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Dirichlet form analysis of the Jacobi process

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Abstract

We construct and analyze the Jacobi process – in mathematical biology referred to as Wright–Fisher diffusion – using a Dirichlet form. The corresponding Dirichlet space takes the form of a Sobolev space with different weights for the function itself and its derivative and can be rewritten in a canonical form for strongly local Dirichlet forms in one dimension. Additionally to the statements following from the general theory on these forms, we obtain orthogonal decompositions of the Dirichlet space, derive Sobolev embeddings, verify functional inequalities of Hardy type and analyze the long time behavior of the associated semigroup. We deduce corresponding properties of the Markov process and show that it is up to minor technical modifications a solution to the Jacobi SDE. We also provide uniqueness statements for this SDE, such that properties of general solutions follow.

MSC: 60J46; 46E35; 92D25; 33C05

Keywords: Jacobi process; Dirichlet form; Wright–Fisher diffusion; Hypergeometric functions

1. Introduction

The Jacobi process is a $[0, d]$-valued solution to the stochastic differential equation

$$dY_t = (a - bY_t)\,dt + \sigma\sqrt{Y_t(d - Y_t)}\,dW_t. \tag{1.1}$$

Here, a, b and $\sigma, d > 0$ are parameters and W is a Brownian motion. The Jacobi process arises in different applications, most prominently as a model for allele frequencies in mathematical biology, see [10, Section 10.2], where it is commonly referred to as Wright–Fisher diffusion.

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Moreover, the Jacobi processes can be used as a model for membrane depolarization [8] and interest rates [7] or in the modeling of electricity prices [6,28].

The parameters

$$\alpha = \frac{2b}{\sigma^2} - \frac{2a}{\sigma^2 d} - 1 \text{ and } \beta = \frac{2a}{\sigma^2 d} - 1 \tag{1.2}$$

capture the behavior of the process close to the boundary points. Indeed, if Y_t is close to 0, the drift is approximately given by adt and the stochastic fluctuation by $\sigma(Y_t d)^{\frac{1}{2}} dW_t$ which has variance $\sigma^2(Y_t d)dt$. Therefore, the ratio $\frac{2a}{\sigma^2 d}$ quantifies how much the drift pushes the process back into the state space $[0, d]$ compared to the stochastic fluctuations. This demonstrates the importance of the parameter β and an analogous argument shows that α quantifies the boundary behavior near d.

If $\alpha, \beta > -1$, the Jacobi process is one of three types of real-valued diffusions, which are associated to a family of orthogonal polynomials, see [21]. This allows for an explicit expression of the transition semigroup, which can be used as in [7] to analyze the process. The Jacobi process can be constructed by its associated Feller semigroup [10] and belongs to the larger class of Pearson diffusions [13]. For an extensive study of the associated differential operator we refer to [9]. Examples of how to analyze the Jacobi process using classical methods for one-dimensional diffusions can be found in [15]. There are many other works on the Jacobi process, but we hope that this selection gives an overview of the main tools, which were used to construct and analyze the Jacobi process up to now. In the current article, we take the new approach to construct and analyze solutions to (1.1) by Dirichlet form methods. We stress that we allow for the case $\alpha \leq -1$ or $\beta \leq -1$ which leads to additional mathematical challenges.

As state space of the Dirichlet form we define X as the union of the interval $(0, d)$ with the right boundary point $\{d\}$ if $-1 < \alpha < 0$ and with the left boundary $\{0\}$ if $-1 < \beta < 0$. We write $\mathfrak{B}$ for the Borel σ-field on X and dx for the Lebesgue measure. The generator of (1.1) is given by

$$Gf(x) = \frac{1}{2}\sigma^2 x(d-x)f''(x) + (a - bx)f'(x) \tag{1.3}$$

for $x \in [0, d]$. The stationary solution to the corresponding Kolmogorov forward equation on $(0, d)$ is

$$m(x) = \frac{x^\beta (d-x)^\alpha}{d^{\alpha+\beta+1}} \tag{1.4}$$

and is the natural candidate for the density of the invariant measure of the Jacobi process. Therefore, we equip X with the measure with density m with respect to dx. Then, $dm = mdx$ is a positive Radon measure on X with full support and we define $dcm = cmdx$ using the additional density $c(x) = \frac{1}{2}\sigma^2 x(d-x)$. The calculation

$$(cm)'(x) = \frac{1}{2}\sigma^2 \left[\frac{2a}{\sigma^2 d}(d-x) - \left(\frac{2b}{\sigma^2} - \frac{2a}{\sigma^2 d} \right) x \right] \frac{x^\beta (d-x)^\alpha}{d^{\alpha+\beta+1}} = (a - bx)m(x) \tag{1.5}$$

implies that

$$(f'cm)'(x) = \left[c(x)f''(x) + (a-bx)f'(x) \right] m(x) = m(x)Gf(x) \tag{1.6}$$

for $f \in C^2((0, d))$. Consequently, the operator $(G, C_c^\infty((0, d)))$ is of the form [14, Eq. (3.3.17)]. By [14, Theorem 3.3.1]

$$\begin{aligned} D(\mathcal{E}) &= \left\{ f \in L^2(X, dm) \,\middle|\, f' \in L^2(X, dcm) \right\}, \\ \mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) &\to \mathbb{R}, (f, g) \mapsto \int_X f'(x) g'(x) \, dcm(x) \end{aligned} \tag{1.7}$$

defines a strongly local Dirichlet form corresponding to the maximal Markovian self-adjoint extension $(L, D(L))$ of $(G, C_c^\infty((0,d)))$ in $L^2(X, dm)$.

One can rewrite $(\mathcal{E}, D(\mathcal{E}))$ in the form of [5, Section 2.2.3] by introducing the measure $ds = \frac{1}{2cm}dx$ and the corresponding distribution function s determined up to an additive constant by

$$s(y) - s(z) = ds((z, y]) = \int_z^y \frac{1}{2cm(x)}\,dx. \tag{1.8}$$

Indeed, a function f absolutely continuous with respect to ds, in the sense that

$$f(x) - f(y) = \int_x^y \frac{df}{ds}\,ds$$

for a density function $\frac{df}{ds} \in L^1_{loc}((0,d), ds)$ and all $x, y \in (0,d)$, admits the density $\frac{df}{ds}\frac{1}{2cm}$ with respect to dx. By [4, Lemma 8.2], f has $\frac{df}{ds}\frac{1}{2cm}$ as weak derivative on $(0,d)$ and hence

$$f' \in L^2(X, dcm) \iff \frac{1}{2}\int_0^d \left(\frac{df}{ds}\right)^2 \frac{1}{2cm}\,dx < \infty \iff \frac{df}{ds} \in L^2(X, ds). \tag{1.9}$$

Using [4, Theorem 8.2] on compact subintervals shows that any weakly differentiable function f on $(0,d)$ has $2cmf'$ as density with respect to ds and (1.9) holds again. Hence, $D(\mathcal{E})$ consists of all $f \in L^2(X, dm)$ absolutely continuous with respect to ds sufficing $\frac{df}{ds} \in L^2(X, ds)$ and

$$\mathcal{E}(f, g) = \frac{1}{2}\int_X \frac{df}{ds}\frac{dg}{ds}\,ds, \tag{1.10}$$

i.e. up to the factor $\frac{1}{2}$, $(\mathcal{E}, D(\mathcal{E}))$ coincides with the Dirichlet form [5, Eq. (2.2.30)]. In this context dm is called the speed measure and s is called the scale function of $(\mathcal{E}, D(\mathcal{E}))$. For a general decomposition theorem of strongly local Dirichlet forms in one dimension into Dirichlet forms of this type, we refer to [19].

In the following, we denote the Markovian, symmetric, strongly continuous semigroup generated by $(L, D(L))$ by $(T_t)_{t>0}$ and write

$$\mathcal{E}_\lambda(f, g) = \mathcal{E}(f, g) + \lambda(f, g)_{L^2(X,dm)}$$

for $\lambda > 0$ and $f, g \in D(\mathcal{E})$. We equip $D(\mathcal{E})$ with the topology induced by any of the inner products $\mathcal{E}_\lambda$ and, if considered as a Hilbert space, we equip it with $\mathcal{E}_1$ unless stated otherwise. We write $\mathcal{F}$ for the closure of $C_c^\infty((0,d))$ in $D(\mathcal{E})$.

In the preliminary Section 2 we introduce and analyze a notion of solutions to (1.1), which serve later on to translate our findings on Hunt processes to statements on general solutions to (1.1). In Section 3 we consider the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. In particular, many basic properties of $(\mathcal{E}, D(\mathcal{E}))$ follow by the general theory from [5, Section 2.2.3]. Moreover, we provide orthogonal decompositions of $D(\mathcal{E})$, prove embedding theorems and functional inequalities and analyze the asymptotic behavior of $(T_t)_{t>0}$ depending on the parameters α and β. Finally, in Section 4 we translate the findings on $(\mathcal{E}, D(\mathcal{E}))$ and $(T_t)_{t>0}$ into properties of an associated Hunt process. We show under which assumption on the parameters, the process is recurrent, ergodic, conservative or a variant of transitive. Furthermore, we show how a Markov process associated to $(\mathcal{E}, D(\mathcal{E}))$ as well as its restriction to $(0,d)$ is related to general solutions to (1.1). This allows to transfer the gathered results from this article as well as future findings on $(\mathcal{E}, D(\mathcal{E}))$ into statements on (1.1).

2. Local solutions to the Jacobi SDE

Throughout this article we fix parameters $a, b \in \mathbb{R}$ and $\sigma, d > 0$. Unless stated otherwise in the assumptions, all theorems hold for any choice of these parameters. Moreover, equality of random variables is meant almost surely unless stated otherwise.

Definition 2.1. A local solution to the stochastic differential equation (1.1) is a quadruple, consisting of a filtered probability space $(\Omega, \mathfrak{A}, P, \mathfrak{F})$, a Brownian motion W, a real-valued, adapted process Y and a stopping time ζ, such that $\mathfrak{F}$ satisfies the usual conditions and the following conditions are satisfied.

(i) The mapping $Y_\cdot(\omega)\colon [0, \zeta(\omega)] \to \mathbb{R}$ is $[0, d]$-valued and continuous for every $\omega \in \Omega$.
(ii) For all $t \geq 0$ we have that

$$Y_{t\wedge\zeta} = Y_0 + \int_0^{t\wedge\zeta} (a - bY_s)\, ds + \int_0^{t\wedge\zeta} \sigma\sqrt{Y_s(d - Y_s)}\, dW_s. \tag{2.1}$$

(iii) It holds $P(\{Y_\zeta \notin \{0, d\}\} \cap \{\zeta < \infty\}) = 0$.

We sometimes call (Y, ζ) a local solution and $(\Omega, \mathfrak{A}, P, \mathfrak{F}, W)$ its stochastic basis or do not even specify the latter.

Remark 2.2. We understand the first integral in (2.1) as the continuous and adapted process

$$\int_0^{\cdot} \mathbb{1}_{[0,\zeta]}(s)(a - bY_s)\, ds.$$

at time t. The second integral is meant as the stochastic integral

$$\int_0^t \mathbb{1}_{[0,\zeta]}(s)\sigma\sqrt{Y_s(d - Y_s)}\, dW_s,$$

which is well-defined, since the integrand is left-continuous, adapted and bounded.

In the next definition (Y, ζ) and $(\tilde{Y}, \tilde{\zeta})$ are local solutions on the same stochastic basis.

Definition 2.3. The solution $(\tilde{Y}, \tilde{\zeta})$ is an extension of (Y, ζ), if $\tilde{\zeta} \geq \zeta$ and $Y_{t\wedge\zeta} = \tilde{Y}_{t\wedge\zeta}$ for all $t \geq 0$. If additionally $\zeta = \tilde{\zeta}$, we identify the local solutions.

If $(\tilde{Y}, \tilde{\zeta})$ is an extension of (Y, ζ), we write $(Y, \zeta) \lesssim (\tilde{Y}, \tilde{\zeta})$. In the case of equality we write $(Y, \zeta) = (\tilde{Y}, \tilde{\zeta})$. The relation $\lesssim$ is then a partial ordering on the equivalence classes of local solutions on a fixed stochastic basis.

Definition 2.4. A local solution (Y, ζ) is called minimal (maximal), if it is a minimal (maximal) element with respect to the ordering $\lesssim$ of local solutions.

To use the theory of stochastic differential equations on $\mathbb{R}$ we let $\mu \in C_c^\infty(\mathbb{R})$ such that $\mu(x) = (a - bx)$ on a neighborhood of $[0, d]$ and $\nu(x) = \mathbb{1}_{[0,d]}(x)\sigma\sqrt{x(d - x)}$ for $x \in \mathbb{R}$. Then μ is Lipschitz continuous and ν is $1/2$-Hölder continuous. Therefore, the stochastic differential equation

$$dZ_t = \mu(Z_t)dt + \nu(Z_t)dW_t \tag{2.2}$$

is well-posed. Indeed, existence of weak solutions follows by the Skorohod existence theorem [17, Theorem 18.7; Theorem 18.9] and pathwise uniqueness holds due to the Yamada–Watanabe condition [17, Theorem 20.3]. Consequently, the Yamada–Watanabe theorem [17, Lemma 18.17] implies that strong existence and uniqueness in law hold for arbitrary initial distributions on $\mathbb{R}$.

Theorem 2.5. *Let (Y, ζ) be a local solution with respect to a stochastic basis $(\Omega, \mathfrak{A}, P, \mathfrak{F}, W)$. Then the unique solution Z to (2.2) with initial value Y_0 satisfies*

$$Z_{t\wedge\zeta} = Y_{t\wedge\zeta} \tag{2.3}$$

for all $t \geq 0$. Moreover,

(i) (Y, ζ) is minimal iff $\zeta = \inf\{t \geq 0 | Z_t \in \{0, d\}\}$,
(ii) (Y, ζ) is maximal iff $\zeta = \inf\{t \geq 0 | Z_t \notin [0, d]\}$.

Proof. Let (Y, ζ) be a local solution, then the first part of the claim follows by a localized Yamada–Watanabe condition, see Lemma A.1. We define $\tilde{\zeta}$ as the infimum in (i), which defines a stopping time. In particular $(Z, \tilde{\zeta})$ is a local solution to (1.1). Due to condition (iii) of Definition 2.1 we have that

$$P(\{\tilde{\zeta} > \zeta\}) = P(\{\tilde{\zeta} > \zeta\} \cap \{Y_\zeta \in \{0, d\}\}).$$

By (2.3) it follows that $P(\{Z_\zeta \neq Y_\zeta\} \cap \{\zeta < \infty\}) = 0$ and therefore $P(\{\tilde{\zeta} > \zeta\})$ is dominated by

$$P(\{\tilde{\zeta} > \zeta\} \cap \{Z_\zeta \in \{0, d\}\}) = 0.$$

Hence $(Z, \tilde{\zeta}) \lesssim (Y, \zeta)$. Therefore, if $\zeta \neq \tilde{\zeta}$, the local solution (Y, ζ) is not minimal. Conversely, if $\zeta = \tilde{\zeta}$, it holds $(Z, \tilde{\zeta}) = (Y, \zeta)$ and by our previous considerations it follows $(Z, \tilde{\zeta}) \lesssim (\hat{Y}, \hat{\zeta})$ for every other local solution $(\hat{Y}, \hat{\zeta})$.

Next, let $\tilde{\zeta}$ be the infimum from (ii) instead. The identity (2.3) implies that

$$P(\{\tilde{\zeta} < \zeta\}) \leq P(\{\inf\{t | Z_{t\wedge\zeta} \notin [0, d]\} < \infty\}) \leq P(\{\exists t \geq 0 : Y_{t\wedge\zeta} \notin [0, d]\}) = 0.$$

In the latter equality we used condition (i) of Definition 2.1. We conclude $(Y, \zeta) \lesssim (Z, \tilde{\zeta})$ and obtain (ii) analogously to (i). □

The previous statement implies pathwise existence and uniqueness of minimal and maximal local solutions to (1.1) with a prescribed initial value. To formulate a uniqueness statement concerning their laws, we introduce the space $[0, d]_\Delta$ as the set $[0, d] \cup \{\Delta\}$, where Δ is topologically adjoined as a separate point. We equip the space $\mathcal{X} = ([0, d]_\Delta)^{[0,\infty)}$ with the corresponding product of Borel σ-fields.

Corollary 2.6. *Let $((\Omega^{(i)}, \mathfrak{A}^{(i)}, P^{(i)}), \mathfrak{F}^{(i)}, W^{(i)}, Y^{(i)}, \zeta^{(i)})$ for $i \in \{1, 2\}$ be two minimal (maximal) local solutions to (3.18) with the same initial distribution on $[0, d]$. Then the laws $P^{(i)} \circ (\tilde{Y}^{(i)})^{-1}$ on $\mathcal{X}$ coincide, where*

$$\tilde{Y}_t^{(i)}(\omega) = \begin{cases} Y_t(\omega), & t \leq \zeta^{(i)}(\omega) \\ \Delta, & t > \zeta^{(i)}(\omega), \end{cases} \qquad \omega \in \Omega^{(i)}.$$

In particular, the laws of their lifetimes $P^{(i)} \circ (\zeta^{(i)})^{-1}$ coincide.

Proof. We first consider the case that the two solutions are maximal. Let $Z^{(i)}$ be the solution to (2.2) with the same initial value and $\tilde{\zeta}^{(i)}$ the stopping time as in Theorem 2.5 (ii) for $i \in \{1, 2\}$.

By the uniqueness in law of (2.2), we have that $P^{(1)} \circ (Z^{(1)})^{-1} = P^{(2)} \circ (Z^{(2)})^{-1}$ as probability measures on $C([0,\infty),\mathbb{R})$, where we equip the latter space with its Borel σ-field. We note that

$$\tau : C([0,\infty),\mathbb{R}) \to \mathbb{R} \cup \{\infty\},\ f \mapsto \inf\{t \geq 0 | f(t) \notin [0,d]\}$$

is measurable. Therefore, the mapping $\phi : C([0,\infty),\mathbb{R}) \to \mathcal{X}$, where

$$\phi(f)(t) = \begin{cases} f(t), & t \leq \tau, \\ \Delta, & t > \tau \end{cases}$$

is measurable as well. Consequently, if we denote the measure $P^{(i)} \circ (Z^{(i)})^{-1}$ by Q, which is independent of i, we obtain that

$$P^{(i)} \circ (\tilde{Y}^{(i)})^{-1} = Q \circ \phi^{-1}, \tag{2.4}$$

by Theorem 2.5. The statement for minimal solutions follows analogously by replacing τ by

$$f \mapsto \inf\{t \geq 0 | f(t) \in \{0,d\}\}. \quad \square$$

3. The Dirichlet space

We recall that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ was defined in (1.7) and can be rewritten in the form [5, Eq. (2.2.30)] by (1.9) and (1.10). Following [5, pp. 65–66], the right boundary d is called approachable, if the function s defined by (1.8) satisfies

$$\lim_{x \nearrow d} s(x) < \infty. \tag{3.1}$$

Since

$$s'(x) = \frac{1}{2cm(x)} = \frac{d^{\alpha+\beta+1}}{\sigma^2 x^{\beta+1}(d-x)^{\alpha+1}},$$

(3.1) is satisfied if and only if $\alpha < 0$. Analogously, the left boundary 0 is approachable iff $\beta < 0$. By [5, p. 65], every function $f \in D(\mathcal{E})$ admits a dm-version that is continuous on $\tilde{X}$, where we define $\tilde{X}$ as the union of $(0,d)$ with the approachable boundary points. In the following, we denote this dm-version of f by $\tilde{f}$. Equipping $C(\tilde{X})$ with the topology of uniform convergence on compact subsets, we obtain the following continuous embedding.

Proposition 3.1. *The mapping*

$$D(\mathcal{E}) \to C(\tilde{X}),\ f \mapsto \tilde{f} \tag{3.2}$$

is a continuous embedding.

Proof. The inclusion $D(\mathcal{E}) \subset C(\tilde{X})$ follows by the preceding considerations. Moreover, the space $C(\tilde{X})$ is a complete metric vector space by [12, pp. 167–168]. Since every $D(\mathcal{E})$-convergent sequence admits a dm-almost everywhere convergent subsequence it follows that (3.2) has closed graph and therefore is continuous by the closed graph theorem, see [25, Theorem 2.3, p. 78]. $\square$

Following [5, p. 66], d is a regular boundary point, if d is approachable and $dm((c,d)) < \infty$ for every $c \in (0,d)$. Since dm has density (1.4) with respect to dx, d is regular iff $-1 < \alpha < 0$ and analogously 0 is regular iff $-1 < \beta < 0$. Consequently, the choice of the state space X coincides with the one in [5, Theorem 2.2.11] and using the previous boundary classification, we obtain the following result. For definitions of the appearing properties, we refer to [14, Section 1.1], [14, p. 55] and [14, Eq. (1.5.4)].

Theorem 3.2. *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is a regular, strongly local and irreducible Dirichlet form on $L^2(X, dm)$. Moreover, $(\mathcal{E}, D(\mathcal{E}))$ is recurrent if and only if $\alpha, \beta > -1$ and transient otherwise.*

As shown in [5, p. 68] the quasi notions with respect to $(\mathcal{E}, D(\mathcal{E}))$ trivialize as stated in the following proposition. For definitions of the 1-capacity and quasi notions, see [14, Eq. (2.1.3)] and [14, pp. 68–69].

Proposition 3.3. *Every non-empty subset of X has positive 1-capacity and a function on X is quasi-continuous with respect to $(\mathcal{E}, D(\mathcal{E}))$, iff f is continuous on X.*

In particular, the quasi-continuous version of a function $f \in D(\mathcal{E})$, which exists by [14, Theorem 2.1.3], coincides with the previously introduced version $\tilde{f}$. Finally, using [19, Theorem 3.2], we obtain the following characterization of the space $\mathcal{F}$ introduced in the introduction, which is useful later on.

Theorem 3.4. *The space $\mathcal{F}$ is given by*

$$\Big\{ f \in \mathcal{D}(\mathcal{E}) : \ \tilde{f}(d) = 0, \text{if} -1 < \alpha < 0 \ \text{ and } \ \tilde{f}(0) = 0, \ \text{if } -1 < \beta < 0 \Big\}. \tag{3.3}$$

We conclude this part by observing when the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is conservative, for a definition of this property see [14, p. 56]. The case in which $(\mathcal{E}, D(\mathcal{E}))$ is conservative can be straightforwardly treated using [14, Theorem 1.6.6]. For the remaining cases we use the characterization of conservativeness of a one dimensional Dirichlet form in terms of a scale function and speed measure from [11, Theorem 5.1].

Proposition 3.5. *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is conservative if and only if $\alpha, \beta > -1$.*

Proof. If $\alpha, \beta > -1$ we have $\mathbb{1} \in D(\mathcal{E})$, such that $(\mathcal{E}, D(\mathcal{E}))$ is conservative by [14, Theorem 1.6.6]. We assume next that $\alpha \leq -1$. In this case, $d \notin X$, (3.1) holds and

$$\begin{aligned}
\int_{\frac{d}{2}}^{d} \int_{\frac{d}{2}}^{x} dm(y)\, ds(x) &\leq C_{\beta,\sigma,d} \int_{\frac{d}{2}}^{d} \int_{\frac{d}{2}}^{x} (d-y)^{\alpha}\, dy\, (d-x)^{-(\alpha+1)}\, dx \\
&= C_{\beta,\sigma,d} \int_{\frac{d}{2}}^{d} \int_{y}^{d} (d-x)^{-(\alpha+1)} dx\, (d-y)^{\alpha}\, dy \\
&= C_{\beta,\sigma,d} \int_{\frac{d}{2}}^{d} \frac{-1}{\alpha}\, dy \ < \ \infty,
\end{aligned}$$

where we used that $\alpha < 0$ in the latter equality. That $(\mathcal{E}, D(\mathcal{E}))$ is not conservative follows now from [11, Lemma 5.1; Theorem 5.1]. The case $\beta \leq -1$ can be treated analogously. □

3.1. Orthogonal decompositions of the domain

This subsection is devoted to calculating the orthogonal complements of the spaces

$$\Big\{ f \in D(\mathcal{E}) \big| \ \tilde{f}(d) = 0 \Big\} \quad \text{and} \quad \Big\{ f \in D(\mathcal{E}) \big| \ \tilde{f}(0) = 0 \Big\} \tag{3.4}$$

in $(D(\mathcal{E}), \mathcal{E}_\lambda)$ for $\lambda > 0$, which we denote by $\mathcal{H}^{\lambda}_{\{d\}}$ and $\mathcal{H}^{\lambda}_{\{0\}}$ respectively. They are of interest since they are connected to the hitting distributions of the corresponding boundary point, see [14, Section 4.3]. To motivate the following considerations, we assume that $f \in \mathcal{H}^{\lambda}_{\{d\}}$. Then, in particular,

$$\forall \varphi \in C_c^{\infty}((0,d)) : \int_0^d f'(x)\varphi'(x)\, dcm(x) + \lambda \int_0^d f(x)\varphi(x)\, dm(x) = 0.$$

By (1.6) this is a weak formulation of the differential equation $Gf = \lambda f$, where we recall that G was defined as (1.3). For a general study of solutions to the resolvent equation of the generator of a diffusion, we refer to [20, Chapter II] or [16, Section 5.12]. In our situation, solutions to this differential equation are proper candidates to span $\mathcal{H}^{\lambda}_{\{d\}}$. To simplify the notation we introduce the λ-dependent parameter

$$\gamma = \frac{\sqrt{\sigma^4 - 4b\sigma^2 - 8\lambda\sigma^2 + 4b^2}}{2\sigma^2}$$

such that by definition

$$\left(\frac{\alpha+\beta+1}{2}\right)^2 - \gamma^2 = \frac{2\lambda}{\sigma^2}. \tag{3.5}$$

Using ${}_2F_1$ as notation for hypergeometric functions, we define the real-valued function

$$\xi_\lambda(x) = \begin{cases} {}_2F_1\left(\frac{\alpha+\beta+1}{2} + \gamma, \frac{\alpha+\beta+1}{2} - \gamma; \beta+1; \frac{x}{d}\right), & \beta > -1, \\ \left(\frac{x}{d}\right)^{-\beta} {}_2F_1\left(\frac{\alpha-\beta+1}{2} + \gamma, \frac{\alpha-\beta+1}{2} - \gamma; 1-\beta; \frac{x}{d}\right), & \beta \le -1 \end{cases}$$

on $(0,d)$. Since ξ_λ is a rescaled version of the solution to a hypergeometric differential equation, see [18, p. 163], it satisfies indeed $G\xi_\lambda = \lambda\xi_\lambda$ on $(0,d)$. In course of this subsection, we prove the following result.

Theorem 3.6. *Let $\lambda > 0$ and $-1 < \alpha < 0$, then we have*

$$\mathcal{H}^{\lambda}_{\{d\}} = \operatorname{span}\{\xi_\lambda\}.$$

Since the proof of Theorem 3.6 relies on an integration by parts argument, we first investigate the boundary values of ξ_λ based on the following general formulae for boundary values of hypergeometric functions. They are the key tool to perform the explicit calculations and can be found in [2, Theorem 2.1.3; Theorem 2.2.2]. The appearing function Γ is the well-known gamma function and we write $\Re$ for the real part of an imaginary number.

Lemma 3.7. *Let $\kappa, \iota, \upsilon \in \mathbb{C}$ with $-\upsilon \notin \mathbb{N}_0$.*

(i) If $\Re(\upsilon - \kappa - \iota) > 0$, then

$$\lim_{x \nearrow 1} {}_2F_1(\kappa, \iota; \upsilon; x) = \frac{\Gamma(\upsilon)\Gamma(\upsilon-\kappa-\iota)}{\Gamma(\upsilon-\kappa)\Gamma(\upsilon-\iota)}. \tag{3.6}$$

(ii) If $\upsilon - \kappa - \iota = 0$, then

$$\lim_{x \nearrow 1} \frac{{}_2F_1(\kappa, \iota; \upsilon; x)}{-\log(1-x)} = \frac{\Gamma(\upsilon)}{\Gamma(\kappa)\Gamma(\iota)}. \tag{3.7}$$

(iii) If $\Re(\upsilon - \kappa - \iota) < 0$, then

$$\lim_{x \nearrow 1} \frac{{}_2F_1(\kappa, \iota; \upsilon; x)}{(1-x)^{\upsilon-\kappa-\iota}} = \frac{\Gamma(\upsilon)\Gamma(\kappa+\iota-\upsilon)}{\Gamma(\kappa)\Gamma(\iota)}. \tag{3.8}$$

They result in the following statements on ξ_λ. The detailed calculations leading to it are contained in Appendix A.2.

Lemma 3.8. *The function ξ_λ admits the following properties.*

(i) It holds that

$$\lim_{x\searrow 0}\xi_\lambda(x) = \begin{cases} 1, & \beta > -1,\\ 0, & \beta \le -1.\end{cases}$$

(ii) It holds that

$$\lim_{x\nearrow d}\xi_\lambda(x) = \begin{cases} \frac{\Gamma(\beta+1)\Gamma(-\alpha)}{\Gamma\left(\frac{-\alpha+\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha+\beta+1}{2}-\gamma\right)}, & \alpha<0,\ \beta>-1,\\ \frac{\Gamma(1-\beta)\Gamma(-\alpha)}{\Gamma\left(\frac{-\alpha-\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2}-\gamma\right)}, & \alpha<0,\ \beta\le -1,\\ \infty, & \alpha\ge 0\end{cases}$$

and the above limit is positive if it is finite.

(iii) It holds that

$$\lim_{x\searrow 0}\xi_\lambda' cm(x) = \begin{cases} 0, & \beta > -1,\\ \frac{-\beta\sigma^2}{2}, & \beta \le -1.\end{cases}$$

(iv) Under the additional assumption

$$\lambda > \frac{\sigma^2}{2}\left(\frac{\alpha+\beta+1}{2}\right)^2 \tag{3.9}$$

for $\alpha < -1$ and $\beta \le -1$ it holds that

$$\lim_{x\nearrow d}\xi_\lambda' cm(x) = \begin{cases} \frac{\lambda\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha+\beta+3}{2}+\gamma\right)\Gamma\left(\frac{\alpha+\beta+3}{2}-\gamma\right)}, & \alpha>-1,\ \beta>-1,\\ \left[\frac{2\lambda}{\sigma^2}-\beta(\alpha+1)\right]\frac{\sigma^2\Gamma(1-\beta)\Gamma(\alpha+1)}{2\Gamma\left(\frac{\alpha-\beta+3}{2}+\gamma\right)\Gamma\left(\frac{\alpha-\beta+3}{2}-\gamma\right)}, & \alpha>-1,\ \beta\le -1,\\ \infty, & \alpha\le -1\end{cases}$$

and the above limit is positive if it is finite.

Lemma 3.9. *Let $\lambda > 0$ and $-1 < \alpha < 0$, then $\xi_\lambda \in D(\mathcal{E})$.*

Proof. Since $G\xi_\lambda = \lambda\xi_\lambda$ on $(0,d)$, the identity (1.6) yields that

$$(\xi_\lambda' cm)'(x) = \lambda m(x)\xi_\lambda(x) \tag{3.10}$$

for all $x\in(0,d)$. Hence, integration by parts yields that

$$\int_\epsilon^{d-\epsilon}\xi_\lambda'\xi_\lambda' cm(x)\,dx + \lambda\int_\epsilon^{d-\epsilon}\xi_\lambda\xi_\lambda m(x)\,dx = \left[\xi_\lambda\xi_\lambda' cm\right]_\epsilon^{d-\epsilon}$$

for $\epsilon > 0$. By letting $\epsilon\searrow 0$ we get

$$\int_X \xi_\lambda'\xi_\lambda'\,dcm(x) + \lambda\int_X \xi_\lambda\xi_\lambda\,dm(x) = \lim_{\epsilon\searrow 0}\left[\xi_\lambda\xi_\lambda' cm\right]_\epsilon^{d-\epsilon}. \tag{3.11}$$

Parts (i) and (iii) of Lemma 3.8 imply that $\xi_\lambda\xi_\lambda' cm(x)$ converges as $x\searrow 0$. If we assume $-1<\alpha<0$, parts (ii) and (iv) yield that $\xi_\lambda\xi_\lambda' cm(x)$ converges as well as $x\nearrow d$. In particular, (3.11) is finite in this case and $\xi_\lambda\in D(\mathcal{E})$. □

To complete the proof of Theorem 3.6, we also need the following observation. Since the density functions m and cm are both bounded from below on every compactly contained subset of $(0, d)$, $D(\mathcal{E})$ embeds continuously in the local Sobolev space $H^1_{\text{loc}}((0, d))$, for an introduction to these spaces we refer to [3]. Hence, we obtain the following integration by parts formula

$$\tilde{f}(r)g(r) - \tilde{f}(l)g(l) = \int_l^r \left[f'g(x) + fg'(x)\right] dx \tag{3.12}$$

for any function g which is continuously differentiable on $[l, r] \subset (0, d)$ as a consequence of [4, Corollary 8.10].

Proof of Theorem 3.6. Since the point evaluation

$$\delta_{\{d\}} : D(\mathcal{E}) \to \mathbb{R}$$

is a rank one operator for $-1 < \alpha < 0$,

$$\left\{ f \in D(\mathcal{E}) \middle| \tilde{f}(d) = 0 \right\} \tag{3.13}$$

has codimension one and it suffices to check that $\text{span}\{\xi_\lambda\}$ and (3.13) are orthogonal to each other in $(D(\mathcal{E}), \mathcal{E}_\lambda)$ by Lemma 3.9. To this end, we distinguish different cases of β. For $\beta \le -1$ or $\beta \ge 0$ the space (3.13) coincides with (3.3). Hence it suffices to check that

$$\forall \varphi \in C_c^\infty((0, d)) : \quad \mathcal{E}_\lambda(\xi_\lambda, \varphi) = 0.$$

For $\varphi \in C_c^\infty((0, d))$, we choose l, r such that $\text{supp}(\varphi) \subset (l, r)$ and then

$$\mathcal{E}_\lambda(\xi_\lambda, \varphi) = \int_l^r \varphi' \xi_\lambda' cm(x)\, dx + \lambda \int_l^r \varphi \xi_\lambda m(x)\, dx = 0$$

by integration by parts and (3.10). This finishes the proof for these cases of β. For $-1 < \beta < 0$, we use the integration by parts formula (3.10) and (3.12) to conclude that

$$\left[\tilde{f}\xi_\lambda' cm\right]_\epsilon^{d-\epsilon} = \int_\epsilon^{d-\epsilon} f'\xi_\lambda' cm(x) + \lambda f \xi_\lambda m(x)\, dx$$

for f in (3.13) and $\epsilon > 0$. Hence,

$$\left[\tilde{f}\xi_\lambda' cm\right]_\epsilon^{d-\epsilon} \to \mathcal{E}_\lambda(f, \xi_\lambda)$$

as $\epsilon \searrow 0$. By Lemma 3.8 (iv) together with $\tilde{f}(d) = 0$ we get $\lim_{x \nearrow d} \tilde{f}\xi_\lambda' cm(x) = 0$. Lemma 3.8 (iii) together with the continuity of $\tilde{f}$ at 0 implies that $\lim_{x \searrow 0} \tilde{f}\xi_\lambda' cm(x) = 0$. Hence $\mathcal{E}_\lambda(f, \xi_\lambda) = 0$, which completes the proof also in this case. □

To also calculate $\mathcal{H}^\lambda_{\{0\}}$ we define the dual set of parameters

$$(d^\dagger, a^\dagger, b^\dagger, \sigma^\dagger) = (d, bd - a, b, \sigma).$$

Then, accordingly

$$G^\dagger f(x) = \frac{1}{2}\sigma^2 x(d - x) f''(x) + ((bd - a) - ax) f'(x)$$

such that $Gf = \lambda f$ is equivalent to $G^\dagger(f(d-\cdot)) = \lambda f(d-\cdot)$. Hence, the function $\eta_\lambda = \xi_\lambda^\dagger(d-\cdot)$ satisfies $G\eta_\lambda = \lambda\eta_\lambda$. It is straightforward to verify that $\alpha^\dagger = \beta$, $\beta^\dagger = \alpha$ and $\gamma^\dagger = \gamma$ such that we can write explicitly

$$\eta_\lambda(x) = \begin{cases} {}_2F_1\left(\frac{\alpha+\beta+1}{2} + \gamma, \frac{\alpha+\beta+1}{2} - \gamma; \alpha + 1; 1 - \frac{x}{d}\right), & \alpha > -1, \\ \left(1 - \frac{x}{d}\right)^{-\alpha} {}_2F_1\left(\frac{-\alpha+\beta+1}{2} + \gamma, \frac{-\alpha+\beta+1}{2} - \gamma; 1 - \alpha; 1 - \frac{x}{d}\right), & \alpha \le -1 \end{cases}$$

for $x \in (0, d)$. In the following corollary we collect all the properties of η_λ, which follow immediately from the respective properties of $\xi_\lambda^\dagger$.

Corollary 3.10. *The function η_λ admits the following properties.*

(i) We have $G\eta_\lambda = \lambda\eta_\lambda$ on $(0, d)$.

(ii) It holds that

$$\lim_{x \nearrow d} \eta_\lambda(x) = \begin{cases} 1, & \alpha > -1, \\ 0, & \alpha \le -1. \end{cases}$$

(iii) It holds that

$$\lim_{x \searrow 0} \eta_\lambda(x) = \begin{cases} \frac{\Gamma(\alpha+1)\Gamma(-\beta)}{\Gamma\left(\frac{\alpha-\beta+1}{2}+\gamma\right)\Gamma\left(\frac{\alpha-\beta+1}{2}-\gamma\right)}, & \beta < 0,\ \alpha > -1, \\ \frac{\Gamma(1-\alpha)\Gamma(-\beta)}{\Gamma\left(\frac{-\alpha-\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2}-\gamma\right)}, & \beta < 0,\ \alpha \le -1, \\ \infty, & \beta \ge 0 \end{cases}$$

and the above limit is positive if it is finite.

(iv) It holds that

$$\lim_{x \nearrow d} \eta_\lambda' cm(x) = \begin{cases} 0, & \alpha > -1, \\ \frac{\alpha\sigma^2}{2}, & \alpha \le -1. \end{cases}$$

(v) Under the additional assumption

$$\lambda > \frac{\sigma^2}{2}\left(\frac{\alpha+\beta+1}{2}\right)^2$$

for $\alpha \le -1$ and $\beta < -1$ it holds that

$$\lim_{x \searrow 0} \eta_\lambda' cm(x) = \begin{cases} \frac{-\lambda\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma\left(\frac{\alpha+\beta+3}{2}+\gamma\right)\Gamma\left(\frac{\alpha+\beta+3}{2}-\gamma\right)}, & \beta > -1,\ \alpha > -1, \\ \left[\frac{2\lambda}{\sigma^2} - \alpha(\beta+1)\right] \frac{-\sigma^2\Gamma(1-\alpha)\Gamma(\beta+1)}{2\Gamma\left(\frac{-\alpha+\beta+3}{2}+\gamma\right)\Gamma\left(\frac{-\alpha+\beta+3}{2}-\gamma\right)}, & \beta > -1,\ \alpha \le -1, \\ -\infty, & \beta \le -1 \end{cases}$$

and the above limit is negative if it is finite.

Using the same line of arguments as in the proof of Theorem 3.6 one shows the following.

Theorem 3.11. *Let $\lambda > 0$ and $-1 < \beta < 0$, then we have*

$$\mathcal{H}_{\{0\}}^\lambda = \operatorname{span}\{\eta_\lambda\}.$$

3.2. Sobolev type embeddings

In this subsection we analyze the embedding properties of $D(\mathcal{E})$ in the spaces $L^q(X, dm)$ for $q > 2$. By Proposition 3.1 we have the embedding $D(\mathcal{E}) \hookrightarrow L^\infty(X, dm)$, if $\alpha, \beta < 0$. Since trivially $D(\mathcal{E}) \hookrightarrow L^2(X, dm)$ also the embeddings $D(\mathcal{E}) \hookrightarrow L^q(X, dm)$ for $2 \le q \le \infty$ follow by Hölder's inequality. To derive a similar result in the case that $\alpha \ge 0$ or $\beta \ge 0$ we state a special case of the characterization of embeddings of weighted Sobolev spaces from [24]. To formulate it, we introduce for $s \in \mathbb{R}$, $1 \le q \le \infty$ the weighted spaces $L^q_s(\mathbb{R})$ as the L^q space on $\mathbb{R}$ equipped with the measure with density $|x|^s$ with respect to the Lebesgue measure. Moreover, we set

$$\|f\|^2_{W^{1,(2,2)}_{s,s+1}(\mathbb{R}\setminus\{0\})} = \|f\|^2_{L^2_s(\mathbb{R})} + \|f'\|^2_{L^2_{s+1}(\mathbb{R})}, \tag{3.14}$$

for functions f, which are weakly differentiable on $\mathbb{R}\setminus\{0\}$ and define $W^{1,(2,2)}_{s,s+1}(\mathbb{R}\setminus\{0\})$ as the space of all f such that (3.14) is finite. As a consequence of the completeness of the involved spaces $L^2_s(\mathbb{R})$ and $L^2_{s+1}(\mathbb{R})$ the space $W^{1,(2,2)}_{s,s+1}(\mathbb{R}\setminus\{0\})$ is complete as well.

Lemma 3.12. *Let $s \ge 0$ and $1 \le q < \infty$. If we define*

$$q_s = \begin{cases} 2\left(1+\frac{1}{s}\right), & s > 0, \\ \infty, & s = 0, \end{cases}$$

then $W^{1,(2,2)}_{s,s+1}(\mathbb{R}\setminus\{0\})$ embeds continuously in $L^q_s(\mathbb{R})$ if and only if $q \in [2, q_s]$.

Proof. The stated embedding holds if and only if any of the cases (i)–(vi) of [24, Theorem 1.1] holds. The cases (ii), (v) and (vi) are impossible and the case (iii) corresponds to the trivial case $q = 2$. The case (iv) is impossible if $s = 0$, else it corresponds to $q = q_s$. Finally, we note that the case (i) is equivalent to $q \in (2, q_s)$. □

Using a cutoff argument we can transfer these findings to the setting of a bounded interval, which results in the following theorem. During its proof we denote by $K_{\dots}$ a constant only depending on its indices, which may change from line to line.

Theorem 3.13. *We define $q_* = \min\{q_\alpha, q_\beta\}$, where*

$$q_\alpha = \begin{cases} 2\left(1+\frac{1}{\alpha}\right), & \alpha > 0, \\ \infty, & \alpha \le 0, \end{cases} \quad \text{and} \quad q_\beta = \begin{cases} 2\left(1+\frac{1}{\beta}\right), & \beta > 0, \\ \infty, & \beta \le 0. \end{cases}$$

Then there is a continuous embedding $D(\mathcal{E}) \hookrightarrow L^q(X, dm)$ for $q \in [2, q_*)$. If additionally $\alpha, \beta \neq 0$, then the embedding is also true for $q = q_*$.

Proof. Let $\alpha < 0$. Then the restriction mapping

$$C(\tilde{X}) \to C\left(\left[\frac{d}{2}, d\right]\right), f \mapsto f|_{\left[\frac{d}{2},d\right]}$$

is continuous. Therefore, by Proposition 3.1 we can estimate

$$\|f\|_{L^\infty\left(\left[\frac{d}{2},d\right],dm\right)} = \|\tilde{f}\|_{C\left(\left[\frac{d}{2},d\right]\right)} \le K_{\alpha,\beta,d}\sqrt{\mathcal{E}_1(f,f)}$$

for every $f \in D(\mathcal{E})$. Due to Hölder's inequality we conclude that

$$\|f\|_{L^q\left(\left[\frac{d}{2},d\right],dm\right)} \le \|f\|^{\frac{2}{q}}_{L^2\left(\left[\frac{d}{2},d\right],dm\right)} \|f\|^{\frac{q-2}{q}}_{L^\infty\left(\left[\frac{d}{2},d\right],dm\right)} \le K_{\alpha,\beta,d}\sqrt{\mathcal{E}_1(f,f)}.$$

for any $q \in [2, q_\alpha]$.

For $\alpha \geq 0$ we choose a smooth cutoff-function $\varphi \in C_c^\infty(\mathbb{R}_{\geq 0})$, $0 \leq \varphi \leq 1$ such that $\varphi(x) = 1$ for $0 \leq x < \frac{d}{2}$ and $\varphi(x) = 0$ for $x > \frac{3d}{4}$. For $f \in D(\mathcal{E})$ we define the function

$$f_\varphi(x) = \begin{cases} f(d-x)\varphi(x), & x \in (0,d), \\ 0, & \text{else.} \end{cases}$$

Then for any $1 \leq q < \infty$ we have the estimate

$$\begin{aligned} \|f\|_{L^q\left(\left[\frac{d}{2},d\right],dm\right)} &= K_{\alpha,\beta,d} \left(\int_{\frac{d}{2}}^{d} |f(x)|^q x^\beta (d-x)^\alpha \, dx \right)^{\frac{1}{q}} \\ &\leq K_{\alpha,\beta,d} \left(\int_0^{\frac{d}{2}} |f_\varphi(x)|^q x^\alpha \, dx \right)^{\frac{1}{q}} \leq K_{\alpha,\beta,d} \|f_\varphi\|_{L^q_\alpha(\mathbb{R})}. \end{aligned} \tag{3.15}$$

Moreover, for any $\psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$ we have $\varphi\psi \in C_c^\infty((0,d))$, where we use the convention $\varphi(x) = 0$ for $x < 0$, and consequently

$$\begin{aligned} -\int_{\mathbb{R}} f_\varphi(x)\psi'(x)\,dx &= -\int_0^d f(d-x)\left[(\varphi\psi)'(x) - \varphi'(x)\psi(x)\right] dx \\ &= \int_0^d f(d-x)\varphi'(x)\psi(x)\,dx - \int_0^d f'(d-x)(\varphi\psi)(x)\,dx \\ &= \int_{\mathbb{R}} \left[f(d-x)\varphi'(x) - f'(d-x)\varphi(x)\right]\psi(x)\,dx. \end{aligned}$$

Hence f_φ is weakly differentiable on $\mathbb{R}\setminus\{0\}$ with the ψ-independent part of the latter integrand as weak derivative. We can estimate

$$\|f_\varphi\|^2_{L^2_\alpha(\mathbb{R})} = \int_0^d f^2(d-x)\varphi^2(x)x^\alpha\,dx \leq \int_{\frac{d}{4}}^d f^2(x)(d-x)^\alpha\,dx \leq K_{\alpha,\beta,d}\|f\|^2_{L^2(X,dm)}.$$

Similarly, we obtain that

$$\begin{aligned} \|f'_\varphi\|^2_{L^2_{\alpha+1}(\mathbb{R})} &\leq \int_0^d 2\left[f^2(d-x)\varphi'^2(x) + f'^2(d-x)\varphi^2(x)\right]x^{\alpha+1}\,dx \\ &\leq 2\left[\sup_{y\in\mathbb{R}}|\varphi'(y)|^2 \int_{\frac{d}{4}}^{\frac{d}{2}} f^2(x)(d-x)^{\alpha+1}\,dx + \int_{\frac{d}{4}}^d f'^2(x)(d-x)^{\alpha+1}\,dx\right] \\ &\leq K_{\alpha,\beta,d,\varphi}\|f\|^2_{L^2(X,dcm)} + K_{\alpha,\beta,d}\|f'\|^2_{L^2(X,dcm)} \leq K_{\alpha,\beta,d,\varphi}\mathcal{E}_1(f,f). \end{aligned}$$

In the last line we used that the function c is bounded. Combining now (3.15), Lemma 3.12 and the previous two estimates, we obtain that

$$\|f\|_{L^q\left(\left[\frac{d}{2},d\right],dm\right)} \leq K_{\alpha,\beta,d}\|f_\varphi\|_{L^q_\alpha(\mathbb{R})} \leq K_{\alpha,\beta,d,q}\|f_\varphi\|_{W^{1,(2,2)}_{\alpha,\alpha+1}(\mathbb{R}\setminus\{0\})} \leq K_{\alpha,\beta,d,\varphi,q}\sqrt{\mathcal{E}_1(f,f)}.$$

for any finite $q \in [2, q_\alpha]$. Analogously, we find that

$$\|f\|_{L^q\left(\left[0,\frac{d}{2}\right],dm\right)} \leq K_{\alpha,\beta,d,\varphi,q}\sqrt{\mathcal{E}_1(f,f)}$$

for $q \in [2, q_\beta]$, where we additionally require q to be finite for $\beta = 0$. The claim follows now by

$$\|f\|_{L^q(X,dm)} \leq \|f\|_{L^q\left(\left[0,\frac{d}{2}\right],dm\right)} + \|f\|_{L^q\left(\left[\frac{d}{2},d\right],dm\right)}. \quad \square$$

3.3. Hardy type inequalities

In this subsection we provide Hardy-type inequalities for the integrability pairs $(p, q) = (2, 1)$ and $(2, 2)$. The former gives rise to reference functions for $(\mathcal{E}, D(\mathcal{E}))$ and the latter to an estimate on the spectral gap of $(L, D(L))$ later on. The proofs rely on verifying conditions which are derived similarly to the more general situations [22, Theorem 1, p. 50] and [23, Theorem 1]. As a preliminary step, we need the following observation regarding the boundary behavior of functions from $D(\mathcal{E})$.

Proposition 3.14. *Let $f \in D(\mathcal{E})$. Then it holds*

$$\begin{cases} \lim_{x \nearrow d} \tilde{f} cm(x) = 0, & \alpha \geq 0, \\ \lim_{x \nearrow d} \tilde{f}(x) \in \mathbb{R}, & -1 < \alpha < 0, \\ \lim_{x \nearrow d} \tilde{f}(x) = 0, & \alpha \leq -1, \end{cases}$$

as well as

$$\begin{cases} \lim_{x \searrow 0} \tilde{f} cm(x) = 0, & \beta \geq 0, \\ \lim_{x \searrow 0} \tilde{f}(x) \in \mathbb{R}, & -1 < \beta < 0, \\ \lim_{x \searrow 0} \tilde{f}(x) = 0, & \beta \leq -1. \end{cases}$$

Proof. We prove the first part of the statement, the second part can be shown analogously. The case $-1 < \alpha < 0$ follows from the continuity of (3.2). We recall that for $\alpha \leq -1$, the boundary point d is approachable but non-regular, such that

$$\lim_{x \nearrow d} \tilde{f}(x) = 0$$

by [5, Eq. (2.2.39)] and the fact that $D(\mathcal{E})$ is contained in its extended Dirichlet space. Lastly, we assume that $\alpha \geq 0$ and first show that

$$\lim_{x \nearrow d} \tilde{f} cm(x) = 0 \tag{3.16}$$

for $f \in \mathcal{F}$. By an approximation argument, this follows if we can verify continuity of

$$\left(C_c^\infty((0, d)), \mathcal{E}_1\right) \to C\left(\left[\frac{d}{2}, d\right]\right), \ \varphi \mapsto cm\varphi|_{\left[\frac{d}{2}, d\right]}. \tag{3.17}$$

By (1.5) we have that

$$(\varphi cm)'(x) = cm(x)\varphi'(x) + (a - bx)m(x)\varphi(x) \tag{3.18}$$

for every $\varphi \in C_c^\infty((0, d))$ and therefore

$$\begin{aligned} |cm\varphi(x)| &\leq \int_{\frac{d}{2}}^{d} |\varphi'| \, dcm + \left(|a| + |b|d\right) \int_{\frac{d}{2}}^{d} |\varphi| \, dm \\ &\leq \left[\left(\int_{\frac{d}{2}}^{d} \mathbb{1} \, dcm \right)^{\frac{1}{2}} + (|a| + |b|d) \left(\int_{\frac{d}{2}}^{d} \mathbb{1} \, dm \right)^{\frac{1}{2}} \right] \sqrt{\mathcal{E}_1(\varphi, \varphi)} \end{aligned}$$

for any $x \in \left[\frac{d}{2}, d\right]$. The prefactor on the right-hand side is finite since $\alpha > -1$ and consequently (3.17) is indeed bounded. Now, by Theorem 3.4, $D(\mathcal{E}) = \mathcal{F}$ if $\beta \leq -1$ or $\beta \geq 0$, such that the claim follows in this case. If $-1 < \beta < 0$, $\mathcal{F}$ coincides with

$$\{f \in D(\mathcal{E}) | \, \tilde{f}(0) = 0\}$$

such that

$$D(\mathcal{E}) = \mathcal{F} \oplus \operatorname{span}\{\eta_\lambda\}$$

orthogonally with respect to $\mathcal{E}_\lambda$ by Theorem 3.11. But by Corollary 3.10 (ii) η_λ attains a limit at d such that

$$\lim_{x \nearrow d} \eta_\lambda cm(x) = 0$$

by $\alpha \geq 0$. Consequently, (3.16) holds for all $f \in D(\mathcal{E})$, which finishes the proof. □

Lemma 3.15. *If $\alpha \leq -1$ or $\beta \leq -1$ and $r, s \in \mathbb{R}$ such that either*

(i) $\beta \leq -1$, $\alpha > -1$, $r > \frac{-2-\beta}{2}$ and $s > \max\{-1-\alpha, \frac{-2-\alpha}{2}\}$,
(ii) $\beta > -1$, $\alpha \leq -1$, $r > \max\{-1-\beta, \frac{-2-\beta}{2}\}$ and $s > \frac{-2-\alpha}{2}$ or
(iii) $\beta \leq -1$, $\alpha \leq -1$, $r > \frac{-2-\beta}{2}$ and $s > \frac{-2-\alpha}{2}$.

Then there is a constant $C_{\alpha,\beta,\sigma,d,r,s} < \infty$ such that

$$\int_X |f(x)| x^r (d-x)^s \, dm(x) \leq C_{\alpha,\beta,\sigma,d,r,s} \sqrt{\mathcal{E}(f,f)} \tag{3.19}$$

for every $f \in D(\mathcal{E})$.

Proof. Let $f \in D(\mathcal{E})$. In the case of (i) we have $\tilde{f}(0) = 0$ by Proposition 3.14 such that an application of [4, Theorem 8.2] and Fubini's theorem yields

$$\begin{aligned} \int_0^d |f(x)| x^r (d-x)^s \, dm(x) &\leq \int_0^d \int_0^x |f'(y)| \, dy \, \frac{x^{\beta+r}(d-x)^{\alpha+s}}{d^{\alpha+\beta+1}} \, dx \\ &= \int_0^d \int_y^d \frac{x^{\beta+r}(d-x)^{\alpha+s}}{d^{\alpha+\beta+1}} \, dx \, |f'(y)| \, dy. \end{aligned}$$

If we can show that

$$\int_0^d g(y)^2 \frac{2d^{\alpha+\beta+1}}{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}} \, dy \tag{3.20}$$

is finite, where

$$g(y) = \int_y^d \frac{x^{\beta+r}(d-x)^{\alpha+s}}{d^{\alpha+\beta+1}} \, dx,$$

an application of Hölder's inequality yields that

$$\begin{aligned} &\int_X |f(x)| x^r (d-x)^s \, dm(x) \\ &\leq \left(\int_0^d g(y)^2 \frac{2d^{\alpha+\beta+1}}{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}} \, dy \right)^{\frac{1}{2}} \left(\int_0^d |f'(y)|^2 \frac{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}}{2d^{\alpha+\beta+1}} \, dy \right)^{\frac{1}{2}} \end{aligned}$$

and the claim follows. To estimate (3.20) we observe that for $y \geq \frac{d}{2}$

$$g(y) \leq C_{\alpha,\beta,d,r} \int_y^d (d-x)^{\alpha+s} \, dx = C_{\alpha,\beta,d,r} \int_0^{d-y} x^{\alpha+s} \, dx = C_{\alpha,\beta,d,r,s} (d-y)^{\alpha+s+1}$$

by the assumption $s > -1-\alpha$. Using additionally $s > \frac{-2-\alpha}{2}$ we conclude that

$$\begin{aligned}\int_{\frac{d}{2}}^{d} g(y)^2 \frac{2d^{\alpha+\beta+1}}{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}}\,dy &\le C_{\alpha,\beta,\sigma,d,r,s} \int_{\frac{d}{2}}^{d} \frac{(d-y)^{2\alpha+2s+2}}{(d-y)^{\alpha+1}}\,dy \\ &= C_{\alpha,\beta,\sigma,d,r,s} < \infty. \end{aligned} \tag{3.21}$$

For $y < \frac{d}{2}$ we obtain instead the bound

$$g(y) \le g\left(\frac{d}{2}\right) + C_{\alpha,\beta,d,s} \int_y^{\frac{d}{2}} x^{\beta+r}\,dx \le C_{\alpha,\beta,d,r,s} \begin{cases} (1+y^{\beta+r+1}), & \beta+r \le -1, \\ (1+|\log(y)|), & \beta+r=-1, \\ 1, & \beta+r > -1. \end{cases} \tag{3.22}$$

Therefore, by Young's inequality and the assumptions $\beta \le -1$ and $r > \frac{-2-\beta}{2}$

$$\begin{aligned}\int_0^{\frac{d}{2}} g(y)^2 \frac{2d^{\alpha+\beta+1}}{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}}\,dy &\le C_{\alpha,\beta,\sigma,d,r,s} \begin{cases} \int_0^{\frac{d}{2}} \frac{1+y^{2\beta+2r+2}}{y^{\beta+1}}\,dy, & \beta+r \le -1, \\ \int_0^{\frac{d}{2}} \frac{1+\log(y)^2}{y^{\beta+1}}\,dy, & \beta+r=-1, \\ \int_0^{\frac{d}{2}} \frac{1}{y^{\beta+1}}\,dy, & \beta+r > -1. \end{cases} \\ &= C_{\alpha,\beta,\sigma,d,r,s} < \infty. \end{aligned} \tag{3.23}$$

Adding up (3.21) and (3.23) we conclude that (3.20) is indeed finite. The case (ii) can be treated analogously. For the last case (iii) we use that $\tilde{f}(d) = \tilde{f}(0) = 0$ and obtain that

$$\begin{aligned}&\int_X |f(x)| x^r (d-x)^s\,dm(x) \\ &\le \int_0^{\frac{d}{2}} \int_0^x |f'(y)|\,dy\, \frac{x^{\beta+r}(d-x)^{\alpha+s}}{d^{\alpha+\beta+1}}\,dx + \int_{\frac{d}{2}}^{d} \int_x^d |f'(y)|\,dy\, \frac{x^{\beta+r}(d-x)^{\alpha+s}}{d^{\alpha+\beta+1}}. \end{aligned} \tag{3.24}$$

As before, we estimate the first integral of the right-hand side by

$$\begin{aligned}&\int_0^{\frac{d}{2}} \int_y^{\frac{d}{2}} \frac{x^{\beta+r}(d-x)^{\alpha+s}}{d^{\alpha+\beta+1}}\,dx\, |f'(y)|\,dy \\ &\le \left(\int_0^{\frac{d}{2}} h(y)^2 \frac{2d^{\alpha+\beta+1}}{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}}\,dy\right)^{\frac{1}{2}} \left(\int_0^{\frac{d}{2}} |f'(y)|^2 \frac{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}}{2d^{\alpha+\beta+1}}\,dy\right)^{\frac{1}{2}}, \end{aligned}$$

where we set this time

$$h(y) = \int_y^{\frac{d}{2}} \frac{x^{\beta+r}(d-x)^{\alpha+s}}{d^{\alpha+\beta+1}}\,dx.$$

We estimate the function h as before by

$$h(y) \le C_{\alpha,\beta,d,s} \int_y^{\frac{d}{2}} x^{\beta+r}\,dx,$$

which is a term appearing in (3.22). In particular, continuing as in (3.23) we conclude finiteness of

$$\int_0^{\frac{d}{2}} h(y)^2 \frac{2d^{\alpha+\beta+1}}{\sigma^2 y^{\beta+1}(d-y)^{\alpha+1}}\,dy$$

due to $\beta \leq -1$ and $r > \frac{-2-\beta}{2}$. The second term of the right-hand side of (3.24) can be estimated analogously and therefore the claim follows. □

Lemma 3.16. *The following holds if either $\alpha \leq -1$, $\beta > -1$ or $\alpha > -1$, $\beta \leq -1$.*

$$\int_X f^2(x)\,dm(x) \leq \mathcal{E}(f,f)\cdot \begin{cases} \frac{2}{\sigma^2 \min\{|\alpha+1|,|\beta+1|\}^2}, & \alpha,\beta \neq -1,\\ \frac{8}{\sigma^2}\sup_{x\in(0,1)} x(1-x)\left[\frac{1}{\alpha+1}-\log(x)\right]^2, & \beta=-1,\\ \frac{8}{\sigma^2}\sup_{x\in(0,1)} x(1-x)\left[\frac{1}{\beta+1}-\log(x)\right]^2, & \alpha=-1. \end{cases} \tag{3.25}$$

Remark 3.17. It would be interesting to show an estimate (3.25) with a good constant also for the case $\alpha,\beta \leq -1$, which requires probably a different approach. While the calculation in (3.26) is still possible in this case, the hypergeometric function from (3.27) is not well-defined if $-(\alpha+2) \in \mathbb{N}$. Even if it is well-defined an application of Euler's integral representation (3.29) requires $\alpha+1>0$.

Proof. We assume that $\beta \leq -1$, $\alpha > -1$ and calculate using [4, Corollary 8.10]

$$\begin{aligned} \int_X f^2(x)\,dm(x) &\leq \int_0^d \int_0^x 2|f(y)f'(y)|\,dy\,m(x)\,dx \\ &= 2\int_0^d \int_y^d m(x)\,dx\,|f(y)f'(y)|\,dy \\ &\leq 2\left(\int_0^d f'(y)^2 cm(y)\,dy\right)^{\frac{1}{2}} \left(\int_0^d f^2(y)\left(\int_y^d m(x)\,dx\right)^2 cm(y)^{-1}\,dy\right)^{\frac{1}{2}} \\ &\leq 2\left(\int_0^d f'(y)^2 cm(y)\,dy\right)^{\frac{1}{2}} \left(\int_0^d f^2(y)m(y)\,dy\right)^{\frac{1}{2}} \\ &\quad\times \sup_{y\in(0,d)}\left[\left(\int_y^d m(x)\,dx\right)^2 cm(y)^{-1}m(y)^{-1}\right]^{\frac{1}{2}}. \end{aligned} \tag{3.26}$$

Therefore, we have

$$\int_X f^2(x)\,dm(x) \leq 4\sup_{y\in(0,d)}\left[\left(\int_y^d m(x)\,dx\right)^2 cm(y)^{-1}m(y)^{-1}\right]\mathcal{E}(f,f)$$

and it is left to estimate the supremum in front of $\mathcal{E}(f,f)$. If we define

$$M(y) = \frac{(1-\frac{y}{d})^{\alpha+1}}{(\alpha+1)}\,{}_2F_1\left(-\beta,\alpha+1;\alpha+2;1-\frac{y}{d}\right), \tag{3.27}$$

we see directly that $M(d)=0$ since $\alpha+1>0$. Moreover, term-wise differentiation yields

$$M'(y) = -\frac{(1-\frac{y}{d})^{\alpha}}{d}\,{}_2F_1\left(-\beta,\alpha+1;\alpha+1;1-\frac{y}{d}\right) \tag{3.28}$$

The appearing hypergeometric function is of the form

$$\sum_{n=0}^{\infty}\frac{(-\beta)_n}{n!}\left(1-\frac{y}{d}\right)^n = \left(\frac{y}{d}\right)^{\beta}$$

by Taylor expansion of the function $(1-x)^\beta$ in $x = 0$. Inserting this into (3.28) yields $M'(y) = -m(y)$ such that

$$M(y) = \int_y^d m(x)\,dx.$$

To estimate $M(y)$, we use Euler's transformation formula and Euler's integral representation, see [2, Theorem 2.2.1; Theorem 2.2.5] to obtain that

$$\begin{aligned} {}_2F_1\left(-\beta, \alpha+1; \alpha+2; 1-\frac{y}{d}\right) &= \left(\frac{y}{d}\right)^{\beta+1} {}_2F_1\left(\alpha+\beta+2, 1; \alpha+2; 1-\frac{y}{d}\right) \\ &= (\alpha+1)\left(\frac{y}{d}\right)^{\beta+1} \int_0^1 (1-t)^\alpha \left(1-\left(1-\frac{y}{d}\right)t\right)^{-(\alpha+\beta+2)} dt. \end{aligned} \tag{3.29}$$

If $\alpha+\beta+2 \geq 0$ the hypergeometric function

$${}_2F_1\left(\alpha+\beta+2, 1; \alpha+2; x\right)$$

is monotonously increasing in x and we can estimate it under the additional assumption $\beta < -1$ by its limit

$$\lim_{x \nearrow 1} {}_2F_1\left(\alpha+\beta+2, 1; \alpha+2; x\right) = \frac{\alpha+1}{-(\beta+1)},$$

due to (3.6). For $\beta = -1$ we have

$$\begin{aligned} {}_2F_1\left(-\beta, \alpha+1; \alpha+2; 1-\frac{y}{d}\right) &= \sum_{n=0}^\infty \left(1-\frac{y}{d}\right)^n \frac{(\alpha+1)_n}{(\alpha+2)_n} \\ &\leq 1 + (\alpha+1)\sum_{n=1}^\infty \left(1-\frac{y}{d}\right)^n \frac{1}{\alpha+n+1} \leq 1 + (\alpha+1)\sum_{n=1}^\infty \left(1-\frac{y}{d}\right)^n \frac{1}{n} \\ &= 1 - (\alpha+1)\log\left(\frac{y}{d}\right). \end{aligned}$$

If $\alpha+\beta+2 < 0$, which implies in particular $\beta < -1$, we can instead estimate

$$\int_0^1 (1-t)^\alpha \left(1-\left(1-\frac{y}{d}\right)t\right)^{-(\alpha+\beta+2)} dt \leq \int_0^1 (1-t)^\alpha\,dt = \frac{1}{\alpha+1}.$$

All in all we obtained the estimate

$${}_2F_1\left(-\beta, \alpha+1; \alpha+2; 1-\frac{y}{d}\right) \leq \begin{cases} \left(\frac{y}{d}\right)^{\beta+1}, & \alpha+\beta+2 < 0, \\ \frac{\alpha+1}{-(\beta+1)}\left(\frac{y}{d}\right)^{\beta+1}, & \alpha+\beta+2 \geq 0,\ \beta \neq -1, \\ 1-(\alpha+1)\log\left(\frac{y}{d}\right), & \beta = -1. \end{cases}$$

This implies

$$M(y) \leq \begin{cases} \frac{\left(\frac{y}{d}\right)^{\beta+1}\left(1-\frac{y}{d}\right)^{\alpha+1}}{\min\{a+1, -(\beta+1)\}}, & \beta \neq -1, \\ \left(1-\frac{y}{d}\right)^{\alpha+1}\left[\frac{1}{\alpha+1} - \log\left(\frac{y}{d}\right)\right], & \beta = -1, \end{cases}$$

which in the case $\beta \neq -1$ leads us to

$$\begin{aligned} \frac{M(y)^2}{cm(y)m(y)} &\leq \frac{2d^{2(\alpha+\beta+1)}\left(\frac{y}{d}\right)^{2\beta+2}\left(1-\frac{y}{d}\right)^{2\alpha+2}}{\sigma^2 y^{2\beta+1}(d-y)^{2\alpha+1}\min\{a+1, -(\beta+1)\}^2} \\ &= \frac{2\left(\frac{y}{d}\right)\left(1-\frac{y}{d}\right)}{\sigma^2\min\{a+1, -(\beta+1)\}^2} \leq \frac{1}{2\sigma^2\min\{a+1, -(\beta+1)\}^2}. \end{aligned}$$

If $\beta = -1$ we obtain instead

$$\frac{M(y)^2}{cm(y)m(y)} \leq \frac{2\left(\frac{y}{d}\right)\left(1-\frac{y}{d}\right)\left[\frac{1}{\alpha+1} - \log\left(\frac{y}{d}\right)\right]^2}{\sigma^2}$$
$$\leq \frac{2}{\sigma^2} \sup_{x\in(0,1)} x(1-x)\left[\frac{1}{\alpha+1} - \log(x)\right]^2.$$

The case $\alpha \leq -1$, $\beta > -1$ can be treated analogously. □

3.4. Spectral gap and asymptotics of the semigroup

We recall that $(T_t)_{t>0}$ is the Markovian, symmetric operator semigroup associated to $(L, D(L))$. In this last subsection we analyze its properties as $t \to \infty$. If $\alpha, \beta > -1$, the (rescaled) family of Jacobi polynomials is given by

$$Q_n(x) = {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; 1-\frac{x}{d}\right), \quad n \in \mathbb{N}_0.$$

This is a complete orthogonal basis of $L^2(X, dm)$ and moreover we have

$$GQ_n(x) = \frac{-\sigma^2 n(n+\alpha+\beta+1)}{2} Q_n(x)$$

for $x \in (0, d)$, see [2, Theorem 6.4.3; Theorem 6.5.2] and [2, Eq. (6.3.8)]. The following lemma shows, that $(Q_n)_{n\in\mathbb{N}}$ is even an orthogonal system of eigenfunctions of $(L, D(L))$ in this case.

Lemma 3.18. *The following holds.*

(i) If $\alpha, \beta > -1$, it holds $(G, C^\infty([0,d])) \subset (L, D(L))$.
(ii) If $\alpha > -1$, it holds $(G, C_c^\infty((0,d])) \subset (L, D(L))$.
(iii) If $\beta > -1$, it holds $(G, C_c^\infty([0,d))) \subset (L, D(L))$.

Proof. We provide a proof of (i), the remaining parts can be shown analogously. We assume $\alpha, \beta > -1$ and let $f \in C^\infty([0,d])$. Then we have $Gf \in C([0,d])$ and in particular $f, Gf \in L^2(X, dm)$ since dm has finite total mass. Similarly, we conclude $f \in D(\mathcal{E})$ and the claim follows if we can verify that

$$\mathcal{E}(f, g) = -(Gf, g)_{L^2(X,dm)} \tag{3.30}$$

for any $g \in D(\mathcal{E})$ by [26, Proposition 10.4 (ii)]. Due to (1.5) we have

$$(f'cm)'(x) = \left[c(x)f''(x) + (a-bx)f'(x)\right]m(x)$$

for all $x \in (0, d)$. The integration by parts formula (3.12) yields that

$$\int_\epsilon^{d-\epsilon} g'(x)f'(x)\,dcm(x) = -\int_\epsilon^{d-\epsilon} g(x)Gf(x)\,dm(x) + \left[\tilde{g}f'cm\right]_\epsilon^{d-\epsilon} \tag{3.31}$$

for $\epsilon > 0$. Observe that $f'(x)$ converges as $x \searrow 0$ and $x \nearrow d$. Furthermore, we get $\lim_{x\searrow 0} \tilde{g}cm(x) = \lim_{x\nearrow d} \tilde{g}cm(x) = 0$ by Proposition 3.14. Hence, taking $\epsilon \searrow 0$ in (3.31) yields (3.30). □

Using spectral theory, we obtain the explicit representation of the semigroup

$$T_t f = \sum_{n=0}^{\infty} \frac{e^{\frac{-\sigma^2 n(n+\alpha+\beta+1)t}{2}} (f, Q_n)_{L^2(X,dm)}}{\| Q_n \|^2_{L^2(X,dm)}} Q_n \tag{3.32}$$

for $f \in L^2(X, dm)$ by [26, Proposition 5.12] whenever $\alpha, \beta > -1$.

Corollary 3.19. *The following holds for every $t > 0$ and $f \in L^2(X, dm)$.*

(i) If $\alpha, \beta > -1$ we have

$$\left\| T_t f - \frac{\int_X f\, dm}{dm(X)} \mathbb{1} \right\|_{L^2(X,dm)} \leq e^{-bt} \left\| f - \frac{\int_X f\, dm}{dm(X)} \mathbb{1} \right\|_{L^2(X,dm)}. \tag{3.33}$$

(ii) If $\alpha, \beta > -1$ we have

$$\left\| T_t f - \frac{\int_X f\, dm}{dm(X)} \mathbb{1} \right\|_{D(\mathcal{E})} \to 0 \tag{3.34}$$

as $t \to \infty$. If additionally $f \in D(\mathcal{E})$, the estimate

$$\left\| T_t f - \frac{\int_X f\, dm}{dm(X)} \mathbb{1} \right\|_{D(\mathcal{E})} \leq e^{-bt} \sqrt{\mathcal{E}_1(f, f)} \tag{3.35}$$

holds as well.

(iii) If $\alpha \leq -1$, $\beta > -1$ or $\alpha > -1$, $\beta \leq -1$ we have

$$\| T_t f \|_{L^2(X,dm)} \leq e^{\frac{-t}{C(\alpha,\beta,\sigma)}} \| f \|_{L^2(X,dm)},$$

where $C(\alpha, \beta, \sigma)$ is the constant from the right-hand side of (3.25).

Proof. For (i) we observe that $Q_0 = \mathbb{1}$ and consequently (3.32) yields

$$\begin{aligned}
\left\| T_t f - \frac{\int_X f\, dm}{dm(X)} \mathbb{1} \right\|^2_{L^2(X,dm)} &= \sum_{n=1}^{\infty} e^{-\sigma^2 n(n+\alpha+\beta+1)t} \frac{(f, Q_n)^2_{L^2(X,dm)}}{\| Q_n \|^2_{L^2(X,dm)}} \\
&\leq e^{-2bt} \sum_{n=1}^{\infty} \frac{(f, Q_n)^2_{L^2(X,dm)}}{\| Q_n \|^2_{L^2(X,dm)}} \\
&= e^{-2bt} \left\| f - \frac{\int_X f\, dm}{dm(X)} \mathbb{1} \right\|^2_{L^2(X,dm)}.
\end{aligned}$$

In the inequality we used additionally that $\sigma^2(\alpha + \beta + 2) = 2b$ by (1.2). We conclude that (3.33) holds by taking the square-root. For (ii) we use [26, Eqs. (10.7), (10.8)] to calculate

$$\left\| T_t f - \frac{\int_X f\, dm}{dm(X)} \mathbb{1} \right\|^2_{D(\mathcal{E})} = \sum_{n=1}^{\infty} \left(1 + \frac{\sigma^2 n(n + \alpha + \beta + 1)}{2} \right) e^{-\sigma^2 n(n+\alpha+\beta+1)t} \frac{(f, Q_n)^2_{L^2(X,dm)}}{\| Q_n \|^2_{L^2(X,dm)}}. \tag{3.36}$$

By introducing

$$C = \sup_{x>0} \left(1 + \frac{x}{2} \right) e^{-x} < \infty$$

we can estimate

$$\left(1+\frac{\sigma^2 n(n+\alpha+\beta+1)}{2}\right) e^{-\sigma^2 n(n+\alpha+\beta+1)t} \le C$$

for all $t \ge 1$ and positive $n \in \mathbb{N}$. Since also

$$\left(1+\frac{\sigma^2 n(n+\alpha+\beta+1)}{2}\right) e^{-\sigma^2 n(n+\alpha+\beta+1)t} \frac{(f, Q_n)^2_{L^2(X,dm)}}{\| Q_n\|^2_{L^2(X,dm)}} \to 0$$

as $t \to \infty$ for every positive $n \in \mathbb{N}$, the dominated convergence theorem yields (3.34). The quantitative version (3.35) follows by continuing in (3.36) as in the proof of (i). Part (iii) is an immediate consequence from Lemma 3.16 together with [27, Theorem 1.1.1]. □

4. The corresponding process

In this section we analyze a dm-symmetric Hunt process, which is associated to the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ in the sense that its transition semigroup determines $(T_t)_{t>0}$ as in [14, Lemma 1.4.3]. To this end, we adjoin the cemetery Δ as Alexandroff point to the state space X and write $X_\Delta = X \cup \{\Delta\}$. Moreover, we let $\mathbf{M} = \left(\Omega, \mathfrak{A}, (Y_t)_{t\in[0,\infty]}, (P_x)_{x\in X_\Delta}\right)$ be a Hunt process associated to $(\mathcal{E}, D(\mathcal{E}))$, for details on Hunt processes see [14, Appendix A.2]. We denote its transition function on X by $(\rho_t)_{t>0}$ and its life time by ζ.

Remark 4.1. The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is regular by Theorem 3.2 and therefore there exists an associated Hunt process. A construction of it can be found in [14, Chapter 7].

4.1. Basic properties

We use the convention that $f(\Delta) = 0$ for any function f, which is a priori defined on X and we write $Y_{\zeta-}$ for the left limit of the process Y at ζ. We recall that a statement holds quasi-everywhere in X iff it holds for all $x \in X$ and a function is quasi-continuous iff it is continuous on X by Proposition 3.3. Moreover, as a consequence of [14, Theorem 4.2.1 (ii)], there are no (properly) exceptional sets with respect to $\mathbf{M}$, for a definition see [14, pp. 152–153].

Theorem 4.2. *The following holds.*

(i) The transition probability $\rho_t(x,\cdot)$ is absolutely continuous with respect to dm for every $t > 0$ and $x \in X$.
(ii) If $\alpha, \beta > -1$, we have $P_x(\{\zeta < \infty\}) = 0$ for every $x \in X$.
(iii) The path $[0,\zeta) \to X, t \mapsto Y_t$ is P_x-almost surely continuous for every $x \in X$.
(iv) It holds $P_x\left(\{Y_{\zeta-} \in X\} \cap \{\zeta < \infty\}\right) = 0$ for every $x \in X$.
(v) If $\alpha \le -1$ or $\beta \le -1$, we have

$$P_x\left(\left\{\lim_{t\nearrow\infty} Y_t = \Delta\right\} \cap \{\zeta = \infty\}\right) = P_x(\{\zeta = \infty\})$$

for every $x \in X$.
(vi) If $\alpha, \beta > -1$, then

$$\int_0^\infty 1_B(Y_s)\, ds = \infty$$

P_x-almost surely for every $x \in X$ and $B \in \mathfrak{B}(X)$ with $dm(B) > 0$.

(vii) If $\alpha \leq -1$ or $\beta \leq -1$ and $r, s \in \mathbb{R}$ according to Lemma 3.15, then

$$\int_0^\infty Y_t^r (d - Y_t)^s \, dt < \infty \tag{4.1}$$

P_x-almost surely for every $x \in X$.

(viii) If $\alpha, \beta > -1$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(Y_s)\, ds = \frac{1}{dm(X)} \int_X f \, dm$$

P_x-almost surely for every $x \in X$ and $\mathfrak{B}$-measurable $f \in L^1(X, dm)$.

(ix) If $\alpha, \beta > -1$, then

$$\lim_{t\to\infty} E_x(f(Y_t)) = \frac{1}{dm(X)} \int_X f \, dm \tag{4.2}$$

for every $x \in X$ and universally measurable $f \in L^2(X, dm)$. If moreover $f \in D(\mathcal{E})$, then

$$\left| E_x(f(Y_t)) - \frac{1}{dm(X)} \int_X f \, dm \right| \leq C_{\alpha,\beta,\sigma,d,x} \, e^{-bt} \sqrt{\mathcal{E}_1(f, f)}. \tag{4.3}$$

Remark 4.3. We note that due to the convention $f(\Delta) = 0$ the integrand in (4.1) vanishes as soon as $Y_t = \Delta$.

Proof. Part (i) follows from Theorem 3.13 and [14, Theorem 4.2.7]. By Proposition 3.5 and [14, Exercise 4.5.1] we obtain (ii). Since $(\mathcal{E}, D(\mathcal{E}))$ is strongly local as remarked in the introductory section, parts (iii) and (iv) follow from [14, Theorem 4.5.3]. If $\alpha \leq -1$ or $\beta \leq -1$, $(\mathcal{E}, D(\mathcal{E}))$ is transient by Theorem 3.2 and hence [5, Theorem 3.5.2] yields (v).

Part (vi) follows from [14, Lemma 4.8.1] together with irreducible recurrence of $(\mathcal{E}, D(\mathcal{E}))$ in this case by Theorem 3.2. For (vii) we choose strictly positive functions $(\varphi_n)_{n\in\mathbb{N}}$ on X with $\varphi_n(x) \nearrow 1$ for every $x \in X$ such that $f_n(x) = \varphi_n(x) f(x)$ is bounded on X and integrable with respect to dm for every $n \in \mathbb{N}$, where $f(x) = x^r (d - x)^s$. If we choose $C_{\alpha,\beta,\sigma,d,r,s}$ as in (3.19) the function

$$\frac{f_n}{C_{\alpha,\beta,\sigma,d,r,s}}$$

is a reference function of $(\mathcal{E}, D(\mathcal{E}))$ for every $n \in \mathbb{N}$, see [14, p. 40] for a definition. By [14, Theorem 1.5.1] it holds

$$\int_X f_n \left(\lim_{\lambda \searrow 0} R_\lambda f_n \right) dm \leq C^2_{\alpha,\beta,\sigma,d,r,s},$$

where $(R_\lambda)_{\lambda>0}$ denotes the resolvent associated to $(T_t)_{t>0}$, for details see [14, Section 1.3], and the limit is attained almost everywhere. We have also

$$R_\lambda f_n(x) = E_x \left(\int_0^\infty e^{-\lambda t} f_n(Y_s)\, ds \right) \to E_x \left(\int_0^\infty f_n(Y_s)\, ds \right)$$

as $\lambda \searrow 0$ for almost every $x \in X$ by [14, Theorem 4.2.3] and monotone convergence. Employing again the monotone convergence theorem we conclude that

$$\begin{aligned} &C^2_{\alpha,\beta,\sigma,d,r,s} \\ &\geq \int_X f_n(x) E_x\left(\int_0^\infty f_n(Y_s)\,ds\right) dm(x) \to \int_X f(x) E_x\left(\int_0^\infty f(Y_s)\,ds\right) dm(x). \end{aligned}$$

Therefore, transience of $(\mathcal{E}, D(\mathcal{E}))$ and [14, Theorem 4.2.6] yield that

$$E_x\left(\int_0^\infty f(Y_s)\,ds\right)$$

is quasi-continuous in x and in particular finite for every $x \in X$. We conclude that the integrand

$$\int_0^\infty f(Y_s)\,ds$$

is finite P_x-almost surely. Part (viii) is a consequence of irreducible recurrence of $(\mathcal{E}, D(\mathcal{E}))$ in this case together with [14, Theorem 4.7.3 (iii)]. Finally, for (ix) let $f \in L^2(X, dm)$ be universally measurable and $x \in X$, then Corollary 3.19 (ii) and Proposition 3.1 yield that

$$\widetilde{T_t f}(x) \to \frac{1}{dm(X)} \int_X f\,dm$$

as $t \to \infty$. Since

$$E_x(f(Y_t)) = \widetilde{T_t f}(x) \tag{4.4}$$

by [14, Theorem 4.2.3 (i)], (4.2) follows. If $f \in D(\mathcal{E})$, (3.35) and Proposition 3.1 yield that

$$\left|\widetilde{T_t f}(x) - \frac{1}{dm(X)} \int_X f\,dm\right| \leq C_{\alpha,\beta,\sigma,d,x}\, e^{-bt} \sqrt{\mathcal{E}_1(f, f)},$$

where the constant denotes the operator norm of the point evaluation δ_x on $D(\mathcal{E})$. The estimate (4.3) follows by (4.4). □

By parts (iv) and (v) of the preceding Theorem, the process Y approaches Δ as $t \nearrow \zeta$ in the transient case $\alpha \leq -1$ or $\beta \leq -1$ P_x-almost surely for every $x \in X$. Considering X as a subset of $[0, d]$, this means that Y converges to an element of $[0, d] \setminus X$ due to the continuity of paths (iii). The following corollary gives a stronger statement.

Corollary 4.4. *Let $\alpha \leq -1$ or $\beta \leq -1$, then we have*

$$P_x\left(\left\{\lim_{t \nearrow \zeta} Y_t \in \tilde{X} \setminus X\right\}\right) = 1$$

for every $x \in X$.

Proof. Since $\tilde{X} = [0, d]$ for $\alpha, \beta < 0$, this case is treated by the preceding considerations. For the remaining cases $\alpha \geq 0, \beta \leq -1$ and $\alpha \leq -1, \beta \geq 0$ [5, Eq. (3.5.14)] yields the claim. □

For a nearly Borel set $B \subset X$, see [14, p. 392] for the definition, we define the λ-order hitting distribution

$$H^\lambda_B(x, E) = E_x(e^{-\lambda \tau_B} \mathbb{1}_E(X_{\tau_B})) \tag{4.5}$$

for positive $\lambda > 0$ and universally measurable subsets E of X. The appearing random time τ_B is defined as

$$\tau_B = \inf\{t > 0 | X_t \in B\}$$

and is a stopping time with respect to the minimum completed admissible filtration of $\mathbf{M}$, see [14, Theorem A.2.3]. Using the relation between orthogonal projections on the spaces $\mathcal{H}^{\lambda}_{\{d\}}$ and $\mathcal{H}^{\lambda}_{\{0\}}$ from Section 3.1 and the λ-order hitting distribution of the corresponding sets $\{d\}$ and $\{0\}$ given by [14, Theorem 4.3.1], we calculate the hitting probabilities of the boundary points.

Theorem 4.5. *The following holds.*

(i) If $-1 < \alpha < 0$, we have for every $x \in X$ that

$$P_x\left(\{\tau_{\{d\}} < \infty\}\right) = \begin{cases} 1, & \beta > -1, \\ 1, & \beta \le -1,\ x = d, \\ \frac{\Gamma(-\alpha-\beta)}{\Gamma(1-\beta)\Gamma(-\alpha)} \left(\frac{x}{d}\right)^{-\beta} {}_2F_1\left(\alpha+1, -\beta; 1-\beta; \frac{x}{d}\right), & \beta \le -1,\ x < d. \end{cases}$$

(ii) If $-1 < \beta < 0$, we have for every $x \in X$ that

$$P_x\left(\{\tau_{\{0\}} < \infty\}\right) = \begin{cases} 1, & \alpha > -1, \\ 1, & \alpha \le -1,\ x = 0, \\ \frac{\Gamma(-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(-\beta)} \left(1-\frac{x}{d}\right)^{-\alpha} {}_2F_1\left(\beta+1, -\alpha; 1-\alpha; 1-\frac{x}{d}\right), & \alpha \le -1,\ x > 0. \end{cases}$$

Proof. We assume that $-1 < \alpha < 0$ and additionally that $\beta > -1$. Then the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is irreducible recurrent by Theorem 3.2, such that the claim follows by [14, Theorem 4.7.1 (iii)]. Next, we assume $\beta \le -1$ and let $f \in D(\mathcal{E})$ with $\tilde{f}(d) = 1$. In the following, we first identify $P_{\mathcal{H}^{\lambda}_{\{d\}}} f$, the orthogonal projection of f onto $\mathcal{H}^{\lambda}_{\{d\}}$ in $(D(\mathcal{E}), \mathcal{E}_\lambda)$. Due to [14, Theorem 4.3.1] the function

$$H^{\lambda}_{\{d\}} \tilde{f}(x) = E_x(e^{-\lambda \tau_{\{d\}}} \tilde{f}(X_{\tau_{\{d\}}})) \tag{4.6}$$

exists for every $x \in X$ and defines a continuous version of $P_{\mathcal{H}^{\lambda}_{\{d\}}} f$. Knowing (4.6) allows us then by $\tilde{f}(X_{\tau_{\{d\}}}) = 1$ to calculate the limit

$$H^{\lambda}_{\{d\}} \tilde{f}(x) \to P_x(\{\tau_{\{d\}} < \infty\}) \tag{4.7}$$

as $\lambda \searrow 0$. Take $\lambda > 0$, then the space $\mathcal{H}^{\lambda}_{\{d\}}$ coincides with $\operatorname{span}\{\xi_\lambda\}$ by Theorem 3.6. Since $f - P_{\mathcal{H}^{\lambda}_{\{d\}}} f$ is an element of (3.13) we have $\tilde{f}(d) - \widetilde{P_{\mathcal{H}^{\lambda}_{\{d\}}} f}(d) = 0$. Lemma 3.8 (ii) yields

$$\lim_{x \nearrow d} \xi_\lambda(x) = \frac{\Gamma(1-\beta)\Gamma(-\alpha)}{\Gamma\left(\frac{-\alpha-\beta+1}{2} + \gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2} - \gamma\right)} > 0,$$

such that

$$P_{\mathcal{H}^{\lambda}_{B}} f = \frac{\Gamma\left(\frac{-\alpha-\beta+1}{2} + \gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2} - \gamma\right)}{\Gamma(1-\beta)\Gamma(-\alpha)} \xi_\lambda.$$

It follows that $H^{\lambda}_{\{d\}}\tilde{f}(x)$ equals

$$\begin{cases} \frac{\Gamma\left(\frac{-\alpha-\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2}-\gamma\right)}{\Gamma(1-\beta)\Gamma(-\alpha)} \left(\frac{x}{d}\right)^{-\beta} {}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma, \frac{\alpha-\beta+1}{2}-\gamma; 1-\beta; \frac{x}{d}\right), & x < d, \\ 1, & x = d. \end{cases}$$

If we let $\lambda \searrow 0$ to calculate (4.7) the corresponding parameter

$$\gamma = \sqrt{\left(\frac{\alpha+\beta+1}{2}\right)^2 - \frac{2\lambda}{\sigma^2}} \tag{4.8}$$

tends towards $\left|\frac{\alpha+\beta+1}{2}\right|$. Analyticity of the gamma function and ${}_2F_1$ in its first two parameters, see [2, p. 65], yields that

$$P_x(\{\tau_{\{d\}} < \infty\}) = \begin{cases} \frac{\Gamma(-\alpha-\beta)}{\Gamma(1-\beta)\Gamma(-\alpha)} \left(\frac{x}{d}\right)^{-\beta} {}_2F_1\left(\alpha+1, -\beta; 1-\beta; \frac{x}{d}\right), & x < d, \\ 1, & x = d, \end{cases}$$

for every $x \in X$. This finishes the proof of (i), part (ii) can be shown analogously. □

4.2. Maximal local solutions to the Jacobi SDE

In this subsection we draw a connection between **M** and maximal local solutions to the Jacobi stochastic differential equation (1.1). As we show, this is quite immediate whenever $\alpha, \beta > -1$, but requires some technical work in any other case. We denote the minimum completed admissible filtration of **M** by $\mathfrak{F}$, its last element by $\mathfrak{F}_\infty$ and the completion of $\mathfrak{F}$ with respect to P_x by $\mathfrak{F}^{P_x}$, for details see [14, p. 386]. Then for every $f \in D(L)$ the process

$$\tilde{f}(Y_t) - \tilde{f}(Y_0) - \int_0^t Lf(Y_s)\,ds \tag{4.9}$$

is a martingale under P_x for every $x \in X$, see [1, Remark 3.2]. We point out that the convention $Lf(\Delta) = 0$ is used again. For the remainder of this subsection we fix an $x \in X$ and choose $\Lambda \in \mathfrak{F}_\infty$ as a set of full measure under P_x such that

(i) $Y_\cdot(\omega) : [0, \zeta(\omega)) \to X$ is continuous,
(ii) $Y_{\zeta-}(\omega) = \Delta$ if $\zeta(\omega) < \infty$ and
(iii) $\zeta(\omega) > 0$

for every $\omega \in \Lambda$. Such a set exists due to Theorem 4.2 (iii), (iv) and the fact that **M** is a normal Markov process. We recall the functions μ, ν introduced in Section 2 and define $u(t)$ and $v(t)$ as the solutions to the differential equation $u'(t) = \mu(u(t))$ with initial value $u(0) = d$ and $v(0) = 0$, respectively.

Remark 4.6. Under the assumption $\alpha \le -1$ we have $\mu(d) = a - bd \ge 0$ and therefore $u(t) \ge d$ for all $t \ge 0$. Indeed, if $\mu(d) = 0$, $u(t)$ is constant and equal to d. If $\mu(d) > 0$, $u'(0) > 0$ and hence $u(t) > d$ for $t \in (0, \epsilon)$ for some $\epsilon > 0$. Hence, if $u(t) = d$ again for some $t > 0$, there is a smallest strictly positive time t_* with $u(t_*) = d$. But then $u'(t_*) \le 0$ contradicts

$$u'(t_*) = \mu(u(t_*)) = \mu(d) > 0.$$

Similarly, $v(t) \le 0$ for $t \ge 0$ whenever $\beta \le -1$.

We introduce the modified process

$$Z_t(\omega) = \begin{cases} Y_t(\omega), & \omega \in \Lambda, t < \zeta(\omega), \\ u(t-\zeta(\omega)), & \omega \in \Lambda, t \geq \zeta(\omega), \lim_{s\nearrow\zeta} Y_s(\omega) = d, \\ v(t-\zeta(\omega)), & \omega \in \Lambda, t \geq \zeta(\omega), \lim_{s\nearrow\zeta} Y_s(\omega) = 0, \\ 0, & \Omega \setminus \Lambda \end{cases} \tag{4.10}$$

and the modified life time

$$\tilde{\zeta}(\omega) = \begin{cases} \infty, & \omega \in \Lambda, \alpha = -1, \zeta < \infty, \lim_{t\nearrow\zeta} Y_t(\omega) = d, \\ \infty, & \omega \in \Lambda, \beta = -1, \zeta < \infty, \lim_{t\nearrow\zeta} Y_t(\omega) = 0, \\ 0, & \omega \in \Omega \setminus \Lambda, \\ \zeta(\omega), & \text{else}. \end{cases} \tag{4.11}$$

Notice that we interpret here the limit $\lim_{t\nearrow\zeta} Y_t$ as an element of $\mathbb{R}$ instead of the topological space X_Δ, such that we can distinguish between the boundary points at which Y dies. We make some technical observations.

Lemma 4.7. *The following holds.*

(i) The process Z has continuous paths, is $\mathfrak{F}^{P_x}$-adapted and satisfies $Z_{t\wedge\tilde{\zeta}} \in [0,d]$ for all $t \geq 0$.
(ii) $\tilde{\zeta}$ is a stopping time with respect to $\mathfrak{F}^{P_x}$.
(iii) It holds P_x-almost surely $\tilde{\zeta} = \inf\{t \geq 0 | Z_t \notin [0,d]\}$.

Proof. The continuity assertion and the claim that $Z_{t\wedge\tilde{\zeta}} \in [0,d]$ of part (i) follow by the definition of Λ, Z and $\tilde{\zeta}$ together with the observation, that $u = d$ ($v = 0$) is constant whenever $\alpha = -1$ ($\beta = -1$). The claim regarding adaptedness in (i) reduces to verifying that

$$\{u(t-\zeta) \in B\} \cap \{0 < \zeta \leq t\} \cap \{\lim_{s\nearrow\zeta} Y_s = d\} \in \mathfrak{F}_t^{P_x} \tag{4.12}$$

and

$$\{v(t-\zeta) \in B\} \cap \{0 < \zeta \leq t\} \cap \{\lim_{s\nearrow\zeta} Y_s = 0\} \in \mathfrak{F}_t^{P_x}$$

for every $B \in \mathfrak{B}(\mathbb{R})$. We observe that

$$\{u(t - t\wedge\zeta) \in B\} \in \mathfrak{F}_t^{P_x}, \tag{4.13}$$

since $t\wedge\zeta$ is $\mathfrak{F}_t^{P_x}$-measurable. Moreover, we have

$$\{0 < \zeta \leq t\} \cap \left(\bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} \left\{ Y_{t\wedge(\zeta\vee\frac{1}{n}-\frac{1}{n})} > \frac{d}{2} \right\} \right) \in \mathfrak{F}_t^{P_x}, \tag{4.14}$$

as a consequence of $Y_{t\wedge(\zeta\vee\frac{1}{n}-\frac{1}{n})}$ being $\mathfrak{F}_{t+\frac{1}{n}}^{P_x}$-measurable and the right-continuity of $\mathfrak{F}^{P_x}$, for the latter see [14, Lemma A.2.2]. Since the left-hand side of (4.12) is the intersection of (4.13) and (4.14), (4.12) follows. Similarly, (ii) reduces to showing that

$$\{0 < \zeta \leq t\} \cap \left\{ \lim_{s\nearrow\zeta} Y_s = d \right\} \in \mathfrak{F}_t^{P_x} \tag{4.15}$$

and

$$\{0 < \zeta \leq t\} \cap \left\{ \lim_{s\nearrow\zeta} Y_s = 0 \right\} \in \mathfrak{F}_t^{P_x}.$$

Both statements can be verified by rewriting the events analogously to (4.14). To also verify (iii) we notice that as a consequence of part (i) we have

$$P_x(\{\tilde{\zeta} = \infty\} \cap \{\inf\{t \geq 0 | Z_t \notin [0,d]\} = \infty\}) = P_x(\{\tilde{\zeta} = \infty\}).$$

Hence, it is sufficient to verify that $\tilde{\zeta} = \inf\{t \geq 0 | Z_t \notin [0,d]\}$ P_x-almost surely on the set $\{\tilde{\zeta} < \infty\}$. Therefore, we only have to consider the cases $\alpha < -1$ or $\beta < -1$ by Theorem 4.2 (iii) and the definition of $\tilde{\zeta}$. We assume that $\alpha \geq -1$ and $\beta < -1$ and let $\omega \in \{\tilde{\zeta} < \infty\} \cap \Lambda$. It follows that $\lim_{s \nearrow \zeta} Y_s(\omega) = 0$. Indeed for $\alpha > -1$ this follows by the choice of Λ and for $\alpha = -1$ by definition of $\tilde{\zeta}$ and the assumption $\omega \in \{\tilde{\zeta} < \infty\}$. Since $v(t) < 0$ for $t > 0$ we conclude that

$$\inf\{t \geq 0 | Z_t(\omega) \notin [0,d]\} = \zeta(\omega) = \tilde{\zeta}(\omega). \tag{4.16}$$

It follows that

$$P_x(\{\tilde{\zeta} < \infty\} \cap \{\inf\{t \geq 0 | Z_t \notin [0,d]\} = \tilde{\zeta}\}) = P_x(\{\tilde{\zeta} < \infty\}) \tag{4.17}$$

as desired. The case $\alpha < -1$, $\beta \geq -1$ can be treated analogously. Lastly, we assume that $\alpha, \beta < -1$. In this case we additionally have that $u(t) > 0$ for $t > 0$ by Remark 4.6. Hence, for each $\omega \in \{\tilde{\zeta} < \infty\} \cap \Lambda$ we conclude again (4.16), which yields (4.17). This finishes the proof. □

In the final theorem of this subsection we prove that the tuple $(Z, \tilde{\zeta})$ is a maximal local solution to (2.2) with initial value x. The main ingredients are Theorem 2.5 and the martingale problem characterization of the auxiliary equation (2.2). To apply the latter we let W' be a Brownian motion on a probability space $(\Omega', \mathfrak{A}', P')$ with respect to a right-continuous filtration $\mathfrak{F}'$. We define the enriched probability space by $\Omega^\dagger = \Omega \times \Omega'$, $P^\dagger = P_\nu \times P'$ and the completed σ-field $\mathfrak{A}^\dagger = \overline{\mathfrak{F}_\infty^{P_x} \times \mathfrak{A}'}^{P^\dagger}$. We equip it with the filtration $\mathfrak{F}^\dagger$, which we define as the $P^\dagger$-completion of $\mathfrak{F}_t^{P_x} \times \mathfrak{F}_t'$ in $\mathfrak{A}^\dagger$. In particular, $\mathfrak{F}^\dagger$ satisfies the usual conditions by [17, Lemma 6.8]. We note that any random variable defined on Ω or Ω' extends canonically to the enriched space. Moreover, we denote the generator of (2.2) by G, i.e. we set

$$Gf(x) = \frac{\nu^2(x)}{2} f''(x) + \mu(x) f'(x) \tag{4.18}$$

for $f \in C^2(\mathbb{R})$. Note that this is consistent with (1.3), which was introduced for functions on $[0,d]$.

Theorem 4.8. *There exists a Brownian motion $W^\dagger$ on $(\Omega^\dagger, \mathfrak{A}^\dagger, P^\dagger)$ such that Z is a solution to (2.2) with initial value x. In particular, $(Z, \tilde{\zeta})$ is a maximal local solution to (1.1).*

Proof. If we prove the first part of the statement, the second one follows by Lemma 4.7 (iii) together with Theorem 2.5 (ii). For the first part it is again sufficient to show that Z solves the G-martingale problem, i.e. that

$$f(Z_t) - f(Z_0) - \int_0^t Gf(Z_s) ds \tag{4.19}$$

is for every $f \in C_c^\infty(\mathbb{R})$ a martingale with respect to $\mathfrak{F}^{P_x}$, due to [10, Theorem 3.3, p.293]. To verify this we distinguish different cases. We assume first that $\alpha, \beta > -1$. In this case $\zeta = \infty$ P_x-almost surely by Theorem 4.2 (iii) and therefore Y and Z are indistinguishable.

Since (4.9) is an $\mathfrak{F}^{P_x}$-martingale it follows that indeed Z solves the G-martingale problem by Lemma 3.18 (i). Secondly, we assume that $\alpha > -1$ and $\beta \leq -1$ such that $(L, D(L)) \supset (G, C_c^\infty((0,d]))$ by Lemma 3.18 (ii). Let $f \in C_c^\infty(\mathbb{R})$, then we decompose (4.19) into

$$f(Z_{\zeta\wedge t}) - f(Z_0) - \int_0^{\zeta\wedge t} Gf(Z_s)\,ds + f(Z_t) - f(Z_{\zeta\wedge t}) - \int_{\zeta\wedge t}^t Gf(Z_s)\,ds. \tag{4.20}$$

By definition of Z we can replace the last three terms of (4.20) by

$$\mathbb{1}_{\{t>\zeta\}}\left[f(v(t-\zeta)) - f(v(0)) - \int_0^{t-\zeta} \mu(v(s))f'(v(s))\,ds\right],$$

because $\nu = 0$ on $(-\infty, 0]$. Since $v'(s) = \mu(v(s))$ an application of the fundamental theorem of calculus shows that the above term vanishes. To also treat the remaining part of (4.20) we introduce

$$\zeta_n = \inf\left\{t \geq 0 \middle| Y_t \leq \frac{1}{n}\right\},$$

which is a stopping time for every n by [14, Theorem A.2.3]. It follows then that $\lim_{n\to\infty}\zeta_n = \zeta$ P_x-almost surely by Corollary 4.4. Using additionally Lemma 4.7 (i) we conclude that

$$M_t^{(n)} = f(Y_{\zeta_n\wedge t}) - f(Y_0) - \int_0^{\zeta_n\wedge t} Gf(Y_s)\,ds = f(Z_{\zeta_n\wedge t}) - f(Z_0) - \int_0^{\zeta_n\wedge t} Gf(Z_s)\,ds \tag{4.21}$$

converges P_x-almost surely to the first three terms of (4.20) as $n \to \infty$. We can replace f with a function $g \in C_c^\infty((0,d])$, which coincides with f on $[-\frac{1}{n}, d]$, without changing the value of (4.21). It follows by Lemma 3.18 (ii) that $M^{(n)}$ is a stopped version of (4.9) and therefore an $\mathfrak{F}^{P_x}$-martingale for every $n \in \mathbb{N}$. Since moreover $M_t^{(n)}$ is uniformly bounded for fixed t it follows that its limit is an $\mathfrak{F}^{P_x}$-martingale as well. We conclude that Z indeed solves the G-martingale problem. The cases $\alpha \leq -1$ and $\beta > -1$ as well as $\alpha, \beta \leq -1$ can be treated analogously, the latter by using the approximating sequence of stopping times

$$\zeta_n = \inf\left\{t \geq 0 \middle| Y_t \leq \frac{1}{n} \vee Y_t \geq d - \frac{1}{n}\right\}$$

instead. □

Remark 4.9. By definition, the process Z and its life time $\tilde{\zeta}$ can be constructed from Y and ζ. Considering also uniqueness in law of maximal local solutions as shown in Corollary 2.6 one obtains properties of a general maximal local solution to (1.1) by transferring the properties of **M**.

4.3. Minimal local solutions to the Jacobi SDE

In this last subsection we consider the restriction of the Hunt process **M** to the open interval $\hat{X} = (0, d)$. The restricted process is obtained by defining the stopping time

$$\dot{\tau}_{X\setminus\hat{X}} = \inf\{t \geq 0 | Y_t \in X \setminus \hat{X}\}$$

and stopping the process at this time, i.e. by setting

$$\hat{Y}_t(\omega) = \begin{cases} Y_t(\omega), & t < \dot{\tau}_{X\setminus\hat{X}}(\omega), \\ \Delta, & t \geq \dot{\tau}_{X\setminus\hat{X}}(\omega). \end{cases}$$

Then $\hat{\mathbf{M}} = (\Omega, \mathfrak{A}, (P_x)_{x\in\hat{X}_\Delta}, (\hat{Y}_t)_{t\geq 0})$ is a Hunt process again, see [14, Theorem A.2.10]. We note that, as shown in the proof of the cited theorem, $\hat{\mathbf{M}}$ is quasi-left continuous and a strong Markov process with respect to the minimum completed admissible filtration $\mathfrak{F}$ of $\mathbf{M}$. Its life time is given by $\hat{\zeta} = \zeta \wedge \dot{\tau}_{X\setminus\hat{X}}$ and its transition function by

$$\hat{\rho}_t(x, B) = P_x(\{X_t \in B\} \cap \{t < \dot{\tau}_{X\setminus\hat{X}}\}) \tag{4.22}$$

for $B \in \mathfrak{B}(\hat{X})$. We restrict the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ as well by replacing its domain by

$$\{f \in D(\mathcal{E}) | \tilde{f} = 0 \text{ on } X \setminus \hat{X}\}.$$

We denote the restricted form by $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ and obtain as a consequence of Theorem 3.4 that $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}})) = (\mathcal{E}, \mathcal{F})$. In particular, it is the form corresponding to the Friedrichs extension of the operator $(G, C_c^\infty((0,d)))$. By [14, Theorem 4.4.3 (i)] $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ is a regular Dirichlet form on $L^2(\hat{X}, dm)$. Since $\hat{X}$ is an open subset of X [14, Theorem 4.4.2] yields that $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ is associated to $\hat{\mathbf{M}}$. Since $\hat{Y} = Y$ unless $-1 < \alpha < 0$ or $-1 < \beta < 0$, we obtain the following proposition trivially.

Proposition 4.10. *We assume that neither $-1 < \alpha < 0$ nor $-1 < \beta < 0$. Then the properties stated in Theorem 4.2 hold also for $\hat{\mathbf{M}}$.*

In any other case, we conclude the following.

Theorem 4.11. *Let $-1 < \alpha < 0$ or $-1 < \beta < 0$, then the following holds.*

(i) The transition probability $\hat{\rho}_t(x, \cdot)$ is absolutely continuous with respect to dm for every $t > 0$ and $x \in \hat{X}$.
(ii) The path $[0, \hat{\zeta}) \to \hat{X}, t \mapsto \hat{Y}_t$ is P_x-almost surely continuous for every $x \in \hat{X}$.
(iii) It holds $P_x\left(\left\{\hat{Y}_{\hat{\zeta}-} \in \hat{X}\right\} \cap \left\{\hat{\zeta} < \infty\right\}\right) = 0$ for every $x \in \hat{X}$.
(iv) We have for every $x \in \hat{X}$ that

$$P_x(\{\hat{\zeta} < \infty\}) \geq \begin{cases} 1, & \alpha, \beta > -1, \\ \frac{\Gamma(-\alpha-\beta)}{\Gamma(1-\beta)\Gamma(-\alpha)} \left(\frac{x}{d}\right)^{-\beta} {}_2F_1\left(\alpha+1, -\beta; 1-\beta; \frac{x}{d}\right), & \alpha > -1, \beta \leq -1, \\ \frac{\Gamma(-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(-\beta)} \left(1-\frac{x}{d}\right)^{-\alpha} {}_2F_1\left(\beta+1, -\alpha; 1-\alpha; 1-\frac{x}{d}\right), & \beta > -1, \alpha \leq -1. \end{cases}$$

Proof. Part (i) follows from (4.22) together with Theorem 4.2 (i). Part (ii) is a direct consequence of Theorem 4.2 (iii) and the definitions of $\hat{Y}$ and $\hat{\zeta}$. We decompose the probability from (iii) into the parts

$$P_x\left(\left\{\hat{Y}_{\hat{\zeta}-} \in \hat{X}\right\} \cap \left\{\hat{\zeta} = \zeta < \infty\right\}\right) + P_x\left(\left\{\hat{Y}_{\hat{\zeta}-} \in \hat{X}\right\} \cap \left\{\hat{\zeta} < \zeta\right\}\right).$$

The first term vanishes by Theorem 4.2 (iv). The second one vanishes as a consequence of part (iii) of the same Theorem. Part (iv) follows from Theorem 4.5 and the fact that $\hat{\zeta} \leq \tau_{\{d\}} \wedge \tau_{\{0\}}$. □

Finally, we prove a statement similar to Theorem 4.8 relating the restricted process $\hat{\mathbf{M}}$ to minimal solutions to the Jacobi stochastic differential equations. Therefore let $x \in \hat{X}$ and $\Lambda \in \mathfrak{F}_\infty$ and Z be the corresponding set and process from the preceding subsection.

Lemma 4.12. *It holds the following.*

(i) We have P_x-almost surely for every $t \geq 0$ that

$$Z_{t\wedge\hat{\zeta}} = \begin{cases} \hat{Y}_t, & t < \hat{\zeta}, \\ d, & t \geq \hat{\zeta}, \lim_{t\nearrow\hat{\zeta}} \hat{Y}_t = d, \\ 0, & t \geq \hat{\zeta}, \lim_{t\nearrow\hat{\zeta}} \hat{Y}_t = 0. \end{cases} \tag{4.23}$$

(ii) We have that $\hat{\zeta} = \inf\{t \geq 0 | Z_t \in \{0, d\}\}$ P_x-almost surely.

Proof. Due to $u(0), v(0) \in \{0, d\}$, part (ii) follows by the definition of $\hat{\zeta}$ and Z. Since we have

$$\mathbb{1}_{\{t<\hat{\zeta}\}} Z_t = \mathbb{1}_{\{t<\hat{\zeta}\}} Y_t = \mathbb{1}_{\{t<\hat{\zeta}\}} \hat{Y}_t$$

P_x-almost surely and Z has continuous paths, the right-hand side of (4.23) is nothing but the process Z stopped as it hits the set $\{0, d\}$. Therefore, (i) is a consequence of (ii). □

Finally, we consider the enriched probability space $(\Omega^\dagger, \mathfrak{A}^\dagger, P^\dagger)$ with the filtration $\mathfrak{F}^\dagger$.

Corollary 4.13. $(Z, \hat{\zeta})$ *is a minimal local solution to (1.1) with initial value x.*

Proof. By Theorem 4.8 the process Z is a solution to (2.2) with initial value x on $(\Omega^\dagger, \mathfrak{A}^\dagger, P^\dagger)$ with respect to a Brownian motion $W^\dagger$. The claim follows by Lemma 4.12 (ii) and Theorem 2.5 (i). □

Remark 4.14. By (4.23) and the uniqueness in law statement from Corollary 2.6 the properties of a general minimal local solution to (1.1) can be derived from the properties of $\hat{\mathbf{M}}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

A.1. A localized Yamada–Watanabe condition

We provide a localized version of the Yamada–Watanabe condition. The proof translates verbatim from the classical setting, see [17, Theorem 20.3], and is contained for convenience of the reader.

Lemma A.1. *Let $\mu, \nu: \mathbb{R} \to \mathbb{R}$ be mappings such that μ is Lipschitz and ν is $\frac{1}{2}$-Hölder continuous. Moreover, let $(\Omega, \mathfrak{A}, P)$ be a probability space equipped with a filtration satisfying the usual conditions and W a Brownian motion. If there are two adapted processes $Y^{(1)}$ and $Y^{(2)}$ and a stopping time ζ such that*

(i) $Y_0^{(1)} = Y_0^{(2)}$,
(ii) $Y_{\cdot\wedge\zeta}^{(i)}$ *has continuous paths and*
(iii) $Y_{t\wedge\zeta}^{(i)} = \int_0^{t\wedge\zeta} \mu(Y^{(i)})ds + \int_0^{t\wedge\zeta} \nu(Y^{(i)})dW_s$ *for all* $t \geq 0$, $i \in \{1, 2\}$,

then $Y_{t\wedge\zeta}^{(1)} = Y_{t\wedge\zeta}^{(2)}$ *for all* $t \geq 0$.

Proof. We define the process

$$D_t = Y_{t\wedge\zeta}^{(1)} - Y_{t\wedge\zeta}^{(2)} = \int_0^{t\wedge\zeta} \mu(Y_s^{(1)}) - \mu(Y_s^{(2)})\,ds + \int_0^{t\wedge\zeta} \nu(Y_s^{(1)}) - \nu(Y_s^{(2)})\,dW_s.$$

It suffices to show that $D = 0$. By a localization argument we can assume that D is uniformly bounded. Let $(L_t^x)_{t\geq 0, x\in\mathbb{R}}$ be a cadlag version of the local time of D, for existence of such a version see [17, Theorem 19.4]. Then [17, Theorem 19.5] yields for any $t \geq 0$ that

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)L_t^x\,dx &= \int_0^t f(D_s)\,d\,\langle D\rangle_s \\ &= \int_0^{t\wedge\zeta} f(Y_s^{(1)} - Y_s^{(2)})(\nu(Y_s^{(1)}) - \nu(Y_s^{(2)}))^2\,ds \leq C^2 t,\end{aligned}$$

where we choose $f(x) = \frac{1}{|x|}$ for $x \neq 0$ and $f(0) = 1$ and C to be the $\frac{1}{2}$-Hölder seminorm of ν. By taking the expectation we obtain that

$$\infty > E\left[\int_{-\infty}^{\infty} f(x)L_t^x\,dx\right] = \int_{-\infty}^{\infty} f(x)E\left[L_t^x\right]dx.$$

Due to the right-continuity of L_t^x in x we can employ Fatou's lemma to conclude that

$$E(L_t^0) \leq \liminf_{x\searrow 0} E(L_t^x) = 0.$$

Using the defining property of the local time we obtain

$$|D_t| = \int_0^{t\wedge\zeta} \operatorname{sign}(D_s)(\mu(Y_s^{(1)}) - \mu(Y_s^{(2)}))\,ds + \int_0^{t\wedge\zeta} \operatorname{sign}(D_s)(\nu(Y_s^{(1)}) - \nu(Y_s^{(2)}))\,dW_s,$$

where the convention $\operatorname{sign}(0) = -1$ is used. We note that the integrand of the stochastic integral is uniformly bounded by the boundedness of D and the Hölder continuity of ν. Therefore, the stochastic integral is a martingale and taking the expectation yields that

$$E(|D_t|) \leq C \int_0^t E(|D_s|)\,ds,$$

where we let here C be the Lipschitz coefficient of μ. Since this holds for any $t \geq 0$ it is left to apply Grönwall's Lemma. □

A.2. Boundary values of hypergeometric functions

In this part of the appendix we perform the tedious steps leading to Lemma 3.8.

Proof of Lemma 3.8 (i). This is a direct consequence of the definition of ξ_λ. □

Proof of Lemma 3.8 (ii). First, we consider the case $\beta > -1$. To make use of Lemma 3.7 we distinguish between different signs of $-\alpha$. If $\alpha < 0$ the identity (3.6) applies and yields

$$\lim_{x \nearrow d} \xi_\lambda(x) = \frac{\Gamma(\beta+1)\Gamma(-\alpha)}{\Gamma\left(\frac{-\alpha+\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha+\beta+1}{2}-\gamma\right)}. \tag{A.1}$$

To conclude that this limit is positive we first note that its numerator is positive by the assumptions on α, β and the fact that $\Gamma(z) > 0$ for $z > 0$. Furthermore, as a consequence of (3.5) we have that

$$\left(\frac{-\alpha+\beta+1}{2}\right)^2 - \gamma^2 > \left(\frac{\alpha+\beta+1}{2}\right)^2 - \gamma^2 > 0.$$

It follows that either γ is purely imaginary or satisfies $0 \leq \gamma < \frac{-\alpha+\beta+1}{2}$. In the latter case positivity of the denominator follows as for the numerator. In the former case positivity follows instead by $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ and that Γ is non-zero outside of the negative real axis. Next, we assume that $\alpha = 0$. Then (3.7) applies and yields

$$\lim_{x \nearrow d} \frac{\xi_\lambda(x)}{-\log\left(1-\frac{x}{d}\right)} = \frac{\Gamma(\beta+1)}{\Gamma\left(\frac{\beta+1}{2}+\gamma\right)\Gamma\left(\frac{\beta+1}{2}-\gamma\right)}.$$

By analogous arguments as before we conclude that the right-hand side is positive. Since $-\log\left(1-\frac{x}{d}\right)$ converges to infinity as $x \nearrow d$, it follows that $\lim_{x \nearrow d} \xi_\lambda(x) = \infty$. Lastly, we assume $\alpha > 0$. The identity (3.8) gives us then

$$\lim_{x \nearrow d} \frac{\xi_\lambda(x)}{\left(1-\frac{x}{d}\right)^{-\alpha}} = \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma\left(\frac{\alpha+\beta+1}{2}+\gamma\right)\Gamma\left(\frac{\alpha+\beta+1}{2}-\gamma\right)}.$$

The same reasoning as before applies to argue that the right-hand side is positive. Since $\left(1-\frac{x}{d}\right)^{-\alpha}$ approaches infinity as $x \nearrow d$, we obtain $\lim_{x \nearrow d} \xi_\lambda(x) = \infty$.

Now we consider the case $\beta \leq -1$, where it is sufficient to consider the hypergeometric part of ξ_λ. To make use of Lemma 3.7 we distinguish between the signs of $-\alpha$ again. If $\alpha < 0$, (3.6) gives us

$$\lim_{x \nearrow d} \xi_\lambda(x) = \frac{\Gamma(1-\beta)\Gamma(-\alpha)}{\Gamma\left(\frac{-\alpha-\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2}-\gamma\right)}.$$

If $\alpha = 0$, (3.7) applies and yields

$$\lim_{x \nearrow d} \frac{\xi_\lambda(x)}{-\log\left(1-\frac{x}{d}\right)} = \frac{\Gamma(1-\beta)}{\Gamma\left(\frac{1-\beta}{2}+\gamma\right)\Gamma\left(\frac{1-\beta}{2}-\gamma\right)}.$$

Finally, if $\alpha > 0$, we get by (3.8) that

$$\lim_{x \nearrow d} \frac{\xi_\lambda(x)}{\left(1-\frac{x}{d}\right)^{-\alpha}} = \frac{\Gamma(1-\beta)\Gamma(\alpha)}{\Gamma\left(\frac{\alpha-\beta+1}{2}+\gamma\right)\Gamma\left(\frac{\alpha-\beta+1}{2}-\gamma\right)}.$$

Analogous considerations as in the case $\beta > -1$ yield positivity of these three limits. In particular, for $\alpha \geq 0$ we can conclude that $\lim_{x \nearrow d} \xi_\lambda(x) = \infty$. □

Before proceeding with the remaining statements we note that the product rule, termwise differentiation of the hypergeometric function as well as the identities (3.5) and

$$\left(\frac{\alpha-\beta+1}{2}\right)^2-\gamma^2=\left(\frac{\alpha+\beta+1}{2}-\beta\right)^2-\gamma^2=\frac{2\lambda}{\sigma^2}-\beta(\alpha+1)$$

yield the explicit expression

$$\xi_\lambda'(x)=\begin{cases}\frac{2\lambda}{\sigma^2 d(\beta+1)}\,{}_2F_1\left(\frac{\alpha+\beta+3}{2}+\gamma,\frac{\alpha+\beta+3}{2}-\gamma;\beta+2;\frac{x}{d}\right), & \beta>-1,\\ \frac{\frac{2\lambda}{\sigma^2}-\beta(\alpha+1)}{d(1-\beta)}\left(\frac{x}{d}\right)^{-\beta}{}_2F_1\left(\frac{\alpha-\beta+3}{2}+\gamma,\frac{\alpha-\beta+3}{2}-\gamma;2-\beta;\frac{x}{d}\right) & \\ -\frac{\beta}{d}\left(\frac{x}{d}\right)^{-(\beta+1)}{}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma,\frac{\alpha-\beta+1}{2}-\gamma;1-\beta;\frac{x}{d}\right), & \beta\le-1\end{cases}$$

for $x\in(0,d)$.

Proof of Lemma 3.8 (iii). For $\beta>-1$ the claim follows, because $cm(x)$ converges to 0 as $x\searrow 0$. If $\beta\le-1$, the derivative of ξ_λ consists out of two summands. For the first one we observe that

$$\frac{\sigma^2x^{\beta+1}(d-x)^{\alpha+1}}{2d^{\alpha+\beta+1}}\left(\frac{x}{d}\right)^{-\beta}{}_2F_1\left(\frac{\alpha-\beta+3}{2}+\gamma,\frac{\alpha-\beta+3}{2}-\gamma;2-\beta;\frac{x}{d}\right)\to 0$$

as $x\searrow 0$. Hence,

$$\begin{aligned}&\lim_{x\searrow 0}\xi_\lambda' cm(x)\\&=\lim_{x\searrow 0}-\frac{\beta\sigma^2(d-x)^{\alpha+1}}{2d^{\alpha+1}}\,{}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma,\frac{\alpha-\beta+1}{2}-\gamma;1-\beta;\frac{x}{d}\right)\\&=\frac{-\beta\sigma^2}{2}.\quad\square\end{aligned}$$

Proof of Lemma 3.8 (iv). We consider $\beta>-1$. In this case we have

$$\begin{aligned}&\lim_{x\nearrow d}\xi_\lambda' cm(x)\\&=\lim_{x\nearrow d}\frac{\lambda x^{\beta+1}(d-x)^{\alpha+1}}{d^{\alpha+\beta+2}(\beta+1)}\,{}_2F_1\left(\frac{\alpha+\beta+3}{2}+\gamma,\frac{\alpha+\beta+3}{2}-\gamma;\beta+2;\frac{x}{d}\right)\\&=\frac{\lambda}{\beta+1}\lim_{x\nearrow d}\left(1-\frac{x}{d}\right)^{\alpha+1}{}_2F_1\left(\frac{\alpha+\beta+3}{2}+\gamma,\frac{\alpha+\beta+3}{2}-\gamma;\beta+2;\frac{x}{d}\right).\end{aligned}$$

To make use of Lemma 3.7 we distinguish between the signs of $-(\alpha+1)$. We assume first $\alpha>-1$, i.e. $-(\alpha+1)<0$. Then (3.8) applies and yields

$$\lim_{x\nearrow d}\xi_\lambda' cm(x)=\frac{\lambda\Gamma(\beta+2)\Gamma(\alpha+1)}{(\beta+1)\Gamma\left(\frac{\alpha+\beta+3}{2}+\gamma\right)\Gamma\left(\frac{\alpha+\beta+3}{2}-\gamma\right)}.$$

The claimed identity follows since $\Gamma(\beta+2)=(\beta+1)\Gamma(\beta+1)$. The numerator of the limit is positive, again by our assumptions on α,β. Note that

$$\left(\frac{\alpha+\beta+3}{2}\right)^2-\gamma^2>\left(\frac{\alpha+\beta+1}{2}\right)^2-\gamma^2>0$$

due to (3.5). Hence either γ is purely imaginary or $0\le\gamma<\frac{\alpha+\beta+3}{2}$ and positivity of the denominator follows as in the proof of Lemma 3.8 (ii). Next we assume that $\alpha=-1$, i.e. $-(\alpha+1)=0$. Then (3.7) gives us that

$$\lim_{x\nearrow d}\frac{{}_2F_1\left(\frac{\beta+2}{2}+\gamma,\frac{\beta+2}{2}-\gamma;\beta+2;\frac{x}{d}\right)}{-\log\left(1-\frac{x}{d}\right)}=\frac{\Gamma(\beta+2)}{\Gamma\left(\frac{\beta+2}{2}+\gamma\right)\Gamma\left(\frac{\beta+2}{2}-\gamma\right)}.$$

Positivity of the right-hand side follows as before and consequently we have $\lim_{x\nearrow d}\xi_\lambda' cm(x)=\infty$. Lastly, we assume that $\alpha<-1$, i.e. $-(\alpha+1)>0$. Applying (3.6) results in

$$\begin{aligned}&\lim_{x\nearrow d}{}_2F_1\left(\frac{\alpha+\beta+3}{2}+\gamma,\frac{\alpha+\beta+3}{2}-\gamma;\beta+2;\frac{x}{d}\right)\\&=\frac{\Gamma(\beta+2)\,\Gamma(-(\alpha+1))}{\Gamma\left(\frac{-\alpha+\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha+\beta+1}{2}-\gamma\right)}\end{aligned}$$

and analogous arguments as before yield that the right-hand side is positive. Since $\left(1-\frac{x}{d}\right)^{\alpha+1}$ approaches infinity as $x\nearrow d$ we conclude $\lim_{x\nearrow d}\xi_\lambda' cm(x)=\infty$ also in this case.

Now we consider $\beta\le -1$. Then we have

$$\begin{aligned}&\lim_{x\nearrow d}\xi_\lambda' cm(x)\\&=\lim_{x\nearrow d}\left[\frac{2\lambda}{\sigma^2}-\beta(\alpha+1)\right]\frac{\sigma^2(d-x)^{\alpha+1}}{2(1-\beta)d^{\alpha+1}}{}_2F_1\left(\frac{\alpha-\beta+3}{2}+\gamma,\frac{\alpha-\beta+3}{2}-\gamma;2-\beta;\frac{x}{d}\right)\\&\quad-\frac{\sigma^2\beta(d-x)^{\alpha+1}}{2d^{\alpha+1}}{}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma,\frac{\alpha-\beta+1}{2}-\gamma;1-\beta;\frac{x}{d}\right).\end{aligned}\tag{A.2}$$

We start by investigating the limit of

$$-\frac{\sigma^2\beta}{2}\left(1-\frac{x}{d}\right)^{\alpha+1}{}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma,\frac{\alpha-\beta+1}{2}-\gamma;1-\beta;\frac{x}{d}\right)\tag{A.3}$$

as $x\nearrow d$. To make use of Lemma 3.7 we distinguish between the signs of $-\alpha$. If $\alpha>0$, (3.8) yields that the limit

$$\lim_{x\nearrow d}\left(1-\frac{x}{d}\right)^{\alpha}{}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma,\frac{\alpha-\beta+1}{2}-\gamma;1-\beta;\frac{x}{d}\right)$$

exists. Because of the additional factor $\left(1-\frac{x}{d}\right)$ the term (A.3) converges to 0 as $x\nearrow d$. If $\alpha=0$, (3.7) yields that the limit

$$\lim_{x\nearrow d}\frac{{}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma,\frac{\alpha-\beta+1}{2}-\gamma;1-\beta;\frac{x}{d}\right)}{-\log\left(1-\frac{x}{d}\right)}$$

exists and since $\lim_{x\nearrow d}\left(1-\frac{x}{d}\right)\log\left(1-\frac{x}{d}\right)=0$ we get that (A.3) converges to 0 for $x\nearrow d$. If $\alpha<0$, (3.6) implies that

$$\begin{aligned}&\lim_{x\nearrow d}{}_2F_1\left(\frac{\alpha-\beta+1}{2}+\gamma,\frac{\alpha-\beta+1}{2}-\gamma;1-\beta;\frac{x}{d}\right)\\&=\frac{\Gamma(1-\beta)\,\Gamma(-\alpha)}{\Gamma\left(\frac{-\alpha-\beta+1}{2}+\gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2}-\gamma\right)}.\end{aligned}\tag{A.4}$$

For $-1 < \alpha < 0$ we conclude as before that (A.3) converges to 0 as $x \nearrow d$ because of the additional prefactor. For $\alpha = -1$ the term (A.3) instead converges to a real number which we do not specify and instead denote by y during this proof. For $\alpha < -1$ the prefactor $\left(1 - \frac{x}{d}\right)^{\alpha+1}$ converges to infinity as $x \nearrow d$. Since the right-hand side of (A.4) is positive by analogous arguments as before we see that (A.3) tends to infinity as $x \nearrow d$ as well. We combine all cases in the following formula.

$$\begin{aligned}&\lim_{x \nearrow d} \frac{-\sigma^2 \beta}{2} \left(1 - \frac{x}{d}\right)^{\alpha+1} {}_2F_1\left(\frac{\alpha-\beta+1}{2} + \gamma, \frac{\alpha-\beta+1}{2} - \gamma; 1-\beta; \frac{x}{d}\right) \\ &= \begin{cases} 0, & \alpha > -1, \\ y, & \alpha = -1, \\ \infty, & \alpha < -1. \end{cases}\end{aligned} \tag{A.5}$$

We proceed by investigating the limiting behavior of

$$\left[\frac{2\lambda}{\sigma^2} - \beta(\alpha+1)\right] \frac{\sigma^2 (d-x)^{\alpha+1}}{2(1-\beta) d^{\alpha+1}} {}_2F_1\left(\frac{\alpha-\beta+3}{2} + \gamma, \frac{\alpha-\beta+3}{2} - \gamma; 2-\beta; \frac{x}{d}\right) \tag{A.6}$$

as $x \nearrow d$ using Lemma 3.7 and therefore distinguish between the signs of $-(\alpha+1)$.

For $\alpha > -1$, i.e. $-(\alpha+1) < 0$, we get by (3.8) that

$$\begin{aligned}&\lim_{x \nearrow d} \left[\frac{2\lambda}{\sigma^2} - \beta(\alpha+1)\right] \frac{\sigma^2 (d-x)^{\alpha+1}}{2(1-\beta) d^{\alpha+1}} \\ &\times {}_2F_1\left(\frac{\alpha-\beta+3}{2} + \gamma, \frac{\alpha-\beta+3}{2} - \gamma; 2-\beta; \frac{x}{d}\right) \\ &= \left[\frac{2\lambda}{\sigma^2} - \beta(\alpha+1)\right] \frac{\sigma^2 \Gamma(2-\beta)\Gamma(\alpha+1)}{2(1-\beta)\Gamma\left(\frac{\alpha-\beta+3}{2} + \gamma\right)\Gamma\left(\frac{\alpha-\beta+3}{2} - \gamma\right)}.\end{aligned}$$

Substituting $\Gamma(2-\beta) = (1-\beta)\Gamma(1-\beta)$ together with (A.5) yields the claimed identity in this case. Note that the fraction on the right-hand side is positive by analogous reasoning as before. Also the prefactor is positive in this particular case of α and β.

Next, we consider the case $\alpha = -1$. Then (3.7) yields

$$\lim_{x \nearrow d} \frac{{}_2F_1\left(\frac{2-\beta}{2} + \gamma, \frac{2-\beta}{2} - \gamma; 2-\beta; \frac{x}{d}\right)}{-\log\left(1 - \frac{x}{d}\right)} = \frac{\Gamma(2-\beta)}{\Gamma\left(\frac{2-\beta}{2} + \gamma\right)\Gamma\left(\frac{2-\beta}{2} + \gamma\right)}.$$

By analogous arguments as before we conclude that this limit is positive. Since also the prefactor of the expression (A.6) is positive due to $\alpha = -1$ it follows that (A.6) tends to infinity as $x \nearrow d$. Together with (A.5) this implies that (A.2) equals infinity in this case.

It remains to consider $\alpha < -1$, i.e. $-(\alpha+1) > 0$. Then (3.6) yields

$$\begin{aligned}&\lim_{x \nearrow d} {}_2F_1\left(\frac{\alpha-\beta+3}{2} + \gamma, \frac{\alpha-\beta+3}{2} - \gamma; 2-\beta; \frac{x}{d}\right) \\ &= \frac{\Gamma(2-\beta)\,\Gamma(-(\alpha+1))}{\Gamma\left(\frac{-\alpha-\beta+1}{2} + \gamma\right)\Gamma\left(\frac{-\alpha-\beta+1}{2} + \gamma\right)}.\end{aligned}$$

Analogously as before we get that the right-hand side of the above equality is positive. By $\lim_{x \nearrow d}(d-x)^{\alpha+1} = \infty$ it is sufficient to show $\frac{2\lambda}{\sigma^2} - \beta(\alpha+1) > 0$ to conclude that (A.6) tends to infinity as $x \nearrow d$. We assumed in this particular case that (3.9) holds which implies that

$$\frac{2\lambda}{\sigma^2} - \beta(\alpha+1) > \left(\frac{\alpha+\beta+1}{2}\right)^2 - \beta(\alpha+1) = \left(\frac{\beta-(\alpha+1)}{2}\right)^2 \geq 0.$$

The claimed identity follows by employing (A.5). □

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