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Multilinear transference of Fourier and Schur multipliers acting on noncommutative L_p-spaces

Martijn Caspers, Amudhan Krishnaswamy-Usha, and Gerrit Vos

Abstract. Let G be a locally compact unimodular group, and let ϕ be some function of n variables on G. To such a ϕ, one can associate a multilinear Fourier multiplier, which acts on some n-fold product of the noncommutative L_p-spaces of the group von Neumann algebra. One may also define an associated Schur multiplier, which acts on an n-fold product of Schatten classes $S_p(L_2(G))$. We generalize well-known transference results from the linear case to the multilinear case. In particular, we show that the so-called "multiplicatively bounded $(p_1, \ldots, p_n)$-norm" of a multilinear Schur multiplier is bounded above by the corresponding multiplicatively bounded norm of the Fourier multiplier, with equality whenever the group is amenable. Furthermore, we prove that the bilinear Hilbert transform is not bounded as a vector-valued map $L_{p_1}(\mathbb{R}, S_{p_1}) \times L_{p_2}(\mathbb{R}, S_{p_2}) \to L_1(\mathbb{R}, S_1)$, whenever p_1 and p_2 are such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. A similar result holds for certain Calderón–Zygmund-type operators. This is in contrast to the nonvector-valued Euclidean case.

1 Introduction

In recent years, the analysis of Fourier multipliers on noncommutative L_p-spaces has seen a rapid development. In particular, several multiplier theorems have been established for the noncommutative L_p-spaces of a group von Neumann algebra (see, e.g., [CCP22, CGPTb, JMP14, MeRi17, MRX22, PRS22]). Here, the symbol of the multiplier is a function on a locally compact group and the multiplier acts on the noncommutative L_p-space. In particular, the group plays the role of the frequency side.

In several of these approaches, Schur multipliers are used to estimate the bounds of Fourier multipliers and vice versa. For instance, upper bounds on the norms of Fourier multipliers in terms of Schur multipliers play a crucial role in [PRS22]. Conversely, transference from Fourier to Schur multipliers was used by Pisier [Pi98] to provide examples of bounded multipliers on L_p-spaces that are not completely bounded. In [LaSa11], analogous transference techniques were used to provide examples of noncommutative L_p-spaces without the completely bounded approximation property. Furthermore, the use of multilinear Schur multipliers and operator integrals led to

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several surprising results such as the resolution of Koplienko's conjecture on higher-order spectral shift [PSS13] (see also [PSST17]).

Bożejko and Fendler proved the following in [BoFe84]. Let G be a locally compact group. Let $\phi \in C_b(G)$ and set $\widetilde{\phi}(s,t) = \phi(st^{-1}), s,t \in G$. Then the Schur multiplier $M_{\widetilde{\phi}} : B(L_2(G)) \to B(L_2(G))$ is bounded if and only if it is completely bounded if and only if the Fourier multiplier $T_\phi : \mathcal{L}G \to \mathcal{L}G$ is completely bounded.

Several papers have treated the extension of the Bożejko–Fendler result to non-commutative L_p-spaces. In particular, in [NeRi11], Neuwirth and Ricard proved for a discrete group G that

$$\|M_{\widetilde{\phi}} : S_p(L_2(G)) \to S_p(L_2(G))\|_{cb} \leq \|T_\phi : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)\|_{cb}.$$

If G is moreover amenable, then this is an equality. The same result was then obtained for G a locally compact group in [CaSa15]. An analogous result was obtained for actions and crossed products by González-Perez [Gon18], and in an ad hoc way in the bilinear discrete setting, a similar result was obtained for the discrete Heisenberg group in [CJKM, Section 7].

The purpose of this paper is to prove transference results for Fourier and Schur multipliers in the multilinear setting for arbitrary unimodular locally compact groups. We confine ourselves to the unimodular setting for reasons further discussed in Remark 4.4.

Now, we describe in more detail the contents of the paper. Our first main result (Theorem 3.1) is the following multilinear extension of [CaSa15, Theorem 4.2]. The definition of $(p_1,\ldots,p_n)$-multiplicatively bounded maps is given in Section 2.7. If $n=1$, then we are in the linear case and by a well-known theorem of Pisier [Pi98] a map is "p-multiplicatively bounded" if and only if it is completely bounded as a map on the L_p-space with the natural operator space structure that was also introduced in [Pi98].

Theorem A *Let G be a locally compact second countable unimodular group, and let $1 \leq p \leq \infty$, $1 < p_1,\ldots,p_n \leq \infty$ be such that $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ and set $\widetilde{\phi} \in C_b(G^{\times n+1})$ by*

$$\widetilde{\phi}(s_0,\ldots,s_n) = \phi(s_0s_1^{-1}, s_1s_2^{-1},\ldots,s_{n-1}s_n^{-1}), \qquad s_i \in G. \tag{1.1}$$

If ϕ is the symbol of a $(p_1,\ldots,p_n)$-multiplicatively bounded Fourier multiplier T_ϕ of G, then $\widetilde{\phi}$ is the symbol of a $(p_1,\ldots,p_n)$-multiplicatively bounded Schur multiplier $M_{\widetilde{\phi}}$ of G. Moreover,

$$\begin{aligned}\|M_{\widetilde{\phi}} &: S_{p_1}(L_2(G)) \times \cdots \times S_{p_n}(L_2(G)) \to S_p(L_2(G))\|_{(p_1,\ldots,p_n)-mb}\\ &\leq \|T_\phi : L_{p_1}(\mathcal{L}G) \times \cdots \times L_{p_n}(\mathcal{L}G) \to L_p(\mathcal{L}G)\|_{(p_1,\ldots,p_n)-mb}.\end{aligned}$$

Note that in the linear (unimodular, second countable) case of $n=1$, this result is actually a strengthening of [CaSa15, Theorem 4.2]. Namely, the symbol here is only assumed to define an L_p-Fourier multiplier at a single exponent p, whereas [CaSa15, Theorem 4.2] requires that the multiplier is (completely) bounded for all $1 \leq p \leq \infty$ simultaneously. The current proof takes a different route and uses results that appeared after [CaSa15] in the papers [CJKM, CPPR15, CPR18].

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For transference in the other direction, we need our group G to be amenable, just as in [CaSa15, NeRi11]. In fact, amenability is a necessary requirement for our proof strategy (see [CaSa15, Theorem 2.1]). The following is our second main result (Corollary 4.2). Note here the strict bounds $1 < p < \infty$, caused by the requirement that the maps i_q from Theorem 4.1 are complete isometries.

Theorem B *Let G be a locally compact unimodular amenable group, and let $1 < p, p_1, \dots, p_n < \infty$ be such that $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ and define $\widetilde{\phi}$ as in (1.1). If $\widetilde{\phi}$ is the symbol of a $(p_1, \dots, p_n)$-bounded (resp. multiplicatively bounded) Schur multiplier, then ϕ is the symbol of a $(p_1, \dots, p_n)$-bounded (resp. multiplicatively bounded) Fourier multiplier. Moreover,*

$$\|T_\phi\|_{(p_1,\dots,p_n)} \leq \|M_{\widetilde{\phi}}\|_{(p_1,\dots,p_n)}, \qquad \|T_\phi\|_{(p_1,\dots,p_n)-mb} \leq \|M_{\widetilde{\phi}}\|_{(p_1,\dots,p_n)-mb},$$

with equality in the $(p_1, \dots, p_n)$-mb norm when G is second countable.

The proof is a multilinear version of the ultraproduct techniques from [CaSa15, Theorem 5.2] and [NeRi11].

In the final section, which can mostly be read separately from the rest of the paper, we consider the case of vector-valued bilinear Fourier multipliers on $\mathbb{R}$. Lacey and Thiele have shown in [LaTh99] that the bilinear Hilbert transform is bounded from $L_{p_1}(\mathbb{R}) \times L_{p_2}(\mathbb{R}) \to L_p(\mathbb{R})$, when $\frac{2}{3} < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The vector-valued bilinear Hilbert transform is bounded as a map from

$$L_{p_1}(\mathbb{R}, S_{q_1}) \times L_{p_2}(\mathbb{R}, S_{q_2}) \to L_p(\mathbb{R}, S_q)$$

whenever $1 < \frac{1}{\max\{q,q'\}} + \frac{1}{\max\{q_1,q_1'\}} + \frac{1}{\max\{q_2,q_2'\}}$, as shown by Amenta and Uraltsev in [AmUr20] and Di Plinio et al. in [DMLV22]. In particular, this class does not include Hölder combinations of q_i. We show that this result does not extend to the case when $p_i = q_i$, $p = q = 1$, using a transference method similar to the ones used in earlier sections. To be precise, we prove the following result (Theorem 5.2).

Theorem C *Let $1 < p_1, p_2 < \infty$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and set $h(s,t) = \chi_{\geq 0}(s-t)$. There exists an absolute constant $C > 0$ such that, for every $N \in \mathbb{N}_{\geq 1}$, we have*

$$\|T_h^{(N)} : L_{p_1}(\mathbb{R}, S_{p_1}^N) \times L_{p_2}(\mathbb{R}, S_{p_2}^N) \to L_1(\mathbb{R}, S_1^N)\| > C \log(N).$$

Additionally, we show a similar result for Calderón–Zygmund operators. Here, Grafakos and Torres [GrTo02] have shown that for a class of Calderón–Zygmund operators, we have boundedness $L_1 \times L_1 \to L_{\frac{1}{2},\infty}$ in the Euclidean case. Later, a vector-valued extension was obtained in [DLMV20]. Here, for a class of Calderón–Zygmund operators, the boundedness of the vector-valued map was obtained for $L_{p_1} \times L_{p_2} \to L_p$ with $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Theorem 5.3 shows that the latter result cannot be extended to the case when $p = 1$.

The structure of this paper is as follows. In Section 2, we treat the necessary preliminaries and establish the definitions of multilinear Fourier and Schur multipliers. We also look briefly at transference in the case that $p_i = \infty$ for all i. In Section 3, we prove the transference from Fourier to Schur multipliers for general p_i; the other direction for amenable G is proved in Section 4. Finally, in Section 5, we give the counterexamples for the vector-valued bilinear Hilbert transform and Calderón–Zygmund operators.

2 Preliminaries

2.1 Notational conventions

$\mathbb{N}$ denotes the natural numbers starting from 0, and $\mathbb{N}_{\geq 1}$ denotes $\mathbb{N}\backslash\{0\}$. $M_n := M_n(\mathbb{C})$ denotes the complex $n \times n$ matrices. We denote $B(H)$ for the bounded operators on a Hilbert space H. We denote by $E_{st}, 1 \leq s, t \leq n$, the matrix units of M_n. Likewise, E_{st}, $s, t \in F$ denote the matrix units of $B(\ell_2(F))$ whenever F is a finite set. We also use 1_F to denote the indicator function on the set F.

2.2 Locally compact groups

Let G be a locally compact group, which we assume to be unimodular with Haar measure μ_G (see Remark 4.4). Integration against the Haar measure is denoted by $\int \cdot\, ds$. For $1 \leq p < \infty$, we let $L_p(G)$ be the p-integrable functions with norm determined by $\|f\|_p^p = \int |f(s)|^p ds$. $C_c(G)$ denotes the continuous and compactly supported functions on G. $L_1(G)$ is a $*$-algebra with multiplication given by convolution $(f * g)(t) = \int f(s)g(s^{-1}t)ds$ and involution given by $f^*(s) = \overline{f(s^{-1})}$. λ denotes the left regular representation of G on $L_2(G)$, i.e., $(\lambda_s f)(t) = f(s^{-1}t)$. λ also determines a representation of $L_1(G)$ by the strongly convergent integral $\lambda(f) = \int f(s)\lambda_s ds$. The Fourier algebra [Eym64] is defined as

$$A(G) := L_2(G) * L_2(G) = \{s \mapsto \langle \lambda_s \xi, \eta\rangle \mid \xi, \eta \in L_2(G)\}. \tag{2.1}$$

Set the group von Neumann algebra

$$\mathcal{L}G = \{\lambda_s \mid s \in G\}'' = \{\lambda(f) \mid f \in L_1(G)\}''.$$

It comes equipped with a natural weight φ called the *Plancherel weight* that is given, for $x \in \mathcal{L}G$, by

$$\varphi(x^*x) = \begin{cases} \|f\|_2^2, & \text{if } \exists f \in L_2(G) \text{ s.t. } \forall \xi \in C_c(G) : x\xi = f * \xi, \\ \infty, & \text{otherwise.} \end{cases}$$

φ is tracial since (in fact if and only if) G is unimodular, i.e., $\varphi(x^*x) = \varphi(xx^*)$.

2.3 Noncommutative L_p-spaces

Let $L_p(\mathcal{L}G)$ denote the noncommutative L_p-space associated with $\mathcal{L}G$ and the Plancherel weight φ. Since φ is a trace, this space can be viewed as the completion of the set of elements in $\mathcal{L}G$ with finite $\|.\|_{L_p(\mathcal{L}G)}$ norm:

$$L_p(\mathcal{L}G) = \overline{\left\{x \in \mathcal{L}G : \|x\|_{L_p(\mathcal{L}G)} = \varphi(|x|^p)^{1/p} < \infty\right\}}^{\|\cdot\|_{L_p(\mathcal{L}G)}}.$$

Let $C_c(G) \star C_c(G)$ denote the span of the set of functions of the form $f_1 * f_2, f_i \in C_c(G)$. Then $\lambda(C_c(G) \star C_c(G))$ is dense in $L_p(\mathcal{L}G)$ for every $1 \leq p < \infty$ and is weak* dense in $L_\infty(\mathcal{L}G) = \mathcal{L}G$.

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2.4 Multilinear Fourier multipliers

Let $\phi \in C_b(G^{\times n})$. The Fourier multiplier T_ϕ associated with the symbol ϕ is the multilinear map defined for $\lambda(f_i), f_i \in C_c(G) \star C_c(G)$ by

$$T_\phi(\lambda(f_1), \dots, \lambda(f_n)) = \int_{G^{\times n}} \phi(t_1, \dots, t_n) f_1(t_1) \dots f_n(t_n) \lambda(t_1 \dots t_n) dt_1 \dots dt_n.$$

Let $1 \leq p_1, \dots, p_n, p < \infty$ with $p^{-1} = \sum_i p_i^{-1}$. Assume that T_ϕ maps $\lambda(C_c(G) \star C_c(G)) \times \cdots \times \lambda(C_c(G) * C_c(G))$ into $L_p(\mathcal{L}G)$. Equip the ith copy of $\lambda(C_c(G) \star C_c(G))$ with the $\|.\|_{L_{p_i}(\mathcal{L}G)}$ topology. If T_ϕ is a continuous multilinear map with respect to the above topologies, we extend T_ϕ to a map $L_{p_1}(\mathcal{L}G) \times \cdots \times L_{p_n}(\mathcal{L}G) \to L_p(\mathcal{L}G)$. By mild abuse of notation, we also denote this map by T_ϕ, and call it the $(p_1, \dots, p_n)$-Fourier multiplier associated with ϕ. When some or all of the p_i, p are equal to ∞, we equip the corresponding copy of $\lambda(C_c(G) \star C_c(G))$ with the norm topology from $C^*_\lambda(G)$, and replace $L_{p_i}(\mathcal{L}G)$ by $C^*_\lambda(G)$.

By a closed graph argument, T_ϕ is then a bounded multilinear map, and we denote its norm by $\|T_\phi\|_{(p_1, \dots, p_n)}$.

2.5 Schatten p-operators

For $1 \leq p < \infty$, let $S_p(H)$ denote the Schatten p-operators on a Hilbert space H consisting of all $x \in B(H)$ such that $\|x\|_{S_p} := \mathrm{Tr}(|x|^p)^{1/p} < \infty$ where Tr is the usual trace on $B(H)$. $S_\infty(H)$ denotes the compact operators on H. For $1 \leq p \leq q \leq \infty$, we have the dense inclusions $S_p(H) \subseteq S_q(H)$.

Again, let G be a unimodular locally compact group. For $F \subset G$ a relatively compact set with positive measure, let $P_F : L_2(G) \to L_2(F)$ be the orthogonal projection. Then, for $1 \leq p \leq \infty$, and $x \in L_{2p}(\mathcal{L}G)$, xP_F defines an operator in $S_{2p}(L_2(G))$ (see [CaSa15, Proposition 3.1]). For $x \in L_p(\mathcal{L}G)$ with polar decomposition $x = u|x|$, we will abusively denote by $P_F x P_F$ the operator $(|x|^{1/2} u^* P_F)^* |x|^{1/2} P_F$, which lies in $S_p(L_2(G))$ whenever $x \in L_p(\mathcal{L}G)$. We will additionally use the fact that the map

$$x \mapsto \mu_G(F)^{-1/p} P_F x P_F$$

defines a contraction from $L_p(\mathcal{L}G)$ to $S_p(L_2(G))$ [CaSa15, Theorem 5.1].

Let E be an operator space. For $N \in \mathbb{N}_{\geq 1}$, and $1 \leq p \leq \infty$, $S_p^N[E]$ will denote the space $M_N(E)$ equipped with the "operator-valued Schatten p-norm." When $E = \mathbb{C}$, this is the Schatten p-class associated with a Hilbert space of dimension n, or equivalently, the noncommutative L_p-space associated with M_N equipped with the normalized trace, and is denoted by just S_p^N. When $p = \infty$, the norm on $S_p[E]$ is the operator space norm on $M_N(E)$; for $p = 1$, this is the projective operator space norm on $S_1^N \otimes E$. The rest are constructed via interpolation. The particulars will not be used in what follows; we refer to [Pi98] for the details. If E is the noncommutative L_p-space associated with some tracial von Neumann algebra $\mathcal{M}$, $S_p^N[E]$ can be identified with the noncommutative L_p-space corresponding to $M_N \otimes \mathcal{M}$.

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2.6 Multilinear Schur multipliers

Let X be some measure space. We identify $S_2(L_2(X))$ linearly and isometrically with the integral operators given by kernels in $L_2(X \times X)$. This way $A \in L_2(X \times X)$ corresponds to $(A\xi)(t) = \int A(t,s)\xi(s)ds$. Throughout what follows, we will make no distinction between a Hilbert–Schmidt operator on $L_2(X)$ and its kernel in $L_2(X \times X)$.

For $\phi \in L_\infty(X^{\times n+1})$, the associated Schur multiplier is the multilinear map $S_2(L_2(X)) \times \cdots \times S_2(L_2(X)) \to S_2(L_2(X))$ determined by

$$
\begin{aligned}
&M_\phi(A_1, \dots, A_n)(t_0, t_n) \\
&\quad = \int_{X^{\times n-1}} \phi(t_0, \dots, t_n) A_1(t_0, t_1) A_2(t_1, t_2) \dots A_n(t_{n-1}, t_n) dt_1 \dots dt_{n-1}.
\end{aligned}
$$

That M_ϕ indeed takes values in $S_2(L_2(X))$, or rather $L_2(X \times X)$, is an easy application of the Cauchy–Schwarz inequality, as we show here in the case of $n = 2$:

$$
\begin{aligned}
&\iint_{X^2} \left| \int_X \phi(r,s,t) A(r,s) B(s,t) ds \right|^2 dr dt \\
&\leq \|\phi\|_\infty^2 \int_X \int_X \left(\int_X |A(r,s)|^2 ds \right) dr \left(\int_X |B(s,t)|^2 ds \right) dt \\
&= \|\phi\|_\infty^2 \|A\|_2^2 \|B\|_2^2.
\end{aligned}
$$

The case of higher-order n is similar to [PSST17, Lemma 2.1].

Let $1 \leq p, p_1, \dots, p_n \leq \infty$, with $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Consider the restriction of M_ϕ where its ith input is restricted to $S_2(L_2(X)) \cap S_{p_i}(L_2(X))$. If the resulting restriction takes values in $S_p(L_2(X))$ and has a bounded extension to $S_{p_1}(L_2(X)) \times \cdots \times S_{p_n}(L_2(X))$, its extension, also denoted by M_ϕ, is called the $(p_1, \dots, p_n)$-Schur multiplier.

2.7 Norms for multilinear maps

If $E_1, \dots, E_n, E$ are Banach spaces and $T : E_1 \times \cdots \times E_n \to E$ is a multilinear map, we recall that $\|T\|$ is the quantity $\sup_{x_i \in E_i, \|x_i\|=1} \|T(x_1, \dots, x_n)\|$.

If $E_1, \dots, E_n, E$ are operator spaces, T is a multilinear map, and $N \in \mathbb{N}_{\geq 1}$, the multiplicative amplification of T refers to the map

$$
T^{(N)} : M_N(E_1) \times \cdots \times M_N(E_n) \to M_N(E), \tag{2.2}
$$

which sends $x_i = \alpha_i \otimes v_i$ where $\alpha_i \in M_N, v_i \in E_i$, to $(\alpha_1 \dots \alpha_n) \otimes T(v_1, \dots, v_n)$. $T^{(N)}$ is viewed as a multilinear map on the space $M_N(E_i)$, which is equipped with the matrix norm from the operator space structure of E_i. We say T is multiplicatively bounded if $\|T\|_{mb} = \sup_N \|T^{(N)}\| < \infty$.

Let $1 \leq p, p_1, \dots, p_n \leq \infty$ with $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Generalizing a definition by [Xu06], we will say a multilinear map $T : E_1 \times \cdots \times E_n \to E$ is $(p_1, \dots, p_n)$-multiplicatively bounded if the multiplicative extensions

$$
T^{(N)} := T^{(N)}_{p_1, \dots, p_n} : S^N_{p_1}[E_1] \times \cdots \times S^N_{p_n}[E_n] \to S^N_p[E]
$$

have uniformly bounded norms. The $(p_1,\dots,p_n)$-multiplicatively bounded norm of T is then $\sup_N \|T^{(N)}\|$. It is denoted by $\|T\|_{(p_1,\dots,p_n)-mb}$.

Remark 2.1 It is unclear if this definition of $(p_1,\dots,p_n)$-multiplicative boundedness corresponds to complete boundedness of some linear map on some appropriate tensor product of the E_i's. In the special case that the range space is $\mathbb{C}$ and $n=2$, such a tensor product has been constructed in [Xu06, Remark 2.7]. However, this tensor norm does not seem to admit a natural operator space structure, nor does it seem to work in the multilinear case.

The norms of multilinear Schur multipliers are determined by the restriction of the symbol to finite sets. This is the multilinear version of [LaSa11, Theorem 1.19] and [CaSa15, Theorem 3.1].

Theorem 2.2 *Let μ be a Radon measure on a locally compact space X, and let $\phi : X^{n+1} \to \mathbb{C}$ be a continuous function. Let $K>0$. The following are equivalent for $1 \le p_1,\dots,p_n,p \le \infty$:*

(i) *ϕ defines a bounded Schur multiplier $S_{p_1}(L_2(X)) \times \cdots \times S_{p_n}(L_2(X)) \to S_p(L_2(X))$ with norm less than K.*
(ii) *For every σ-finite measurable subset X_0 in X, ϕ restricts to a bounded Schur multiplier $S_{p_1}(L_2(X_0)) \times \cdots \times S_{p_n}(L_2(X_0)) \to S_p(L_2(X_0))$ with norm less than K.*
(iii) *For any finite subset $F=\{s_1,\dots,s_N\} \subset X$ belonging to the support of μ, the symbol $\phi|_{F^{\times(n+1)}}$ defines a bounded Schur multiplier $S_{p_1}(\ell_2(F)) \times \cdots \times S_{p_2}(\ell_2(F)) \to S_p(\ell_2(F))$ with norm less than K.*

The same equivalence is true for the $(p_1,\dots,p_n)-mb$ norms.

Proof $(i) \Rightarrow (ii)$ is trivial. The implication $(ii) \Rightarrow (i)$ remains exactly the same as in [CaSa15, Theorem 3.1] except for the fact that we have to take $x_i \in S_{p_i}(L_2(X))$ and take into account the support projections of $x_1,x_1^*,\dots,x_n,x_n^*$ when choosing X_0. The equivalence $(ii) \Leftrightarrow (iii)$ is mutatis mutandis the same as in [LaSa11, Theorem 1.19]. For the $(p_1,\dots,p_n)-mb$ norms, we apply the theorem on the space $X_N = X \times \{1,\dots,N\}$ and function $\phi_N((s_0,i_0),\dots,(s_n,i_n)) = \phi(s_0,\dots,s_n)$ and use the isometric identifications

$$S_q(L_2(X_N)) \cong S_q(L_2(X) \otimes \mathbb{C}^N) \cong S_q^N(S_q(L_2(X)))$$

and the fact that under these identifications, we have

$$M_\phi^{(N)}(x_1,\dots,x_n) = M_{\phi_N}(x_1,\dots,x_n), \qquad x_i \in S_{p_i}^N(S_{p_i}(L_2(X))). \qquad \blacksquare$$

2.8 Transference for $p_i = \infty$

Let G be a locally compact group which in this subsection is not required to be unimodular. The following Proposition 2.3 (based on [EfRu90, Theorem 4.1]) was proved in [ToTu10, Theorem 5.5]. This is a multilinear version of the Bożejko–Fendler result [BoFe84], and it yields a transference result between Fourier and Schur multipliers for the case $p_1 = \cdots = p_n = \infty$. We give a proof of the 'if' direction that is slightly different from [ToTu10] by using the transference techniques from Theorem 3.1, which simplify in the current setup.

Proposition 2.3 *For $\phi \in C_b(G^{\times n})$, set*

$$\widetilde{\phi}(s_0, \dots, s_n) = \phi(s_0 s_1^{-1}, s_1 s_2^{-2}, \dots, s_{n-1} s_n^{-1}), \qquad s_i \in G.$$

Then $M_{\widetilde{\phi}}$ is multiplicatively bounded on $S_\infty(L_2(G))^{\times n} \to S_\infty(L_2(G))$ iff T_ϕ defines a multiplicatively bounded multilinear map on $\mathcal{L}G^{\times n} \to \mathcal{L}G$. In this case, we have

$$\|T_\phi\|_{mb} = \|M_{\widetilde{\phi}}\|_{mb}.$$

Proof of the "if" direction Assume that T_ϕ is multiplicatively bounded. Let $F \subseteq G$ be finite with $|F| = N$. Let $p_s \in B(\ell_2(F))$ be the projection on the one-dimensional space spanned by the delta function δ_s. Let $\widetilde{\phi}_F := \widetilde{\phi}|_{F^{\times n+1}}$. By Theorem 2.2 (using that ϕ is continuous), it suffices to prove that

$$M_{\widetilde{\phi}_F} : B(\ell_2(F))^{\times n} \to B(\ell_2(F))$$

and its matrix amplifications are bounded by $\|T_\phi\|_{mb}$. Define the unitary $U = \sum_{s \in F} p_s \otimes \lambda_s \in B(\ell_2(F)) \otimes \mathcal{L}G$ and the isometry

$$\pi : B(\ell_2(F)) \to B(\ell_2(F)) \otimes \mathcal{L}G, \quad \pi(x) = U(x \otimes \mathrm{id})U^*.$$

Note that π satisfies $\pi(E_{st}) = E_{st} \otimes \lambda_{st^{-1}}$. For $s_0, \dots, s_n \in F$,

$$\begin{aligned}\pi(M_{\widetilde{\phi}_F}(E_{s_0 s_1}, E_{s_1 s_2}, \dots, E_{s_{n-1} s_n})) &= \pi(\widetilde{\phi}(s_0, \dots, s_n) E_{s_0 s_n}) \\ &= \phi(s_0 s_1^{-1}, \dots, s_{n-1} s_n^{-1}) E_{s_0 s_n} \otimes \lambda_{s_0 s_n^{-1}},\end{aligned}$$

whereas

$$\begin{aligned}T_\phi^{(N)}(\pi(E_{s_0 s_1}), \dots, \pi(E_{s_{n-1} s_n})) &= T_\phi^{(N)}(E_{s_0 s_1} \otimes \lambda_{s_0 s_1}^{-1}, \dots, E_{s_{n-1} s_n} \otimes \lambda_{s_{n-1} s_n^{-1}}) \\ &= E_{s_0 s_n} \otimes T_\phi(\lambda_{s_0 s_1^{-1}}, \dots, \lambda_{s_{n-1} s_n^{-1}}) \\ &= \phi(s_0 s_1^{-1}, \dots, s_{n-1} s_n^{-1}) E_{s_0 s_n} \otimes \lambda_{s_0 s_n^{-1}}.\end{aligned}$$

It follows that $T_\phi^{(N)} \circ \pi^{\times n} = \pi \circ M_{\widetilde{\phi}_F}$. This implies that

$$\|M_{\widetilde{\phi}_F}\| = \|\pi \circ M_{\widetilde{\phi}_F}\| = \|T_\phi^{(N)} \circ \pi\| \le \|T_\phi^{(N)}\| \le \|T_\phi\|_{mb}.$$

By taking matrix amplifications, we prove similarly that $\|M_{\widetilde{\phi}_F}\|_{mb} \le \|T_\phi\|_{mb}$. ∎

Remark 2.4 A multilinear map on the product of some operator spaces is multiplicatively bounded iff its linearization is completely bounded as a map on the corresponding Haagerup tensor product. However, as [JTT09, Lemma 3.3] shows, for Schur multipliers $M_{\widetilde{\phi}}$ on $S_\infty(L_2(G))^{\times n}$, just boundedness on the Haagerup tensor product is sufficient to guarantee that $M_{\widetilde{\phi}}$ is multiplicatively bounded. Note that even in the linear case, when $p < \infty$, it is unknown whether a bounded Schur multiplier on $S_p(L_2(\mathbb{R}))$ is necessarily completely bounded unless ϕ has continuous symbol (we refer to [Pi98, Conjecture 8.1.12], [LaSa11, Theorem 1.19], and [CaWi19]).

3 Transference from Fourier to Schur multipliers

Let G be a locally compact group, which is again assumed to be unimodular. We will prove that symbols of $(p_1, \dots, p_n)$-Fourier multipliers are also symbols of

$(p_1, \ldots, p_n)$-Schur multipliers using a multilinear transference method. This yields a multilinear version of [CaSa15, Theorem 4.2]. We stipulate that in the proofs of [CaSa15, Section 4], the transference is carried out for rational exponents p. In order to treat the general multilinear case, we present an alternative proof that transfers multipliers directly for every real exponent $p \in [1, \infty)$. In fact, this gives an improvement of [CaSa15, Theorem 4.2], which is stated under the stronger assumption that the multiplier acts boundedly on the Fourier algebra (equivalently is a p-multiplier at $p = 1$). The fundamental difference in the proof is that we base ourselves on the methods from [CPPR15, Claim B, p. 24] and [CJKM, Lemma 4.6].

As before, for $\phi \in C_b(G^{\times n})$, we set $\widetilde{\phi} \in C_b(G^{\times n+1})$ by

$$\widetilde{\phi}(s_0, \ldots, s_n) = \phi(s_0 s_1^{-1}, s_1 s_2^{-1}, \ldots, s_{n-1} s_n^{-1}), \qquad s_i \in G.$$

Theorem 3.1 *Let G be a locally compact, second-countable, unimodular group, and let $1 \le p \le \infty, 1 < p_1, \ldots, p_n \le \infty$ be such that $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ be the symbol of a $(p_1, \ldots, p_n)$-multiplicatively bounded Fourier multiplier of G. Then $\widetilde{\phi}$ is the symbol of a $(p_1, \ldots, p_n)$-multiplicatively bounded Schur multiplier of G. Moreover,*

$$\begin{aligned} &\| M_{\widetilde{\phi}} : S_{p_1}(L_2(G)) \times \cdots \times S_{p_n}(L_2(G)) \to S_p(L_2(G)) \|_{(p_1,\ldots,p_n)-mb} \\ &\quad \le \| T_\phi : L_{p_1}(\mathcal{L}G) \times \cdots \times L_{p_n}(\mathcal{L}G) \to L_p(\mathcal{L}G) \|_{(p_1,\ldots,p_n)-mb}. \end{aligned}$$

Proof Let $F \subseteq G$ be finite. Consider the Hilbert space $\ell_2(F)$, and for $s \in F$, let p_s be the orthogonal projection onto the one-dimensional space spanned by the delta function δ_s. By Theorem 2.2, it suffices to show that the norm of

$$M_{\widetilde{\phi}} : \mathcal{S}_{p_1}(\ell_2(F)) \times \cdots \times \mathcal{S}_{p_n}(\ell_2(F)) \to \mathcal{S}_p(\ell_2(F))$$

and its matrix amplifications are bounded by $\| T_\phi \|_{(p_1,\ldots,p_n)-mb}$.

The proof requires the introduction of coordinatewise convolutions as follows. Fix functions $\varphi_k \in A(G)$ such that $\varphi_k \ge 0$, $\|\varphi_k\|_{L_1(G)} = 1$ and such that the support of φ_k shrinks to the identity of G; from (2.1), it is clear that such functions exist. Then, for any function $\phi \in C_b(G^{\times n})$, we set

$$\begin{aligned} &\phi_k(s_1, \ldots, s_n) \\ &:= \int_{G^{\times n}} \phi(t_1^{-1} s_1 t_2, t_2^{-1} s_2 t_3, \ldots, t_{n-2}^{-1} s_{n-2} t_{n-1}, t_{n-1}^{-1} s_{n-1}, s_n t_n^{-1}) \left(\prod_{j=1}^n \varphi_k(t_j) \right) dt_1 \ldots dt_n. \end{aligned} \tag{3.1}$$

For the particular case $n = 1$, this expression becomes by definition

$$\phi_k(s) := \int_G \phi(st^{-1}) \varphi_k(t) dt = \int_G \phi(t^{-1}) \varphi_k(ts) dt.$$

Let $(U_\alpha)_\alpha$ be a symmetric neighborhood basis of the identity of G consisting of relatively compact sets. Set

$$k_\alpha = |U_\alpha|^{-\frac{1}{2}} \lambda(1_{U_\alpha}),$$

with polar decomposition $k_\alpha = u_\alpha h_\alpha$. Then k_α is an element in $L_2(\mathcal{L}G)$ with $\|k_\alpha\|_2 = 1$. Consequently, $h_\alpha^{2/q}$ is in $L_q(\mathcal{L}G)$ for $1 \le q < \infty$ with $\|h_\alpha^{2/q}\| = 1$. In case $q = \infty$ by

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mild abuse of notation, we set $h_\alpha^{2/q} = 1$. Set the unitary

$$U = \sum_{s \in F} p_s \otimes \lambda_s \in B(\ell_2(F)) \otimes \mathcal{L}G.$$

Now, let $a_i \in S_{p_i}(\ell_2(F))$. Since ϕ_k converges to ϕ pointwise, we have

$$M_{\widetilde{\phi}}(a_1, \dots, a_n) = \lim_k M_{\widetilde{\phi}_k}(a_1, \dots, a_n).$$

So, with $N = |F|$,

$$\begin{aligned}
\|M_{\widetilde{\phi}}(a_1, \dots, a_n)\|_{S_p^N} &= \lim_k \limsup_\alpha \|M_{\widetilde{\phi}_k}(a_1, \dots, a_n) \otimes h_\alpha^{\frac{2}{p}}\|_{S_p^N \otimes L_p(\mathcal{L}G)} \\
&= \lim_k \limsup_\alpha \|U(M_{\widetilde{\phi}_k}(a_1, \dots, a_n) \otimes h_\alpha^{\frac{2}{p}})U^*\|_{S_p^N \otimes L_p(\mathcal{L}G)} \qquad (3.2)\\
&\leq A + B,
\end{aligned}$$

where

(3.3)

$$A = \limsup_k \limsup_\alpha \|T_{\phi_k}^{(N)}(U(a_1 \otimes h_\alpha^{\frac{2}{p_1}})U^*, \dots, U(a_n \otimes h_\alpha^{\frac{2}{p_n}})U^*)\|_{S_p^N \otimes L_p(\mathcal{L}G)},$$

$$\begin{aligned}
B = \limsup_k \limsup_\alpha \|T_{\phi_k}^{(N)}(U(a_1 \otimes h_\alpha^{\frac{2}{p_1}})U^*, \dots, U(a_n \otimes h_\alpha^{\frac{2}{p_n}})U^*) - \\
U(M_{\widetilde{\phi}_k}(a_1, \dots, a_n) \otimes h_\alpha^{\frac{2}{p}})U^*\|_{S_p^N \otimes L_p(\mathcal{L}G)}.
\end{aligned}$$

Below, we prove that $B = 0$. Therefore,

(3.4)

$$\begin{aligned}
&\|M_{\widetilde{\phi}}(a_1, \dots, a_n)\|_{S_p^N} \\
&\leq \limsup_k \limsup_\alpha \|T_{\phi_k}^{(N)}(U(a_1 \otimes h_\alpha^{\frac{2}{p_1}})U^*, \dots, U(a_n \otimes h_\alpha^{\frac{2}{p_n}})U^*)\|_{S_p^N \otimes L_p(\mathcal{L}G)} \\
&\leq \limsup_k \limsup_\alpha \|T_{\phi_k}^{(N)}\| \|U(a_1 \otimes h_\alpha^{\frac{2}{p_1}})U^*\|_{S_{p_1}^N \otimes L_{p_1}(\mathcal{L}G)} \dots \|U(a_n \otimes h_\alpha^{\frac{2}{p_n}})U^*\|_{S_{p_n}^N \otimes L_{p_n}(\mathcal{L}G)} \\
&= \limsup_k \|T_{\phi_k}^{(N)}\| \|a_1\|_{S_{p_1}^N} \dots \|a_n\|_{S_{p_n}^N},
\end{aligned}$$

where the norm of $T_{\phi_k}^{(N)}$ is understood as in (2.2) for the map $T_{\phi_k} : L_{p_1}(\mathcal{L}G) \times \dots \times L_{p_n}(\mathcal{L}G) \to L_p(\mathcal{L}G)$. By [CJKM, Lemma 4.3] and the fact that $\|\varphi_k\|_{L_1(G)} = 1$, it follows then that $\|T_{\phi_k}^{(N)}\| \leq \|T_\phi^{(N)}\|$. Hence,

$$\|M_{\widetilde{\phi}}(a_1, \dots, a_n)\|_{S_p^N} \leq \|T_\phi^{(N)}\| \|a_1\|_{S_{p_1}^N} \dots \|a_n\|_{S_{p_n}^N}. \qquad (3.5)$$

This finishes the proof. The multiplicatively bounded case follows by taking matrix amplifications.

Now, let us prove that the last term in (3.2) goes to 0. By the triangle inequality, it suffices to prove that the limits of the following terms are 0. For $r_0, \dots, r_n \in F$ with matrix units $E_{r_i,r_{i+1}}$,

(3.6)
$$T_{\phi_k}^{(N)}(U(E_{r_0,r_1}\otimes h_\alpha^{\frac{2}{p_1}})U^*,\dots,U(E_{r_{n-1},r_n}\otimes h_\alpha^{\frac{2}{p_n}})U^*)-U(M_{\widetilde{\phi_k}}(E_{r_0,r_1},\dots,E_{r_{n-1},r_n})\otimes h_\alpha^{\frac{2}{p}})U^*$$
$$=E_{r_0,r_n}\otimes\left(T_{\phi_k}(\lambda_{r_0}h_\alpha^{\frac{2}{p_1}}\lambda_{r_1}^*,\dots,\lambda_{r_{n-1}}h_\alpha^{\frac{2}{p_n}}\lambda_{r_n}^*)-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})\lambda_{r_0}h_\alpha^{\frac{2}{p}}\lambda_{r_n}^*\right).$$

Applying the transformation formula [CJKM, Lemma 4.3] to the T_{ϕ_k} term,

$$T_{\phi_k}(\lambda_{r_0}h_\alpha^{\frac{2}{p_1}}\lambda_{r_1}^*,\dots,\lambda_{r_{n-1}}h_\alpha^{\frac{2}{p_n}}\lambda_{r_n}^*)=\lambda_{r_0}T_{\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_\alpha^{\frac{2}{p_1}},\dots,h_\alpha^{\frac{2}{p_n}})\lambda_{r_n}^*.$$

Taking the norm of the expression in equation (3.6),

(3.7)
$$\|(3.6)\|_{S_p^N\otimes L_p(\mathcal{L}G)}$$
$$=\|T_{\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_\alpha^{\frac{2}{p_1}},\dots,h_\alpha^{\frac{2}{p_n}})-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})h_\alpha^{\frac{2}{p}}\|_{L_p(\mathcal{L}G)}.$$

We now claim that $\lim_k\limsup_\alpha$ of this expression yields 0, by almost identical arguments as those used in [CJKM, Lemma 4.6]. Since we have a couple of differences, namely that we have a translated function $\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})$ and we do not use the small almost invariant neighborhoods (SAIN) condition (see [CPPR15, Definition 3.1]), we spell out some of the details here.

Let $\zeta:G\to\mathbb{R}_{\geq 0}$ be a continuous compactly supported positive definite function in $A(G)$ with $\zeta(e)=1$, so that T_ζ is contractive. For $1\leq j\leq n$, let $\zeta_j(s)=\zeta(r_{j-1}^{-1}sr_j)$ and let $\phi(\zeta_1,\dots,\zeta_n)(s_1,\dots,s_n)=\phi(s_1,\dots,s_n)\zeta_1(s_1)\dots\zeta_n(s_n)$. Then

$$\begin{aligned}
&\|T_{\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_\alpha^{\frac{2}{p_1}},\dots,h_\alpha^{\frac{2}{p_n}})-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})h_\alpha^{\frac{2}{p}}\|_{L_p(\mathcal{L}G)}\\
&\leq\|\left((\phi(\zeta_1,\dots,\zeta_n))_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})-\phi_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})\right)h_\alpha^{\frac{2}{p}}\|_{L_p(\mathcal{L}G)}\\
&\quad+\|T_{(\phi(\zeta_1,\dots,\zeta_n))_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_\alpha^{\frac{2}{p_1}},\dots,h_\alpha^{\frac{2}{p_n}})-(\phi(\zeta_1,\dots,\zeta_n))_k(r_0r_1^{-1},\dots,r_{n-1}r_n^{-1})h_\alpha^{\frac{2}{p}}\|_{L_p(\mathcal{L}G)}\\
&\quad+\|T_{(\phi(\zeta_1,\dots,\zeta_n))_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_\alpha^{\frac{2}{p_1}},\dots,h_\alpha^{\frac{2}{p_n}})-T_{\phi_k(r_0\cdot r_1^{-1},\dots,r_{n-1}\cdot r_n^{-1})}(h_\alpha^{\frac{2}{p_1}},\dots,h_\alpha^{\frac{2}{p_n}})\|_{L_p(\mathcal{L}G)}\\
&=A_{k,\alpha}+B_{k,\alpha}+C_{k,\alpha}.
\end{aligned}$$

The terms $A_{k,\alpha}$, $B_{k,\alpha}$, and $C_{k,\alpha}$ should be compared to the terms occurring in [CJKM, equation (4.7)]. Since φ_k have support shrinking to e, and $\zeta_j(r_{j-1}r_j^{-1})=1$, it follows that $\lim_k A_{k,\alpha}=0$.

For fixed $t_1,\dots,t_n\in G$, define

$$C_\alpha(t_1,\dots,t_n):=\|T_{\phi-\phi(\zeta_1,\dots,\zeta_n)(t_1^{-1}r_0\cdot r_1^{-1}t_2,\dots,t_{n-1}^{-1}r_{n-2}\cdot r_{n-1}^{-1},r_{n-1}\cdot r_n^{-1}t_n^{-1})}(h_\alpha^{\frac{2}{p_1}},\dots,h_\alpha^{\frac{2}{p_n}})\|_{L_p(\mathcal{L}G)},$$

and for $1\leq j\leq n-2$,

$$y_{j,\alpha}=\lambda_{t_j^{-1}r_{j-1}}h_\alpha^{\frac{2}{p_j}}\lambda_{r_j^{-1}t_{j+1}},$$
$$y_{n-1,\alpha}=\lambda_{t_{n-1}^{-1}r_{n-2}}h_\alpha^{\frac{2}{p_{n-1}}}\lambda_{r_{n-1}^{-1}},$$
$$y_{n,\alpha}=\lambda_{r_{n-1}}h_\alpha^{\frac{2}{p_n}}\lambda_{r_n^{-1}t_n^{-1}}.$$

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By the same arguments as in [CJKM, Lemma 4.6, after equation (4.8)], we get, for $1 \leq j \leq n-2$,

$$
\begin{aligned}
\lim_{\alpha} \|(T_{\zeta_j} - \mathrm{id})(y_{j,\alpha})\|_{L_{p_j}(\mathcal{L}G)} &= |\zeta(r_{j-1}^{-1} t_j^{-1} r_{j-1} r_j^{-1} t_{j+1} t_j) - 1|,\\
\lim_{\alpha} \|(T_{\zeta_{n-1}} - \mathrm{id})(y_{n-1,\alpha})\|_{L_{p_{n-1}}(\mathcal{L}G)} &= |\zeta(r_{n-2}^{-1} t_{n-1}^{-1} r_{n-2}) - 1|,\\
\lim_{\alpha} \|(T_{\zeta_n} - \mathrm{id})(y_{n,\alpha})\|_{L_{p_n}(\mathcal{L}G)} &= |\zeta(r_n^{-1} t_n^{-1} r_n) - 1|.
\end{aligned} \tag{3.8}
$$

Crucially, here we require this argument from [CJKM, Lemma 4.6, after equation (4.8)] only for $x_j = 1$. Hence, we do not require the use of [CJKM, Lemma 3.15], which uses the SAIN condition. Furthermore, note that in this case, the above equalities are trivially true when $p_j = \infty$ (as $h_\alpha^{2/p_j} = 1$ in that case), so we do not require the use of [CJKM, Proposition 3.9], which holds only for $1 < p_j < \infty$. Integrating $C_\alpha(t_1, \ldots, t_n)$ against $\prod_{j=1}^n \varphi_k(t_j)$, we can show that $\lim_k \limsup_\alpha C_{k,\alpha} = 0$, just as in [CJKM, Lemma 4.6].

To show that $\lim_k \lim_\alpha B_{k,\alpha} = 0$, we only note the modifications from [CJKM, Lemma 4.6]. As before, $x_j = 1$ in the proof of [CJKM, Lemma 4.6]. The operators T_j appearing in that proof are to be replaced with $T_{\varphi_k(r_{j-1} \cdot r_j^{-1} t_j)}$ for $1 \leq j \leq n-1$ and T_n is $T_{\varphi_k(t_n r_{n-1} \cdot r_n^{-1})}$. Since all x_j are 1, the term $S_{j+1}(x_{j+1} S_{j+2}(x_{j+2} \ldots S_{n-1}(x_{n-1}) \ldots))$ in the definition of $R_{j,V}$ that appears in [CJKM, Lemma 4.6] is now just the scalar

$$\prod_{i=j+1}^{n-1} \varphi_k(r_{i-1} r_i^{-1} t_i).$$

Now, in equation (4.13) of [CJKM], the commutator terms vanish, as one of the terms in every case is a scalar. Additionally, since the $\widetilde{S_j}$ in the estimate for the first summand in equation (4.14) of [CJKM] is a scalar, we can once more avoid [CJKM, Lemma 3.15] and the SAIN condition. Once again, note that since $x_j = 1$, the proof remains valid even when some of the $p_j = \infty$. ∎

Remark 3.2 Theorem 3.1 assumes G to be second-countable since its proof relies on [CJKM], which assumes second countability.

Remark 3.3 Fix $1 \leq i \leq n$. In case $p_i = p = 1$ and $p_j = \infty$ for all $1 \leq j \leq n$, $i \neq j$, we do not know whether Theorem 3.1 holds. The reason is that we do not know whether the limits (3.8) (at index i) hold and neither do we know if the two further applications of Proposition 3.9 in [CJKM, Proof of Lemma 4.6] hold.

4 Amenable groups: transference from Schur to Fourier multipliers

Recall [BHV08, Section G] that G is amenable iff it satisfies the Følner condition: for any $\varepsilon > 0$ and any compact set $K \subseteq G$, there exists a compact set F with nonzero measure such that $\frac{\mu_G(s.F \Delta F)}{\mu_G(F)} < \varepsilon$ for all $s \in K$. Here, Δ is the symmetric difference. This allows us to construct a net $F_{(\varepsilon,K)}$ of such Følner sets using the ordering $(\varepsilon_1, K_1) \leq (\varepsilon_2, K_2)$ if $\varepsilon_1 \geq \varepsilon_2$, $K_1 \subseteq K_2$.

Theorem 4.1 *Let G be a locally compact, unimodular, amenable group, and let $1 \leq p, p_1, \ldots, p_n \leq \infty$ be such that $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ be such that $\widetilde{\phi}$ is the*

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symbol of a $(p_1,\dots,p_n)$*-Schur multiplier of* G*. Then there is an ultrafilter* $\mathcal{U}$ *on a set* I *and there are complete contractions (resp. complete isometries if* $1<q<\infty$*)* $i_q : L_q(\mathcal{L}G) \to \prod_{\mathcal{U}} S_q(L_2(G))$ *such that, for all* $f_i, f \in C_c(G) \star C_c(G)$*,*

(4.1)

$$\langle i_p(T_\phi(x_1,\dots,x_n)), i_{p'}(y^*)\rangle_{p,p'} = \Big\langle \Big(M_{\tilde{\phi}}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n))\Big)_{\alpha\in I}, i_{p'}(y^*)\Big\rangle_{p,p'},$$

where $x_i = \lambda(f_i)$, $y=\lambda(f)$*, and* $\frac{1}{p'}+\frac{1}{p}=1$*. In a similar way, the matrix amplifications of* i_q *intertwine the multiplicative amplifications of the Fourier and Schur multipliers.*

Proof Let $F_\alpha, \alpha\in I$ be a Følner net for G, where I is the index set consisting of pairs (ε,K) for $\varepsilon>0$, $K\subseteq G$ compact. It has the ordering as described above. Let $P_\alpha = P_{F_\alpha}$ be the projection onto $L_2(F_\alpha)$. Let $\mathcal{U}$ be an ultrafilter refining the net I, and consider the map $i_p : L_p(\mathcal{L}G)\to \prod_{\mathcal{U}} S_p(L_2(G))$ defined by $i_p(x) = (i_{p,\alpha}(x))_{\alpha\in I} = (\frac{1}{\mu_G(F_\alpha)^{1/p}} P_\alpha x P_\alpha)_{\alpha\in I}$. From [CaSa15, Theorem 5.1], i_p is a complete contraction (and even a complete isometry for $1<p<\infty$ [CaSa15, Theorem 5.2]); here, the Følner condition is used.

Fix α and let $f\in C_c(G) * C_c(G)$. We first observe that by [CaSa15, Theorem 5.1] applied to the bounded operator $x=\lambda(f)$ we have $P_{F_\alpha}\lambda(f)P_{F_\alpha}\in S_q(L_2(G))$ for all $1\le q\le\infty$ and the kernel of this operator is given by the function

$$(s,t)\mapsto 1_{F_\alpha}(s) f(st^{-1}) 1_{F_\alpha}(t).$$

So we have

$$\begin{aligned} &M_{\tilde{\phi}}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n))(t_0,t_n)\\ &\quad= \frac{1}{\mu_G(F_\alpha)^{1/p}} 1_{F_\alpha}(t_0)1_{F_\alpha}(t_n)\int_{F_\alpha^{\times n-1}} \phi(t_0t_1^{-1},\dots,t_{n-1}t_n^{-1}) f_1(t_0t_1^{-1})\dots f_n(t_{n-1}t_n^{-1})dt_1\dots dt_{n-1}. \end{aligned}$$

Moreover, after some change of variables, we see that $P_{F_\alpha}T_\phi(x_1,\dots,x_n)P_{F_\alpha}$ is given by the kernel

$$\begin{aligned} &(t_0,t_n)\\ &\quad\mapsto 1_{F_\alpha}(t_0)\int_{G^{\times n-1}} \phi(t_0t_1^{-1},\dots,t_{n-1}t_n^{-1}) f_1(t_0t_1^{-1})\dots f_n(t_{n-1}t_n^{-1}) 1_F(t_n) dt_1\dots dt_{n-1}. \end{aligned}$$

Let Φ denote the function

$$\Phi(t_0,\dots,t_n) = \phi(t_0t_1^{-1},\dots,t_{n-1}t_n^{-1}) f_1(t_0t_1^{-1})\dots f_n(t_{n-1}t_n^{-1}) f(t_nt_0^{-1}),$$

and let Ψ_α be defined by

$$\Psi_\alpha(t_0,\dots,t_n) = 1_{F_\alpha}(t_0)1_{F_\alpha}(t_n) - 1_{F_\alpha^{\times n+1}}(t_0,\dots,t_n).$$

Let K be some compact set such that $\mathrm{supp}(f_j), \mathrm{supp}(f)\subseteq K$. Let $t_0,\dots,t_n$ be such that both $\Phi(t_0,\dots,t_n)$ and $\Psi_\alpha(t_0,\dots,t_n)$ are nonzero. Since $\Psi_\alpha(t_0,\dots,t_n)$ is nonzero, we must have $t_0,t_n\in F_\alpha$ and $t_1,\dots,t_{n-1}\notin F_\alpha$. Since $\Phi(t_0,\dots,t_n)$ is nonzero, there are $k_1,\dots,k_n\in K$ such that $t_{n-1}=k_nt_n,\ t_{n-2}=k_{n-1}k_nt_n,\ \dots,\ t_0 = k_1\dots k_nt_n$. Hence, we find that t_n belongs to the set

$$F_\alpha\cap F_\alpha.(k_1\dots k_n)^{-1}\backslash\big(F_\alpha.(k_2\dots k_n)^{-1}\cup\dots\cup F_\alpha.k_n^{-1}\big).$$

Using these facts, along with some change of variables, we get

(4.2)
$$
\begin{aligned}
&|\langle i_{p,\alpha}(T_\phi(x_1,\dots,x_n)), i_{p',\alpha}(y^*)\rangle_{p,p'} - \langle M_{\widetilde{\phi}}(i_{p_1,\alpha}(x_1),\dots,i_{p_n,\alpha}(x_n)), i_{p',\alpha}(y^*)\rangle_{p,p'}| \\
&= \left| \frac{1}{\mu_G(F_\alpha)} \int_{G^{n+1}} \Phi(t_0,\dots,t_n)\Psi_\alpha(t_0,\dots,t_n) dt_0\dots dt_n \right| \\
&= \left| \frac{1}{\mu_G(F_\alpha)} \int_{K^n}\int_G \Phi(k_1\dots k_n t_n,\dots,k_n t_n, t_n)\Psi_\alpha(k_1\dots k_n t_n,\dots,k_n t_n, t_n) dk_1\dots dk_n dt_n \right| \\
&\leq \|\Phi\|_\infty \int_{K^n} \frac{1}{\mu_G(F_\alpha)} \mu_G \\
&\quad \times \left(F_\alpha \cap (F_\alpha.(k_1\dots k_n)^{-1}) \cap (F_\alpha.(k_2\dots k_n)^{-1})^c \cap \dots \cap (F_\alpha.(k_n)^{-1})^c\right) dk_1\dots dk_n \\
&\leq \|\Phi\|_\infty \int_{K^n} \frac{1}{\mu_G(F_\alpha)} \mu_G\left(F_\alpha \cap (F_\alpha.(k_n)^{-1})^c\right) dk_1\dots dk_n.
\end{aligned}
$$

Using the ordering described earlier, if the index $\alpha \geq (\varepsilon \times (\|\Phi\|_\infty \mu_G(K^n))^{-1}, K^{-1})$, then the Følner condition implies that (4.2) is less than ε, and hence equation (4.1) is true.

A direct modification of this argument now shows that the i_p also intertwine the multiplicative amplifications of the Fourier and Schur multipliers. That is, for $\beta_i \in S^N_{p_i}$, $\beta \in S^N_{p'}$, we have

(4.3)
$$
\begin{aligned}
&\langle \mathrm{id} \otimes i_p(T^{(N)}_\phi(\beta_1 \otimes x_1,\dots,\beta_n \otimes x_n)), \mathrm{id} \otimes i_{p'}(\beta \otimes y^*)\rangle_{p,p'} = \\
&\quad \left\langle \left(M^{(N)}_{\widetilde{\phi}}(\mathrm{id} \otimes i_{p_1,\alpha}(\beta_1 \otimes x_1),\dots,\mathrm{id} \otimes i_{p_n,\alpha}(\beta_n \otimes x_n))\right)_\alpha, \mathrm{id} \otimes i_{p'}(\beta \otimes y^*)\right\rangle.
\end{aligned}
$$

■

Combining this with Theorem 3.1, we get the multilinear version of [CaSa15, Corollary 5.3]

Corollary 4.2 *Let $1 < p, p_1, \dots, p_n < \infty$ be such that $p^{-1} = \sum_{i=1}^n p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ and assume that G is amenable. If $\widetilde{\phi}$ is the symbol of a $(p_1,\dots,p_n)$-bounded (resp. multiplicatively bounded) Schur multiplier, then ϕ is the symbol of a $(p_1,\dots,p_n)$-bounded (resp. multiplicatively bounded) Fourier multiplier. Moreover,*

$$
\|T_\phi\|_{(p_1,\dots,p_n)} \leq \|M_{\widetilde{\phi}}\|_{(p_1,\dots,p_n)}, \qquad \|T_\phi\|_{(p_1,\dots,p_n)-mb} \leq \|M_{\widetilde{\phi}}\|_{(p_1,\dots,p_n)-mb},
$$

with equality in the $(p_1,\dots,p_n)$-mb norm when G is second-countable.

Proof For $1 < p < \infty$, i_p is a (complete) isometry. Hence, for x_i as in the hypothesis of Theorem 4.1, we get

$$
\begin{aligned}
&\|T_\phi(x_1,\dots,x_n)\|_{L_p(\mathcal{L}G)} \\
&\quad = \|i_p \circ T_\phi(x_1,\dots,x_n)\|_{L_p(\mathcal{L}G)} = \|M_{\widetilde{\phi}}(i_{p_1}(x_1),\dots,i_{p_n}(x_n))\|_{L_p(\mathcal{L}G)} \\
&\quad \leq \|M_{\widetilde{\phi}}\|_{(p_1,\dots p_n)} \prod_{i=1}^n \|x_i\|_{L_{p_i}(\mathcal{L}G)}.
\end{aligned}
$$

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For $1 < p_i < \infty$, such x_i are norm-dense in $L_{p_i}(\mathcal{L}G)$, so we get the bound on $\|T_\phi\|_{(p_1,\ldots,p_n)}$. The multiplicatively bounded version follows similarly, with the other inequality coming from Theorem 3.1. ■

Let $H \le G$ be a subgroup. Clearly, from Theorem 2.2, the restriction of ϕ to H also determines a bounded Schur multiplier, with

$$\|M_{\widetilde{\phi_{|H}}}\|_{(p_1,\ldots,p_n)} \le \|M_{\tilde{\phi}}\|_{(p_1,\ldots,p_n)}.$$

Combining this observation with Corollary 4.2 gives us the multiplicatively bounded version of the multilinear de Leeuw restriction theorem [CJKM, Theorem 4.5] for amenable discrete subgroups of second-countable groups. Note that the SAIN condition used in [CJKM, Theorem 4.5] is implicit here since the subgroup is amenable.

Corollary 4.3 *Let G be a locally compact, unimodular, second-countable group, and let $1 < p, p_1, \ldots, p_n < \infty$ with $p^{-1} = \sum_i p_i^{-1}$. Let $\phi \in C_b(G^{\times n})$ be a symbol of a $(p_1, \ldots, p_n)$-multiplicatively bounded Fourier multiplier. If H is an amenable discrete subgroup of G, then we have*

$$\|T_{\phi|_H}\|_{(p_1,\ldots,p_n)-mb} \le \|T_\phi\|_{(p_1,\ldots,p_n)-mb}.$$

Remark 4.4 We now discuss some difficulties one encounters when modifying the above methods to the nonunimodular case, meaning that the Plancherel weight φ is no longer a trace. In this case, $\lambda(C_c(G) \star C_c(G))$ is no longer a common dense subset of the $L_p(\mathcal{L}G)$. Rather, we have embeddings $\eta_{t,p} : \lambda(C_c(G) \star C_c(G)) \to L_p(\mathcal{L}G)$ with dense image given by $\lambda(f) \mapsto \Delta^{(1-t)/p}\lambda(f)\Delta^{t/p}$, where $0 \le t \le 1$ and Δ is the multiplication operator with the modular function, which we denote also by Δ (see [Ter81, Ter82] for details).

This raises the question how to define the $(p_1, \ldots, p_n)$-Fourier multiplier. A possible choice would be to take some $t \in [0,1]$ and define

$$T_\phi(\eta_{t,p_1}(x_1), \ldots, \eta_{t,p_n}(x_n)) = \eta_{t,p}(T_\phi(x_1, \ldots, x_n)).$$

However, with this definition, the intertwining property in the ultralimit of (4.2) will no longer hold. This can be illustrated by considering the case $n = 2$ and $\phi(x, y) = \phi_1(x)\phi_2(y)$. In order for the intertwining property to hold, the Fourier multiplier would have to satisfy

$$T_\phi(\eta_{t,p_1}(x_1), \eta_{t,p_2}(x_2)) = T_{\phi_1}(\eta_{t,p_1}(x_1))T_{\phi_2}(\eta_{t,p_2}(x_2)),$$

where on the right-hand side we use the linear definition of the Fourier multiplier from [CaSa15]. This is not the case with the above definition. As a consequence, we no longer have nice relations between nested linear Fourier multipliers and multilinear Fourier multipliers, as in [CJKM, Lemma 4.4]. As the proof of the multilinear restriction theorem in [CJKM] repeatedly uses such formulae, it is also unclear if these de Leeuw-type theorems are still valid in the nonunimodular case.

5 Domain of the completely bounded bilinear Hilbert transform and Calderón–Zygmund operators

In this final section, we prove a result about nonboundedness of the bilinear Hilbert transform based on our multilinear transference techniques. We prove an analogous result for examples of Calderón–Zygmund operators. This shows that the main results from [AmUr20, DMLV22] about L_p-boundedness of certain Fourier multipliers cannot be extended to range spaces with $p \leq 1$. This is in contrast with the Euclidean (nonvector-valued) case.

5.1 Lower bounds for the vector-valued bilinear Hilbert transform

For $0 < p < \infty$, let $S_p^N = S_p(\mathbb{C}^N)$ be the Schatten L_p-space associated with linear operators on $\mathbb{C}^N$. For $0 < p < 1$, we have that S_p^N is a quasi-Banach space satisfying the quasi-triangle inequality:

$$\|x+y\|_p \leq 2^{\frac{1}{p}-1}(\|x\|_p + \|y\|_p), \qquad x, y \in S_p^N.$$

We set

$$h(\xi_1, \xi_2) = \chi_{\geq 0}(\xi_1 - \xi_2), \qquad \xi_1, \xi_2 \in \mathbb{R}.$$

The first statement of the following theorem is the main result of [LaTh99], and the latter statement of this theorem for $1 < p < \infty$ was proved in [AmUr20, DMLV22].

Theorem 5.1 *For every $1 < q_1, q_2, q, p_1, p_2 < \infty$, $\frac{2}{3} < p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $N \in \mathbb{N}_{\geq 1}$, there exists a bounded linear map*

$$T_h^{(N)} : L_{p_1}(\mathbb{R}, S_{q_1}^N) \times L_{p_2}(\mathbb{R}, S_{q_2}^N) \to L_p(\mathbb{R}, S_q^N), \tag{5.1}$$

which is determined by

$$T_h^{(N)}(f_1, f_2)(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2) h(\xi_1, \xi_2) e^{is(\xi_1+\xi_2)} d\xi_1 d\xi_2,$$

where $s \in \mathbb{R}$ and $f_i, i = 1, 2$ are functions in $L_{p_i}(\mathbb{R}, S_{q_i}^N)$ whose Fourier transforms $\widehat{f_i}$ are continuous compactly supported functions $\mathbb{R} \to S_{q_i}^N$. If $1 < p := (\frac{1}{p_1} + \frac{1}{p_2})^{-1} < \infty$ and $\frac{1}{\max\{q,q'\}} + \frac{1}{\max\{q_1,q_1'\}} + \frac{1}{\max\{q_2,q_2'\}} > 1$, we have that this operator is moreover uniformly bounded in N.

Note that the map $T_h^{(N)}$ as defined above coincides with the multiplicative amplification of the map $T_h := T_h^{(1)}$ as defined in Section 2.7, so this notation is consistent.

Our aim is to show that the results of [AmUr20, DMLV22] cannot be extended to the case $p_i = q_i$, $q = p = 1 = \frac{1}{p_1} + \frac{1}{p_2}$; i.e., the bound of (5.1) is not uniform in N. In particular, we show that the bound can be estimated from below by $C \log(N)$ for some constant C independent of N.

For a function $\phi : \mathbb{R}^2 \to \mathbb{C}$, we recall the definition

$$\widetilde{\phi}(\lambda_0, \lambda_1, \lambda_2) = \phi(\lambda_0 - \lambda_1, \lambda_1 - \lambda_2), \qquad \lambda_i \in \mathbb{R}.$$

Theorem 5.2 *Let* $1 < p_1, p_2 < \infty$ *be such that* $\frac{1}{p_1} + \frac{1}{p_2} = 1$. *There exists an absolute constant* $C > 0$ *such that, for every* $N \in \mathbb{N}_{\geq 1}$, *we have*

$$A_{p_1,p_2,N} := \| T_h^{(N)} : L_{p_1}(\mathbb{R}, S_{p_1}^N) \times L_{p_2}(\mathbb{R}, S_{p_2}^N) \to L_1(\mathbb{R}, S_1^N) \| > C \log(N).$$

Proof In the proof, let $\mathbb{Z}_N = [-N, N] \cap \mathbb{Z}$. We may naturally identify S_p^{2N+1} with $S_p(\ell_2(\mathbb{Z}_N))$. Let $\varphi \in C_c(\mathbb{R})$, $\varphi \geq 0$ be such that $\varphi(t) = \varphi(-t)$, $t \in \mathbb{R}$, $\|\varphi\|_{L_1(G)} = 1$ and its support is contained in $[-\frac{1}{2}, \frac{1}{2}]$. Set, for $s_1, s_2 \in \mathbb{R}$,

$$H(s_1, s_2) = \int_{\mathbb{R}} h(s_1 + t, -t + s_2)\varphi(t)dt.$$

Then H is continuous and H equals h on $\mathbb{Z} \times \mathbb{Z}$. As a consequence of [CJKM, Lemma 4.3], we find

$$\| T_H^{(2N+1)} : L_{p_1}(\mathbb{R}, S_{p_1}^{2N+1}) \times L_{p_2}(\mathbb{R}, S_{p_2}^{2N+1}) \to L_1(\mathbb{R}, S_1^{2N+1}) \| \leq A_{p_1,p_2,2N+1}.$$

By the multilinear De Leeuw restriction theorem [CJKM, Theorem C], we have

$$\| T_{H|_{\mathbb{Z}\times\mathbb{Z}}}^{(2N+1)} : L_{p_1}(\mathbb{T}, S_{p_1}^{2N+1}) \times L_{p_2}(\mathbb{T}, S_{p_2}^{2N+1}) \to L_1(\mathbb{T}, S_1^{2N+1}) \| \leq A_{p_1,p_2,2N+1}. \tag{5.2}$$

Let $\zeta_l(z) = z^l, z \in \mathbb{T}, l \in \mathbb{Z}$. Set the unitary $U = \sum_{l=-N}^{N} p_l \otimes \zeta_l$ and for any $1 < p < \infty$ the isometric map

$$\pi_p : S_p^{2N+1} \to S_p^{2N+1} \otimes L_p(\mathbb{T}) : x \mapsto U(x \otimes 1)U^*.$$

Then,

$$T_{H|_{\mathbb{Z}\times\mathbb{Z}}}^{(2N+1)} \circ (\pi_{p_1} \times \pi_{p_2}) = \pi_p \circ M_{\widetilde{H}|_{\mathbb{Z}_N\times\mathbb{Z}_N}}.$$

This together with (5.2) implies that

$$\| M_{\widetilde{H}|_{\mathbb{Z}_N\times\mathbb{Z}_N}} : S_{p_1}^{2N+1} \times S_{p_2}^{2N+1} \to S_1^{2N+1} \| \leq A_{p_1,p_2,2N+1}. \tag{5.3}$$

Now, set $H_j(s,t) = \widetilde{H}|_{\mathbb{Z}\times\mathbb{Z}}(s, j, t), s, t \in \mathbb{Z}$. Note that

$$H_j(s,t) = \chi_{\geq 0}(s + t - 2j).$$

By [PSST17, Theorem 2.3], we find that

$$\max_{-N\leq j\leq N} \| M_{H_j} : S_1^{2N+1} \to S_1^{2N+1} \| \leq \| M_{\widetilde{H}|_{\mathbb{Z}_N\times\mathbb{Z}_N}}^{(N)} : S_{p_1}^{2N+1} \times S_{p_2}^{2N+1} \to S_1^{2N+1} \|. \tag{5.4}$$

For $j = 0$, we have that M_{H_j} is the triangular truncation map and therefore by [Dav88, Proof of Lemma 10] (apply M_{H_0} to the matrix consisting of only 1's) there is a constant $C > 0$ such that

$$C \log(2N + 1) \leq \| M_{H_0} : S_1^{2N+1} \to S_1^{2N+1} \|. \tag{5.5}$$

Combining (5.3)–(5.5) yields the result for $2N + 1$. Since the norm of $T_h^{(N)}$ is increasing in N, the result for even N also follows. ■

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5.2 Lower bounds for Calderón–Zygmund operators

The aim of this section is to show a result similar to Theorem 5.2 for Calderón–Zygmund operators by considering an example. This shows that the results from [DLMV20] cannot be extended to the case where the range space is $p = 1$. This is in contrast with the commutative situation where Grafakos and Torres [GrTo02] have shown boundedness of a class of Calderón–Zygmund operators with natural size and smoothness conditions as maps $L_p \times \cdots \times L_p \to L_{p/n}$ for $p \in (1, \infty)$.

Consider any symbol m that is smooth on $\mathbb{R}^2 \backslash \{0\}$, homogeneous, and which is determined on one of the quadrants by

$$m(s,t) = \frac{s}{s-t}, \qquad s \in \mathbb{R}_{>0}, t \in \mathbb{R}_{<0}. \tag{5.6}$$

Here, homogeneous means that $m(\lambda s, \lambda t) = m(s,t)$, $s, t \in \mathbb{R}$, $\lambda > 0$. We assume moreover that m is regulated at 0, by which we mean that

$$m(0) = \pi^{-1} r^{-2} \int_{\|(t_1,t_2)\|_2 \le r} m(t_1, t_2) dt_1 dt_2, \qquad r > 0.$$

As m is homogeneous, this expression is independent of r. This type of symbol m is important as it occurs naturally in the analysis of divided difference functions; for instance, it plays a crucial role in [CSZ21].

Theorem 5.3 *Let $1 < p_1, p_2 < \infty$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. There exists an absolute constant $C > 0$ such that*

$$B_{p_1,p_2,N} := \| T_m^{(N)} : L_{p_1}(\mathbb{R}, S_{p_1}^N) \times L_{p_2}(\mathbb{R}, S_{p_2}^N) \to L_1(\mathbb{R}, S_1^N) \| > C \log(N).$$

Proof By [Dav88, Lemma 10] (and the proof of [Dav88, Corollary 11]), there exist constants $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N$ such that the function

$$\phi(i,j) = \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}, \qquad 1 \le i, j \le N,$$

is the symbol of a linear Schur multiplier $M_\phi : S_1^N \to S_1^N$ whose norm is at least $C \log(N)$ for some absolute constant $C > 0$. Without loss of generality, we may assume that $\lambda_i \in K_N^{-1}\mathbb{Z}$ for some $K_N \in \mathbb{N}_{\ge 1}$ by an approximation argument. Then, in this proof, let $\Lambda_N = \{\lambda_0, \lambda_1, \ldots, \lambda_N\}$. We may naturally identify S_p^{N+1} with $S_p(\ell_2(\Lambda_N))$ by identifying $E_{i,j}$ with E_{λ_i, λ_j}. We proceed as in the proof of Theorem 5.2.

For $\lambda \in K_N^{-1}\mathbb{Z}$, let p_λ be the orthogonal projection of $\ell_2(K_N^{-1}\mathbb{Z})$ onto $\mathbb{C}\delta_\lambda$. Furthermore, for $\lambda \in K_N^{-1}\mathbb{Z}$, set $\zeta_\lambda : \mathbb{T} \to \mathbb{C}$ by $\zeta_\lambda(z) = z^{K_N \lambda}$, $\theta \in \mathbb{R}$. This way every $z \in \mathbb{T}$ determines a representation $\lambda \mapsto \zeta_\lambda(z)$ of $K_N^{-1}\mathbb{Z}$ and this identifies $\mathbb{T}$ with the Pontrjagin dual of $K_N^{-1}\mathbb{Z}$. Set the unitary $U = \sum_{\lambda \in \Lambda_N} p_\lambda \otimes \zeta_\lambda$ and for any $1 < p < \infty$ the isometric map

$$\pi_p : S_p^{N+1} \to S_p^{N+1} \otimes L_p(\mathbb{T}) : x \mapsto U(x \otimes 1)U^*.$$

For $r > 0$, consider the function

$$m_r(s_1, s_2) = \frac{1}{\pi r^2} \int_{\|(s_1 - t_1, s_2 - t_2)\|_2 \le r} m(t_1, t_2) dt_1 dt_2.$$

This function is continuous and bounded, and hence we may apply the bilinear De Leeuw restriction theorem [CJKM, Theorem C] to get

$$\begin{aligned}(5.7)\qquad &\| T^{(N+1)}_{m_r|_{(K_N^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{T}, S^{N+1}_{p_2}) \to L_1(\mathbb{T}, S^{N+1}_1)\| \\ &\quad \leq \| T^{(N+1)}_{m_r} : L_{p_1}(\mathbb{R}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{R}, S^{N+1}_{p_2}) \to L_1(\mathbb{R}, S^{N+1}_1)\|.\end{aligned}$$

Since $m_r|_{(K_N^{-1}\mathbb{Z})^2}$ converges to $m|_{(K_N^{-1}\mathbb{Z})^2}$ pointwise, we obtain (by considering the action of the multiplier on functions with finite frequency support)

$$\begin{aligned}(5.8)\qquad &\lim_{r\searrow 0} \| T^{(N+1)}_{m_r|_{(K_N^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{T}, S^{N+1}_{p_2}) \to L_1(\mathbb{T}, S^{N+1}_1)\| \\ &\quad = \| T^{(N+1)}_{m|_{(K_N^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{T}, S^{N+1}_{p_2}) \to L_1(\mathbb{T}, S^{N+1}_1)\|.\end{aligned}$$

Furthermore, viewing m_r as a convolution of m with an $L_1(\mathbb{R}^2)$ function, from [CJKM, Lemma 4.3],

$$\begin{aligned}(5.9)\qquad &\| T^{(N+1)}_{m_r} : L_{p_1}(\mathbb{R}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{R}, S^{N+1}_{p_2}) \to L_1(\mathbb{R}, S^{N+1}_1)\| \\ &\quad \leq \| T^{(N+1)}_{m} : L_{p_1}(\mathbb{R}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{R}, S^{N+1}_{p_2}) \to L_1(\mathbb{R}, S^{N+1}_1)\| = B_{p_1,p_2,N}.\end{aligned}$$

Combining the estimates (5.7)–(5.9), we find that

$$(5.10)\qquad \| T^{(N+1)}_{m|_{(K_N^{-1}\mathbb{Z})^2}} : L_{p_1}(\mathbb{T}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{T}, S^{N+1}_{p_2}) \to L_1(\mathbb{T}, S^{N+1}_1)\| \leq B_{p_1,p_2,N}.$$

We view $\widetilde{m}|_{\Lambda_N\times\Lambda_N\times\Lambda_N}$ as the symbol of a Schur multiplier $M_{\widetilde{m}|_{\Lambda_N\times\Lambda_N\times\Lambda_N}} : S^{N+1}_{p_1} \times S^{N+1}_{p_2} \to S^{N+1}_1$. Then,

$$T^{(N+1)}_{m|_{K_N^{-1}\mathbb{Z}}} \circ (\pi_{p_1} \times \pi_{p_2}) = \pi_p \circ M_{\widetilde{m}|_{\Lambda_N\times\Lambda_N\times\Lambda_N}}.$$

It follows with (5.10) that

$$\begin{aligned}(5.11)\qquad &\| M_{\widetilde{m}|_{\Lambda_N\times\Lambda_N\times\Lambda_N}} : S^{N+1}_{p_1} \times S^{N+1}_{p_2} \to S^{N+1}_1\| \\ &\leq \| T^{(N+1)}_{m|_{K_N^{-1}\mathbb{Z}}} : L_{p_1}(\mathbb{T}, S^{N+1}_{p_1}) \times L_{p_2}(\mathbb{T}, S^{N+1}_{p_2}) \to L_1(\mathbb{T}, S^{N+1}_p)\| \leq B_{p_1,p_2,N+1}.\end{aligned}$$

By [PSST17, Theorem 2.3], we find that

$$(5.12)\quad \| M_{\widetilde{m}|_{\Lambda_N\times\Lambda_N\times\Lambda_N}(\,\cdot\,,0\,\cdot\,)} : S^{N+1}_1 \to S^{N+1}_1\| \leq \| M_{\widetilde{m}|_{\Lambda_N\times\Lambda_N\times\Lambda_N}} : S^{N+1}_{p_1} \times S^{N+1}_{p_2} \to S^{N+1}_p\|.$$

Now, for $s, t \in \mathbb{R}_{>0}$, we find

$$\widetilde{m}(s,0,t) = m(s-0, 0-t) = \frac{s}{s+t} = \frac{1}{2}(1 + \frac{s-t}{s+t}) = \frac{1}{2}(1 + \phi(s,t)).$$

It follows therefore by the first paragraph that for some constant $C > 0$,

$$C\log(N) \leq \| M_{\widetilde{m}|_{\Lambda_N\times\Lambda_N\times\Lambda_N}(\,\cdot\,,0\,\cdot\,)} : S^{N+1}_1 \to S^{N+1}_1\|.$$

The combination of the latter estimate with (5.11) yields the result. ∎

Remark 5.4 In [GrTo02] it is shown that for a natural class of Calderón–Zygmund operators, the associated convolution operator is bounded as a map $L_1 \times L_1 \to L_{\frac{1}{2},\infty}$

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as well as $L_{p_1} \times L_{p_2} \to L_p$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{2} < p < \infty, 1 < p_1, p_2 < \infty$. This applies in particular to the map T_m with symbol m as in (5.6) (see [GrTo02, Proposition 6]). Our example shows that this result does not extend to the vector-valued setting in case $\frac{1}{2} < p \leq 1$. On the other hand, affirmative results in case $1 < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ were obtained in [DLMV20]. The question remains open whether a weak L_1-bound $L_{p_1} \times L_{p_2} \to L_{1,\infty}$, $\frac{1}{p_1} + \frac{1}{p_2} = 1$ holds, even in the case $p_1 = p_2 = 2$.

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