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AN ALTERNATING FREQUENCY-TIME HARMONIC BALANCE METHOD FOR FAST-SLOW DYNAMICAL SYSTEMS

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The Alternating Frequency-Time (AFT) Harmonic Balance method has been widely applied in the analysis of non-linear mechanical systems under periodic excitation. Customarily, a periodic displacement is considered as ansatz in a harmonic balance analysis. In the present work, a deviation from the latter ansatz is realized and the periodicity is assumed in the velocity, leading to a linear term in the displacement of the system. The latter approach aims to facilitate the analysis of a certain class of systems, which are characterized by a fast periodic motion and a slow non-periodic motion. The motivation of this study originates in the area of offshore engineering and more specifically in the topic of monopile installation. During vibratory pile installation, the pile is forced into the soil under the combined action of a periodic excitation at the pile top and the self-weight of the pile and the vibratory device. As a result, the pile simultaneously penetrates into the soil as a rigid body (slow motion) and vibrates in the driving frequency and its super-harmonics both as a rigid and a flexible body (fast motion). In this study, the AFT harmonic balance with the ansatz of periodic velocity is implemented in different problem cases. A set of non-linear mechanical systems are analysed, ranging from a single-degree-of-freedom to a continuum, to showcase the potential application of the method and to verify its accuracy.
Keywords: harmonic balance, nonlinear vibrations, Galerkin method, Coulomb friction

1. Introduction

For the analysis of systems subjected to periodic excitations, the Harmonic Balance (HB) method comprises one of the most advantageous and widely used numerical techniques. In the vast range of problems that have been addressed with the HB method, an increasing number involves cases with non-linear forces that are not explicit expansions of displacement and/or velocity. The latter problems are treated with the Alternating Frequency-Time (AFT) HB method, which was proposed by Cameron and Griffin [1]. The AFT-HB method is based on the computation of the non-linear forces in the time domain and their numerical transform to the frequency domain via a Discrete Fourier Transform (DFT), in an iterative scheme. Since its inception, a wide range of modifications and enhancements have been introduced in the AFT-HB method and applied to problems of structures with bolted connections [2], interfaces of turbomachinery blades [3] and various other applications.

In the present paper, a new version of the AFT-HB method is formulated that addresses a specific class of systems which combine fast (periodic) and slow (non-periodic) motions. The motivation lies in the process of vibratory pile installation, during which the pile performs simultaneously periodic flexible vibrations (fast motion) and a gradual non-periodic progression into the soil (slow motion). The present approach differs from the customary implementation of the HB method in that the velocity of the system is assumed periodic instead of the displacement. Two mechanical systems are analyzed by this method. First, the AFT-HB method is formulated for an elastic-perfectly plastic oscillator. Subsequently, the method is employed to solve the non-linear vibration problem of a rod on a frictional surface. Similar models have been used to study the frictional behavior in mechanical joints [4, 5] and pile-soil analogues [6, 7]. Numerical results are presented for both examples and benchmarked against numerical integration to validate the accuracy of the proposed method.

2. A non-linear oscillator with fast-slow dynamics

2.1 Model description and governing equations

We consider a single-degree-of-freedom (SDOF) system (Fig. 1) governed by the following equation of motion:

$$m\ddot{y}(t) + f_{\mathrm{d}}(y(t)) = p_{\mathrm{s}} + p_{\mathrm{h}}\sin(\Omega_{\mathrm{d}}t) \tag{1}$$

where m is the mass of the oscillator, $y(t)$ is the displacement of the oscillator, $f(y(t))$ is the non-linear restoring force of the elastic-perfectly plastic element, p_{s} is the static force and p_{h} is the amplitude of the harmonic force of frequency Ω_{d}. Hereafter, the overdot denotes temporal differentiation, i.e. $\dot{(\cdot)} = \partial(\cdot)/\partial t$. As regards to the elastic-perfectly plastic element, the non-linear restoring force is defined as [8]:

$$f_{\mathrm{d}}(y(t)) = k(y(t) - y_{\mathrm{pl}}(t)), \quad \dot{y}_{\mathrm{pl}}(t) = \begin{cases} 0, \ |f_{\mathrm{d}}(y(t))| < f_{\mathrm{d,u}} \\ \dot{y}(t), \ |f_{\mathrm{d}}(y(t))| = f_{\mathrm{d,u}} \end{cases} \tag{2}$$

where k is the linear spring stiffness, $y_{\mathrm{pl}}(t)$ is the plastic displacement and $f_{\mathrm{d,u}}$ is the yield force of the elastic-perfectly plastic element.

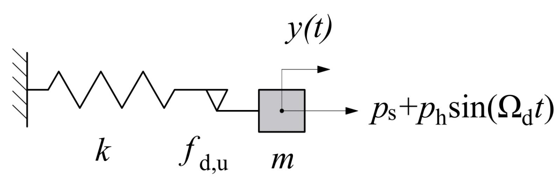

Figure 1: A non-linear single-degree-of-freedom (SDOF) system with elastic-perfectly plastic restoring force.

2.2 AFT-HB method

For the considered problem an ansatz of periodic displacement cannot lead to the correct solution. In this work, we apply the HB method assuming that the velocity is periodic and the following approximate solution for the displacement is obtained:

$$y(t) = c_0 t + \sum_{n=1}^{N_h} \left(c_n \cos(n\Omega_\mathrm{d} t) + s_n \sin(n\Omega_\mathrm{d} t)\right) \tag{3}$$

The latter ansatz leads to the following velocity and acceleration forms:

$$\dot{y}(t) = c_0 + \sum_{n=1}^{N_h} \left(-n\Omega_\mathrm{d} c_n \sin(n\Omega_\mathrm{d} t) + n\Omega_\mathrm{d} s_n \cos(n\Omega_\mathrm{d} t)\right) \tag{4}$$

$$\ddot{y}(t) = \sum_{n=1}^{N_h} \left(-n^2\Omega_\mathrm{d}^2 c_n \cos(n\Omega_\mathrm{d} t) - n^2\Omega_\mathrm{d}^2 s_n \sin(n\Omega_\mathrm{d} t)\right) \tag{5}$$

We substitute the assumed solution in Eq. (1), which results to the residual $R_d(t)$. Then according to the harmonic balance method, we require that the residual is orthogonal to each harmonic function (up to truncation order N_h) over one fundamental period T_d:

$$\int_0^{T_\mathrm{d}} R_\mathrm{d}(t) h_n(t) \,\mathrm{d}\,t = 0, \quad n = 0, \ldots, 2N_h \tag{6}$$

where $h_n(t)$ denotes the n-th harmonic test function:

$$h_n(t) = \frac{1}{2}\left[(1 + (-1)^n)\cos\left(\frac{n}{2}\Omega_\mathrm{d} t\right) + \left(1 + (-1)^{n+1}\right)\sin\left(\frac{n+1}{2}\Omega_\mathrm{d} t\right)\right], \quad n = 0, \ldots, 2N_h \tag{7}$$

It is evident that the operation of Eq. (6) cannot provide an analytical expression for the non-linear force $f_\mathrm{d}(y(t))$ in the frequency domain. For that purpose, the Alternating Frequency-Time Harmonic Balance (AFT-HB) method is employed. For the evaluation of non-linear forcing terms in time domain and the subsequent transform to the frequency domain the DFT is performed via the Fast Fourier Transform (FFT) algorithm.

2.3 Numerical results

The numerical results obtained by the AFT-HB method are benchmarked against numerical integration. Specifically, Eq. (1) was solved via the explicit Runge-Kutta (RK45) method of accuracy $\mathcal{O}(\Delta t^4)$ [9]. The initial conditions were considered equal to zero for the analyses with the RK45 method. The properties of the SDOF system are given in Table 1. The comparison between the results of the RK45 approach and the proposed AFT-HB method are shown in Fig. 2. As can be seen the displacement obtained by the two approaches differs by a constant term, which is evidently the result of the transient phase until the response becomes stationary. Naturally, such an effect is not addressed in HB analyses, since the stationary periodic response is directly obtained. This hypothesis is verified by the velocity results in Fig. 2, as both approaches converge to the same velocity after the passing of one (fundamental) vibration period. It is noted that the great agreement between the results of the two methods in terms of velocity indicates the accuracy of the proposed method as the high-frequency components are more pronounced in this case.

Table 1: Parameters of the SDOF example

m	k	$f_{\mathrm{d,u}}$	p_{s}	p_{h}	Ω_{d}
1 kg	900 N/m	5 N	0.4 N	4 N	20 rad/s

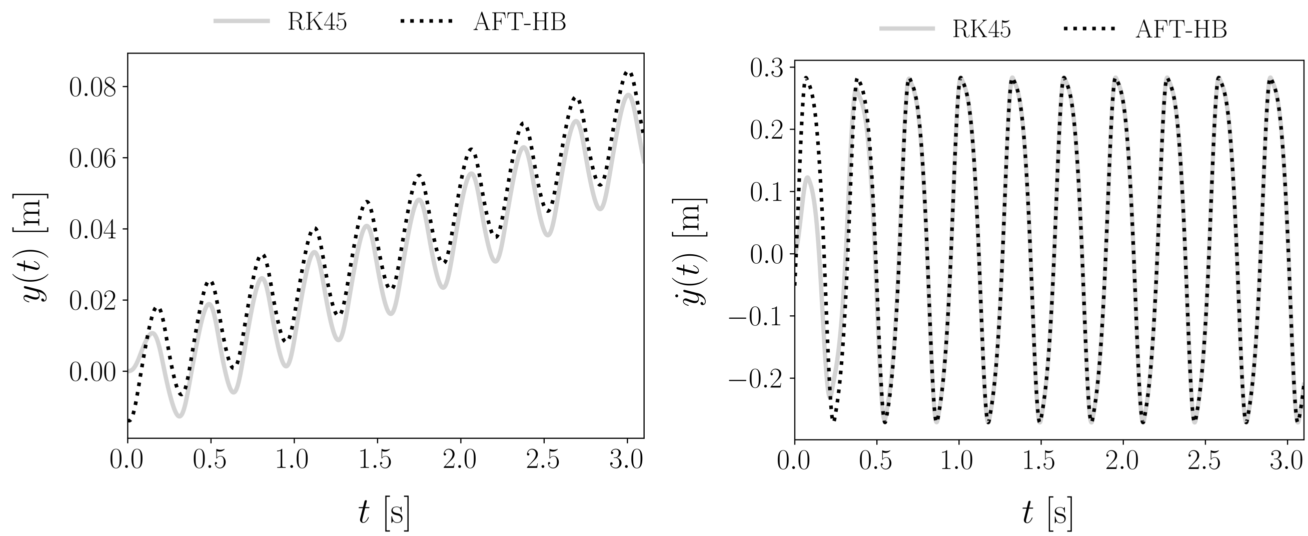

Figure 2: Comparison between the RK45 and the AFT-HB method in terms of the SDOF displacement (left) and velocity (right).

3. Vibrations of a rod on a frictional surface

3.1 Model description and governing equations

A linear homogeneous elastic rod is considered, occupying the domain $0 \leq x \leq L$, where L denotes the length of the rod as shown in Fig. 3. The equation of motion of the rod reads:

$$\rho A\ddot{u}(x,t) = EAu''(x,t) + f_c(\dot{u}(x,t)) \tag{8}$$

where ρ is the mass density of the rod, A is the area of the rod cross-section, E is the Young's modulus of the rod, $f_c(\dot{u}(x,t))$ is a distributed dry friction force and $u(x,t)$ is the axial displacement of the rod, which is a function of the spatial coordinate x and time t. Hereafter, the prime denotes spatial differentiation, i.e. $(\cdot)' = \partial(\cdot)/\partial x$. The friction force $f_{\mathrm{c}}(\dot{u}(x,t))$ obeys the regularized Coulomb friction law by Threlfall [10]; the static and kinetic friction amplitudes are identical and equal to $f_{\mathrm{c,u}}$. Furthermore, the boundary conditions read:

$$N(0,t) = -P_{\mathrm{s}} - P_{\mathrm{h}}\sin(\Omega_{\mathrm{c}}t), \quad N(L,t) = -f_{\mathrm{t}}(u(L,t)) - c_{\mathrm{t}}\dot{u}(L,t) \tag{9}$$

where N is the axial force, P_{s} is the static force, P_{h} is the amplitude of the harmonic force of frequency Ω_{c}, $f_{\mathrm{t}}(u(L,t))$ is the non-linear restoring force of the elastic-perfectly plastic element and c_{t} is the viscous dashpot coefficient. The non-linear restoring force $f_{\mathrm{t}}(u(L,t))$ is defined as:

$$f_{\mathrm{t}}(u(L,t)) = k_{\mathrm{t}}(u(L,t) - u_{\mathrm{pl}}(t)), \quad \dot{u}_{\mathrm{pl}}(t) = \begin{cases} 0, \ |f_{\mathrm{t}}(u(L,t))| < f_{\mathrm{t,u}} \\ \dot{u}(L,t), \ |f_{\mathrm{t}}(u(L,t))| = f_{\mathrm{t,u}} \end{cases} \tag{10}$$

where k_{t} is the linear spring stiffness, $u_{\mathrm{pl}}(t)$ is the plastic displacement and $f_{\mathrm{t,u}}$ is the yield force of the elastic-perfectly plastic element.

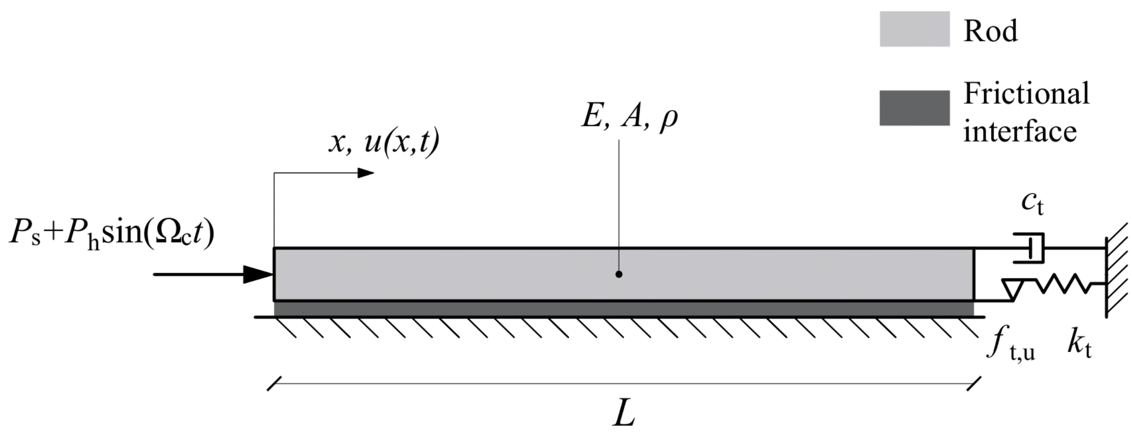

Figure 3: A rod on a frictional surface.

3.2 Galerkin/AFT-HB method

The Galerkin method is employed for the spatial discretization of the rod [11]. For that purpose, the concentrated body force method (CBFM) is applied to render the two boundaries force-free and translate the boundary forces into Eq. (8) via the Dirac delta function $\delta(\cdot)$ [12]. Subsequently, the free vibration modes of the free-free rod *in vacuo* are found and employed as trial and test functions. Therefore, the solution of Eq. (8) is approximated by the series:

$$u(x,t) = \sum_{m=0}^{N_m} U_m(x)\, q_m(t) \tag{11}$$

where $U_m(x)$ denotes the m-th free vibration mode, $q_m(t)$ is the m-th generalized coordinate and N_m is the upper limit of the truncated summation. The residual is obtained by substituting Eq. (11) into Eq. (8) and upon multiplication with the test functions and integration over the rod length, a set of weighted residuals $R_m(t)$ is derived. By setting the latter equal to zero a set of N_m non-linear coupled ordinary differential equations (ODEs) is obtained.

The problem is now reduced into a set of temporal ODEs that describe the generalized coordinates. The latter are solved by means of the AFT-HB method. For the system under consideration, rigid body motion is admissible and in our solution approach this is addressed through the term $U_0(x)\, q_0(t)$, which corresponds to the rigid-body component of the rod response. In that system, the periodic velocity ansatz needs to be used as solution of $q_0(t)$. Therefore, we need to employ two different assumed solution forms for the generalized coordinates:

$$q_0(t) = c_{0,0}t + \sum_{n=1}^{N_h} \left(c_{0,n}\cos(n\Omega_{\mathrm{c}}t) + s_{0,n}\sin(n\Omega_{\mathrm{c}}t)\right) \tag{12}$$

$$q_m(t) = c_{m,0} + \sum_{n=1}^{N_h} \left(c_{m,n} \cos(n\Omega_c t) + s_{m,n} \sin(n\Omega_c t) \right), \quad m = 1, \ldots, N_m \tag{13}$$

Similarly to the case of the SDOF system, we require that the residual $R_m(t)$ of each generalized coordinate is orthogonal to each harmonic test function $h_n(t)$ over one fundamental period T_c:

$$\int_0^{T_c} R_m(t) h_n(t) \, \mathrm{d}\, t = 0 \tag{14}$$

Overall, the above operation leads to a system of $N_m \cdot (2N_h + 1)$ coefficients which can be found via an optimization procedure that aims to satisfy Eq. (14) for $n = 0, \ldots, 2N_h$ and $m = 0, \ldots, N_m$. It is remarked that the number of harmonics employed in the solution can be different for each generalized coordinate and an adaptive harmonic selection scheme can potentially increase the computational efficiency of the method.

3.3 Numerical results

Similarly to the SDOF example, the AFT-HB results are benchmarked against numerical integration. Specifically, a system of coupled non-linear temporal ODEs is solved via the RK45 method in this example. In Table 2 the parameters of the rod case are provided.

Table 2: Parameters of the rod example

ρA	EA	L	$f_{c,u}$	k_t	c_t	$f_{t,u}$	P_s	P_h	Ω_c
320 kg/m	$8 \cdot 10^8$ N	10 m	$5 \cdot 10^3$ N/m	$3 \cdot 10^8$ N/m	10^6 N·s/m	10^5 N	$3 \cdot 10^4$ N	$3 \cdot 10^5$ N	rad/s

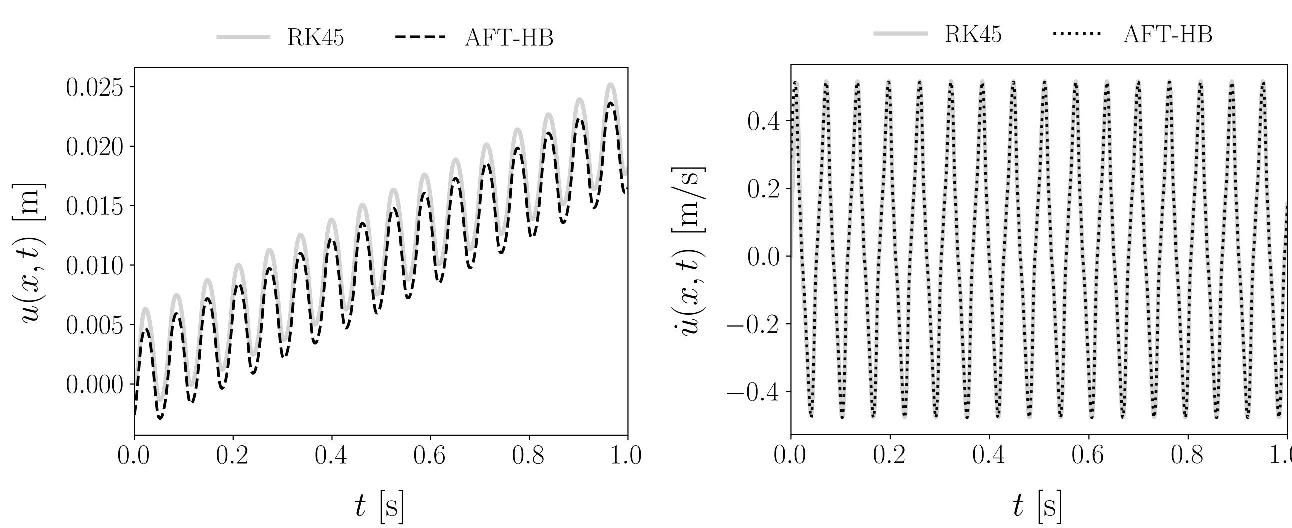

Figure 4: Comparison between the RK45 and the AFT-HB method in terms of the rod displacement (left) and velocity (right) at $x = 0$.

For the comparison of the two numerical approaches, i.e. RK45 and AFT-HB, two positions along the rod length are examined at $x = 0$ and $x = L$, respectively. In Fig. 4 the response at $x = 0$ is shown. Similarly to Fig. 2, the rod displacement is characterized by a combination of a periodic motion and a (presumably) linearly increasing non-periodic motion. The offset between the RK45 and the AFT-HB results is due to the effect of the free vibrations triggered by the motion initiation, while examination of the velocity indicates that the overall is response is captured to remarkable accuracy. Fig. 5 presents similar results to Fig. 4, albeit the amplitudes of both displacement and velocity are reduced compared to the left end of the rod. This observation is mainly attributed to the presence of the non-linear element at $x = L$.

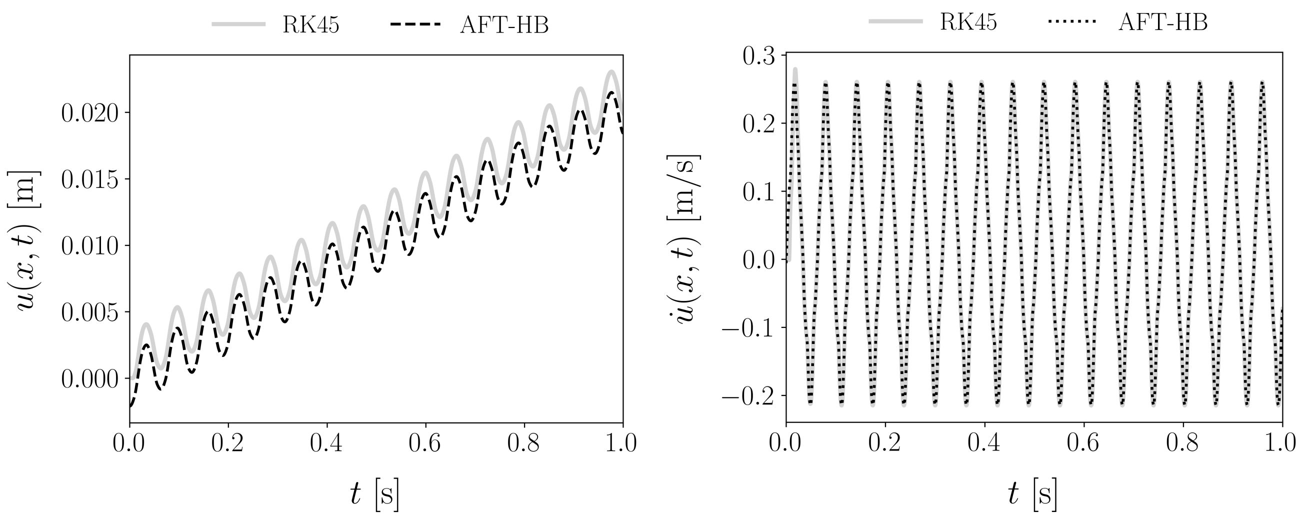

Figure 5: Comparison between the RK45 and the AFT-HB method in terms of the rod displacement (left) and velocity (right) at $x = L$.

4. Conclusions

This paper presents an Alternating Frequency-Time Harmonic Balance method that is based on a periodic velocity ansatz, such that specific cases of fast-slow dynamical systems can be addressed. In that manner, both fast and slow motions are captured adequately with the proposed assumed solutions and the use of a frequency-time method has been proved to be feasible. Numerical examples of a single-degree-of-freedom and a continuous system are provided and the results by the Alternating Frequency-Time Harmonic Balance method are validated against numerical integration. The present approach can be extended into an iterative scheme such that arbitrary slow motions may be approximated in a piecewise manner, given that a clear separation between fast and slow motion components exists.

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