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DOI
10.1080/00207179.2022.2085180

Publication date
2022

Document Version
Final published version

Published in
International Journal of Control

International Journal of Control

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

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INTERNATIONAL JOURNAL OF CONTROL
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Output feedback stabilisation of an axially moving string subject to a spring-mass-dashpot

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ABSTRACT

In this paper, we consider the output feedback stabilisation of an axially moving string system subject to a spring-mass-dashpot boundary condition. By constructing an invertible backstepping transformation, we design a state feedback controller to stabilise the system. Next, we present an observer to estimate the states of the system, and based on the estimated states, we design an output-feedback controller. The closed-loop system is proved to be exponentially stable by Lyapunov analysis. Numerical simulations are presented to verify the effectiveness of the proposed controller.

ARTICLE HISTORY
Received 31 December 2021
Accepted 29 May 2022

KEYWORDS
Moving string; spring-mass-dashpot; PDE-ODE; backstepping; vibration control

1. Introduction

The moving cable systems, due to their low bending and torsional stiffness, and due to their better performance and high-speed automation, have been widely applied in many practical devices, such as conveyor belts (L. Q. Chen, 2005; Sack, 2002), elevator cables (Gaiko & van Horssen, 2018; Sandilo & van Horssen, 2015), hoisting systems (Kaczmarczyk & Ostachowicz, 2003a, 2003b) and so on. These systems vibrations may lead to structural failure by excessive strain in the moving process. Therefore, moving cable modelling and vibration stabilisation have been a research hot spot in recent years. E. W. Chen and Ferguson (2014) used Lagrange's equation to establish the model of an axially moving string and analysed energy dissipation in a moving string with a viscous damper at one end. Further, E. W. Chen et al. (2017), E. W. Chen et al. (2019) and E. W. Chen et al. (2021) investigated a reflected wave superposition method for moving string vibration with classical and nonclassical boundaries at two ends and analysed the total mechanical energy.

There are many methods to achieve the vibration stabilisation of axially moving strings or beams. One of the most useful methods for boundary controller is based on the Lyapunov method, by which control laws to reduce vibration energy to zero are derived using Lyapunov function candidates constructed by the total mechanical energy of the moving system. Nguyen and Hong (2010) investigated an adaptive boundary control based on Lyapunov's method for a nonlinear axially moving string. Nguyen and Hong (2012) presented simultaneous controls of longitudinal and transverse vibrations of an axially moving string with velocity tracking. Tebou (2019) studied the boundary stabilisation of an axially moving Euler–Bernoulli beam. In the literature, the controllers are required to follow the end causing vibration excitation, which is sometimes difficult to achieve in practical implementation due to inconvenient installation. Hence, it is necessary to study the control system where control is applied at the end opposite to the instability. This is a more challenging task than the classical collocated 'boundary damper' feedback control (Krstic et al., 2008). The backstepping approach, which is proposed by Krstic, can deal with the proposed non-collocated stabilisation problem. Ren et al. (2013) analysed boundary stability of an ODE-Schrödinger cascade. Krstic (2009) provided an explicit feedback law that compensates the wave PDE dynamics at the input of an LTI ODE and stabilises the overall system. In Susto and Krstic (2010), a ODE-PDE cascade system was extended from the Dirichlet type interconnections to Neumann type interconnections. Wang et al. (2018) designed an observer-based output-feedback control law for the stability of the axial vibration in the ascending mining cable elevator. For more information on vibration suppression problems of axially moving strings, the reader is referred to Zhu et al. (2001), He et al. (2015) and He et al. (2016).

In this paper, we consider a moving string system with constant speed on a finite spatial domain subject to a spring-mass-dashpot attached at one end of the string as shown in Figure 1. This model arises from conveyor belts, cranes or elevators devices for suppressing large vibrations, and the spring-mass-dashpot boundary causes vibration excitations. The objective of the paper is to design an observer-based output feedback controller at the free boundary to stabilise the system.

The remaining part of this paper is organised as follows. Section 2 formulates the problem by extended Hamilton's principle. Section 3 designs a controller based state feedback to stabilise the system exponentially. Section 4 concludes the output feedback law based observer. Section 5 presents some numerical

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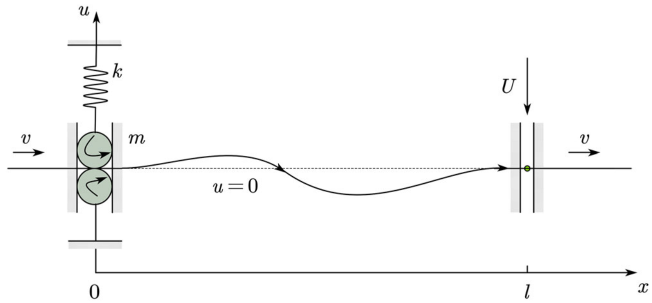


Figure 1. An axially moving string with a spring-mass-dashpot boundary.

approximations by using a central finite difference scheme to validate the theoretical results, and in the last section we draw some conclusions.

2. Formulation of the problem

2.1 Modelling of the physical system

Nomenclature

$u(x,t)$	the transverse displacement of the string at the coordinate x and the time t
l	the distance between two boundary ends
v	the travelling speed of the moving string
ρ	the mass density of the string
m	the mass of the spring-mass
T	the uniform tension of the string
k	the stiffness of the spring

According to Figure 1, we can obtain the partial differential equation (PDE) for the moving string by applying Hamilton's principle in the following form (Meirovitch, 1997):

$$\int_{t_1}^{t_2} (\delta E_k(t) - \delta E_p(t) + \delta W(t))\mathrm{d}t = 0. \tag{1}$$

The kinetic energy $E_k(t)$ is given by

$$E_k(t) = \frac{1}{2}\rho \int_0^l (u_t + vu_x)^2 \mathrm{d}x + \frac{1}{2} m u_t^2(0,t), \tag{2}$$

where $u_t + vu_x$ is the instantaneous transverse velocity of a material particle. The potential energy $E_p(t)$ is given by

$$E_p(t) = \frac{1}{2}\int_0^l T u_x^2 \mathrm{d}x + \frac{1}{2} k u^2(0,t), \tag{3}$$

and the difference of $\delta E_k(t)$ and $\delta E_p(t)$ is

$$\begin{aligned} &\delta E_k(t) - \delta E_p(t) \\ &= \rho \int_0^l (u_t + vu_x)\delta(u_t + vu_x)\mathrm{d}x + m u_t(0,t)\delta u_t(0,t) \\ &\quad - \left[\int_0^l T u_x \delta u_x \mathrm{d}x + k u(0,t)\delta u(0,t)\right]. \end{aligned} \tag{4}$$

The virtual work $\delta W(t)$ is written as

$$\delta W(t) = U(t)\delta u(l,t). \tag{5}$$

Substituting Equations (4)–(5) into (1) yields:

$$\begin{aligned} &\int_{t_1}^{t_2}\int_0^l \rho(u_t + vu_x)\delta(u_t + vu_x)\mathrm{d}x\mathrm{d}t \\ &+ \int_{t_1}^{t_2} m u_t(0,t)\delta u_t(0,t)\mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_0^l T u_x \delta u_x \mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2} k u(0,t)\delta u(0,t)\mathrm{d}t \\ &+ \int_{t_1}^{t_2} U(t)\delta u(l,t)\mathrm{d}t = 0. \end{aligned} \tag{6}$$

Integrating (6) by parts with respect to the spatial variable (refer to E. W. Chen et al., 2021) yields:

$$\begin{cases} \rho(u_{tt} + 2vu_{xt} + v^2 u_{xx}) - T u_{xx} = 0, \ 0 \le x \le l, \quad t > 0, \\ m u_{tt}(0,t) + T u_x(0,t) + k u(0,t) \\ -\rho v u_t(0,t) - \rho v^2 u_x(0,t) = 0, \quad t > 0, \\ T u_x(l,t) - \rho v u_t(l,t) - \rho v^2 u_x(l,t) = U(t), \quad t > 0. \end{cases} \tag{7}$$

For simplicity, we introduce the following dimensionless parameters: $u^* = \frac{u}{l}$, $x^* = \frac{x}{l}$, $t^* = \frac{t}{l}\sqrt{\frac{T}{\rho}}$, $v^* = v\sqrt{\frac{\rho}{T}}$, $m^* = \frac{m}{\rho l}$, $k^* = \frac{kl}{T}$, $U^* = \frac{U}{T}$. The problem (7) then becomes

$$\begin{cases} u_{tt} + 2vu_{xt} + (v^2 - 1)u_{xx} = 0, \quad 0 \le x \le 1, \ t > 0, \\ m u_{tt}(0,t) - (v^2 - 1)u_x(0,t) + k u(0,t) \\ -v u_t(0,t) = 0, \quad t > 0, \\ (1 - v^2)u_x(1,t) - v u_t(1,t) = U(t), \quad t > 0, \end{cases} \tag{8}$$

where the asterisks are omitted in problem (8) for convenience, and $0 < v < 1$.

2.2 Simplified model for controller design

Define the control force as

$$U(t) = -v u_t(1,t) + (1 - v^2)U_2(t), \tag{9}$$

where $U_2(t)$ is a new control. Then, problem (8) can be rewritten as

$$\begin{cases} u_{tt} + 2vu_{xt} + (v^2 - 1)u_{xx} = 0, \quad 0 \le x \le 1, \ t > 0, \\ m u_{tt}(0,t) - (v^2 - 1)u_x(0,t) \\ + k u(0,t) - v u_t(0,t) = 0, \quad t > 0, \\ u_x(1,t) = U_2(t), \quad t > 0. \end{cases} \tag{10}$$

Notice that the axially moving problem (10) is a wave PDE with a second-order derivative in time boundary condition, we introduce new variables $x_1(t)$ and $x_2(t)$:

$$x_1(t) = u(0,t), \quad x_2(t) = u_t(0,t). \tag{11}$$

Substituting (11) into the boundary condition at $x = 0$ in problem (10), we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{v^2-1}{m}u_x(0,t) - \frac{k}{m}u(0,t) + \frac{v}{m}u_t(0,t). \end{aligned} \tag{12}$$

Let $X(t) \in \mathbb{R}^{2\times 1}$ be a state variable:

$$X(t) = [x_1(t), x_2(t)]^T, \tag{13}$$

then we rewrite problem (10) as the following coupled ODE-PDE system:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu_x(0,t), & t > 0, \\ u_{tt} + 2vu_{xt} + (v^2-1)u_{xx} = 0, & 0 \le x \le 1,\ t > 0, \\ u(0,t) = CX(t), & t > 0, \\ u_x(1,t) = U_2(t), & t > 0, \end{cases} \tag{14}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \frac{v}{m} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{v^2-1}{m} \end{pmatrix}, \quad C = \begin{pmatrix} 1, 0 \end{pmatrix}. \tag{15}$$

3. State feedback control

In this section, we construct an invertible transformation to make system (14) equivalent to an ODE-PDE cascade target system. For the target system, we present the well-posedness and stability results in a suitable space.

First, we consider the backstepping transformation of the form (Krstic, 2009):

$$\begin{aligned} w(x,t) = u(x,t) - \int_0^x b(x,y)u(y,t)\mathrm{d}y - \int_0^x c(x,y)u_t(y,t)\mathrm{d}y \\ - \gamma(x)X(t), \end{aligned} \tag{16}$$

where the kernel functions $b(x,y) \in \mathbb{R}$, $c(x,y) \in \mathbb{R}$ and $\gamma(x) \in \mathbb{R}^{1\times 2}$ need to be chosen to transform system (14) into the system of the ODE-PDE cascade

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw_x(0,t), & t > 0, \\ w_{tt} + 2vw_{xt} + (v^2-1)w_{xx} = 0, & 0 \le x \le 1,\ t > 0, \\ w(0,t) = 0, & t > 0, \\ w_x(1,t) = 0, & t > 0, \end{cases} \tag{17}$$

where $K = (k_1, k_2)$ is chosen to make $A + BK$ Hurwitz, and

$$\begin{aligned} U_2(t) = b(1,1)u(1,t) + c(1,1)u_t(1,t) + \gamma'(1)X(t) \\ + \int_0^1 b_x(1,y)u(y,t)\mathrm{d}y + \int_0^1 c_x(1,y)u_t(y,t)\mathrm{d}y. \end{aligned} \tag{18}$$

3.1 Kernels of $b(x,y)$, $c(x,y)$ and $\gamma(x)$

In this subsection, we compute the kernels of $b(x,y)$, $c(x,y)$ and $\gamma(x)$. Differentiating (16) with respect to t and to x, we get

$$\begin{aligned} &w_{tt} + 2vw_{xt} + (v^2-1)w_{xx} \\ &= 2(1-v^2)\left(\frac{\mathrm{d}}{\mathrm{d}x}b(x,x)\right)u(x,t) \\ &+ (1-v^2)\int_0^x (b_{xx}(x,y) - b_{yy}(x,y))u(y,t)\mathrm{d}y \\ &\quad - 2v\int_0^x (b_x(x,y) + b_y(x,y))u_t(y,t)\mathrm{d}y \\ &\quad + 2(1-v^2)\left(\frac{\mathrm{d}}{\mathrm{d}x}c(x,x)\right)u_t(x,t) + (1-v^2) \\ &\quad \int_0^x (c_{xx}(x,y) - c_{yy}(x,y))u_t(y,t)\mathrm{d}y \\ &\quad - 2v\int_0^x (c_x(x,y) + c_y(x,y))u_{tt}(y,t)\mathrm{d}y \\ &\quad + [-2v\gamma'(x)B + (1-v^2)b(x,0) \\ &- 2vb(x,0)CB - \gamma(x)AB \\ &\quad - (1-v^2)c_y(x,0)CB - 2vc(x,0)CAB]u_x(0,t) \\ &\quad + [-\gamma(x)B - 2vc(x,0)CB + (1-v^2)c(x,0)]u_{xt}(0,t) \\ &\quad + [-\gamma(x)A^2 + (1-v^2)\gamma''(x) - 2v\gamma'(x)A \\ &\quad - 2vb(x,0)CA - (1-v^2)b_y(x,0)C \\ &\quad - (1-v^2)c_y(x,0)CA - 2vc(x,0)CA^2]X(t) = 0, \end{aligned} \tag{19}$$

which together with $CB = 0$ yields

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}b(x,x) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}c(x,x) = 0, \\ b_{xx}(x,y) - b_{yy}(x,y) = 0, \quad c_{xx}(x,y) - c_{yy}(x,y) = 0, \\ b_x(x,y) + b_y(x,y) = 0, \quad c_x(x,y) + c_y(x,y) = 0, \\ -2v\gamma'(x)B + (1-v^2)b(x,0) - \gamma(x)AB \\ -2vc(x,0)CAB = 0, \\ -\gamma(x)B - 2vc(x,0)CB + (1-v^2)c(x,0) = 0, \\ -\gamma(x)A^2 + (1-v^2)\gamma''(x) - 2v\gamma'(x)A - 2vb(x,0)CA \\ -(1-v^2)b_y(x,0)C - (1-v^2)c_y(x,0)CA \\ -2vc(x,0)CA^2 = 0. \end{cases} \tag{20}$$

Substituting transformation (16) into the first and third equations of system (17), we derive

$$\begin{aligned} \gamma(0) &= C, \\ \gamma'(0) &= K - b(0,0)C - c(0,0)CA, \end{aligned} \tag{21}$$

for which, the solutions $b(x,y)$, $c(x,y)$ and $\gamma(x)$ of (20) can be presented as follows:

$$\gamma(x) = [\gamma(0), \gamma'(0)]e^{Dx}\begin{pmatrix} I \\ 0 \end{pmatrix},$$

$$b(x,y) = \frac{2v}{1-v^2}\gamma'(x-y)B + \frac{1}{1-v^2}\gamma(x-y)AB$$

$$+\frac{2\nu}{(1-\nu^2)^2}\gamma(x-y)BCAB,$$
$$c(x,y)=\frac{1}{1-\nu^2}\gamma(x-y)B, \tag{22}$$

where

$$\gamma(0)=(1,0),$$
$$\gamma'(0)=\left(k_1+\frac{1+2\nu k_2}{m},k_2\right),$$
$$D=\begin{pmatrix}0 & 0 & \frac{1+3\nu^2}{m(1-\nu^2)} & \frac{2\nu}{1-\nu^2}\\ 0 & 0 & \frac{2km\nu+\nu(1-5\nu^2)}{m^2(1-\nu^2)} & \frac{1-3\nu^2}{m(1-\nu^2)}\\ 1 & 0 & -\frac{km+2\nu^2}{m^2(1-\nu^2)} & -\frac{\nu}{m(1-\nu^2)}\\ 0 & 1 & \frac{-km\nu+2\nu^3}{m^3(1-\nu^2)} & \frac{\nu^2-mk}{m^2(1-\nu^2)}\end{pmatrix}.$$

In the same deduction, we seek the inverse transformation $w(x,t)\to u(x,t)$:

$$u(x,t)=w(x,t)-\int_0^x\varphi(x,y)w(y,t)\mathrm{d}y$$
$$-\int_0^x\lambda(x,y)w_t(y,t)\mathrm{d}y-\alpha(x)X(t), \tag{23}$$

with

$$\alpha(x)=[-C,-K]e^{Zx}\begin{pmatrix}I\\0\end{pmatrix},$$
$$\varphi(x,y)=\frac{1}{1-\nu^2}\alpha(x-y)(A+BK)B+\frac{2\nu}{1-\nu^2}\alpha'(x-y)B,$$
$$\lambda(x,y)=\alpha(x-y)B,$$
$$Z=\begin{pmatrix}0 & 0 & \frac{k_1\nu^2-k_1-k}{m(1-\nu^2)} & \frac{k_2\nu^2-k_2+\nu}{m(1-\nu^2)}\\ 0 & 0 & \frac{(k_1\nu^2-k_1-k)(k_2\nu^2-k_2+\nu)}{m^2(1-\nu^2)} & \frac{(k_2\nu^2-k_2+\nu)^2}{m^2(1-\nu^2)}\\ 1 & 0 & 0 & \frac{2\nu}{1-\nu^2}\\ 0 & 1 & \frac{2\nu(k_1\nu^2-k_1-k)}{m(1-\nu^2)} & \frac{2\nu(k_2\nu^2-k_2+\nu)}{m(1-\nu^2)}\end{pmatrix}.$$

3.2 Stability of target system

Firstly, let us reformulate target system (17) in an appropriate Hilbert state space $\mathcal{H}$. Let $\mathcal{H}$ be the following space:

$$\mathcal{H}=\mathbb{R}^2\times V^1(0,1)\times L^2(0,1),$$
$$V^k(0,1)=\{\xi\in H^k(0,1)|\xi(0)=0\}, \tag{24}$$

equipped with an inner product, for $(X_1,w_1,w_2),(X_2,\bar{w}_1,\bar{w}_2)\in\mathcal{H}$:

$$\langle(X_1,w_1,w_2),(X_2,\bar{w}_1,\bar{w}_2)\rangle$$
$$=X_1^TX_2+\int_0^1(w_2+\nu w_{1,x})(\overline{\bar{w}_2}+\nu\overline{\bar{w}_{1,x}})\,\mathrm{d}x$$
$$+\int_0^1 w_{1,x}\overline{\bar{w}_{1,x}}\,\mathrm{d}x. \tag{25}$$

Define a linear operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ as follows:

$$\begin{cases}\mathcal{A}(X,f_1,f_2)=((A+BK)X(t)\\ \qquad +Bf_1'(0),f_2,-2\nu f_2'+(1-\nu^2)f_1''),\\ D(\mathcal{A})=\{(X,f_1,f_2)\in\mathbb{R}^2\times H^2(0,1)\times V^1(0,1)|f_1'(1)=0\}.\end{cases} \tag{26}$$

Then, system (17) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(X(t),w(\cdot,t),w_t(\cdot,t))=\mathcal{A}(X(t),w(\cdot,t),w_t(\cdot,t)). \tag{27}$$

Lemma 3.1: *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$.*

Proof: Define an equivalent inner product:

$$\langle(X_1,w_1,w_2),(X_2,\bar{w}_1,\bar{w}_2)\rangle_1$$
$$=\mu X_1^TP_1X_2+\int_0^1(w_2+\nu w_{1,x})(\overline{\bar{w}_2}+\nu\overline{\bar{w}_{1,x}})\mathrm{d}x$$
$$+\int_0^1 w_{1,x}\overline{\bar{w}_{1,x}}\,\mathrm{d}x, \tag{28}$$

where

$$0<\mu\le\frac{\nu(1-\nu^2)\lambda_{\min}(Q_1)}{2|P_1B|^2}, \tag{29}$$

and the matrix $P_1=P_1^T>0$ is the solution to the equation:

$$P_1(A+BK)+(A+BK)^TP_1=-Q_1, \tag{30}$$

for some $Q_1=Q_1^T>0$. For any $z=(X(t),w(\cdot,t),w_t(\cdot,t))^T\in D(\mathcal{A})$, a straightforward calculation yields

$$\Re\langle\mathcal{A}z,z\rangle_1$$
$$=-\frac{\mu}{2}X(t)^TQ_1X(t)+\mu X(t)^TP_1Bw_x(0,t)-\frac{\nu}{2}w_t^2(1,t)$$
$$-\frac{\nu(1-\nu^2)}{2}w_x^2(0,t).$$

According to Young's inequality $ab\le\frac{\varepsilon a^2}{2}+\frac{b^2}{2\varepsilon}$, we obtain

$$\Re\langle\mathcal{A}z,z\rangle_1$$
$$\le-\frac{\mu\lambda_{\min}(Q_1)}{4}|X(t)|^2$$
$$-\left[-\frac{\mu|P_1B|^2}{\lambda_{\min}(Q_1)}+\frac{\nu(1-\nu^2)}{2}\right]w_x^2(0,t)-\frac{\nu}{2}w_t^2(1,t)\le0,$$

where μ is given by (29). Hence, $\mathcal{A}$ is dissipative in $\mathcal{H}$. Moreover, let $(Y,g_1,g_2)\in\mathcal{H}$, and solve $\mathcal{A}(X,f_1,f_2)=(Y,g_1,g_2)$ for

$(X, f_1, f_2) \in D(\mathcal{A})$, that is,

$$\begin{cases} (A+BK)X(t) + Bf_1'(0) = Y, \\ f_2 = g_1, \\ -2\nu f_2' + (1-\nu^2)f_1'' = g_2, \\ f_1(0) = 0,\ f_1'(1) = 0. \end{cases} \tag{31}$$

A direct computation gives the unique solution

$$\begin{cases} f_2 = g_1, \\ f_1 = -\int_0^x \int_\xi^1 \frac{1}{1-\nu^2}(g_2(\varsigma) + 2\nu g_1'(\varsigma))\,\mathrm{d}\varsigma\,\mathrm{d}\xi, \\ X(t) = (A+BK)^{-1}Y \\ -\left[-\int_0^1 \frac{1}{1-\nu^2}(g_2(\varsigma) + 2\nu g_1'(\varsigma))\mathrm{d}\varsigma\right](A+BK)^{-1}B. \end{cases} \tag{32}$$

Hence, we get the unique solution $(X, f_1, f_2) \in D(\mathcal{A})$ and $\mathcal{A}^{-1}$ exists. The Sobolev embedding theorem (Adams & Fournier, 2003) implies that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, the Lumer–Phillips theorem asserts that $\mathcal{A}$ generates a C_0 semi-group of contractions on $\mathcal{H}$. The proof is complete. ■

Lemma 3.2: *For any initial values $(X(t), w(x, 0), w_t(x, 0))$, which belong to $\mathcal{H}$, the target system (17) is exponentially stable in $\mathcal{H}$.*

Proof: Define

$$\Xi_1(t) = \|w_t(t) + \nu w_x(t)\|^2 + \|w_x(t)\|^2 + |X(t)|^2. \tag{33}$$

Let V_1 be a Lyapunov function written as

$$V_1(t) = X(t)^T P_1 X(t) + a_1 E_1(t), \tag{34}$$

where the matrix P_1 is given by (30). The positive parameter a_1 is to be chosen later and function $E_1(t)$ is defined by

$$E_1(t) = \frac{1}{2}\left[\|w_t(t) + \nu w_x(t)\|^2 + \|w_x(t)\|^2\right] + \delta_1 \int_0^1 (1+y) w_x(y,t)\left[w_t(y,t) + \nu w_x(y,t)\right]\mathrm{d}y. \tag{35}$$

We observe that

$$\theta_{11}\Xi_1(t) \le V_1(t) \le \theta_{12}\Xi_1(t), \tag{36}$$

where

$$\theta_{11} = \min\left\{\lambda_{\min}(P_1), \frac{a_1}{2}[1-2\delta_1]\right\},$$
$$\theta_{12} = \max\left\{\lambda_{\max}(P_1), \frac{a_1}{2}[1+2\delta_1]\right\}. \tag{37}$$

We choose $0 < \delta_1 < \frac{1}{2}$.

$$\begin{aligned} &\dot V_1(t) \\ &= -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t) + a_1 \dot E_1(t) \\ &= -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t) \\ &+ a_1\left[-\frac{\delta_1}{2}((1-\nu^2)\|w_x\|^2 + \|w_t\|^2 + (1-\nu^2)|w_x(0,t)|^2)\right. \\ &\left.-\left(\frac{\nu}{2} - \delta_1\right)|w_t(1,t)|^2 - \frac{\nu(1-\nu^2)}{2}|w_x(0,t)|^2\right] \\ &\le -\frac{\lambda_{\min}(Q_1)}{2}|X(t)|^2 - \left[-\frac{2|P_1B|^2}{\lambda_{\min}(Q_1)} + \frac{a_1\nu(1-\nu^2)}{2}\right. \\ &\left.+\frac{a_1\delta_1(1-\nu^2)}{2}\right]|w_x(0,t)|^2 \\ &- a_1\left(\frac{\nu}{2} - \delta_1\right)|w_t(1,t)|^2 - \frac{a_1\delta_1}{2}((1-\nu^2)\|w_x\|^2 + \|w_t\|^2). \end{aligned} \tag{38}$$

To have $\dot V_1(t) < 0$ we choose

$$a_1 \ge \frac{4|P_1B|^2}{\left[\nu(1-\nu^2) + \delta_1(1-\nu^2)\right]\lambda_{\min}(Q_1)}, \quad 0 < \delta_1 \le \frac{\nu}{2}. \tag{39}$$

We now have

$$\begin{aligned} \dot V_1(t) &\le -\frac{\lambda_{\min}(Q_1)}{2}|X(t)|^2 \\ &\quad - \frac{a_1\delta_1(1-\nu^2)}{2(1+\nu^2+\nu)}\left(\|w_x\|^2 + \|w_t + \nu w_x\|^2\right) \\ &\le -\eta_1 V_1(t), \end{aligned} \tag{40}$$

where

$$\eta_1 = \frac{\min\left\{\frac{\lambda_{\min}(Q_1)}{2}, \frac{a_1\delta_1(1-\nu^2)}{2(1+\nu^2+\nu)}\right\}}{\theta_{12}}. \tag{41}$$

Thus, we arrive at

$$V_1(t) \le e^{-\eta_1 t} V_1(0). \tag{42}$$

The proof is complete. ■

Theorem 3.3: *For initial value $(X(0), u(x, 0), u_t(x, 0))$, which belongs to $\mathcal{H}$, the closed-loop system (14) with state feedback control law $U_2(t)$ in (18) admits a unique solution $(X(t), u(x, t), u_t(x, t))$ that decays to zero exponentially in $\mathcal{H}$ as time t goes to infinity.*

Proof: The equivalent well-posedness and stability property between the target system (17) and the closed-loop system (14) are ensured due to the invertibility of the backstepping transformation. Then by Lemmas 3.1 and 3.2, the proof is complete. ■

4. Observer and output feedback control

In this section we consider an observer-based output feedback control law, and the observation output is given as

$$y_{out}(t) = CX(t), \tag{43}$$

where C and $\mathcal{A}$ are given by (15) and (26), and $(\mathcal{A}, C)$ is observable.

4.1 Observer design

Design the observer of system (14):

$$\begin{cases}\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}_x(0,t) + \bar{L}C(X(t) - \hat{X}(t)), & t > 0,\\ \hat{u}_{tt} = -2v\hat{u}_{xt} + (1-v^2)\hat{u}_{xx}, \quad 0 \le x \le 1, & t > 0,\\ \hat{u}(0,t) = CX(t), & t > 0,\\ \hat{u}_x(1,t) = U_2(t), & t > 0.\end{cases} \tag{44}$$

The observer gain $\bar{L} = (\bar{l}_1, \bar{l}_2)^T$ is chosen to make $A - \bar{L}C$ Hurwitz. Define the observer error as

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \quad \tilde{X}(t) = X(t) - \hat{X}(t). \tag{45}$$

Then the observer error system can be written as

$$\begin{cases}\dot{\tilde{X}}(t) = (A - \bar{L}C)\tilde{X}(t) + B\tilde{u}_x(0,t), & t > 0,\\ \tilde{u}_{tt} = -2v\tilde{u}_{xt} + (1-v^2)\tilde{u}_{xx}, & 0 \le x \le 1,\ t > 0,\\ \tilde{u}(0,t) = 0, & t > 0,\\ \tilde{u}_x(1,t) = 0, & t > 0.\end{cases} \tag{46}$$

Let us reformulate error system (46) in Hilbert state space $\mathcal{H}$, equipped with inner product in (25). Define a linear operator $\tilde{\mathcal{A}} : D(\tilde{\mathcal{A}}) \subset \mathcal{H} \to \mathcal{H}$ as follows:

$$\begin{cases}\tilde{\mathcal{A}}(X, f_1, f_2)\\ = \left((A - \bar{L}C)X(t) + Bf_1'(0), f_2, -2vf_2' + (1-v^2)f_1''\right),\\ D(\tilde{\mathcal{A}}) = \left\{(X, f_1, f_2) \in \mathbb{R}^2 \times H^2(0,1) \times V^1(0,1) | f_1'(1) = 0\right\}.\end{cases} \tag{47}$$

Then, system (46) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)) = \tilde{\mathcal{A}}(\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)). \tag{48}$$

Theorem 4.1: *For initial value $(\tilde{X}(0), \tilde{u}(x,0), \tilde{u}_t(x,0))$, which belongs to $\mathcal{H}$, the error system (46) admits a unique solution $(\tilde{X}(t), \tilde{u}(x,t), \tilde{u}_t(x,t))$ that decays to zero exponentially in $\mathcal{H}$ as time t goes to infinity.*

Proof: The proofs are similar to the proofs for Lemmas 3.1 and 3.2, so we omit the details here. ■

4.2 Output feedback control

Based on the state feedback controller (18) and observer (44), we can naturally design the following output-feedback controller:

$$U_2(t) = b(1,1)\hat{u}(1,t) + \gamma'(1)\hat{X}(t) + \int_0^1 b_x(1,y)\hat{u}(y,t)\mathrm{d}y + \int_0^1 c_x(1,y)\hat{u}_t(y,t)\mathrm{d}y, \tag{49}$$

which leads to the closed-loop system of (14):

$$\begin{cases}\dot{X}(t) = AX(t) + Bu_x(0,t),\\ u_{tt} + 2vu_{xt} + (v^2-1)u_{xx} = 0,\\ u(0,t) = CX(t),\\ u_x(1,t) = b(1,1)\hat{u}(1,t) + \gamma'(1)\hat{X}(t) + \int_0^1 b_x(1,y)\hat{u}(y,t)\mathrm{d}y\\ + \int_0^1 c_x(1,y)\hat{u}_t(y,t)\mathrm{d}y,\\ \dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}_x(0,t) + \bar{L}C(X(t) - \hat{X}(t)),\\ \hat{u}_{tt} = -2v\hat{u}_{xt} + (1-v^2)\hat{u}_{xx}, \quad 0 \le x \le 1,\\ \hat{u}(0,t) = CX(t),\\ \hat{u}_x(1,t) = b(1,1)\hat{u}(1,t) + \gamma'(1)\hat{X}(t) + \int_0^1 b_x(1,y)\hat{u}(y,t)\mathrm{d}y\\ + \int_0^1 c_x(1,y)\hat{u}_t(y,t)\mathrm{d}y.\end{cases} \tag{50}$$

Theorem 4.2: *For any initial state $(X(0), u(x,0), u_t(x,0), \hat{X}(0), \hat{u}(x,0), \hat{u}_t(x,0)) \in \mathcal{H}^2$, the closed-loop system (50) admits a unique solution $(X(t), u(x,t), u_t(x,t), \hat{X}(t), \hat{u}(x,t), \hat{u}_t(x,t))$ that decays to zero exponentially in $\mathcal{H}$ as time t goes to infinity.*

Proof: By using the transformation (16), the closed-loop system (50) can be converted to the following equivalent system:

$$\begin{cases}\dot{X}(t) = (A + BK)X(t) + Bw_x(0,t), & t > 0,\\ w_{tt} + 2vw_{xt} + (v^2-1)w_{xx} = 0, & 0 \le x \le 1,\ t > 0,\\ w(0,t) = 0, & t > 0,\\ w_x(1,t) = \mathcal{F}(\tilde{X}, \tilde{u}, \tilde{u}_t),\\ \dot{\tilde{X}}(t) = (A - \bar{L}C)\tilde{X}(t) + B\tilde{u}_x(0,t), & t > 0,\\ \tilde{u}_{tt} = -2v\tilde{u}_{xt} + (1-v^2)\tilde{u}_{xx}, & 0 \le x \le 1,\ t > 0,\\ \tilde{u}(0,t) = 0, & t > 0,\\ \tilde{u}_x(1,t) = 0, & t > 0,\end{cases} \tag{51}$$

where operator $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ defined by $\mathcal{F}(\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)) = b(1,1)\tilde{u}(1,t) + \gamma'(1)\tilde{X}(t) + \int_0^1 b_x(1,y)\tilde{u}(y,t)\mathrm{d}y + \int_0^1 c_x(1,y)\tilde{u}_t(y,t)\mathrm{d}y$. The proof will be completed if we can prove that (51) has a unique solution and is exponentially stable in $\mathcal{H}$. ■

The closed-loop system (51) can be written as the following evolution equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \mathcal{A}Y(t) + \bar{B}\mathcal{F}\tilde{Y}(t), \tag{52}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}(t) = \tilde{\mathcal{A}}\tilde{Y}(t), \tag{53}$$

where $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are given by (26) and (47), $Y(t) = (X(t), w(\cdot,t), w_t(\cdot,t)) \in \mathcal{H}$, $\tilde{Y}(t) = (\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)) \in \mathcal{H}$, and

$$\bar{B}\mathcal{F}\tilde{Y}(t) = \left[0, 0, \delta(x-1)\mathcal{F}\tilde{Y}(t)\right] \tag{54}$$

with δ being a Dirac function. The operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ generate C_0 semigroup of contractions $e^{\mathcal{A}t}$ and $e^{\tilde{\mathcal{A}}t}$ on $\mathcal{H}$, respectively. Notice that $\bar{B}$ is an unbounded operator, we will show that $\bar{B}$ is admissible to the C_0 semigroup $e^{\mathcal{A}t}$.

Lemma 4.3: *$\bar{B}$ is admissible to the C_0 semigroup $e^{\mathcal{A}t}$.*

Proof: As $\mathcal{A}^*$ is defined by

$$\begin{cases} \mathcal{A}^*(X,f_1,f_2) = \left((A+BK)X(t), -f_2, 2\nu f_2' - (1-\nu^2)f_1''\right), \\ D(\mathcal{A}^*) = \left\{ \begin{array}{l} (X,f_1,f_2) \in \mathbb{R}^2 \times H^2(0,1) \times H^1(0,1), \\ (1-\nu^2)f_1'(1) - \nu f_2(1) = 0, \\ (1-\nu^2)(\nu f_1'(0) + f_2(0)) - B^T X^* = 0. \end{array} \right\}, \end{cases} \tag{55}$$

the dual system of (52) can be written as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(X^*(t), -w^*(\cdot,t), w_t^*(\cdot,t)) \\ = \mathcal{A}^*(X^*(t), -w^*(\cdot,t), w_t^*(\cdot,t)), \\ y(t) = \bar{B}^*(X^*(t), -w^*(\cdot,t), w_t^*(\cdot,t)) = \frac{w_t^*(1,t)}{1-\nu^2}, \end{cases} \tag{56}$$

which means

$$\begin{cases} \dot{X}^*(t) = (A+BK)X^*(t), \quad t>0, \\ w_{tt}^* - 2\nu w_{xt}^* + (\nu^2-1)w_{xx}^* = 0, \quad 0 \le x \le 1,\ t>0, \\ (1-\nu^2)w_x^*(1,t) + \nu w_t^*(1,t) = 0, \quad t>0, \\ (1-\nu^2)(-\nu w_x^*(0,t) + w_t^*(0,t)) - B^T X^* = 0, \quad t>0. \end{cases} \tag{57}$$

The energy function of system is defined by

$$E^*(t) = \frac{1}{2}X^*(t)^T X^*(t) + \frac{1}{2}\int_0^1 (w_t^* - \nu w_x^*)^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 w_x^{*2}\,\mathrm{d}x. \tag{58}$$

Let

$$E_1^*(t) = \frac{\chi}{2}X^*(t)^T P_1 X^*(t) + \frac{1}{2}\int_0^1 (w_t^* - \nu w_x^*)^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 w_x^{*2}\,\mathrm{d}x, \tag{59}$$

where P_1 is given by (30), and β is given by

$$\chi \ge \frac{2|B|^2}{\nu(1-\nu^2)\lambda_{\min}(Q_1)}. \tag{60}$$

A simple computation for the derivative of $E_1^*(t)$ with respect to t along the solution to (57) gives

$$\begin{aligned} \dot{E}_1^*(t) &= -\frac{\chi}{2}X^*(t)^T Q_1 X^*(t) - B^T X^*(t) w_x^*(0,t) - \frac{\nu}{2}|w_t^*(1,t)|^2 \\ &\quad - \frac{\nu(1-\nu^2)}{2}|w_x^*(1,t)|^2 - \frac{\nu}{2}|w_t^*(0,t)|^2 \\ &\quad - \frac{\nu(1-\nu^2)}{2}|w_x^*(0,t)|^2 \\ &\le -\frac{\chi\lambda_{\min}(Q_1)}{4}|X^*(t)|^2 \\ &\quad + \left[\frac{|B|^2}{\chi\lambda_{\min}(Q_1)} - \frac{\nu(1-\nu^2)}{2}\right]|w_x^*(0,t)|^2 \\ &\quad - \frac{\nu}{2}|w_t^*(1,t)|^2 - \frac{\nu(1-\nu^2)}{2}|w_x^*(1,t)|^2 - \frac{\nu}{2}|w_t^*(0,t)|^2. \end{aligned} \tag{61}$$

Hence, $E_1^*(t) \le E_1^*(0)$. Define

$$\rho(t) = \int_0^1 x(w_t^* - \nu w_x^*)w_x^*\,\mathrm{d}x. \tag{62}$$

Then $\rho(t) \le E_1^*(t)$ for $\forall\ t \ge 0$. Noticing that

$$\dot{\rho}(t) = \frac{1}{2(1-\nu^2)}|w_t^*(1,t)|^2 - \frac{1}{2}\int_0^1 (1-\nu^2)w_x^{*2}\,\mathrm{d}x - \frac{1}{2}\int_0^1 w_t^{*2}\,\mathrm{d}x, \tag{63}$$

we have that

$$\begin{aligned} &\int_0^T w_t^{*2}(1,t)\mathrm{d}t \\ &= 2(1-\nu^2)\left[\rho(T) - \rho(0)\right] \\ &\quad + (1-\nu^2)\int_0^T \left[\int_0^1 (1-\nu^2)w_x^{*2} + w_t^{*2}\,\mathrm{d}x\right]\mathrm{d}t \\ &\le \left[4(1-\nu^2) + \frac{T(1-\nu^2)}{1-\nu}\right]E_1^*(0) \\ &\le \left[4(1-\nu^2) + \frac{T(1-\nu^2)}{1-\nu}\right]\eta^* E^*(0), \end{aligned} \tag{64}$$

where $\eta^* = \max\{\chi\lambda_{\max}(P_1), 1\}$. A direct calculation shows that

$$\bar{B}^*\mathcal{A}^{*-1}(Y, g_1, g_2) = -\frac{g_1(1)}{1-\nu^2}, \quad \forall\ (Y, g_1, g_2) \in \mathcal{H}, \tag{65}$$

which tells us that $\bar{B}^*\mathcal{A}^{*-1}$ is bounded. This together with (64) yields that $\bar{B}^*$ is admissible for $e^{\mathcal{A}^* t}$, which means that $\bar{B}$ is admissible for $e^{\mathcal{A}t}$. The proof is completed. ■

To prove the stability of the closed-loop system (51), define

$$\tilde{V}(t) = \tilde{X}(t)^T P_2 \tilde{X}(t) + a_2 E_2(t). \tag{66}$$

The matrix $P_2 = P_2^T > 0$ is the solution to the equation:

$$P_2(A - \bar{L}C) + (A - \bar{L}C)^T P_2 = -Q_2, \tag{67}$$

for some $Q_2 = Q_2^T > 0$. Function $E_2(t)$ is defined by

$$E_2(t) = \frac{1}{2}\left(\|\tilde{u}_t(t) + \nu\tilde{u}_x(t)\|^2 + \|\tilde{u}_x(t)\|^2\right) + \delta_2\int_0^1 (1+y)\tilde{u}_x(y,t)\left(\tilde{u}_t(y,t) + \nu\tilde{u}_x(y,t)\right)\mathrm{d}y. \tag{68}$$

By choosing

$$a_2 \ge \frac{4|P_2 B|^2}{\left[\nu(1-\nu^2) + \delta_2(1-\nu^2)\right]\lambda_{\min}(Q_2)}, \quad 0 < \delta_2 \le \frac{\nu}{2}, \tag{69}$$

and according to the proof the stability of the target system in Section 3.2, we arrive at

$$\tilde{V}(t) \le e^{-\eta_2 t}\tilde{V}(0), \tag{70}$$

where

$$\eta_2 = \frac{\min\left\{\frac{\lambda_{\min}(Q_2)}{2}, \frac{a_2\delta_2(1-\nu^2)}{2(1-\nu^2+\nu)}\right\}}{\theta_{22}}. \tag{71}$$

Let V be a Lyapunov function written as

$$\begin{aligned}
V(t) &= V_1(t) + \beta \tilde{V}(t), \qquad (72)\\
\dot{V}(t) &= -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t)\\
&\quad + a_1 \dot{E}_1(t) + \beta \tilde{V}(t)\\
&= -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t)\\
&\quad + a_1 \Big[-\frac{\delta_1}{2}((1-\nu^2)\|w_x\|^2 + \|w_t\|^2\\
&\quad + (1-\nu^2)|w_x(0,t)|^2)\\
&\quad - \left(\frac{\nu}{2} - \delta_1\right)|w_t(1,t)|^2 - \frac{\nu(1-\nu^2)}{2}|w_x(0,t)|^2 \Big]\\
&\quad + a_1\left(\delta_1 + \frac{\nu}{2}\right)(1-\nu^2)|w_x(1,t)|^2\\
&\quad + a_1(1-\nu^2) w_t(1,t) w_x(1,t) + \beta \dot{\tilde{V}}(t)\\
&\le -\frac{\lambda_{\min}(Q_1)}{2}|X(t)|^2 - \Big[-\frac{2|P_1B|^2}{\lambda_{\min}(Q_1)} + \frac{a_1\nu(1-\nu^2)}{2}\\
&\quad + \frac{a_1\delta_1(1-\nu^2)}{2} \Big] |w_x(0,t)|^2\\
&\quad + a_1(1-\nu^2)\left(\delta_1 + \frac{\nu}{2} + \frac{1}{\nu}\right)|w_x(1,t)|^2\\
&\quad - a_1\left(\frac{\nu}{4} - \delta_1\right)|w_t(1,t)|^2\\
&\quad - \frac{a_1\delta_1}{2}\left[(1-\nu^2)\|w_x\|^2 + \|w_t\|^2\right] - \beta\eta_2 \tilde{V}(t). \qquad (73)
\end{aligned}$$

From the second boundary condition in the closed-loop system (51), we obtain

$$\begin{aligned}
|w_x(1,t)|^2 &\le b^2(1,1)|\tilde{u}(1,t)|^2 + (\gamma'(1))^2 |\tilde{X}(t)|^2\\
&\quad + \left(\int_0^1 b_x(1,y)\tilde{u}(y,t)\mathrm{d}y\right)^2\\
&\quad + \left(\int_0^1 c_x(1,y)\tilde{u}_t(y,t)\mathrm{d}y\right)^2. \qquad (74)
\end{aligned}$$

According to *Agmon's* inequality and *Poincaré* inequality, we obtain

$$\begin{aligned}
\|\tilde{u}(t)\|^2 &\le 2|\tilde{u}(0,t)|^2 + 4\|\tilde{u}_x(t)\|^2,\\
|\tilde{u}(1,t)|^2 &\le 3|\tilde{u}(0,t)|^2 + 5\|\tilde{u}_x(t)\|^2. \qquad (75)
\end{aligned}$$

Thus, by choosing β big enough and

$$a_1 \ge \frac{4|P_1B|^2}{[\nu(1-\nu^2) + \delta_1(1-\nu^2)]\lambda_{\min}(Q_1)}, \quad 0 < \delta_1 \le \frac{\nu}{4}, \tag{76}$$

there exits η_3 such that

$$\dot{V}(t) \le -\eta_3 V(t), \tag{77}$$

Thus, we arrive at

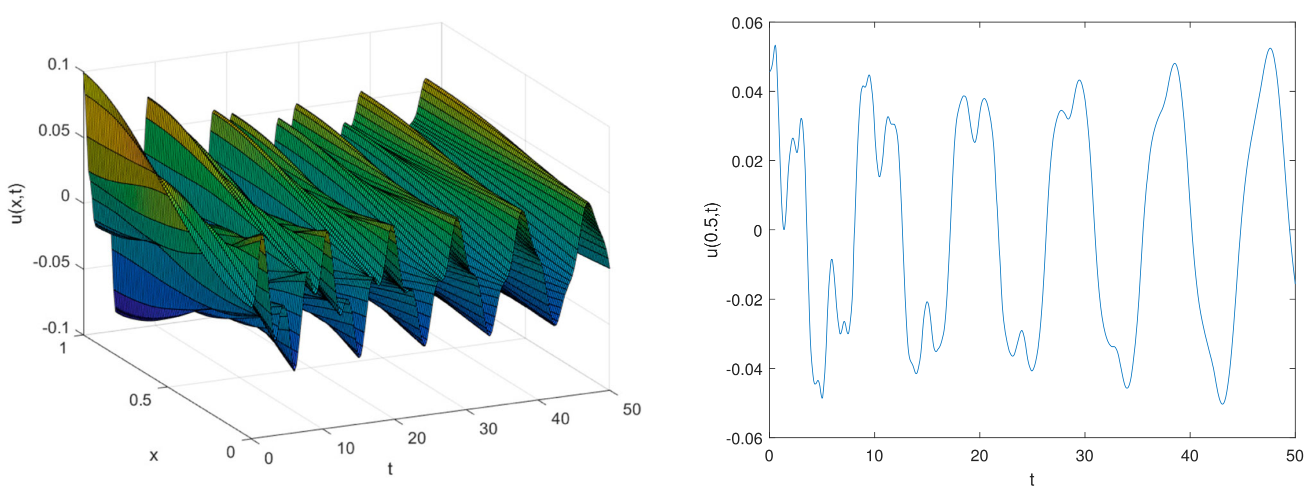
$$V(t) \le e^{-\eta_3 t} V(0). \tag{78}$$

Thus, the closed-loop system of (51) admits a unique solution and decays to zero exponentially in $\mathcal{H}$ as time t goes to infinity. The proof is complete.

5. Numerical simulation

In this section, we give some numerical simulation results for the system (8). The finite difference method is adopted in both the time and the space domain for both PDEs and boundary conditions in (8). In the numerical scheme, we choose the space grid size $N = 200$, time step $\mathrm{d}t = 5 \times 10^{-2}$. The parameter values are set to be

$$\begin{aligned}
&\nu = 0.1,\ k = 1,\ m = 1,\ K = [k_1, k_2] = [1, 1],\\
&\bar{L} = [\bar{l}_1, \bar{l}_2] = [-1, 0.5], \qquad (79)
\end{aligned}$$



(a) (b)

Figure 2. The state of system (8) when $U(t) = 0$. (a) The responses for the whole space domain (0, 1). (b) The responses at the midpoint.

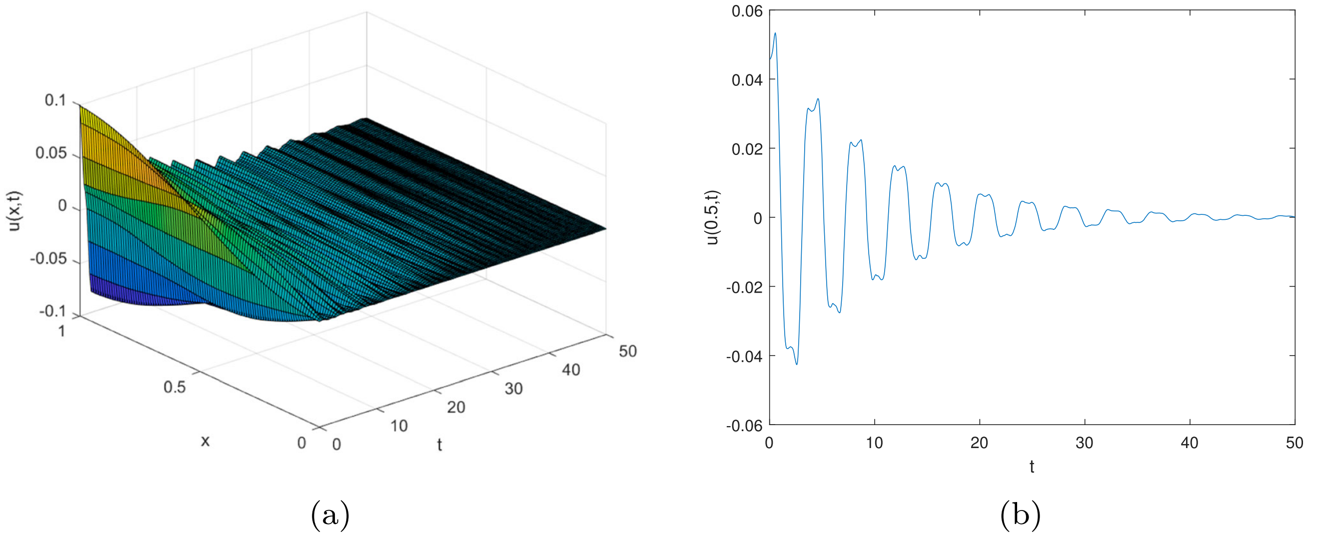


Figure 3. The state $u(x,t)$ of closed-loop system (8) with full-state feedback controller (18). (a) The responses for the whole space domain (0, 1). (b) The responses at the midpoint.

and the initial conditions are taken to be

$$u(x,0) = 0.1\sin(1.5x), \quad u_t(x,0) = 0, \quad 0 \le \xi \le 1. \qquad (80)$$

Figure 2 shows that the displacement of the open-loop system (8) is not convergent to zero. Figure 3 show that the displacement of the closed-loop system (8) with state feedback controller (18) converges to zero. It can be seen that the output feedback controller can make the closed-loop system exponentially stable.

6. Concluding remarks

In this paper, we have presented a control design to stabilise an unstable moving string subject to a spring-mass-dashpot boundary, where the control actuator is located at the other boundary of the string. Firstly, by a transformation for boundary condition, the problem can be convert to a coupled ODE-PDE system. Secondly, by an invertible backstepping transformation, the coupled ODE-PDE system is equivalent to a target system of ODE-PDE cascades, which is shown to be exponentially stable in a suitable Hilbert space. Thirdly, we design the observer-based output feedback controller. It is shown that by using boundary measurements only, the output feedback can make the closed-loop system exponentially stable. Finally, the simulation results illustrate that the proposed control law can efficiently suppress the axial vibrations of the moving string system.

Acknowledgments

We appreciate the anonymous reviewers for their valuable suggestions and comments to improving the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by the National Natural Science Foundation of China (No. 62073037).

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