_j(x^*) = 0$, we obtain $x^* = (1.6, 0)^{\mathrm{T}}$. Let $\iota = 1$. According to (13), the design of control gains in three cases is shown in Table III.



(a)



(b)

Fig. 4. Trajectories of control inputs $u_{i1}, i \in \mathcal{I}_{10}$ in Example 1 with $k_1 = 15.7068$, $k_2 = 5.5628$, $l_1 = 69.8373$, and $l_2 = 34.9186$. (a) Control protocol with continuous weights proposed in our article and (b) control protocol with discontinuous weights.

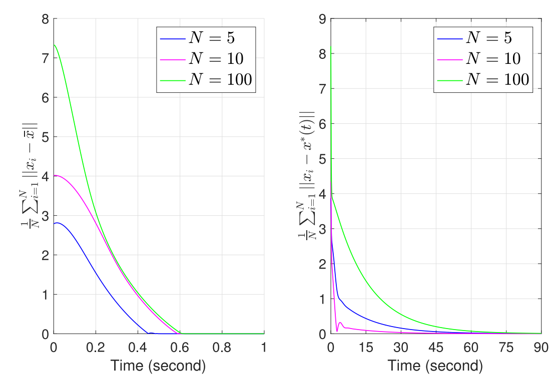


Fig. 5. Profiles of the average consensus errors (left) and the average optimization errors (right) in Example 1 for different N with $k_1 = 15.7068$, $k_2 = 5.5628$, $l_1 = 69.8373$, and $l_2 = 34.9186$. The left figure shows that the larger number of agents, the slower convergence rate of consensus. The right figure shows that the number of agents has minor influence on the convergence rate of optimization.

We choose $k_1 = 34.9526, k_2 = 12.2442, l_1 = 29.3017$, and $l_2 = 14.7008$ such that condition (13) is satisfied for three different cases. The profile of the distances corresponding to the initial adjacent agents in graph $\mathcal{G}_{A(0)}$ under control protocol (19) in three different cases is illustrated in Fig. 6. One can observe that the corresponding distances are always less than the communication range, which demonstrates the

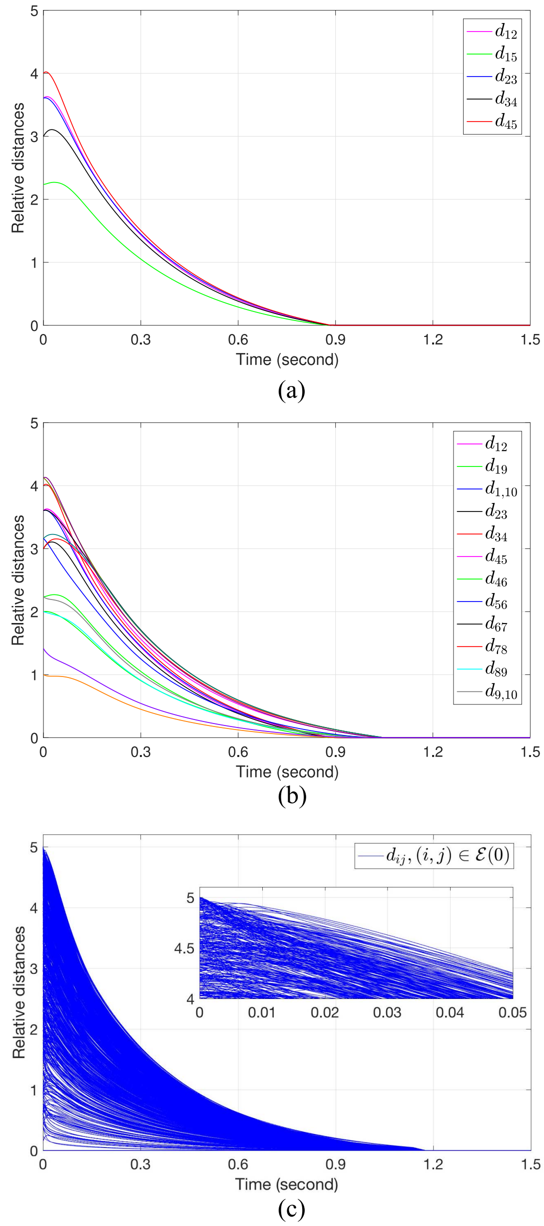


Fig. 6. Evolution of relative distances for adjacent agents in graph $\mathcal{G}(0)$ under control protocol (19) in Example 2 with $k_1 = 34.9526$, $k_2 = 12.2442$, $l_1 = 29.3017$, and $l_2 = 14.7008$ in three different cases: (a) MAS with five agents; (b) MAS with ten agents; and (c) MAS with 100 agents. From this figure, one can observe that the corresponding distances d_{ij}, $(i, j) \in \mathcal{E}(0)$ are always less than the communication range $R = 5$, which means that the initial edges can be well maintained.

effectiveness of the connectivity preservation property of protocol (19).

Fig. 7 shows the trajectories of the average consensus error (left) and the average optimization error (right) for different numbers of agents. From the left figure, it can be seen that the larger the number of agents, the slower the convergence rate of consensus. The right figure illustrates that the number of agents has minor influence on the convergence rate of optimization.

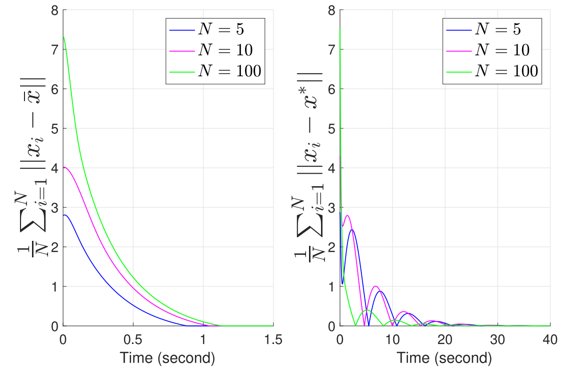


Fig. 7. Profiles of the average consensus errors (left) and the average optimization errors (right) in Example 2 for different N with $k_1 = 34.9526$, $k_2 = 12.2442$, $l_1 = 29.3017$, and $l_2 = 14.7008$. The left figure shows that the larger number of agents, the slower convergence rate of consensus. The right figure shows that the number of agents has minor influence on the convergence rate of optimization.

V. Conclusion

This article has investigated the distributed time-varying optimization problem for second-order MASs with limited communication ranges. First, a connectivity-preserving mechanism was proposed to maintain the initial edges. Then, two optimization algorithms were proposed, which can make the agents reach finite-time consensus and make the consensus value converge to the optimal trajectory asymptotically. Note that the assumption about the gradients and Hessians impose some homogeneity in the local CFs and might be restrictive in some practical applications. Making the algorithms valid for more heterogeneous classes of local CFs is one of our future research goals. Another interesting and challenging work is to extend the algorithm to heterogeneous communication ranges which might prevent pairwise communication. Furthermore, it is an interesting, practical, and challenging open problem to consider that the graph is initially disconnected, that is, some agents outside the sensing range should seek for new connections.

Appendix

This Appendix contains some notations and preliminaries that are used in the article. Two proofs of the lemmas are also provided in this part.

A. Notations

Denote $\mathbb{R}^m$ the m-dimensional real vector space. For simplicity, denote $\mathbb{R}^1 = \mathbb{R}$. Notation $\mathbb{R}^+$ is the set of non-negative real numbers. For a vector $z = (z_1, \dots, z_m)^{\mathrm{T}}$, symbol $\|z\|$ is the 2-norm of z and the superscript T means transpose. Symbol $\otimes$ is the Kronecker product. Let $\mathrm{sgn}(\cdot)$ denote the direction-preserving signum function, that is,

$$\mathrm{sgn}(z) = \begin{cases} \frac{z}{\|z\|}, & \|z\| \neq 0 \\ 0, & \|z\| = 0. \end{cases}$$

Denote $\lambda_2(A)$ the second-smallest eigenvalue of a symmetric matrix $A \in \mathbb{R}^{N \times N}$. Denote ∇f and H the gradient and Hessian matrix of a differentiable function f, respectively. The partial derivative of a vector function $z(s) = z(s_1, \dots, s_n) =$

$(z_1(s), \ldots, z_m(s))^{\mathrm{T}}$ with respect to s_i is defined as $(\partial/\partial s_i)z(s) = ([\partial/\partial s_i]z_1(s), (\partial/\partial s_i)z_2(s), \ldots, (\partial/\partial s_i)z_n(s))^{\mathrm{T}}$, $i = 1, 2, \ldots, n$. Let $\mathcal{I}_N = \{1, 2, \ldots, N\}$.

B. Preliminaries

1) Graph Theory: A weighted undirected time-varying graph $\mathcal{G}_{A(t)}$ can be represented by a triplet $(\mathcal{V}, \mathcal{E}(t), A(t))$, where $\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_N\}$ and $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ represent the node set and the edge set at time instant t. Besides, $A(t) = [a_{ij}(t)]_{N\times N}$ is the adjacency matrix at time instant t. An undirected edge $\epsilon_{ij}(t) \in \mathcal{E}(t)$ is expressed as the unordered pair of nodes (ν_i, ν_j), meaning that nodes ν_i and ν_j can exchange information with each other at time t, that is, $a_{ij}(t) = a_{ji}(t) > 0 \iff (\nu_i, \nu_j) \in \mathcal{E}(t)$. A path between nodes ν_i and ν_j in $\mathcal{G}_{A(t)}$ is a sequence of edges $(\nu_i, \nu_{i_1}), (\nu_{i_1}, \nu_{i_2}), \ldots, (\nu_{i_l}, \nu_j)$ in the graph with distinct nodes ν_{i_k}, $k \in \mathcal{I}_l$. An undirected graph $\mathcal{G}_{A(t)}$ is connected at time t if there is an undirected path between every pair of distinct nodes. The Laplacian matrix $\mathcal{L}_{A(t)} = [l_{ij}(t)]_{N\times N}$ is defined by $l_{ij}(t) = -a_{ij}(t)$, $i \neq j$, $l_{ii}(t) = \sum_{j=1, j\neq i}^{N} a_{ij}(t)$, which has the property that $\sum_{j=1}^{N} l_{ij}(t) = 0$. The Laplacian matrix obeys the following lemma.

Lemma 5 [49]: Let $\mathcal{L}_A \in \mathbb{R}^{N\times N}$ be the Laplacian matrix of an undirected graph $\mathcal{G}_A$. Then, the second-smallest eigenvalue $\lambda_2(\mathcal{L}_A)$ of $\mathcal{L}_A$ satisfies $y^{\mathrm{T}}\mathcal{L}_A y \geq \lambda_2(\mathcal{L}_A)y^{\mathrm{T}}y$, if $\sum_{i=1}^{N} y_i = 0$ and $\mathcal{G}_A$ is connected. Furthermore, $y^{\mathrm{T}}\mathcal{L}_A y = (1/2)\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(y_j - y_i)^2$ $\forall y \in \mathbb{R}^N$.

2) Homogeneity Theory for Uncertain Systems: Consider the following differential equation:

$$\dot{z}(t) = \varphi(z) + \omega(t, z) \tag{24}$$

where $z = (z_1, z_2, \ldots, z_m)^{\mathrm{T}}$ is the state vector, $t \in \mathbb{R}$ is the time variable, and ω can be viewed as a perturbation term. Furthermore, the functions $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)^{\mathrm{T}}$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^{\mathrm{T}}$ are piecewise continuous. If the right-hand side of (24) is discontinuous, the solutions are understood in the Filippov sense [41]. The following definition and lemma clarify when finite-time stability can be obtained for (24).

Definition 2 [50]: A piecewise continuous function $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is called locally homogeneous of degree $\sigma \in \mathbb{R}$ with respect to dilation $(r_1, r_2, \ldots, r_m)$, where $r_i > 0, i = 1, 2, \ldots, m$, if there exists a constant $\varepsilon_0 > 0$ and a ball $B_\delta \subset \mathbb{R}^m$ such that $\varphi_i(\varepsilon^{r_1}z_1, \varepsilon^{r_2}z_2, \ldots, \varepsilon^{r_m}z_m) = \varepsilon^{\sigma + r_i}\varphi_i(z_1, z_2, \ldots, z_m)$ for all $\varepsilon \geq \varepsilon_0$ and almost all $z \in B_\delta$.

Lemma 6 [50]: For system (24), if:

1) φ is a locally homogeneous piecewise continuous function of degree $\sigma < 0$ with respect to dilation $(r_1, r_2, \ldots, r_m)$, and ω is a piecewise continuous function, whose components $\omega_i, i = 1, 2, \ldots, m$, are locally uniformly bounded by constants $M_i \geq 0$ within a ball $B_\delta \subset \mathbb{R}^m$, that is, $|\omega_i(t, z)| \leq M_i$, $i \in \mathcal{I}_m$;
2) $M_i = 0$ whenever $\sigma + r_i > 0$;
3) the uncertain system (24) is globally uniformly asymptotically stable around the origin.

Then, the uncertain system (24) is globally uniformly finite-time stable.

C. Proofs

1) Proof of Lemma 3: We first prove $\lambda_2(\mathcal{L}_{\Lambda(0)}) \geq \lambda_2(\mathcal{L}_0)$. Based on the definition of $\mathcal{L}_{\Lambda(0)}$ and $\mathcal{L}_0$, one obtains that every off-diagonal entries of $\mathcal{L}_{\Lambda(0)} - \mathcal{L}_0$ are $-\varpi_{ij}(0) + a_{ij}(0)$, $i, j \in \mathcal{I}$, $i \neq j$, which are nonpositive based on their definitions in the above section. Because $\mathcal{L}_{\Lambda(0)}$ and $\mathcal{L}_0$ are symmetric Laplacian matrices, $\mathcal{L}_{\Lambda(0)} - \mathcal{L}_0$ is a symmetric Laplacian matrix, which implies that $\mathcal{L}_{\Lambda(0)} - \mathcal{L}_0$ is symmetric positive semidefinite.

Let $z = (z_1, \ldots, z_N)^{\mathrm{T}} \in \mathbb{R}^N$ be the right nonzero eigenvector for $\mathcal{L}_{\Lambda(0)}$ associated with $\lambda_2(\mathcal{L}_{\Lambda(0)})$, which satisfies $\sum_{i=1}^{N} z_i = 0$. Because graphs $\mathcal{G}_{\Lambda(0)}$ and $\mathcal{G}_{A(0)}$ are all undirected and connected, one obtains $\lambda_2(\mathcal{L}_{\Lambda(0)})z^{\mathrm{T}}z = z^{\mathrm{T}}\mathcal{G}_{\Lambda(0)}z \geq z^{\mathrm{T}}\mathcal{G}_{A(0)}z \geq \lambda_2(\mathcal{L}_0)z^{\mathrm{T}}z$ based on Lemma 5. Thus, $\lambda_2(\mathcal{L}_{\Lambda(0)}) \geq \lambda_2(\mathcal{L}_0)$ holds.

Because the initial edges are always preserved, the edges in graph $\mathcal{G}_{\Lambda(t)}$ includes all the edges in graph $\mathcal{G}_{\Lambda(0)}$. Thus, $\lambda_2(\mathcal{L}_{\Lambda(t)}) \geq \lambda_2(\mathcal{L}_{\Lambda(0)})$ based on [51, Th. 2.16]. The inequality $\lambda_2(\mathcal{L}_{\Lambda(t)}) \geq \lambda_2(\mathcal{L}_0)$ holds. Similarly, it is easy to obtain $\lambda_2(\mathcal{L}_{A(t)}) \geq \lambda_2(\mathcal{L}_0)$ and $\lambda_2(\mathcal{L}_{A^2(t)}) \geq \lambda_2(\mathcal{L}_0)$.

2) Proof of Lemma 4: Assume that t_1 is the first finite-time instant satisfying $\lim_{t\to t_1^-} \|x_i(t) - x_j(t)\| = R$ for some $(i, j) \in \mathcal{E}(0)$, that is, at time t_1 the initial edge would be lost. We will prove by contradiction that such instant t_1 does not exist.

The derivative of $V_1(t)$ on $[0, t_1)$ is

$$\begin{aligned}\dot{V}_1(t) = {} & k_1 \sum_{i,j=1}^{N} \varpi_{ij}(t)\big(e_{x_i} - e_{x_j}\big)^{\mathrm{T}}\big(e_{v_i} - e_{v_j}\big) \\ & + \frac{k_2}{2}\sum_{i,j=1}^{N} a_{ij}(t)\big(e_{x_i} - e_{x_j}\big)^{\mathrm{T}}\big(e_{v_i} - e_{v_j}\big) \\ & + l_2 \sum_{i,j=1}^{N} a_{ij}(t)\mathrm{sgn}\big(e_{x_i} - e_{x_j}\big)^{\mathrm{T}}\big(e_{v_i} - e_{v_j}\big).\end{aligned} \tag{25}$$

The derivative of $V_2(t)$ on $[0, t_1)$ is

$$\begin{aligned}\dot{V}_2(t) &= 2e_x^{\mathrm{T}}e_v + e_v^{\mathrm{T}}e_v + (e_x + 2e_v)^{\mathrm{T}}\dot{e}_v \\ &= \sum_{i=1}^{N}\big(e_{x_i} + 2e_{v_i}\big)^{\mathrm{T}}\dot{e}_{v_i} + 2e_x^{\mathrm{T}}e_v + e_v^{\mathrm{T}}e_v \\ &= \sum_{i=1}^{N}\big(e_{x_i} + 2e_{v_i}\big)^{\mathrm{T}}\Bigg(\psi_i(x_i, v_i, t) - \frac{1}{N}\sum_{j=1}^{N}\psi_j\big(x_j, v_j, t\big) \\ &\qquad - k_1\sum_{j=1}^{N}\varpi_{ij}(t)\big(e_{x_i} - e_{x_j}\big) \\ &\qquad - k_2\sum_{j=1}^{N}a_{ij}(t)\big(e_{v_i} - e_{v_j}\big) \\ &\qquad - l_1\sum_{j=1}^{N}a_{ij}(t)\mathrm{sgn}\big(e_{x_i} - e_{x_j}\big) \\ &\qquad - l_2\sum_{j=1}^{N}a_{ij}(t)\mathrm{sgn}\big(e_{v_i} - e_{v_j}\big)\Bigg) \\ &\quad + 2e_x^{\mathrm{T}}e_v + e_v^{\mathrm{T}}e_v\end{aligned} \tag{26}$$

where

$$\sum_{i=1}^{N}\left(e_{x_i}+2e_{v_i}\right)^{\mathrm{T}}\left(\psi_i(x_i,v_i,t)-\frac{1}{N}\sum_{j=1}^{N}\psi_j\left(x_j,v_j,t\right)\right)$$
$$=\sum_{i=1}^{N}\left(e_{x_i}+2e_{v_i}\right)^{\mathrm{T}}\left(g(x_i,v_i,t)-\frac{1}{N}\sum_{j=1}^{N}g\left(x_j,v_j,t\right)\right.$$
$$\left.+\,d_i(x_i,v_i,t)-\frac{1}{N}\sum_{j=1}^{N}d_j\left(x_j,v_j,t\right)\right).$$

Because $\sum_{i=1}^{N}(e_{x_i}+2e_{v_i})=0$, we have $\sum_{i=1}^{N}(e_{x_i}+2e_{v_i})^{\mathrm{T}}(g(x_i,v_i,t)-(1/N)\sum_{j=1}^{N}g(x_j,v_j,t))=\sum_{i=1}^{N}(e_{x_i}+2e_{v_i})^{\mathrm{T}}(g(x_i,v_i,t)-g(\bar{x},\bar{v},t))$. Furthermore, based on Assumption 2, the following inequality can be obtained:

$$\sum_{i=1}^{N}\left(e_{x_i}+2e_{v_i}\right)^{\mathrm{T}}\left(\psi_i(x_i,v_i,t)-\frac{1}{N}\sum_{j=1}^{N}\psi_j\left(x_j,v_j,t\right)\right)$$
$$=\sum_{i=1}^{N}\left(e_{x_i}+2e_{v_i}\right)^{\mathrm{T}}\left(g(x_i,v_i,t)-g(\bar{x},\bar{v},t)\right)$$
$$+\sum_{i=1}^{N}\left(e_{x_i}+2e_{v_i}\right)^{\mathrm{T}}\left(d_i(x_i,v_i,t)-\frac{1}{N}\sum_{j=1}^{N}d_j\left(x_j,v_j,t\right)\right)$$
$$\le\sum_{i=1}^{N}\left\|e_{x_i}+2e_{v_i}\right\|\left(\ell_1\left\|e_{x_i}\right\|+\ell_2\left\|e_{v_i}\right\|\right)+\bar{d}\sum_{i=1}^{N}\left\|e_{x_i}+2e_{v_i}\right\|$$
$$\le\sum_{i=1}^{N}\left(\ell_1\left\|e_{x_i}\right\|^2+2\ell_2\left\|e_{v_i}\right\|^2+(2\ell_1+\ell_2)\left\|e_{v_i}\right\|\cdot\left\|e_{x_i}\right\|\right)$$
$$+\,\bar{d}\sum_{i=1}^{N}\left(\left\|e_{x_i}\right\|+2\left\|e_{v_i}\right\|\right).\tag{27}$$

Based on $\varpi_{ij}(t)=\varpi_{ji}(t)$, $a_{ij}(t)=a_{ji}(t)$ and using (25), we have

$$\Delta\triangleq\sum_{i=1}^{N}\left(e_{x_i}+2e_{v_i}\right)^{\mathrm{T}}\left(-k_1\sum_{j=1}^{N}\varpi_{ij}(t)\left(e_{x_i}-e_{x_j}\right)\right.$$
$$-k_2\sum_{j=1}^{N}a_{ij}(t)\left(e_{v_i}-e_{v_j}\right)$$
$$\left.-l_1\sum_{j=1}^{N}a_{ij}(t)\mathrm{sgn}\left(e_{x_i}-e_{x_j}\right)-l_2\sum_{j=1}^{N}a_{ij}(t)\mathrm{sgn}\left(e_{v_i}-e_{v_j}\right)\right)$$
$$=-\dot{V}_1(t)-\frac{k_1}{2}\sum_{i,j=1}^{N}\varpi_{ij}(t)\left\|e_{x_i}-e_{x_j}\right\|^2$$
$$-\frac{l_1}{2}\sum_{i,j=1}^{N}a_{ij}(t)\left\|e_{x_i}-e_{x_j}\right\|$$
$$-\frac{l_2}{2}\sum_{i,j=1}^{N}a_{ij}(t)\left(e_{x_i}-e_{x_j}\right)^{\mathrm{T}}\mathrm{sgn}\left(e_{v_i}-e_{v_j}\right)$$
$$-k_2\sum_{i,j=1}^{N}a_{ij}(t)\left\|e_{v_i}-e_{v_j}\right\|^2-l_2\sum_{i,j=1}^{N}a_{ij}(t)\left\|e_{v_i}-e_{v_j}\right\|$$
$$\le-\dot{V}_1(t)-\frac{k_1}{2}\sum_{i,j=1}^{N}\varpi_{ij}(t)\left\|e_{x_i}-e_{x_j}\right\|^2$$
$$-\frac{l_1-l_2}{2}\sum_{i,j=1}^{N}a_{ij}(t)\left\|e_{x_i}-e_{x_j}\right\|$$
$$-k_2\sum_{i,j=1}^{N}a_{ij}(t)\left\|e_{v_i}-e_{v_j}\right\|^2-l_2\sum_{i,j=1}^{N}a_{ij}(t)\left\|e_{v_i}-e_{v_j}\right\|.$$

Based on Lemmas 2, 3, and 5, the following inequalities hold:

$$\Delta\le-\dot{V}_1(t)-\frac{k_1}{2}\sum_{i,j=1}^{N}\varpi_{ij}(t)\left\|e_{x_i}-e_{x_j}\right\|^2$$
$$-\frac{l_1-l_2}{2}\left(\sum_{i,j=1}^{N}a_{ij}^2(t)\left\|e_{x_i}-e_{x_j}\right\|^2\right)^{\frac{1}{2}}$$
$$-k_2\sum_{i,j=1}^{N}a_{ij}(t)\left\|e_{v_i}-e_{v_j}\right\|^2-l_2\left(\sum_{i,j=1}^{N}a_{ij}^2(t)\|e_{v_i}-e_{v_j}\|^2\right)^{\frac{1}{2}}$$
$$=-\dot{V}_1(t)-k_1e_x^{\mathrm{T}}\left(\mathcal{L}_{\Lambda(t)}\otimes I_n\right)e_x$$
$$-\frac{l_1-l_2}{2}\left(2e_x^{\mathrm{T}}\left(\mathcal{L}_{A^2(t)}\otimes I_n\right)e_x\right)^{\frac{1}{2}}$$
$$-2k_2e_v^{\mathrm{T}}\left(\mathcal{L}_{A(t)}\otimes I_n\right)e_v-l_2\left(2e_v^{\mathrm{T}}\left(\mathcal{L}_{A^2(t)}\otimes I_n\right)e_v\right)^{\frac{1}{2}}$$
$$\le-\dot{V}_1(t)-k_1\lambda_2\left(\mathcal{L}_{\Lambda(t)}\right)\|e_x\|^2-\frac{l_1-l_2}{2}\sqrt{2}\lambda_2^{\frac{1}{2}}\left(\mathcal{L}_{A^2(t)}\right)\|e_x\|$$
$$-2k_2\lambda_2\left(\mathcal{L}_{A(t)}\right)\|e_v\|^2-\sqrt{2}l_2\lambda_2^{\frac{1}{2}}\left(\mathcal{L}_{A^2(t)}\right)\|e_v\|$$
$$\le-\dot{V}_1(t)-k_1\lambda_2\left(\mathcal{L}_0\right)\|e_x\|^2-\frac{l_1-l_2}{2}\sqrt{2}\lambda_2^{\frac{1}{2}}\left(\mathcal{L}_0\right)\|e_x\|$$
$$-2k_2\lambda_2\left(\mathcal{L}_0\right)\|e_v\|^2-\sqrt{2}l_2\lambda_2^{\frac{1}{2}}\left(\mathcal{L}_0\right)\|e_v\|.\tag{28}$$

Based on (26)–(28), one obtains

$$\dot{V}(t)=\dot{V}_1(t)+\dot{V}_2(t)$$
$$\le-\left(k_1\lambda_2\left(\mathcal{L}_0\right)-\ell_1\right)\|e_x\|^2-\frac{l_1-l_2}{2}\sqrt{2}\lambda_2^{\frac{1}{2}}\left(\mathcal{L}_0\right)\|e_x\|+2e_x^{\mathrm{T}}e_v$$
$$-\left(2k_2\lambda_2\left(\mathcal{L}_0\right)-2\ell_2-1\right)\|e_v\|^2-\sqrt{2}l_2\lambda_2^{\frac{1}{2}}\left(\mathcal{L}_0\right)\|e_v\|$$
$$+(2\ell_1+\ell_2)\sum_{i=1}^{N}\left\|e_{v_i}\right\|\cdot\left\|e_{x_i}\right\|+\bar{d}\sum_{i=1}^{N}\left(\left\|e_{x_i}\right\|+2\left\|e_{v_i}\right\|\right).$$

Due to the basic inequality, one obtains $2e_x^{\mathrm{T}}e_v+(2\ell_1+\ell_2)\sum_{i=1}^{N}\|e_{v_i}\|\cdot\|e_{x_i}\|=2\sum_{i=1}^{N}e_{x_i}^{\mathrm{T}}e_{v_i}+(2\ell_1+\ell_2)\sum_{i=1}^{N}\|e_{v_i}\|\cdot\|e_{x_i}\|\le(2+2\ell_1+\ell_2)\sum_{i=1}^{N}\|e_{v_i}\|\cdot\|e_{x_i}\|\le\iota(2+2\ell_1+\ell_2)\|e_x\|^2+(2+2\ell_1+\ell_2/4\iota)\|e_v\|^2$. Based on Lemma 2, $\sum_{i=1}^{N}(\|e_{x_i}\|+2\|e_{v_i}\|)\le N^{(1/2)}(\sum_{i=1}^{N}\|e_{x_i}\|^2)^{(1/2)}+2N^{(1/2)}(\sum_{i=1}^{N}\|e_{v_i}\|^2)^{(1/2)}=N^{(1/2)}\|e_x\|+2N^{(1/2)}\|e_v\|$. Then

$$\dot{V}(t)\le-(k_1\lambda_2(\mathcal{L}_0)-\ell_1-\iota(2+2\ell_1+\ell_2))\|e_x\|^2$$
$$-\left(\frac{l_1-l_2}{2}\sqrt{2}\lambda_2^{\frac{1}{2}}(\mathcal{L}_0)-\bar{d}N^{\frac{1}{2}}\right)\|e_x\|$$
$$-\left(2k_2\lambda_2(\mathcal{L}_0)-2\ell_2-1-\frac{2+2\ell_1+\ell_2}{4\iota}\right)\|e_v\|^2$$
$$-\left(\sqrt{2}l_2\lambda_2^{\frac{1}{2}}(\mathcal{L}_0)-2\bar{d}N^{\frac{1}{2}}\right)\|e_v\|.\tag{29}$$

By (13), it is possible to obtain $\dot{V}(t)\le0$ on $[0,t_1)$. Moreover, if agents i' and j' satisfy $\lim_{t\to t^-}\|x_{i'}-x_{j'}\|=R$ and $(i',j')\in$

$\mathcal{E}(0)$, one obtains $\int_0^{\|x_{i'}-x_{j'}\|} \varpi(s)sds = \int_0^{\|x_{i'}-x_{j'}\|} \phi(s)sds \to +\infty$ as $t \to t_1^-$ based on the second property of $\phi(s)$. Hence, $\lim_{t\to t_1} V(t) = +\infty$ which is inconsistent with the property $\dot{V}(t) \leq 0$. Hence, the finite time t_1 does not exist, which implies that the initial edges will exist all the time. The proof is thus completed.

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Huifen Hong received the M.Sc. and Ph.D. degrees from Southeast University, Nanjing, China, in 2015 and 2019, respectively.

From 2017 to 2018, she was a visiting student with the School of Engineering, RMIT University, Melbourne, VIC, Australia. From 2019 to 2020, she was a Visiting Fellow with the College of Engineering and Computer Science, Australian National University, Canberra, ACT, Australia. She is currently a Lecturer with the College of Automation, College of Artificial Intelligence, Nanjing University of Posts and Telecommunications, Nanjing, China. Her current research interests include multiagent systems, finite-time control, and distributed optimization.

Simone Baldi (Senior Member, IEEE) received the B.Sc. degree in electrical engineering, and the M.Sc. and Ph.D. degrees in control systems engineering from the University of Florence, Florence, Italy, in 2005, 2007, and 2011, respectively.

He is a Professor with the School of Mathematics, Southeast University, Nanjing, China, and a guest position with Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, where he was an Assistant Professor. His research interests include adaptive and learning systems with applications in networked control and intelligent vehicles.

Prof. Baldi is a Subject Editor of *International Journal of Adaptive Control and Signal Processing* and an Associate Editor of IEEE CONTROL SYSTEMS LETTERS.

Wenwu Yu (Senior Member, IEEE) received the B.Sc. degree in information and computing science and the M.Sc. degree in applied mathematics from the Department of Mathematics, Southeast University, Nanjing, China, in 2004 and 2007, respectively, and the Ph.D. degree from the Department of Electronic Engineering, City University of Hong Kong, Hong Kong, in 2010.

He is currently the Founding Director of Laboratory of Cooperative Control of Complex Systems and the Deputy Associate Director of Jiangsu Provincial Key Laboratory of Networked Collective Intelligence, an Associate Dean with the School of Mathematics, and a Full Professor with the Endowed Chair Honor, Southeast University. He held several visiting positions in Australia, China, Germany, Italy, The Netherlands, and the USA. His research interests include multiagent systems, complex networks and systems, disturbance control, distributed optimization, machine learning, game theory, cyberspace security, smart grids, intelligent transportation systems, and big-data analysis.

Dr. Yu was a recipient of the Second Prize of State Natural Science Award of China in 2016. He servers as an Editorial Board Member of several flag journals, including IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS II, IEEE TRANSACTIONS ON INDUSTRIAL INFORMATICS, IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS: SYSTEMS, *Science China Information Sciences*, and *Science China Technological Sciences*. He was listed by Clarivate Analytics/Thomson Reuters Highly Cited Researchers in Engineering in from 2014 to 2020. He publishes about 100 IEEE Transactions journal papers with more than 18000 citations.

Xinghuo Yu (Fellow, IEEE) received the B.Eng. and M.Eng. degrees in electrical engineering from the University of Science and Technology of China, Hefei, China, in 1982 and 1984, respectively, and the Ph.D. degree from Southeast University, Nanjing, China, in 1988.

He is currently an Associate Deputy Vice-Chancellor and a Distinguished Professor with the Royal Melbourne Institute of Technology, Melbourne (RMIT University), VIC, Australia. His research interests include control systems, complex and intelligent systems, and smart energy systems.

Dr. Yu received a number of awards and honors for his contributions, including the 2013 Dr.-Ing. Eugene Mittelmann Achievement Award of the IEEE Industrial Electronics Society, the 2018 M. A. Sargent Medal from Engineers Australia, and the 2018 Australasian AI Distinguished Research Contribution Award from Australian Computer Society. He was Past President of the IEEE Industrial Electronics Society. He has served as an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS-I: REGULAR PAPERS, the IEEE TRANSACTIONS ON INDUSTRIAL ELECTRONICS, and the IEEE TRANSACTIONS ON INDUSTRIAL INFORMATICS.