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DOI
10.3997/2214-4609.202210061

Publication date
2022

Document Version
Final published version

Published in
Proceedings of the 83rd EAGE Annual Conference & Exhibition

EFFICIENCY OF OLD AND NEW TRIANGULAR FINITE ELEMENTS FOR WAVEFIELD MODELLING IN TIME

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Summary

Finite elements with mass lumping allow for explicit time stepping when modelling wave propagation and can be more efficient than finite differences in complex geological settings. In 2D on quadrilaterals, spectral elements are the obvious choice. Triangles are more flexible for meshing, but the construction of polynomial elements is less straightforward. So far, elements up to degree 9 have been found. Some years ago, an accuracy criterion that is sharper and less restrictive than the customary one led to new tetrahedral elements that are considerably more efficient than those previously known. Applying the same criterion to triangular elements provides infinitely many new elements of degree 5, with the same number of nodes as the old one, and two elements of degree 6 with less nodes than the known ones. Their efficiency, measured in terms of the compute time needed to obtain a solution with a given accuracy, is determined for a homogeneous problem and compared to that of the old elements of degree 1 to 8. For moderate accuracy, elements of degree 3 are the most efficient. For high accuracy, one of the new degree-6 elements performs best.

Efficiency of old and new triangular finite elements for wavefield modelling in time

Introduction

The finite-element method can be attractive for solving the wave equation because irregular topography and large contrasts in material parameters do not have to degrade the spatial accuracy, as is the case with higher-order finite differences. Mass lumping avoids the cost of a lower-upper decomposition of the large sparse mass matrix. However, when applied to the regular polynomial elements on triangles or tetrahedra, mass lumping can lead to non-positive weights, causing instability of the explicit time stepping scheme. In addition, the spatial accuracy may decrease. These two problems can be avoided by enriching the element with higher-degree polynomials in the interior. Unlike spectral elements on quadrilaterals, the construction of higher-order continuous mass-lumped finite elements is not straightforward. So far, elements up to degree 9 on triangles have been found by various authors (Zienkiewicz, 1973; Tordjman, 1995; Mulder, 1996; Chin-Joe-Kong et al., 1999; Cohen et al., 2001; Cui et al., 2017; Liu et al., 2017).

The classic accuracy criterion (Ciarlet, 1978, e.g.) requires that polynomials up to a certain degree should be integrated exactly by the quadrature weights. The degree depends on the degree of the element on the edges, the higher degree in its interior and the order of the partial differential equation. The lumped mass matrix is proportional to the weights. Geevers et al. (2018) proposed a sharper and less restrictive accuracy criterion, enabling the construction of tetrahedral elements that are significantly more efficient than the ones obtained with the classic criterion (Mulder, 1996; Chin-Joe-Kong et al., 1999). When applied to triangular elements, the less restrictive criterion provides the same elements for degree 2 to 4 as with the classic criterion, but leads to infinitely many new elements of degree 5 and two of degree 6 (Mulder, 2022). Here, their relative performance on a homogeneous problem with a point source will be examined, both in terms of the error as a function of the degrees of freedom and as a function of the measured compute time.

New elements

The construction of higher-order polynomial elements on triangles involves several requirements. The maximum degree should be p on the edges and $p' \geq p$ in the interior. Continuity between elements is obtained for $p+1$ distinct points on each edge, two of which are the vertices. The nodes on each element are symmetrically arranged. Accuracy should be preserved after lumping. The lumped mass matrix consists of numerical quadrature weights, apart from a scaling factor, and these should be strictly positive. The set of polynomials U, with degree p on the edges and p' in the interior of an element, should be unisolvent, that is, the values on the nodes should define a unique interpolating polynomial on each element.

The classic accuracy criterion requires numerical quadrature to be exact for polynomials of degree $p+p'-2$. The less restrictive one requires exactness for polynomials in the set $P_{p-2} \otimes U$, where P_{p-2} denotes the set of polynomials in 2D up to degree p. In either case, this leads to a nonlinear system of equations, which is linear in the quadrature weights but has coefficients that are polynomials in the parameters describing the node positions on the reference element. The system may have no solution or a finite or an infinite number of solutions, which are complex in general. Of those, only the real-valued ones with strictly positive weights and nodes not outside the triangle are acceptable. A finite number of solutions is usually but not necessarily obtained if the number of equations equals the number of unknowns, comprising the weights and node parameters. In that case, the number of complex solutions grows very rapidly with degree and problem size. An infinite number can be expected if there are less equations than unknowns. An example is the degree-6 element obtained with the classic criterion (Mulder, 2013).

The less restrictive criterion leads to the same elements as the classic one for degrees up to 4. For degree 5, infinitely many are found, some of which are presented in (Mulder, 2022). They have the same number of nodes as the old element. For degree 6, two elements with less nodes are found. Table 1 lists the new elements of degree 5 (version B) and 6 (version A) from (Mulder, 2022) that have the best performance

Table 1 *Elements of degree $p = 5$ and 6, obtained with the less restrictive accuracy criterion.*

p	p'	K	version	class	weight	node parameter(s)	CFL
5	7	{1,0,2,0,3,1}	B	1	0.0011285083493980018565370 5		0.0660
				3	0.0055378574406123773680510 7	0.1189345636925765855042 31	
					0.0052742629259862257966545 1	0.3645694318725664859978 55	
				5	0.0226509274051637849994858	0.3967800539496765802607 80	
					0.0279597621111092150837526	0.1031526466442512176011 64	
					0.0317808195214495409414580	0.2390008557528951161321 9	
				6	0.0307612042731744587280110	0.0707945818807994555408320, 0.3271496031117195513845 59	
6	8	{1,1,2,0,3,2}	A	1	0.0005351135202816657225305 72		0.0404
				2	0.0042943534602629330666594 7		
				3	0.0030299095092606054429029 9	0.0829411811106452819184084	
					0.0031639631664656317128640 3	0.2686496955927143494807 42	
				5	0.0243035184285235576570195	0.4680597290568147517806 58	
					0.0166312091329395024891160	0.0793088545089875461560173	
					0.0342178857644876540882108	0.3929316366188673338507 38	
				6	0.0173480160090330087919888	0.2481727587094068071340 69, 0.6998121971470497541579 75	
					0.0198004044953264308738093	0.1565820660336875316222 29, 0.2430895923645629887108 41	

Table 2 *Element degree p, number of nodes per element n_p and the estimated CFL number for the old and new elements.*

p	1	2	3	4	5	6	7	8
n_p	3	7	12	18	30	46	57	69
CFL (old)	1.14	0.367	0.210	0.128	0.0512	0.0387	0.0288	0.0180
n_p					30	39		
CFL (new)					0.0660	0.0404		

in terms of their compute cost for a given accuracy. The first column gives the element degree p, the second the degree p' in the interior, and the third the node pattern K. Next the symmetry class of one or more nodes is given with quadrature weight and node position in natural coordinates if not trivial, as for vertices, edge midpoints and centroid. Finally, the estimated CFL number that defines the stability for time stepping is listed. Details can be found in (Mulder, 2013).

A measure for an element's performance is the compute time required for a given accuracy of the solution. The efficiency is determined by several factors. Elements of higher degree have more nodes and a higher cost per element, but the better accuracy allows for larger and less elements. Table 2 shows that the maximum time step for stability, determined by the CFL number, decreases with higher degrees and this will adversely affect the efficiency. The old element of degree 6 listed here is version D from (Mulder, 2013). Among elements of the same degree, the error is expected to behave as Ch^{p+1} with element size h and error constant C, which may be quite different for each version and will depend on the problem. The following section presents results for one problem that offer some insight in the relative efficiency of the various elements.

Performance

The numerical test problem consists in solving the acoustic wave equation

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} + \nabla\cdot\left(\frac{1}{\rho}\nabla u\right) = w(t)\delta(\mathbf{x}-\mathbf{x}_s), \qquad (1)$$

for a homogeneous problem with a point source at $\mathbf{x}_s$. Here, $u(t,\mathbf{x})$ is the solution at time t and position $\mathbf{x}$, $c(\mathbf{x})$ is the sound speed and $\rho(\mathbf{x})$ the density, both taken constant for the performance test. The domain is a square with sides of 2 km. Zero Dirichlet boundary conditions are imposed on all sides. The

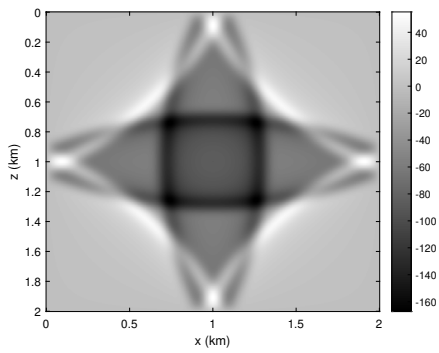


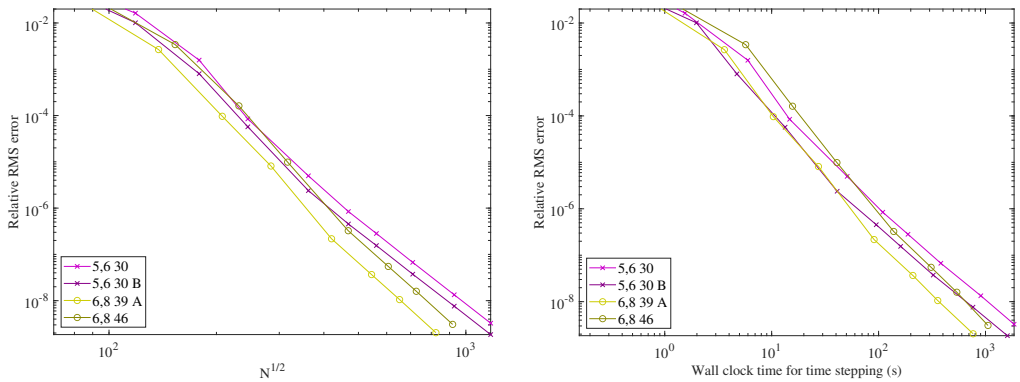
Figure 1 *Exact solution for a point source with reflecting boundaries.*

sound speed is 2 km/s and the density 2 g/cm^3. The point source at the centre has a wavelet $w(t) = [4(t/T_w)\{1-(t/T_w)\}]^{16}$ for time t between 0 and $T_w = 0.2$ s, and zero otherwise. Figure 1 displays the solution at time $t = 1.25$ s.

The error in the numerical solution is measured by the relative RMS error, defined as the approximate L_2 norm of the error divided by the same norm of the solution. This norm is computed as

$$\|u\|_2 = \sqrt{\sum_{j=1}^{n_{\text{tri}}} a_j \sum_{k=1}^{n_{\text{p}}} w_k u_{j,k}^2}, \tag{2}$$

where j runs over all n_{tri} triangles and k over the n_{p} nodes in each triangle with area a_j, quadrature weights w_k and solution values $u_{j,k}$. The expected convergence behaviour for the RMS error is $N^{-(p+1)/2}$ for a number of degrees of freedom N and element degree p. Note that the element size $h \sim N^{-1/2}$. Higher-order time stepping is used rather than Stork's dispersion correction (Stork, 2013). The maximum time step for seond-order time stepping is taken as $\Delta t_{\max} = \sqrt{2/\sigma_{\max}}$, where $\sigma_{\max}$ is the maximum eigenvalue of the spatial operator, the product of the inverse diagonal mass matrix and the stiffness matrix. The power method (von Mises and Pollaczek-Geiringer, 1929) provided an estimate of $\sigma_{\max}$ for each run. The results turned out to be only slightly larger than the crude estimates in Table 2, obtained on a single reference element with natural boundary conditions. Note that $\Delta t_{\max}$ should be increased by a given factor for higher-order time stepping (Mulder, 2013, a.o).

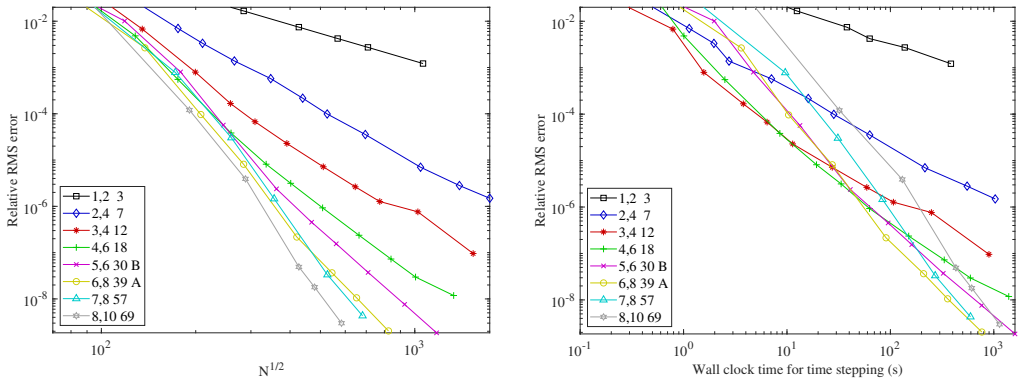


(a) (b)

Figure 2 *(a) Relative RMS error as a function of the square root of the number of degrees of freedom N. and (b) as a function of the measured computer time for the time-stepping part of the code.*

Figure 2(a) confirms the expected convergence behaviour for degree 5 and 6. The legend has three numbers for each case: the degree p of the element, the order M_t of the time stepping scheme, which is chosen as the smallest even number $M_t \geq p+1$, the number of nodes n_{p} per element and a version name, which is absent for the old elements. The new element of degree 5 has the same number of nodes as the old one but a smaller error constant and larger CFL number. The new element of degree 6 has less nodes than the old one. Figure 2(b) shows the measured wall clock time for the time stepping part of the code, averaged over several runs. The C code dates back to 1995 and the observed times are not at all representative for what can be obtained on modern hardware, but are still useful for a comparison. Of the four schemes, the new degree-6 one appears to be the most efficient. It is about 1.7 times faster than the old element.

Figure 3 shows a similar comparison, but now for the most efficient elements found so far for degrees 1 to 8. Figure 3(a) confirms the expected convergence rates. Figure 3(b) shows that for moderate accuracy, the element of degree 3 is the most efficient, whereas for high accuracy, the new element of degree 6 performs best. The large number of nodes together with a small CFL number makes the elements of degree 7 and 8 less attractive. There may, however, exist more efficient elements of higher degrees with the less restrictive accuracy criterion, but these have not yet been found.



(a) (b)

Figure 3 *As Figure 2 but only with the most efficient element found thus far for each degree.*

Conclusions

A less restrictive accuracy criterion for mass lumping provided new and more efficient triangular elements of degree 5 and 6. For lower degrees, the elements are the same as those obtained with the classic criterion. Performance tests show that an element of degree 3 is the most efficient for moderate accuracy, whereas for high accuracy, one of the two new elements of degree 6 performs best.

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