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ON THE ISOMORPHISM CLASS OF q-GAUSSIAN C*-ALGEBRAS FOR INFINITE VARIABLES

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ABSTRACT. For a real Hilbert space $H_{\mathbb{R}}$ and $-1 < q < 1$ Bozejko and Speicher introduced the C*-algebra $A_q(H_{\mathbb{R}})$ and von Neumann algebra $M_q(H_{\mathbb{R}})$ of q-Gaussian variables. We prove that if $\dim(H_{\mathbb{R}}) = \infty$ and $-1 < q < 1, q \neq 0$ then $M_q(H_{\mathbb{R}})$ does not have the Akemann-Ostrand property with respect to $A_q(H_{\mathbb{R}})$. It follows that $A_q(H_{\mathbb{R}})$ is not isomorphic to $A_0(H_{\mathbb{R}})$. This gives an answer to the C*-algebraic part of Question 1.1 and Question 1.2 in raised by Nelson and Zeng [Int. Math. Res. Not. IMRN 17 (2018), pp. 5486–5535].

1. Introduction

In [BoSp91] Bożejko and Speicher introduced a non-commutative version of Brownian motion using a construction that is now commonly known as the q-Gaussian algebra where $-1 \leq q \leq 1$. These algebras range between the extreme Bosonic case $q = 1$ of fields of classical Gaussian random variables and the Fermionic case $q = -1$ of Clifford algebras. For $q = 0$ one obtains Voiculescu's free Gaussian functor. q-Gaussians can be studied on the level of $*$-algebras $\mathcal{A}_q(H_{\mathbb{R}})$, C*-algebras $A_q(H_{\mathbb{R}})$ and von Neumann algebras $M_q(H_{\mathbb{R}})$ starting from a real Hilbert space $H_{\mathbb{R}}$ where $\dim(H_{\mathbb{R}})$ usually refers to the number of variables.

The dependence of q-Gaussian algebras on the parameter q has been an intriguing problem ever since their introduction. The $*$-algebras $\mathcal{A}_q(H_{\mathbb{R}})$ are easily seen to be isomorphic for all $-1 < q < 1$ (see [CIW21, Theorem 4.1, proof]). However, the isomorphisms do not extend to the C*-algebras $A_q(H_{\mathbb{R}})$; one way to see this is that this isomorphism maps generators $W_q(\xi)$ to generators $W_{q'}(\xi)$ with $\xi \in H_{\mathbb{R}}$ (see Section 2 for notation) which is easily seen to be non-isometric. In fact, the isomorphism problem becomes notoriously difficult on the level of the C*-algebras and von Neumann algebras.

A breakthrough result was obtained by Guionnet-Shlyakhtenko in [GuSh14] where free transport techniques were developed to show that in case $\dim(H_{\mathbb{R}}) < \infty$ one has that $A_q(H_{\mathbb{R}}) \simeq A_0(H_{\mathbb{R}})$ and $M_q(H_{\mathbb{R}}) \simeq M_0(H_{\mathbb{R}})$ for a range of q close to 0.

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The range becomes smaller as $\dim(H_{\mathbb{R}})$ increases. The proof is also based on the existence and power series estimates of conjugate variables by Dabrowski [Dab14].

The infinite variable case $\dim(H_{\mathbb{R}}) = \infty$ was then pursued by Nelson-Zeng [NeZe18] where they explicitly ask whether given a fixed Hilbert space $H_{\mathbb{R}}$ one can have isomorphism of the q-Gaussian C*- and von Neumann algebras, see [NeZe18, Questions 1.1 and 1.2]. They already note that the condition $q^2 \dim(H_{\mathbb{R}}) < 1$ is required for the construction of conjugate variables to the free difference quotient [Dab14]. However, by passing to mixed q-Gaussians with sufficient decay on the coefficient array $Q = (q_{ij})_{i,j}$ they show that free transport techniques can still be developed in order to extend the Guionnet-Shlyakhtenko result to this mixed q-Gaussian setting. This approach is in some sense sufficiently close to the case of finite dimensional $H_{\mathbb{R}}$. The main merit of the current note is a rather definite and negative answer to the C*-algebraic part of [NeZe18, Questions 1.1 and 1.2], namely we show that we have $A_0(H_{\mathbb{R}}) \not\simeq A_q(H_{\mathbb{R}}), -1 < q < 1, q \neq 0$ in case the dimension of $H_{\mathbb{R}}$ is infinite.

Our main result is that if $\dim(H_{\mathbb{R}}) = \infty$ then the von Neumann algebra $M_q(H_{\mathbb{R}})$ does not have the Akemann-Ostrand property with respect to the natural C*-subalgebra $A_q(H_{\mathbb{R}})$ for any $-1 < q < 1, q \neq 0$. This will then distinguish $A_0(H_{\mathbb{R}})$ from $A_q(H_{\mathbb{R}})$. The idea of our proof is as follows. In [Con76, Theorem 5.1] Connes proved that a finite von Neumann algebra M is amenable if and only if the map

$$M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \to \mathcal{B}(L_2(M)) : a \otimes b^{\mathrm{op}} \to ab^{\mathrm{op}}$$

is $\otimes_{\min}$-bounded. This characterisation – in combination with a Khintchine inequality – was used by Nou [Nou04] to show that $M_q(H_{\mathbb{R}})$ is not amenable for $-1 < q < 1$ and $\dim(H_{\mathbb{R}}) \geq 2$. We show that if $\dim(H_{\mathbb{R}}) = \infty$ and $-1 < q < 1, q \neq 0$ then we cannot even have that

$$\begin{aligned} A_q(H_{\mathbb{R}}) \otimes_{\mathrm{alg}} A_q(H_{\mathbb{R}})^{\mathrm{op}} \to \mathcal{B}(L_2(M_q(H_{\mathbb{R}})))/\mathcal{K}(L_2(M_q(H_{\mathbb{R}}))) : a \otimes b^{\mathrm{op}} \\ \to ab^{\mathrm{op}} + \mathcal{K}(L_2(M_q(H_{\mathbb{R}}))) \end{aligned}$$

is $\otimes_{\min}$-bounded where we have taken a quotient by compact operators. This is proved in Section 3. We then harvest the non-isomorphism results in Section 4.

2. Preliminaries

2.1. **Von Neumann algebras.** In the following $\mathcal{B}(H)$ denotes the bounded operators on a Hilbert space H and $\mathcal{K}(H)$ denotes the compact operators on H. For a von Neumann algebra M we denote by $(M, L_2(M), J, L_2(M)^+)$ the standard form. For $x \in M$ we write $x^{\mathrm{op}} := Jx^*J$ which is the right multiplication with x on the standard space. This way $L_2(M)$ becomes an M-M-bimodule called the trivial bimodule.

The algebraic tensor product is denoted by $\otimes_{\mathrm{alg}}$ and $\otimes_{\min}$ is the minimal tensor product of C*-algebras which by Takesaki's theorem [Tak02, Theorem IV.4.19] is the spatial tensor product.

2.2. **q-Gaussians.** Let $-1 < q < 1$. Now let $H_{\mathbb{R}}$ be a real Hilbert space with complexification $H := H_{\mathbb{R}} \oplus iH_{\mathbb{R}}$. We define the symmetrization operator P_q^k on $H^{\otimes k}$ by

$$P_q^k(\xi_1 \otimes \ldots \otimes \xi_n) = \sum_{\sigma \in S_k} q^{i(\sigma)} \xi_{\sigma(1)} \otimes \ldots \otimes \xi_{\sigma(n)}, \tag{2.1}$$

where S_k is the symmetric group of permutations of k elements and $i(\sigma) := \#\{(a,b) \mid a < b, \sigma(b) < \sigma(a)\}$ the number of inversions. The operator P_q^k is positive and invertible [BoSp91]. Define a new inner product on $H^{\otimes k}$ by

$$\langle \xi, \eta \rangle_q := \langle P_q^k \xi, \eta \rangle,$$

and call the new Hilbert space $H_q^{\otimes k}$. Set the Hilbert space $F_q(H) := \mathbb{C}\Omega \oplus (\oplus_{k=1}^{\infty} H_q^{\otimes k})$ where Ω is a unit vector called the vacuum vector. For $\xi \in H$ let

$$l_q(\xi)(\eta_1 \otimes \ldots \otimes \eta_k) := \xi \otimes \eta_1 \otimes \ldots \otimes \eta_k, \qquad l_q(\xi)\Omega = \xi,$$

and then $l_q^*(\xi) = l_q(\xi)^*$. These 'creation' and 'annihilation' operators are bounded and extend to $F_q(H)$. We define a $*$-algebra, C*-algebra and von Neumann algebra by

$$\begin{aligned}
\mathcal{A}_q(H_{\mathbb{R}}) &:= *{-}\mathrm{alg}\{l_q(\xi) + l_q^*(\xi) \mid \xi \in H_{\mathbb{R}}\}, \\
A_q(H_{\mathbb{R}}) &:= \overline{\mathcal{A}_q(H_{\mathbb{R}})}^{\|\cdot\|}, \\
M_q(H_{\mathbb{R}}) &:= A_q(H_{\mathbb{R}})'',
\end{aligned}$$

where $*$-alg denotes the unital $*$-algebra in $\mathcal{B}(F_q(H))$ generated by the set. Then $\tau_\Omega(x) := \langle x\Omega, \Omega\rangle$ is a faithful tracial state on $M_q(H_{\mathbb{R}})$ which is moreover normal. Now $F_q(H)$ is the standard form Hilbert space of $M_q(H_{\mathbb{R}})$ and $Jx\Omega = x^*\Omega$.

For $K_{\mathbb{R}}$ a closed subspace of $H_{\mathbb{R}}$ we have that $\mathcal{A}_q(K_{\mathbb{R}})$ is naturally a $*$-subalgebra of $\mathcal{A}_q(H_{\mathbb{R}})$. Further, if $(K_{\mathbb{R},i})_{i\in\mathbb{N}}$ is an increasing sequence of closed subspaces whose span is dense in $H_{\mathbb{R}}$ then $\cup_i \mathcal{A}_q(K_{\mathbb{R},i})$ is dense in $A_q(H_{\mathbb{R}})$.

For vectors $\xi_1, \ldots, \xi_k \in H$ there exists a unique operator $W_q(\xi_1 \otimes \ldots \otimes \xi_k) \in \mathcal{A}_q(H_{\mathbb{R}})$ such that

$$W_q(\xi_1 \otimes \ldots \otimes \xi_k)\Omega = \xi_1 \otimes \ldots \otimes \xi_k.$$

These operators are called Wick operators. It follows that $W_q(\xi)^{\mathrm{op}}\Omega = \xi$. We shall further need the constant

$$C_q := \prod_{i=1}^{\infty} (1 - q^i)^{-1} > 0. \tag{2.2}$$

3. Main theorem: Failure of the Akemann-Ostrand property

3.1. **Failure of AO.** We will work with Definition 3.1 of the Akemann-Ostrand property [BrOz08].

Definition 3.1. A finite von Neumann algebra M has the Akemann-Ostrand property (or AO) if there exists a σ-weakly dense unital C*-subalgebra $A \subseteq M$ such that A is locally reflexive (see [BrOz08]) and such that the multiplication map $\theta : A \otimes_{\mathrm{alg}} A^{\mathrm{op}} \to \mathcal{B}(L_2(M))/\mathcal{K}(L_2(M)) : a \otimes b^{\mathrm{op}} \to ab^{\mathrm{op}} + \mathcal{K}(L_2(M))$ is continuous with respect to the minimal tensor norm. We also say that M has AO with respect to A.

We assumed local reflexivity of the C*-algebra A in Definition 3.1 as part of the usual definition of AO. However, in the current paper local reflexivity does not play a crucial role and all our results hold if we consider Definition 3.1 without the local reflexivity assumption on A.

Note that θ in Definition 3.1 is a $*$-homomorphism, so if it is continuous it is automatically a contraction.

Theorem 3.2. *Let M be a finite von Neumann algebra with a σ-weakly dense unital C^*-subalgebra A. Suppose there exists a unital C^*-subalgebra $B \subseteq A$ and infinitely many mutually orthogonal closed subspaces $H_i \subseteq L_2(M), i \in \mathbb{N}$ that are left and right B-invariant. Suppose moreover that there exist $\delta > 0$ and finitely many operators $b_j, c_j \in B$ such that for every $i \in \mathbb{N}$ we have*

$$\| \sum_j b_j c_j^{op} \|_{\mathcal{B}(H_i)} \geq (1+\delta) \| \sum_j b_j \otimes c_j^{\mathrm{op}} \|_{B \otimes_{\min} B^{\mathrm{op}}}. \tag{3.1}$$

Then M does not have AO with respect to A.

Proof. Since there are infinitely many B-B-invariant spaces H_i we have for any finite rank operator $x \in \mathcal{B}(L_2(M))$ that

$$\| \sum_j b_j c_j^{op} + x \|_{\mathcal{B}(L_2(M))} \geq (1+\delta) \| \sum_j b_j \otimes c_j^{\mathrm{op}} \|_{B \otimes_{\min} B^{\mathrm{op}}}.$$

Taking the infimum over all such x we obtain that

$$\| \sum_j b_j c_j^{op} + \mathcal{K}(L_2(M)) \|_{\mathcal{B}(L_2(M))/\mathcal{K}(L_2(M))} \geq (1+\delta) \| \sum_j b_j \otimes c_j^{\mathrm{op}} \|_{B \otimes_{\min} B^{\mathrm{op}}}. \tag{3.2}$$

But the definition of AO entails the existence of a contraction $\theta : A \otimes_{\min} A^{\mathrm{op}} \to \mathcal{B}(L_2(M))/\mathcal{K}(L_2(M))$ such that $\theta(b \otimes c^{\mathrm{op}}) = bc^{\mathrm{op}} + \mathcal{K}(L_2(M))$ for all $b, c \in A$. Hence

$$\| \sum_j b_j c_j^{op} + \mathcal{K}(L_2(M)) \|_{\mathcal{B}(L_2(M))/\mathcal{K}(L_2(M))} \leq \| \sum_j b_j \otimes c_j^{\mathrm{op}} \|_{B \otimes_{\min} B^{\mathrm{op}}},$$

which contradicts (3.2). □

3.2. The case of q-Gaussians.

Theorem 3.3. *Assume $\dim(H_{\mathbb{R}}) = \infty$ and $-1 < q < 1, q \neq 0$. Then the von Neumann algebra $M_q(H_{\mathbb{R}})$ does not have AO with respect to $A_q(H_{\mathbb{R}})$.*

Proof. Let $d \geq 2$ be so large that $q^2 d > 1$. Let

$$M := M_q(\mathbb{R}^d \oplus H_{\mathbb{R}}), \quad A := A_q(\mathbb{R}^d \oplus H_{\mathbb{R}}), \quad B := A_q(\mathbb{R}^d \oplus 0).$$

We shall prove that M does not have AO with respect to A; since $\mathbb{R}^d \oplus H_{\mathbb{R}} \simeq H_{\mathbb{R}}$ this suffices to conclude the proof.

Let $\{f_i\}_i$ be an orthonormal basis of $0 \oplus H_{\mathbb{R}}$. Let $H_{q,i} := \overline{B f_i B}^{\|\cdot\|}$ as a closed subspace of the Fock space $F_q(\mathbb{R}^d \oplus H_{\mathbb{R}})$. Then $H_{q,i} \perp H_{q,j}$ if $i \neq j$ which can be seen straight from the definition of $\langle \cdot, \cdot \rangle_q$. For $k \in \mathbb{N}$ let

$$\mathcal{B}(k) = \{ W_q(\xi) \mid \xi \in (\mathbb{R}^d \oplus 0)^{\otimes k} \}.$$

Let $\xi, \eta \in (\mathbb{R}^d \oplus 0)^{\otimes k}$ and write $\xi = \xi_1 \otimes \ldots \otimes \xi_k$ with $\xi_i \in \mathbb{R}^d$. We have $W_q(\xi)^* = W_q(\xi^*)$ where $\xi^* = \xi_k \otimes \ldots \otimes \xi_1$. We have that (see [EfPo03])

$$\langle W_q(\xi) f_i W_q(\eta), f_i \rangle_q = \langle f_i W_q(\eta), W_q(\xi)^* f_i \rangle_q = \langle f_i \otimes \eta, \xi^* \otimes f_i \rangle_q = \langle P_q^{k+1} f_i \otimes \eta, \xi^* \otimes f_i \rangle.$$

We examine the right hand side of this expression. The q-symmetrization operator P_q^{k+1} is defined as a sum of permutations $\sigma \in S_{k+1}$ (see (2.1)) and it follows from the fact that $f_i \in 0 \oplus H_{\mathbb{R}}$ and $\xi, \eta \in (\mathbb{R}^d \oplus 0)^{\otimes k}$ that the only summands that contribute a possibly non-zero term are the ones where $\sigma(k+1) = 1$. Note that for such a permutation σ we have

$$i(\sigma) = \# \left(\{ (a, k+1) \mid 1 \leq a \leq k \} \cup \{ (a, b) \mid 1 \leq a < b \leq k, \sigma(b) < \sigma(a) \} \right).$$

Therefore we find

$$\begin{aligned}\langle W_q(\xi) f_i W_q(\eta), f_i\rangle_q &= \sum_{\sigma\in S_k} q^{k+i(\sigma)} \langle \eta_{\sigma(1)}\otimes\ldots\otimes\eta_{\sigma(k)}, \xi_k\otimes\ldots\otimes\xi_1\rangle \\ &= q^k\langle P_q^k\eta,\xi^*\rangle = q^k\langle\eta,\xi^*\rangle_q = q^k\langle W_q(\xi)\Omega W_q(\eta),\Omega\rangle_q.\end{aligned} \tag{3.3}$$

Now from (3.3) we conclude that for $b_j, c_j \in \mathcal{B}(k)$,

$$\|\sum_j b_j c_j^{\rm op}\|_{\mathcal{B}(H_{q,i})} \geq |\langle\sum_j b_j c_j^{\rm op} f_i, f_i\rangle_q| = |\sum_j \langle b_j f_i c_j, f_i\rangle_q| = |\sum_j q^k\langle b_j\Omega c_j,\Omega\rangle_q|. \tag{3.4}$$

Now let $\{e_1,\ldots,e_d\}$ be an orthonormal basis of $\mathbb{R}^d\oplus 0$ and for $j=(j_1,\ldots,j_k)\in\{1,\ldots,d\}^k$ let $e_j = e_{j_1}\otimes\ldots\otimes e_{j_k}$. Let J_k be the set of all such multi-indices of length k. So $\#J_k = d^k$. Set $\xi_j = (P_q^k)^{-\frac{1}{2}} e_j$ so that $\langle\xi_j,\xi_j\rangle_q = \langle P_q^k\xi_j,\xi_j\rangle = 1$.

Now (3.4) yields that for all $k\geq 1$ and all i,

$$\begin{aligned}\|\sum_{j\in J_k} W_q(\xi_j)^* W_q(\xi_j)^{\rm op}\|_{\mathcal{B}(H_{q,i})} &\geq \sum_{j\in J_k} q^k\langle W_q(\xi_j)^*\Omega W_q(\xi_j),\Omega\rangle_q \\ &= \sum_{j\in J_k} q^k\langle \Omega W_q(\xi_j), W_q(\xi_j)\Omega\rangle_q \\ &= \sum_{j\in J_k} q^k\langle\xi_j,\xi_j\rangle_q = q^k d^k.\end{aligned}$$

On the other hand from [Nou04, Proof of Theorem 2] we find

$$\|\sum_{j\in J_k} W_q(\xi_j)^*\otimes W_q(\xi_j)^{\rm op}\|_{B\otimes_{\min}B^{\rm op}} \leq C_q^3(k+1)^2 d^{k/2},$$

where the constant $C_q>0$ was defined in (2.2). Therefore, as $q^2 d>1$ there exists $\delta>0$ such that for k large enough we have for every i,

$$\|\sum_{j\in J_k} W_q(\xi_j)^* W_q(\xi_j)^{\rm op}\|_{\mathcal{B}(H_{q,i})} \geq (1+\delta)\|\sum_{j\in J_k} W_q(\xi_j)^*\otimes W_q(\xi_j)^{\rm op}\|_{B\otimes_{\min}B^{\rm op}}.$$

Hence the assumptions of Theorem 3.2 are witnessed which shows that AO does not hold. □

4. A non-isomorphism result for q-Gaussian C*-algebras

We now turn to the isomorphism question of $A_q(H_\mathbb{R})$ for q close to 0. We first need a result of independent interest which seems not to be proved in the literature. By [Ric05] we know that the von Neumann algebra $M_q(H_\mathbb{R})$ with $\dim(H_\mathbb{R})\geq 2$ is a factor of type II$_1$. This was proven already in the case $\dim(H_\mathbb{R})=\infty$ in [BKS97, Theorem 2.10]. In this section we need a strengthening of the latter result, namely that $A_q(H_\mathbb{R})$ has a unique tracial state. The proof is based again on Nou's Khintchine inequality [Nou04].

Theorem 4.1. *Let $\dim(H_\mathbb{R})=\infty$. Then $A_q(H_\mathbb{R})$ has a unique tracial state and is a simple C*-algebra.*

We will prove the theorem after first proving a lemma. Assume for simplicity that $H_\mathbb{R}$ is separable. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of $H_\mathbb{R}$ and identify $\mathbb{R}^d$

with the span of $\{e_i\}_{i=1}^d$. For $m \in \mathbb{N}$ consider the map $\mathcal{A}_q(H_\mathbb{R}) \to \mathcal{A}_q(H_\mathbb{R})$ given by

$$\Phi_m(X) = \frac{1}{m}\sum_{i=1}^m W_q(e_i)XW_q(e_i).$$

Then Φ_m extends to a bounded map $A_q(H_\mathbb{R}) \to A_q(H_\mathbb{R})$ with bound uniform in m.

Lemma 4.2 is stronger than [BKS97, Theorem 2.10, proof] where only weak convergence was established; the result is used in the proof of [BKS97, Theorem 2.14] but its proof is not given. Therefore we give it here.

Lemma 4.2. *For $X = W_q(\xi), \xi \in H^{\otimes n}$ we have $\Phi_m(X) \to q^n X$ as $m \to \infty$ in the norm of $A_q(H_\mathbb{R})$.*

Proof. First assume that there exists $d \in \mathbb{N}$ such that $\xi \in (\mathbb{C}^d)^{\otimes n} \subseteq H^{\otimes n}$. By density and uniform boundedness of Φ_m in m this suffices to conclude the lemma. Then, for $m > d$, by [EfPo03, Theorem 3.3],

(4.1)

$$\Phi_m(W_q(\xi)) = \frac{1}{m}\sum_{i=1}^d W_q(e_i)W_q(\xi)W_q(e_i) + \frac{1}{m}\sum_{i=d+1}^m q^n W_q(\xi) + \frac{1}{m}\sum_{i=d+1}^m W_q(e_i\otimes\xi\otimes e_i).$$

The first term converges to 0 as $m \to \infty$, whereas the second term converges to $q^n W_q(\xi)$. It thus remains to show that the last term converges to 0 in norm. We have by [Nou04, Lemma 2] (see also [Boz99] where a weaker but sufficient estimate was obtained)

$$\| \sum_{i=d+1}^m W_q(e_i \otimes \xi \otimes e_i)\| \leq (n+3)C_q^{\frac{3}{2}} \| \sum_{i=d+1}^m e_i \otimes \xi \otimes e_i \|_{H_q^{\otimes n+2}}.$$

The vectors $\{e_i \otimes \xi \otimes e_i\}_i$ are orthogonal in $H_q^{\otimes n+2}$ and have the same norm which we denote by C. Therefore,

$$\frac{1}{m}\| \sum_{i=d+1}^m W_q(e_i \otimes \xi \otimes e_i)\| \leq (n+3)C_q^{\frac{3}{2}} C m^{-\frac{1}{2}}.$$

We conclude that the third term in (4.1) converges to 0 as $m \to \infty$ in norm. □

Proof of Theorem 4.1. By Lemma 4.2 for $X \in \mathcal{A}_q(H_\mathbb{R})$ set the norm limit $\Phi(X) := \lim_{m\to\infty} \Phi_m(X)$. Let τ be any tracial state on $A_q(H_\mathbb{R})$. Then, for $X \in \mathcal{A}_q(H_\mathbb{R})$,

$$\begin{aligned}\tau(\Phi(X)) &= \lim_{m\to\infty} \tau(\Phi_m(X)) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m \tau(W_q(e_i)XW_q(e_i)) \\ &= \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m \tau(XW_q(e_i)W_q(e_i)) = \tau(X\Phi(1)) = \tau(X).\end{aligned}$$

Therefore, by Lemma 4.2, $\tau(W_q(\xi)) = \tau(\Phi^k(W_q(\xi))) = q^{kn}\tau(W_q(\xi))$ for $\xi \in H^{\otimes n}$. For $k \to \infty$ the expression converges to 0 for $n \geq 1$. It follows that for $X \in \mathcal{A}_q(H_\mathbb{R})$ we have $\tau(X) = \tau_\Omega(X)$ and by continuity this actually holds for $X \in A_q(H_\mathbb{R})$. So τ_Ω is the unique tracial state on $A_q(H_\mathbb{R})$.

Simplicity was already obtained in [BKS97, Theorem 2.14]; it is also based on Lemma 4.2. □

Proposition 4.3 was also proved in [Hou07, Chapter 4]; the proof uses the same method as [Shl04] where this result was also obtained for finite dimensional $H_\mathbb{R}$.

Proposition 4.3. *For any real Hilbert space $H_\mathbb{R}$ the von Neumann algebra $M_0(H_\mathbb{R})$ satisfies AO with respect to $A_0(H_\mathbb{R})$.*

Theorem 4.4. *Let $H_\mathbb{R}$ be a real Hilbert space with $\dim(H_\mathbb{R}) = \infty$. Then $A_q(H_\mathbb{R})$ with $-1 < q < 1, q \neq 0$ is not isomorphic to $A_0(H_\mathbb{R})$ and neither to $A_{q'}(\mathbb{R}^d)$ with $|q'| < \sqrt{2} - 1$ or $|q'| \leq d^{-\frac{1}{2}}$.*

Proof. If $A_q(H_\mathbb{R})$ were to be isomorphic to $A_{q'}(\mathbb{R}^d)$ then the unique trace property of Theorem 4.1 shows that the pair $(M_q(H_\mathbb{R}),\ A_q(H_\mathbb{R}))$ is isomorphic to $(M_{q'}(\mathbb{R}^d), A_{q'}(\mathbb{R}^d))$, see [CKL21, Lemma 1.1] for the standard argument. However this is not the case by Theorem 3.3 and the fact that $(M_{q'}(\mathbb{R}^d), A_{q'}(\mathbb{R}^d))$ has AO by [CIW21], [Shl04] (the property AO$^+$ in these references directly implies AO). The argument for the non-isomorphism of $A_q(H_\mathbb{R})$ and $A_0(H_\mathbb{R})$ is the same where we use Theorem 3.3 and Proposition 4.3 instead. □

Remark 4.5. Fix a real Hilbert space $H_\mathbb{R}$ with $\dim(H_\mathbb{R}) < \infty$ and complexification H as before. We call the C*-subalgebra of $\mathcal{B}(F_q(H))$ generated by $l_q(\xi), \xi \in H$ the q-CCR algebra. Shortly after completion of this paper it was announced in [Kuz22] that, for $H_\mathbb{R}$ fixed, all q-CCR algebras for $-1 < q < 1$ are isomorphic. In particular these C*-algebras are nuclear. Following the proof of [Shl04, Theorem 4.2] while using that $H_\mathbb{R}$ is finite dimensional, it follows that $(M_{q'}(H_\mathbb{R}), A_{q'}(H_\mathbb{R}))$ has AO for all $-1 < q' < 1$. Consequently, Theorem 4.4 holds for any $-1 < q' < 1$. This also completely classifies when q-Gaussian von Neumann algebras have AO with respect to the underlying q-Gaussian C*-algebra.

Remark 4.6. In principle it is possible to give a purely C*-algebraic proof of Theorem 4.4 as well by considering the following version of AO. We say that a C*-algebra has C*AO if it has a unique faithful tracial state τ and the map $A \otimes_{\text{alg}} A^{\text{op}} \to \mathcal{B}(L_2(A,\tau))/\mathcal{K}(L_2(A,\tau)) : a \otimes b^{\text{op}} \mapsto ab^{\text{op}} + \mathcal{K}(L_2(A,\tau))$ is continuous for the minimal tensor norm. Here $L_2(A,\tau)$ is the GNS-space for τ and b^{op} the right multiplication with b. This property distinguishes the algebras then.

Remark 4.7. The question stays open whether for a real infinite dimensional Hilbert space $H_\mathbb{R}$ one can distinguish the von Neumann algebra $M_0(H_\mathbb{R})$ from $M_q(H_\mathbb{R})$ with $-1 < q < 1, q \neq 0$.

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