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DOI
10.1093/gji/ggad111

Publication date
2023

Document Version
Submitted manuscript

Published in
Geophysical Journal International

submitted to *Geophys. J. Int.*

Minimum-phase property and reconstruction of elastodynamic dereverberation matrix operators

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7 February 2023

SUMMARY

Minimum-phase properties are well-understood for scalar functions where they can be used as physical constraint for phase reconstruction. Existing scalar applications of the latter in geophysics include e.g. the reconstruction of transmission from acoustic reflection data, or multiple elimination via the augmented acoustic Marchenko method. We review scalar minimum-phase reconstruction via the conventional Kolmogorov relation, as well as a less-known factorization method. Motivated to solve practice-relevant problems beyond the scalar case, we investigate (1) the properties and (2) the reconstruction of minimum-phase matrix functions. We consider a simple but non-trivial case of 2×2 matrix response functions associated with elastodynamic wavefields. Compared to the scalar acoustic case, matrix functions possess additional freedoms. Nonetheless, the minimum-phase property is still defined via a scalar function, i.e. a matrix possesses a minimum-phase property if its determinant does. We review and modify a matrix factorization method such that it can accurately reconstruct a 2×2 minimum-phase matrix function related to the elastodynamic Marchenko method. However, the reconstruction is limited to cases with sufficiently small differences between P- and S-wave travel times, which we illustrate with a synthetic example. Moreover, we show that the minimum-phase reconstruction method by factorization shares similarities with the Marchenko method in terms of the algorithm and its limitations. Our results reveal so-far unexplored matrix properties of geophysical responses that open the door towards novel data processing tools. Last but not least, it appears that minimum-phase matrix functions possess additional, still-hidden properties that remain to be exploited e.g. for phase reconstruction.

Key words: Fourier analysis – Numerical solutions – Time-series analysis – Inverse theory – Wave propagation – Wave scattering and diffraction

1 INTRODUCTION

Phase reconstruction can be found in various fields of science and engineering (Shechtman et al. 2015). It is the process of finding a function given its Fourier amplitude spectrum or some multidimensional generalization thereof. The result is not unique but can be better constrained given some a priori knowledge of the function. The focus of this work lies on a special class called minimum-phase reconstruction. It pertains to invertible functions where the function and its inverse are characterized by energy concentrated close to the temporal origin.

In geophysics, minimum-phase is often thought to be a property of the seismic wavelet in marine acquisition (Yilmaz 2001), aside from complications resulting from band-limitation (Lamoureux & Margrave 2007). However, minimum-phase is a more general property which can be a characteristic of response functions that relate wavefields measured at different spatial locations. For example, Sherwood & Trorey (1965) as well as Claerbout (1968) demonstrate that full-bandwidth 1D acoustic transmission responses and their inverses form pairs of minimum-phase signals when measured from the onset of the signal. The aforementioned work distinguishes transmission from reflection responses. This is often reasonable in exploration geophysics when considering a section of the subsurface embedded between top and bottom boundaries. For simplicity, we assume these boundaries are perfectly absorbing. Contrary to transmissions, reflection responses are generally not minimum-phase.

To date, the properties and the reconstruction of multi-dimensional minimum-phase signals remain poorly understood. Here, multi-dimensional signals refer to response functions that are associated with 1.5D elastodynamic or 2D/3D acoustic wavefields as opposed to scalar functions associated with 1.5D acoustic wavefields. This topic remains a relevant geophysics problem which has been studied by only few authors (Claerbout 1998; Fomel et al. 2003). As a result, multi-dimensional minimum-phase signal reconstruction remains a barrier for numerous applications such as retrieving transmission from reflection responses (Wapenaar et al. 2003), or internal multiple elimination using the augmented Marchenko method (e.g. Dukalski et al. 2019). The research of this paper has been motivated by the augmented Marchenko method and its generalization to elastodynamic waves (this method is not discussed here, but details can be found in Reinicke et al. 2020).

In this work, we study the minimum-phase properties and reconstruction of 2×2 matrix response functions. In Section 2, we review existing theory of minimum-phase properties and two reconstruction algorithms for the scalar case. Moreover, we discuss geophysical response functions and show an example of minimum-phase reconstruction for the acoustic dereverberation operator of the Marchenko method. In Section 3, we discuss why elastodynamic response functions are matrices instead of scalars, and analyze the minimum-phase property as well as its reconstruction for the matrix case. In Section 4, we present two numerical examples of minimum-phase matrix reconstruction based on the factorization algorithm by Wilson (1972) with a modification inspired by the Marchenko method. The two examples include a case with an accurate solution as well as another case with artifacts to highlight remaining limitations. Finally, we discuss our insights in Section 5 and highlight similarities between minimum-phase reconstruction and the Marchenko method.

2 MINIMUM-PHASE PROPERTY AND RECONSTRUCTION: SCALAR CASE

In this section, we review existing work to prepare the discussion of the main result of this paper. In particular, we, (2.1) review the scalar minimum-phase property and how it can be used for phase reconstruction via the Kolmogorov relation,

(2.2) show a factorization method for scalar phase-reconstruction under a minimum-phase condition,

(2.3) introduce our notation and geophysical responses.

In part (2.3), we focus on a minimum-phase function that is relevant for the Marchenko method. However, the analysis does not require in depth knowledge of the Marchenko method.

2.1 Minimum-phase in a nutshell

We start by discussing linear time-invariant (LTI) systems. Given an arbitrary input, one can obtain the output of an LTI system via temporal convolution with its impulse response. For example, seismic reflection data can be represented as a temporal convolution of the source signature with the impulse response of the subsurface. This representation assumes that the subsurface remains unchanged during the experiment. For convenience, convolutions in the time (τ) domain are often formulated as multiplications in the frequency (ω) domain, e.g.,

$$output(\omega) = g(\omega)\; input(\omega), \tag{1}$$

where $g(\omega)$ denotes an impulse response. In the following, we imply that all operations, such as products or divisions, are performed per frequency component unless explicitly mentioned. Moreover, we refer to impulse responses as responses or functions, while they may also be known as transfer functions.

The minimum-phase property is a mathematical characteristic associated with a special class of functions. Using a qualitative definition, a function possesses a minimum-phase property if the following conditions are satisfied (Bode et al. 1945; Sherwood & Trorey 1965; Berkhout 1973; Skingle et al. 1977).

(i) The sum of all absolute time components is finite (stability).

(ii) The function vanishes for negative times (causality).

(iii) The inverse exists and satisfies (i) and (ii).

An important consequence is that the product of minimum-phase functions produces a result with a minimum-phase property. The term “minimum-phase” suggests that some attribute is minimized, which is true for special cases, where the group delay is minimized. However, this definition is not used in our analysis.

We illustrate the minimum-phase property using an example. Consider the causal functions (i.e. $\tau_1 > 0$),

$$A(\omega) = 1 + \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1}, \tag{2}$$

$$B(\omega) = \alpha + \mathrm{e}^{-\mathrm{i}\omega\tau_1} = (A(\omega))^* \, \mathrm{e}^{-\mathrm{i}\omega\tau_1}, \tag{3}$$

where α is a constant smaller than one. The variable i and the superscript “*” denote the imaginary unit and complex-conjugation, respectively. Hence, the functions have identical amplitude spectra, $C(\omega) = |A(\omega)| = |B(\omega)|$. Moreover, we use several common operators, which are defined in the appendix (see Table A1). The analysis of causality depends on the definition of the Fourier transform (sign choice of the exponent) which we define according to Eqs. A.1 and A.2. The phase of the functions can be visualized as an angle in the complex plane spanned between a complex number and the real axis (see Figure 1a, where $\alpha = -0.6$ and $\tau = 0.04$ s), or as a function of frequency (see Figure 1b). It can be easily seen that the functions $A(\omega)$ and $B(\omega)$ satisfy conditions (i) and (ii) (see Figure 1c). Their inverses exist and can be found using the geometric series and Eq. 3,

$$(A(\omega))^{-1} = \sum_{k=0}^{\infty} (-\alpha)^k \mathrm{e}^{-\mathrm{i}\omega\tau_1 k}, \tag{4}$$

$$(B(\omega))^{-1} = \left((A(\omega))^{-1}\right)^* \mathrm{e}^{\mathrm{i}\omega\tau_1} = \sum_{k=0}^{\infty} (-\alpha)^k \mathrm{e}^{\mathrm{i}\omega\tau_1 (k+1)}. \tag{5}$$

Moreover, the inverses are stable due to convergence of the geometric series in Eqs. 4 and 5. However, only the inverse $(A(\omega))^{-1}$ is causal whereas the inverse $(B(\omega))^{-1}$ is acausal (see Figure 1c). Hence, the function $A(\omega)$ satisfies conditions (i)-(iii) and possesses a minimum-phase property, but the function $B(\omega)$ does not. The amplitude spectrum $C(\omega)$ has a smaller phase (zero-phase) than the function $A(\omega)$ but it violates the causality condition (ii), and hence is not minimum-phase (see Figure 1c). In the following, we omit the dependency on frequencies except for newly introduced functions.

Minimum-phase reconstruction is the retrieval of a minimum-phase function from its amplitude, or power, spectrum. In general, phase reconstruction carries a degree of freedom $e^{i\Phi(\omega)}$,

$$\left(Ae^{i\Phi(\omega)}\right)^* Ae^{i\Phi(\omega)} = A^*A = |A|^2. \tag{6}$$

However, it can be shown that the aforementioned freedom vanishes under the minimum-phase conditions (i)-(iii). Thus, minimum-phase functions possess a unique amplitude-phase relationship, which can be formulated e.g. via the Kolmogorov relation (e.g. Skingle et al. 1977),

$$\begin{aligned}\log(A) =& \log(|A|) + i\mathrm{Arg}[A] \\ =& \log(|A|) - i\mathcal{H}[\log(|A|)]\,.\end{aligned} \tag{7}$$

Here, we denote the phase by $\mathrm{Arg}[A]$, the natural logarithm by $\log(\cdot)$, and the Hilbert transform by $\mathcal{H}[\cdot]$.

2.2 Minimum-phase reconstruction by factorization

Wilson (1969) formulates minimum-phase reconstruction as a recursive factorization problem, which we call the Wilson algorithm. This method will be important when generalizing the minimum-phase property and reconstruction from scalars to matrices in Section 3.2. Since the Wilson method might be less-known than the Kolmogorov in Eq. 7, we summarize its scalar formulation in more detail.

Consider an arbitrary minimum-phase function $A(\omega)$. The starting point is a relation between the amplitude spectrum $|A|$, an estimate after n iterations A_n, and its update A_{n+1} (see Eq. 6 in Wilson 1969),

$$A_nA_{n+1}^* + A_{n+1}A_n^* = A_nA_n^* + AA^*. \tag{8}$$

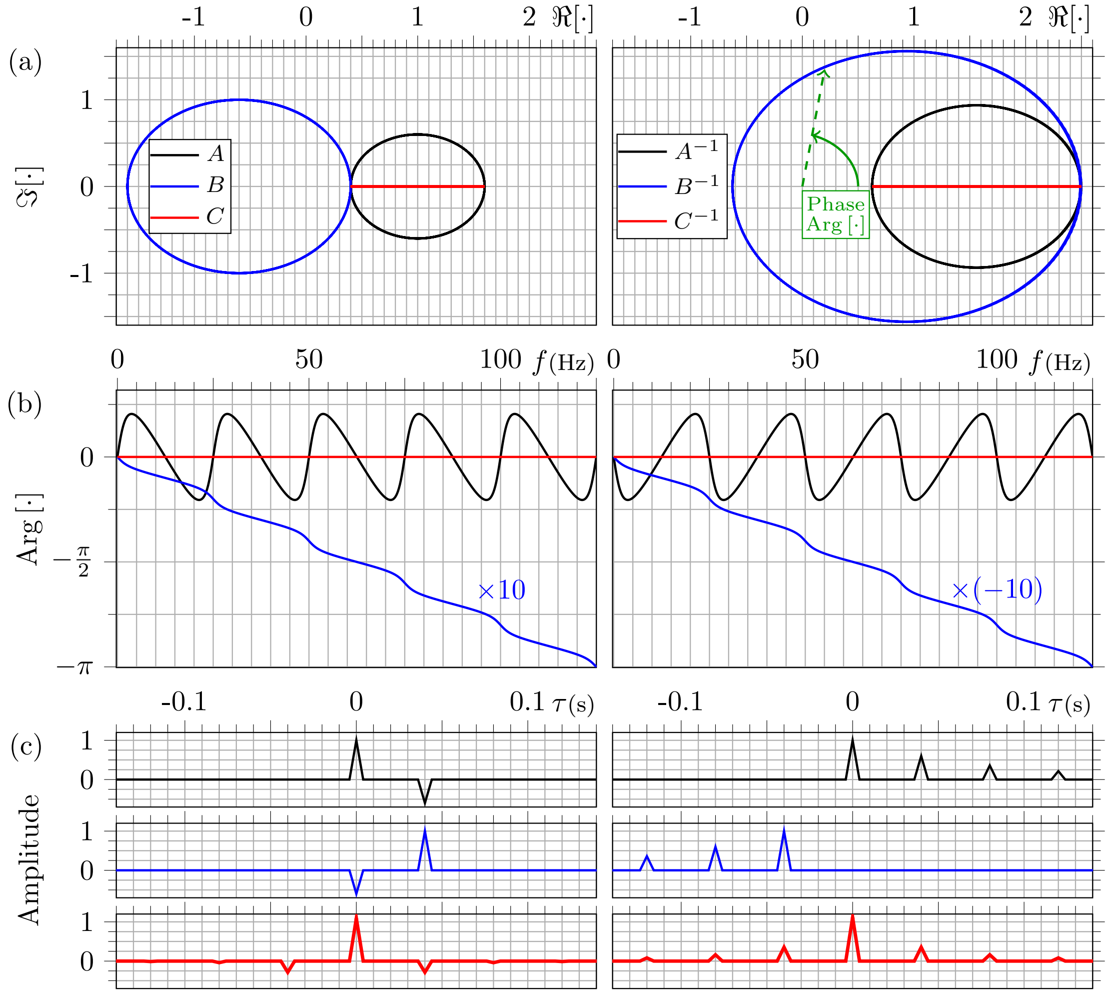

Figure 1. Illustration of the functions A, B, C (left column) and their inverses (right column) defined in Eqs. 2 - 5 using $\alpha = -0.6$ and $\tau_1 = 0.04$ s. The panels show (a) Argand diagrams, (b) phase spectra and (c) time domain representations. The axes of the Argand diagram correspond to the real ($\Re$) and imaginary ($\Im$) part of the functions in the frequency domain. The phase of a complex number is illustrated in the top right panel. Moreover, there is one legend per column and we denote $f = \frac{\omega}{2\pi}$. The minimum-phase function A and its inverse follow trajectories in the complex plane that have winding numbers around the origin equal to zero. However, the trajectory of the function B and its inverse wind five times around the origin of the complex plane (deduced from the phase spectra $\frac{\pi\times 10}{2\pi} = 5$, or $\frac{\omega_{\max}\tau_1}{2\pi} = 125$ Hz $\times$ 0.04 s $= 5$).

Multiplication by $(A_n)^{-1}$ and $(A_n^*)^{-1}$ leads to,

$$A_{n+1}^* \left(A_n^*\right)^{-1} + \left(A_n\right)^{-1} A_{n+1} = 1 + \left(A_n\right)^{-1} AA^* \left(A_n^*\right)^{-1}. \tag{9}$$

It follows from the minimum-phase-property of the desired solution A that Eq. 9 contains a superposition of a strictly causal term, $(A_n)^{-1} A_{n+1}$, with its time-reverse. The acausal term, $\left[(A_n)^{-1} A_{n+1}\right]^*$, can be removed by applying a temporal mute $\Theta[\cdot]$. Next, the result is rearranged to obtain a recursive algorithm,

$$A_{n+1} = A_n \Theta \left[1 + (A_n)^{-1} |A|^2 (A_n^*)^{-1}\right]. \tag{10}$$

Here, the mute represents multiplication by the Heaviside function $\mathrm{H}(\tau)$ in the time domain,

$$\mathrm{H}(\tau) = \begin{cases} 1, & \tau > 0, \\ \frac{1}{2}, & \tau = 0, \\ 0, & \tau < 0. \end{cases} \tag{11}$$

Since most operations in this work are formulated in the frequency domain, the mute operator $\Theta[\cdot]$ includes Fourier transforms between the frequency and time domains. In Section 3, the mute operator will be generalized from a Heaviside function to a more general step function. Wilson (1969) shows that the recursive algorithm in Eq. 10 converges to the desired solution A using the simplest minimum-phase function as initial estimate, $A_0 = 1$ (in the frequency domain). The scaling by $\frac{1}{2}$ at time zero (see Eq. 11) handles the overlap of the causal and acausal terms in Eq. 9. It can also be seen as a termination condition that ensures convergence, i.e. the solution is not updated for $A_n = A$,

$$\begin{aligned} A_{n+1} &= A_n \Theta \left[1 + (A_n)^{-1} |A|^2 (A_n^*)^{-1}\right] \\ &= A_n \Theta [1 + 1] = A_n. \end{aligned} \tag{12}$$

2.3 Geophysical scalar functions and minimum-phase

We briefly introduce our notation, define the dereverberation operator and show a numerical example of the Wilson algorithm.

In geophysics, transfer functions are often used to relate wavefields at different locations. For simplicity, we consider horizontally-layered media in the x-z space, where wavefields decouple per horizontal ray-parameter, $p_x = \frac{sin(\alpha)}{c}$ (see Eq. A.3 for definition of the domain transformation). Here, the angle α is formed by the wave front and the x-axis, and c denotes the local propagation velocity of a given wave type (P, or S which will be relevant in the elastic case).

The term response refers to a Green's function associated with a plane-wave dipole source and a monopole receiver. Hence, a response is a function that relates the wavefields at the source and receiver locations via a product per frequency. We consider an acoustic medium that is homogeneous except for a section between the depth levels z on top, and z' at the bottom. Moreover, the medium is source-free below the upper boundary at depth z. In this configuration, one can relate the wavefields on the boundaries z and z' using a scalar response $D(p_x, z', z, \omega)$ (as opposed to a matrix response) according to,

$$q(p_x, z', \omega) = D(p_x, z', z, \omega) q(p_x, z, \omega). \tag{13}$$

Here, the quantity $q(p_x, z, \omega)$ denotes an acoustic pressure wavefield. We assume all coordinates are fixed except for the frequency and use a detail-hiding notation that omits coordinates, e.g. $q_{below} = D q_{above}$ (similar to Berkhout 1982; Wapenaar 1989).

For all numerical examples in this paper, we consider the four layer model in Figure 2 and a single ray-parameter $p_x = 2 \times 10^{-4}$ s m^{-1}. We use three models that are identical except for the S-wave velocity c_S including an acoustic model ($c_S = 0$) and two elastic ones ($c_S \neq 0$).

Next, we introduce a specific transfer function namely the dereverberation operator which is the desired solution of the Marchenko equation. It can be used to remove internal multiples from seismic reflection data (e.g. van der Neut & Wapenaar 2016; Dukalski & de Vos 2022), however, multiple elimination is not relevant for our analysis. The dereverberation operator is defined via the transmission response $T^{\downarrow}$ that relates the wavefields above and below a scattering medium ($q_{below} = T^{\downarrow} q_{above}$). In the acoustic case, it can be written as,

$$V^{+} = T^{\downarrow -1} T^{\downarrow}_{dir} = 1 + V^{+}_{coda}. \tag{14}$$

Here, the transmission $T^{\downarrow}$ is split in its direct and coda parts indicated by the subscripts

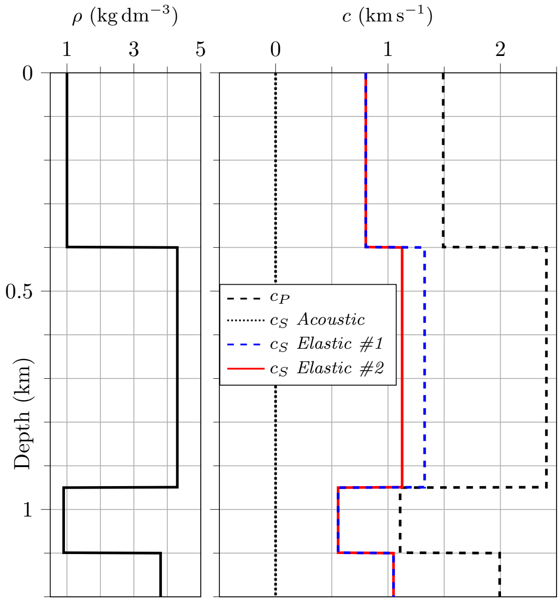

Figure 2. Parameters of the three models used in this work. The density ρ and the P-wave velocity c_P are identical for all models. An *acoustic case* is defined by setting the S-wave velocity to zero $c_S = 0$. The *Elastic #1* case is defined with a non-zero S-wave velocity $c_S \neq 0$. The *Elastic #2* case is defined by reducing the S-wave velocity in one of the layers. The one-way travel times within each layer are integer-multiples of the time sampling interval ($\Delta\tau = 4$ ms) for all models and for P-/S-waves associated with $p_x = 2 \times 10^{-4}$ s m^{-1}. This choice simplifies the interpretation of the medium responses in the time domain because all events perfectly coincide with a time sample, i.e. it avoids smearing of individual events across several time samples. In this setting, we can accurately apply temporal mutes which allows us to verify the accuracy of the discussed algorithms up to numerical noise (in the order of 1×10^{-15} for double-precision).

"*dir*" and "*coda*", respectively,

$$T^{\downarrow} = T^{\downarrow}_{dir} + T^{\downarrow}_{coda}, \tag{15}$$

and the inverse transmission $T^{\downarrow -1}$ is often referred to as a focusing function f^{+} (Wapenaar et al. 2014). Transmissions and their inverses are minimum-phase functions, except for a positive and negative time shift, respectively (Claerbout 1968). These time shifts mutually cancel when evaluating the product in Eq. 14. Hence, the dereverberation operator possesses a minimum-phase property. For example, the function A in Eq. 2 is a dereverberation operator of an acoustic medium with two reflectors that are separated by the travel time $\frac{1}{2}\tau_1$, and the factor α represents the product of the reflection coefficients of the two interfaces.

We illustrate the scalar Wilson algorithm with an example considering the acoustic model shown in Figure 2. The power spectrum of the dereverberation operator $|V^+|^2$ (see Figure 3a) is modeled analytically (Dukalski et al. 2022) and used to evaluate Eq. 10 with $A = V^+$. Figures 3b-f show the solution V_n^+ and its error, $V_n^+ - V^+$, as a function of iterations (n). The convergence in Figure 4 reveals that the Wilson algorithm finds the true solution up to numerical accuracy within seven iterations.

3 MINIMUM-PHASE PROPERTY AND RECONSTRUCTION: MATRIX CASE

In this section, we,

(3.1) introduce matrix functions and their link to elastodynamic wavefields,

(3.2) analyze the minimum-phase property of matrices,

(3.3) review normal products and explore how minimum-phase matrices can be reconstructed from their normal products by factorization. For the reconstruction step, we focus on the special case of the elastodynamic dereverberation operator.

3.1 Geophysical matrix functions

We briefly introduce matrix functions. The literature distinguishes between transfer functions with (1) a single input and a single output (SISO) corresponding to the scalar case discussed above, as well as (2) multi inputs and multi outputs (MIMO) (Johansson 1997). The latter can be represented by frequency-dependent matrices, where the number of rows and columns corresponds to the number of output and input variables, respectively. Hence, they are referred to as matrix functions. Compared to the scalar case, mathematical operations are generalized which can lead to previously unexplored challenges, e.g. scalar products and divisions become matrix multiplications and matrix inverses, respectively.

Elastodynamic responses can be represented by 2×2 matrix functions. Here, we consider the configuration discussed in Section 2.3 but generalize acoustic to elastic media. One can formulate the elastic extension of the wavefield-response relation in Eq. 13 as follows,

$$\mathbf{q}(p_x, z', \omega) = \mathbf{D}(p_x, z', z, \omega)\mathbf{q}(p_x, z, \omega), \tag{16}$$

with,

$$\mathbf{D} = \begin{pmatrix} D_{P,P} & D_{P,S} \\ D_{S,P} & D_{S,S} \end{pmatrix}, \text{ and, } \mathbf{q} = \begin{pmatrix} q_P \\ q_S \end{pmatrix}. \tag{17}$$

The subscripts denote P-/S-waves and we use bold font to distinguish vectors and matrices from scalars. In this context, the matrix function $\mathbf{D}$ is an elastodynamic response defined in the P-S space. The first and second subscripts of its matrix elements denote the wave type at the receiver- and source-side, respectively. For example, the element $D_{P,S}$ relates S-waves at the source location to P-waves at the receiver location. Next, we generalize the temporal mutes to matrices such that they operate, and can differ per matrix element in the P-S space,

$$\boldsymbol{\Theta}\left[\mathbf{D}\right] = \begin{pmatrix} \Theta_{P,P}\left[D_{P,P}\right] & \Theta_{P,S}\left[D_{P,S}\right] \\ \Theta_{S,P}\left[D_{S,P}\right] & \Theta_{S,S}\left[D_{S,S}\right] \end{pmatrix}. \tag{18}$$

Next, we will investigate how to define and reconstruct the minimum-phase property for matrices, e.g. per matrix element or per matrix. Moreover, we will analyze the mathematical behavior of minimum-phase matrices, e.g. whether their property is preserved by matrix products or changes of basis. Despite focusing on 2×2 matrices, we do not exclude generalizations to larger ones.

3.2 Minimum-phase matrix property

The concept of minimum-phase is significantly more difficult beyond scalar functions where several assumptions break. In the following, we discuss the minimum-phase property of matrices by reviewing findings from other areas (e.g. control theory).

Diagonal matrices are a trivial extension from scalars to matrices. Consider the scalar minimum-phase functions, $A_{\pm} = 1 \pm \alpha e^{-i\omega\tau_1}$, with $|\alpha| < 1$ and $\tau_1 > 0$. By arranging them in a diagonal matrix denoted by $\mathrm{diag}(\cdot)$ we obtain the minimum-phase matrix, $\boldsymbol{\Lambda} =$

diag(A_-, A_+). In contrast to this intuitive example, we will show less obvious cases of minimum-phase matrices further onwards.

Existing literature defines matrices as minimum-phase if their determinants are minimum-phase (Wiener 1955; Rosenbrock 1969; Horowitz et al. 1986). Hence, the determinant of a minimum-phase matrix satisfies the Kolmogorov relation (analogously to Eq. 7). This definition is consistent with the special case of scalar functions which are 1×1 matrices. It is also consistent with the simple matrix example above, $\mathbf{\Lambda}$, where the determinant is equal to the product of the minimum-phase diagonal elements, $\det(\mathbf{\Lambda}) = A_- A_+$, producing by definition a minimum-phase result.

In a general case, defining minimum-phase matrices via their determinant has several consequences:

(1) Matrix multiplications and matrix inverses preserve the minimum-phase property. This can be seen by considering the determinants of arbitrary minimum-phase matrix functions $\mathbf{A}$ and $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}), \tag{19}$$

$$\det\left(\mathbf{A}^{-1}\right) = (\det(\mathbf{A}))^{-1}. \tag{20}$$

The determinants, $\det(\mathbf{A})$ and $\det(\mathbf{B})$, are minimum-phase scalar functions. Hence, the right-hand sides of Eqs. 19 and 20 show that the matrix product $\mathbf{AB}$ and the inverse matrix $\mathbf{A}^{-1}$ possess a minimum-phase property.

(2) The minimum-phase property is basis-independent,

$$\det(\mathbf{D}) = \det\left(\mathbf{QDQ}^{-1}\right), \tag{21}$$

where $\mathbf{Q}$ is an arbitrary invertible matrix of the same size as $\mathbf{D}$. Hence, minimum-phase is a physical property that is independent of the coordinate system or domain.

(3) Minimum-phase matrices are not fully consistent with the qualitative conditions (i)-(iii) in Section 2.1. The invertibility criterion (iii) is satisfied because minimum-phase determinants are non-zero. However, it is less clear how to interpret causality and sta-

bility for a matrix (criteria (i) and (ii)). In particular, minimum-phase determinants do not guarantee causality of individual matrix elements. For example, suppose the matrix,

$$\mathbf{Q} = \begin{pmatrix} 1 - 2\alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1} & 1 \\ 1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1} & 1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1} \end{pmatrix}, \tag{22}$$

is used to apply a frequency-dependent basis transformation to the minimum-phase matrix, $\mathbf{\Lambda} = \mathrm{diag}(A_-, A_+)$. The resulting matrix,

$$\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} = \begin{pmatrix} 2 - \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1} & -\frac{1-2\alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1}}{1+\alpha \mathrm{e}^{\mathrm{i}\omega\tau_1}} \\ 1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1} & \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1} \end{pmatrix}, \tag{23}$$

is still minimum-phase but its matrix elements are not such as the acausal element $1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1}$.

(4) Minimum-phase matrices do not necessarily posses minimum-phase eigenvalues. A minimum-phase determinant constrains the phase spectra of the eigenvalues up to a frequency-dependent freedom, $\eta = \eta(\omega)$,

$$\mathrm{Arg}\left[\lambda_1\right] = -\mathcal{H}\left[\log\left(|\lambda_1|\right)\right] + \eta, \tag{24}$$

$$\mathrm{Arg}\left[\lambda_2\right] = -\mathcal{H}\left[\log\left(|\lambda_2|\right)\right] - \eta. \tag{25}$$

There are special cases where all eigenvalues observe a minimum-phase property (i.e. $\eta = 0$), e.g. the aforementioned matrix $\mathbf{\Lambda}$, or transmission-like responses of 2D laterally-invariant acoustic media (see examples by Wapenaar et al. 2003; Elison et al. 2020). This work focuses on more general minimum-phase matrices, where scalar solutions per eigenvalue no longer suffice.

3.3 Minimum-phase reconstruction by normal-product factorization: Matrix case

In this section, we extend minimum-phase reconstruction from scalars to matrices. Firstly, we define normal products as generalized power spectra, and we demonstrate why unique minimum-phase matrix reconstruction is significantly more challenging than its scalar version. Secondly, we modify the minimum-phase matrix reconstruction method by Wilson (1972) considering the special case of the elastodynamic dereverberation operator $\mathbf{V}^+$.

Thirdly, we discuss similarities of this reconstruction method to the Marchenko method. We will illustrate our analysis numerically in Section 4.

3.3.1 Normal products: Generalized power spectra

The normal product is defined as the product of a quantity, with its complex-conjugate transpose, e.g. $|D|^2$ for scalars, or $\mathbf{DD}^\dagger$ for matrices (e.g. Dukalski 2020). Scalar normal products may be better known as auto-correlations in the time domain and are often interpreted physically as power spectra in the frequency domain because their phase vanishes $\mathrm{Arg}\left[|D|^2\right] = 0$. Following this physical interpretation, retrieving the scalar solution D from its normal product $|D|^2$ is often described as a phase reconstruction, while mathematically, it is a factorization problem. In Section 2, we showed that this generally non-unique factorization can be constrained for minimum-phase scalar functions (see Eqs. 6 and 7). However, the matrix case is more complicated.

There are several differences between scalar power spectra and matrix normal products. For example, consider,

$$\mathbf{DD}^\dagger = \begin{pmatrix} D_{P,P} & D_{P,S} \\ D_{S,P} & D_{S,S} \end{pmatrix} \begin{pmatrix} D^*_{P,P} & D^*_{S,P} \\ D^*_{P,S} & D^*_{S,S} \end{pmatrix} = \begin{pmatrix} \delta & \epsilon^* \\ \epsilon & \zeta \end{pmatrix}, \tag{26}$$

with $\delta = |D_{P,P}|^2 + |D_{P,S}|^2$, $\epsilon = D^*_{P,P} D_{S,P} + D^*_{P,S} D_{S,S}$, and $\zeta = |D_{S,P}|^2 + |D_{S,S}|^2$. The off-diagonal elements of the normal product are identical except for a sign-inverted phase that is not necessarily zero $\mathrm{Arg}\left[\epsilon\right] = -\mathrm{Arg}\left[\epsilon^*\right]$. Nonetheless, we keep the physical interpretation from the scalar case, i.e. "power spectra" and "phase reconstruction" refer to normal products and the retrieval of the solution $\mathbf{D}$ from its normal product, respectively. Since matrix multiplications do not commute, there are two normal products, which are generally not equal $\mathbf{DD}^\dagger \neq \mathbf{D}^\dagger\mathbf{D}$. Counting matrix elements as equations, the two normal products provide individually up to three (see Eq. 26), and together up to six independent equations (for 2×2 matrices). Hence, if both normal products are known, there are more equations to constrain the reconstruction of the matrix $\mathbf{D}$. However, we assume only one normal product is available which describes a challenge of the elastodynamic augmented Marchenko method (details are not needed here but can be found in Reinicke et al. 2020).

Compared to the scalar case, the factorization of a (single) normal product has additional degrees of freedom. The normal product of the matrix $\mathbf{D}$ is preserved upon multiplication by an arbitrary unitary 2×2 matrix $\mathbf{U}_2$,

$$\mathbf{D}\mathbf{U}_2 \left(\mathbf{D}\mathbf{U}_2\right)^{\dagger} = \mathbf{D}\mathbf{D}^{\dagger}, \tag{27}$$

due to the unitary property $\mathbf{U}_2 \left[\mathbf{U}_2\right]^{\dagger} = \mathbf{I}$ (here $\mathbf{I}$ denotes an identity matrix). The $\mathbf{U}_2$ element can be represented as follows (the term "element" is commonly used in the relevant literature, e.g. Cornwell 1997),

$$\mathbf{U}_2 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{\gamma+\alpha}{2}} \cos\left[\frac{\beta}{2}\right] & -\mathrm{e}^{\mathrm{i}\frac{\gamma-\alpha}{2}} \sin\left[\frac{\beta}{2}\right] \\ \mathrm{e}^{-\mathrm{i}\frac{\gamma-\alpha}{2}} \sin\left[\frac{\beta}{2}\right] & \mathrm{e}^{\mathrm{i}\frac{\gamma+\alpha}{2}} \cos\left[\frac{\beta}{2}\right] \end{pmatrix} \mathrm{e}^{\mathrm{i}\Phi}, \tag{28}$$

where α, β and γ are Euler angles (Hamada 2015). The freedom $\mathrm{e}^{\mathrm{i}\Phi}$ can be constrained via the minimum-phase property of the determinant $\det(\mathbf{D})$ (shown in chapter 5 of Reinicke 2020),

$$\Phi = -\frac{1}{4}\mathcal{H}\left[\log\left(\left|\det(\mathbf{D}\mathbf{D}^{\dagger})\right|\right)\right]. \tag{29}$$

Unfortunately, the minimum-phase determinant only constrains Φ, i.e. one out of four free parameters. Due to this limitation, we seek for an alternative method, which is discussed next.

3.3.2 Minimum-phase matrix reconstruction by factorization

In the following, we review a minimum-phase matrix reconstruction method, introduce the elastodynamic dereverberation operator and eventually modify the reconstruction method for the dereverberation operator.

The scalar Wilson algorithm can be generalized to matrices. Wilson (1972) proposes a matrix extension of the recursive scalar algorithm which can be written as,

$$\mathbf{D}_{n+1} = \mathbf{D}_n \boldsymbol{\Theta}\left[\mathbf{I} + \left(\mathbf{D}_n\right)^{-1} \mathbf{D}\mathbf{D}^{\dagger}\left(\mathbf{D}_n^{\dagger}\right)^{-1}\right], \tag{30}$$

with $\mathbf{D}_0 = \mathbf{I}$. The function $\mathbf{\Theta}$ element-wise mutes acausal events and scales the time zero components of the diagonal elements by $\frac{1}{2}$. Although the dereverberation operator $\mathbf{V}^+$ has a minimum-phase determinant (shown in the next section), it is not reconstructed correctly by the algorithm in Eq. 30 with $\mathbf{D} = \mathbf{V}^+$. We will show that this limitation is due to the mute $\mathbf{\Theta}\left[\cdot\right]$ and can be overcome using a modified mute.

For better illustration, we briefly define the elastodynamic dereverberation operator. One can generalize the acoustic definition in Eqs. 14 and 15 to the elastic case by replacing scalar with matrix responses in the P-S space (Reinicke et al. 2020),

$$\mathbf{V}^+ = \mathbf{T}^{\downarrow -1}\mathbf{T}^{\downarrow}_{dir} = \mathbf{I} + \mathbf{V}^+_{coda}. \tag{31}$$

The acoustic direct transmission $T^{\downarrow}_{dir}$ generalizes to a forward-scattered transmission $\mathbf{T}^{\downarrow}_{dir}$ that includes all non-reflected events such as transmitted mode-converted waves (Wapenaar 2014). Assuming that many readers are unfamiliar with the dereverberation operator, we explain its properties that are important for our analysis. Firstly, the dereverberation operator has a finite number of events limited by the number of layers. This follows from the finite number of events of the inverse and forward-scattered transmissions (Dukalski et al. 2022). Secondly, all events of the dereverberation operator arrive within a well-defined time window that only depends on the one-way travel times of P- and S-waves within each layer (Reinicke et al. 2020). Lastly, and most importantly, the onset of its matrix elements in the time domain is not always at time zero. In particular, its off-diagonal elements typically have non-zero onset times that can be acausal (shown by Reinicke et al. 2020).

Given these properties, we modify the mute of the matrix Wilson algorithm to reconstruct the dereverberation operator from its normal product. We propose modifying the operator $\mathbf{\Theta}\left[\cdot\right]$ to mute all events in the time domain prior to the onset of the dereverberation operator per matrix component. This differs from the original matrix Wilson algorithm which instead removes acausal events for all matrix elements. Using the modified mute $\mathbf{\Theta}\left[\cdot\right]$ in Eq. 30, it appears that the matrix Wilson algorithm can accurately factorize the normal product of the dereverberation operator (results will be shown in Section 4).

4 NUMERICAL EXAMPLE

In this section, we show two examples of the matrix Wilson method and analyze determinants and eigenvalues numerically. These examples are associated with the models *Elastic #1* and *Elastic #2*, which are identical except for the S-wave velocity in the second layer from the top (see Figure 2). They are designed such that the Wilson method succeeds (*Elastic #1*) and fails (*Elastic #2*) to reconstruct the respective dereverberation operator correctly. In both cases, we model the dereverberation operator analytically (Dukalski et al. 2022) to calculate the normal product, and to provide a reference for the retrieved solution. For the matrix Wilson method, we define the diagonal elements of the mute ($\Theta_{PP}[\cdot]$ and $\Theta_{SS}[\cdot]$) via the Heaviside function in Eq. 11. The off-diagonal elements $\Theta_{PS}[\cdot]$ and $\Theta_{SP}[\cdot]$ mute all events in the time domain prior to the onset of the components V^+_{PS} and V^+_{SP}, respectively.

Firstly, we consider the successful case *Elastic #1*. We use the normal product $\mathbf{V}^+\mathbf{V}^{+\dagger}$ shown in Figures 5a-d to evaluate eight iterations of the matrix Wilson algorithm, resulting in the solution $\mathbf{V}^+_{n=8}$ in Figures 5e-h. The algorithm monotonically converges to the true solution $\mathbf{V}^+$ up to numerical noise (see Figure 4), hence, we do not show the difference plot. Figures 5e-h illustrate that the dereverberation operator has a finite number of events in the time domain that arrive within a well-defined time window as discussed in Section 3.3.2. Here, the responses are zero outside the displayed time window, i.e. all events are shown. Figures 5e-h also show the identity term of the dereverberation operator (see Eq. 31). Moreover, the onset of the off-diagonal elements in the time domain deviates from time zero and is even acausal for the SP element (see Figure 5g).

Secondly, we modify the model until the proposed method for normal-product factorization becomes inaccurate (case *Elastic #2*). Compared to the previous example, the travel-time difference between P- and S-waves increased, leading to acausal events in the diagonal elements V^+_{PP} and V^+_{SS}. As a result, it is no longer clear how to define the diagonal elements of the mute $\Theta_{PP}[\cdot]$ and $\Theta_{SS}[\cdot]$, which also need to scale the time zero element by $\frac{1}{2}$ to ensure convergence (see Eq. 12). Here, we only adjust the off-diagonal elements of the mute, $\Theta_{PS}[\cdot]$ and $\Theta_{SP}[\cdot]$, to account for the changed onset of the dereverberation operator in the time domain. Then we repeat the previous experiment using the normal product of the dereverberation operator shown in Figures 6a-d. Figures 6e-h show the retrieved dereverberation operator after evaluating eight iterations of Eq. 30 $\mathbf{V}^+_{n=8}$, and the difference with respect to the modeled reference $\mathbf{V}^+$. The convergence (see Figure 4) indicates that the relative error of the retrieved solution is in the order of 10 %.

Lastly, we analyze the determinants and eigenvalues of the dereverberation operators. We verify that the determinants of the modeled dereverberation operators $\det(\mathbf{V}^+)$ satisfy the Kolmogorov relation up to numerical noise (relative error in the order of 1×10^{-14}) for both cases, *Elastic #1* and *Elastic #2*. Next, we inspect the determinants of the retrieved dereverberation operators after eight iterations $\det\left(\mathbf{V}^+_{n=8}\right)$ (see Figure 7). We observe that it satisfies the minimum-phase conditions in the case *Elastic #1* but it violates them in the case *Elastic #2*. This violation can be easily verified by the acausal events of the determinant (see close-up in Figure 7c). The phase error of the determinant can be corrected using Eq. 7. However, the retrieved response $\mathbf{V}^+_{n=8}$ carries an additional error represented by the Euler angles (see Eq. 28) that cannot be removed. The eigenvalues of the dereverberation operators do not satisfy the Kolmogorov relation for any of the tested cases. Even in the successful case (*Elastic #1*), the phase spectra of the eigenvalues differ severely from their minimum-phase spectra defined via the Kolmogorov relation in Eq. 7. This can be illustrated via the phase-freedom η defined in Eqs. 24 and 25, which is far from trivial (see Figure 8).

5 DISCUSSION

Our analysis has shown that the causality condition of minimum-phase functions can be less intuitive for matrix functions. The minimum-phase property does not necessarily hold for individual matrix elements but it does for the determinant. Hence, minimum-phase matrix functions can contain acausal matrix elements. Our numerical examples indicate that the matrix Wilson algorithm can accurately handle acausal off-diagonal elements, while acausal diagonal elements appear to be an obstacle. This limitation is not obvious from the algorithm in Eq. 30. In the presented examples, the temporal mute suppresses acausal events on the diagonal, but not on the off-diagonal, elements. Hence, the subsequent matrix multiplication by $\mathbf{D}_n$ could still introduce acausal events on the diagonals (see Eq. 30). It remains undetermined whether normal-product factorization of minimum-phase matrices is limited to cases with strictly causal diagonal elements, or, whether a more general algorithm remains to be discovered.

Our interest in minimum-phase matrices is motivated by the Marchenko method. The latter formulates internal multiple elimination for seismic reflection data as an inverse problem. It aims to retrieve the dereverberation operator and it is often underconstrained in practice. Existing work demonstrates for the scalar case how two additional constraints can be used to accurately reconstruct the dereverberation operator. Firstly, the normal product of the dereverberation operator is retrieved via energy conservation. Secondly, the dereverberation operator is reconstructed from its normal product by exploiting its minimum-phase property (Dukalski et al. 2019; Elison et al. 2020; Peng et al. 2022). In previous work, we tried to generalize this strategy to the elastic case where the dereverberation operator is no longer a scalar but a 2×2 matrix, and identified two challenges (Reinicke et al. 2020):

(1) Once the normal product of the elastodynamic dereverberation operator is retrieved, it remains unclear how to reconstruct the operator uniquely from its normal product using its minimum-phase property.

(2) Energy conservation provides the normal product of the inverse transmission. The dereverberation operator $\mathbf{V}^+$ is minimum-phase but the inverse transmission $\mathbf{T}^{\downarrow -1}$ (also known as $\mathbf{F}^+$) is not. This is not an issue for scalars, because the scalar normal-products of the inverse transmission and the dereverberation operator are identical up to a frequency-independent constant. This holds because the acoustic direct transmission is a single pulse, $T^{\downarrow}_{dir} = \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_{dir}}$, with travel time τ_{dir},

$$V^+ V^{+*} = T^{\downarrow -1} T^{\downarrow}_{dir} T^{\downarrow -1 *} T^{\downarrow *}_{dir} = T^{\downarrow -1} T^{\downarrow -1 *} |\alpha|^2. \tag{32}$$

However, this relation is more complicated for the elastic case where the direct transmission generalizes to a forward-scattered transmission including mode conversions $\mathbf{T}^{\downarrow}_{dir}$.

Moreover, Eq. 32 cannot be extended from the scalar to the matrix case because matrix multiplications do not commute.

In this paper, we focused on the first challenge. Addressing the second one is beyond the scope of this work.

We notice similarities between the Marchenko method and the here-discussed matrix Wilson method. Both methods use the same ingredients including temporal convolutions and correlations as well as temporal mutes. The modified mute of the matrix Wilson method $\boldsymbol{\Theta}\,[\cdot]$ is inspired by, and is nearly identical to, one of the two mutes of the Marchenko method $\mathbf{P}_B[\cdot]$ (see Eq. 16 in Reinicke et al. 2020). The two mutes only differ at time zero of the diagonal elements, where the Wilson mute scales its argument by $\frac{1}{2}$ instead of 1 to ensure convergence. Moreover, both methods face limitations related to the mutes. It has been shown that the Marchenko method fails to reconstruct the desired solution in the presence of fast-multiples. The latter are multiples that have shorter travel times than some of the converted but non-reflected arrivals. As a result, fast-multiples introduce temporal overlaps between signals that the Marchenko method ought to separate with the mute. These temporal overlaps are due to acausal events in the diagonal elements of the dereverberation operator V^+_{PP} and V^+_{SS}. This limitation of the Marchenko method coincides with the cases where the matrix Wilson method fails to retrieve the correct solution. The question is whether fast multiples pose a fundamental limitation, or whether there is another, more robust solution strategy for the Marchenko and matrix Wilson methods. Despite the remaining challenges, the matrix Wilson algorithm could potentially help to retrieve a better estimate of the desired dereverberation operator. For example, Peng et al. (2022) show that the 2D acoustic augmented Marchenko method can reconstruct the correct dereverberation operator, even though they apply a scalar, instead of a matrix minimum-phase reconstruction. They propose a recursive application of the 2D Marchenko method and a scalar minimum-phase correction. Similarly, one could attempt to recursively apply the elastodynamic Marchenko method and the matrix Wilson algorithm ignoring the challenge of fast multiples.

Minimum-phase matrices and normal-product factorization provide physical relationships that remain mostly unexplored, especially in geophysics. For example, the results of this work could bring new momentum to the research on reconstructing transmission from reflection data in the multi-dimensional acoustic or elastic case (i.e. beyond the work of Wapenaar et al. 2003). Moreover, we illustrated that normal-product factorization has four (real-valued) unknown parameters (for 2×2 matrices) but the determinant provides a single phase. Despite the mismatch in number of unknowns and equations, we demonstrated that the modified matrix Wilson algorithm can reconstruct a special class of minimum-phase matrices. This raises the question whether there are additional, so-far unexplored fundamental properties of minimum-phase matrices. If so, the follow up question is whether these properties allow for a unique factorization of normal products in more general cases, e.g. including fast-multiples. Answering these questions is beyond the scope of this paper but it is a matter of ongoing research. Last but not least, we investigated the simplest non-trivial matrix case, i.e. 2×2 matrices, but generalizations are not excluded. It would be particularly interesting to analyze multi-dimensional acoustic cases which will be subject of future work.

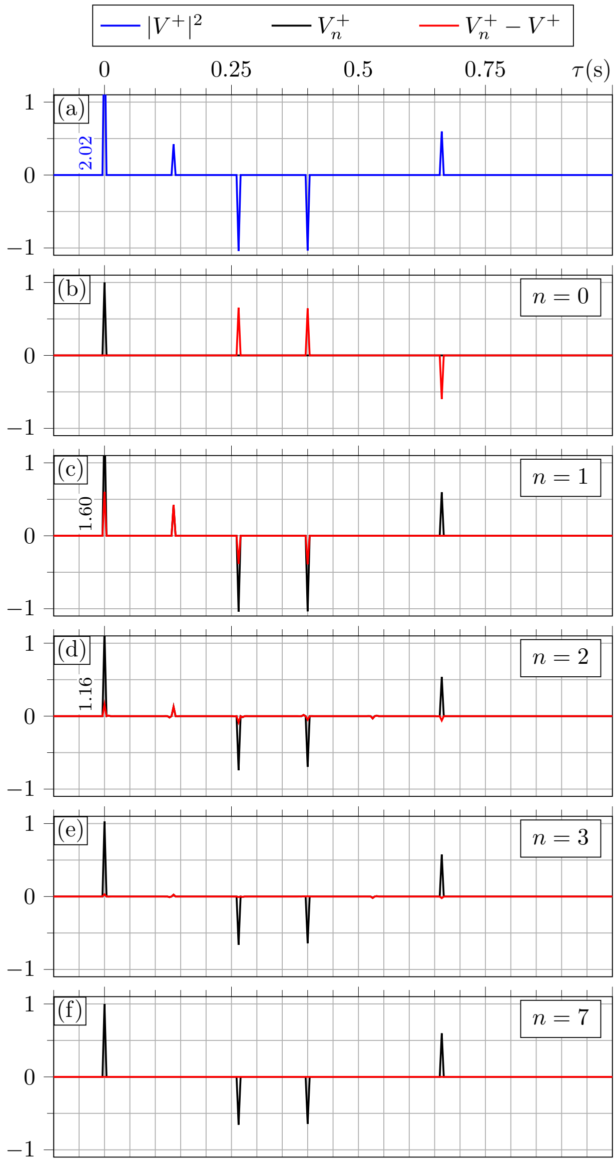

Figure 3. All responses are shown in the time domain. (a) Autocorrelation of the dereverberation operator associated with the acoustic model shown in Figure 2. Negative times are not shown because (scalar) autocorrelations are symmetric in time, $(|V^+|^2)^* = |V^+|^2$. Panels (b)-(f) show the dereverberation operator as it is recursively reconstructed via the Wilson algorithm in Eq. 10 (V_n^+ in black) and its error ($V_n^+ - V^+$ in red). The initial estimate ($n = 0$) is an identity, i.e. a single spike at time zero. After seven iterations the true solution is retrieved up to numerical noise (see Figure 4). For better illustration, strong events are clipped and their amplitudes are indicated with labels.

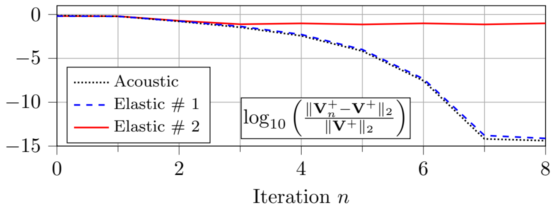

Figure 4. Convergence of the scalar and matrix Wilson algorithms in Eqs. 10 and 30 associated with the dereverberation operators (V^+ and $\mathbf{V}^+$) of the acoustic and elastic models in Figure 2, respectively. The convergence is defined as the relative error with respect to the true solution as indicated by the legend. For the *acoustic* and the *Elastic #1* case, the Wilson algorithm converges up to numerical noise within seven iterations. For the *Elastic #2* case, the relative error converges to approximately 10 %.

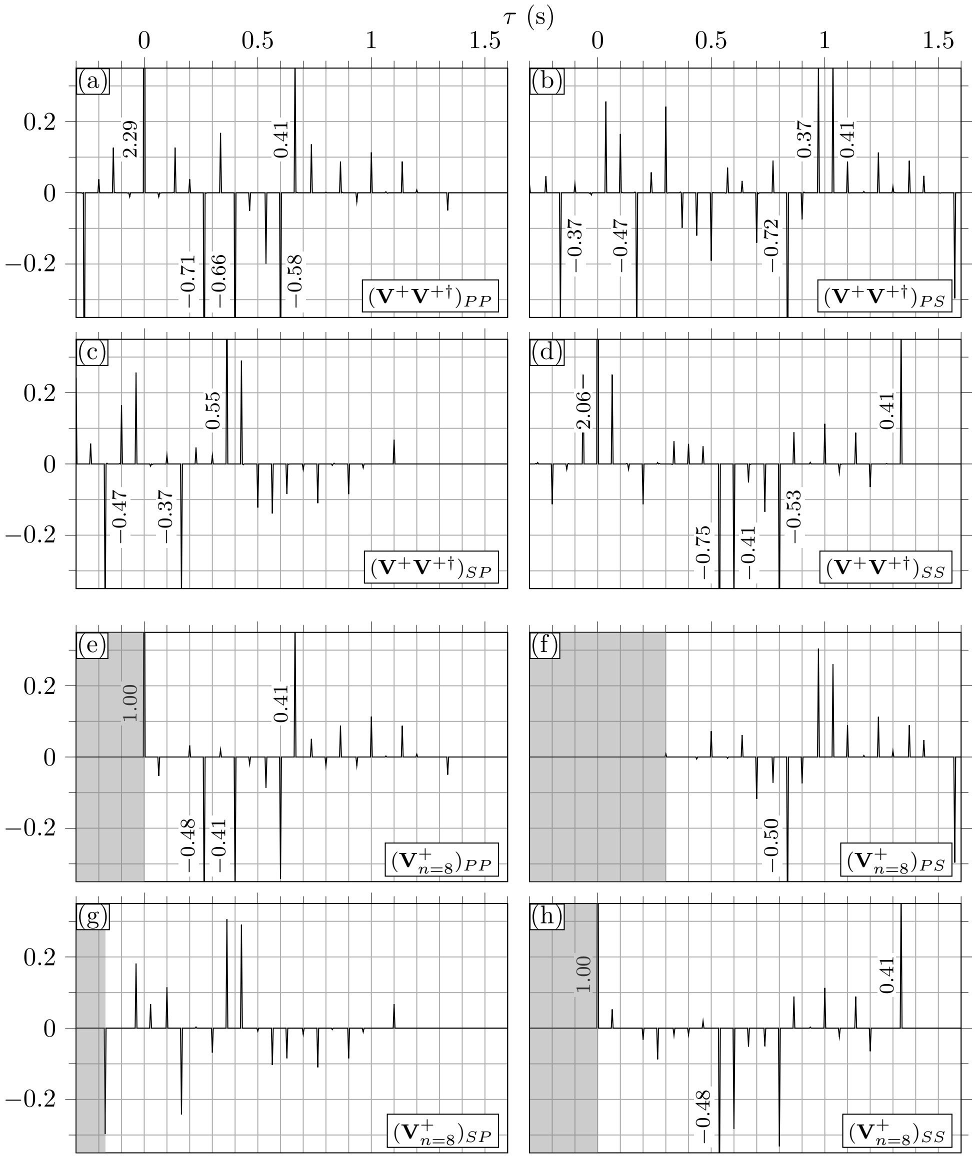

Figure 5. (a)-(d) Normal product $\mathbf{V}^{+}\mathbf{V}^{+\dagger}$ of the dereverberation operator associated with the model *Elastic #1* (see Figure 2). The panels show the four elastic components analogously to the 2×2 matrix in Eq. 17. (e)-(h) Retrieved dereverberation operator after eight iterations. The grey areas indicate the time samples that are muted by the modified operator $\boldsymbol{\Theta}\left[\cdot\right]$ in Eq. 30. We do not show a difference or reference plot because the retrieved and modeled dereverberation operators are identical up to numerical noise (see convergence in Figure 4). All panels show responses in the time domain to facilitate the interpretation.

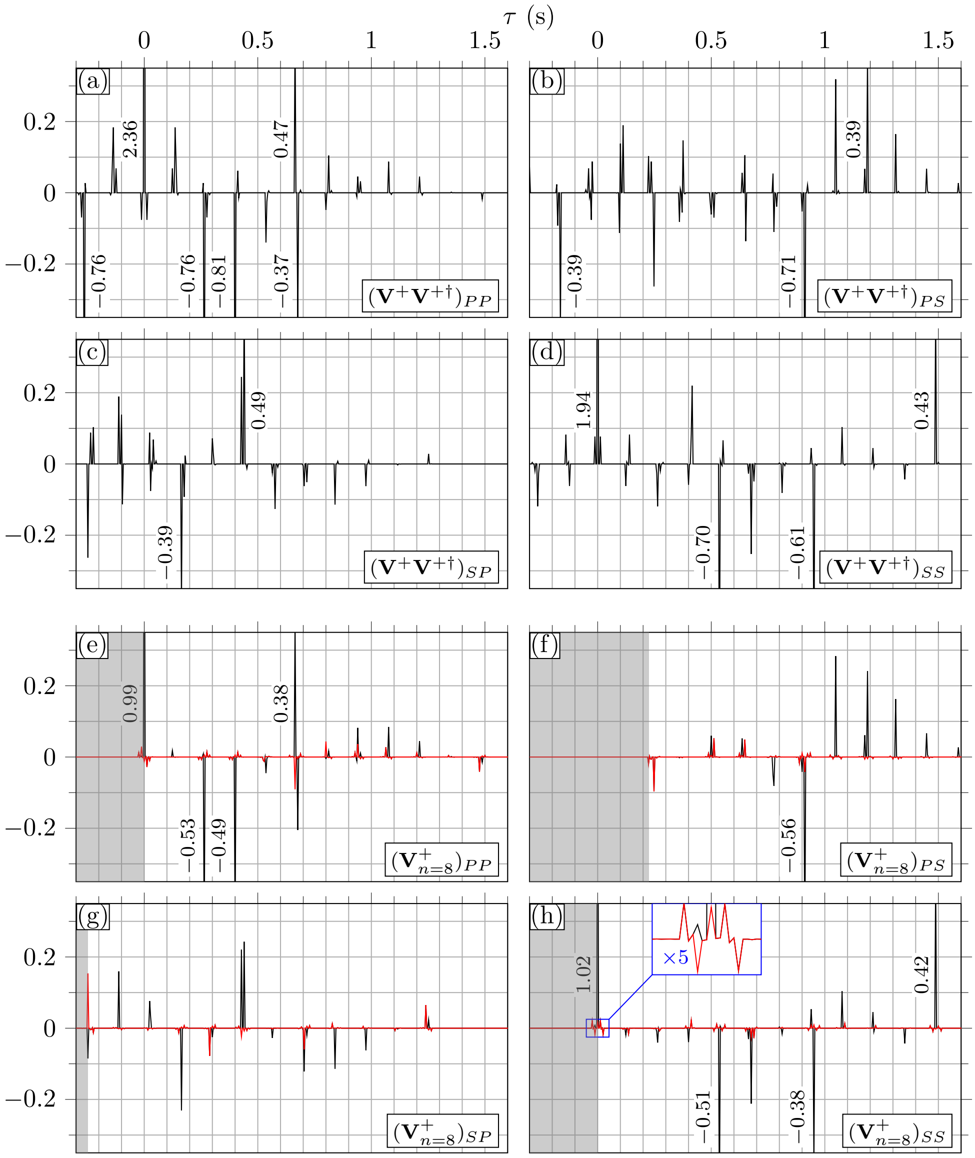

Figure 6. Idem as Figure 5 but associated with the model *Elastic #2* (see Figure 2). In this case, the dereverberation operator has acausal events on the diagonals (PP and SS components). The acausal events on the diagonals appear to be an issue for the Wilson algorithm. The dereverberation operator is reconstructed only up to a relative error in the order of 10 % (see Figure 4), instead of numerical noise as in the previous example in Figure 5.

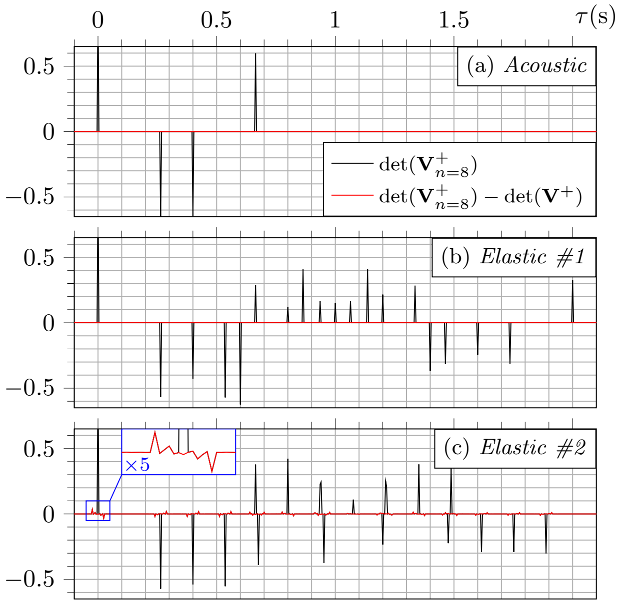

Figure 7. Determinants of the retrieved dereverberation operators (black) and the difference with respect to the modeled solutions (red). The panels are associated with the (a) *Acoustic*, (b) *Elastic #1*, and (c) *Elastic #2*, cases shown in Figures 3, 5 and 6, respectively. In the *Acoustic* case, the dereverberation operator is a scalar function, and hence, identical to its determinant. Nonetheless, it is shown for completeness. For the *Elastic #2* case, the determinant of the retrieved dereverberation operator is not minimum-phase, which can be easily seen via the acausal events shown in the magnified box in blue. The difference plot indicates that acausal events are absent in the determinant of the true solution, which possesses a minimum-phase property.

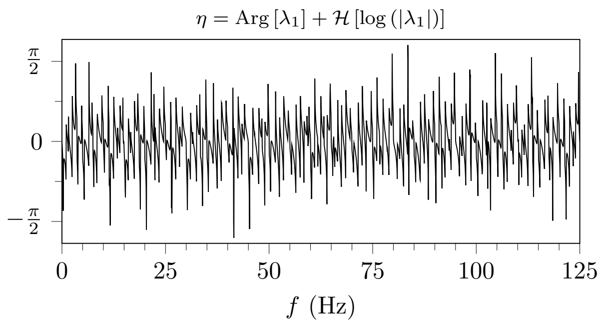

Figure 8. Phase-freedom η of the eigenvalues of the dereverberation operator shown in Figures 5e-h, which is associated with the model *Elastic #1* (also see Eq. 24). The horizontal axis denotes the temporal frequency $f = \frac{\omega}{2\pi}$.

6 CONCLUSION

Minimum-phase properties become significantly more complicated when stepping from scalar to matrix functions. Since the minimum-phase property of a matrix only imposes conditions on its determinant, there are no constraints on individual matrix elements, e.g. they can be acausal.

Our analysis has been motivated by challenges of the Marchenko method. Hence, we focused on the minimum-phase properties of the elastodynamic dereverberation operator, which is a solution of the Marchenko method. We showed that this 2×2 minimum-phase matrix function can be uniquely reconstructed from its normal product using a modified version of the matrix Wilson algorithm. Compared to the original Wilson method, we modified the temporal mute that curiously is identical to one of the two mute operators of the Marchenko method, except for the time zero element.

However, the proposed solution appears to be limited to dereverberation operators with causal diagonal elements. Thus, the method excludes cases with fast-multiples that can occur in the presence of large P- and S-wave velocity differences. Moreover, the dereverberation operator can be seen as a special class of minimum-phase matrices, i.e. the proposed factorization method does not necessarily generalize for other minimum-phase matrices.

The presented results suggest that the minimum-phase property of matrices could play an important role in physics-driven data processing. This work scratches the surface of minimum-phase matrices in the context of geophysics and indicates interesting directions for future research.

DATA AVAILABILITY

The data underlying this article cannot be shared publicly due to company regulations. The data will be shared on reasonable request to the corresponding author.

ACKNOWLEDGMENT

The research of Kees Wapenaar has received funding from the European Research Council (grant no. 742703).

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APPENDIX A: NOTATION

We use the following Fourier transforms (per ray-parameter) where the real-part is denoted by $\Re$,

$$\mathbf{q}(p_x, z, \omega) = \int_{-\infty}^{\infty} \mathbf{q}(p_x, z, \tau) \mathrm{e}^{-\mathrm{i}\omega\tau} \mathrm{d}\tau, \tag{A.1}$$

$$\mathbf{q}(p_x, z, \tau) = \frac{1}{\pi} \Re \left[\int_{0}^{\infty} \mathbf{q}(p_x, z, \omega) \mathrm{e}^{\mathrm{i}\omega\tau} \mathrm{d}\omega \right]. \tag{A.2}$$

In this work, all equations are formulated for plane waves, i.e. per the ray-parameter p_x. We define the transformation from the offset-time domain $\mathbf{q}(x, z, t)$ to the ray-parameter intercept-time domain $\mathbf{q}(p_x, z, \tau)$ as,

$$\mathbf{q}(p_x, z, \tau) = \int_{-\infty}^{\infty} \mathbf{q}(x, z, \tau + p_x x) \mathrm{d}x. \tag{A.3}$$

Table A1. Definition of additional operators used in this paper. All operators are applied per ray-parameter, p_x, and per frequency, ω, except for the Hilbert transform and the L_2 norm which take into account all frequencies. When applied to matrices, the operators act in the P-S space, except for the operations marked with "$\odot$" which act per matrix element. The L_2 norm is calculated using all frequencies and all wavefield components, i.e. a single and four components for acoustic and elastodynamic waves, respectively.

Symbol		**Operation**
Superscript "$*$"		Complex-conjugate
Superscript "$\dagger$"		Complex-conjugate transpose
Superscript "-1"		Inverse
$\log(\cdot)$		Natural logarithm
$\det(\cdot)$		Determinant
$\Vert\cdot\Vert_2$		L_2 norm
$\vert\cdot\vert$	$\odot$	Absolute value
$\mathrm{e}^{[\cdot]}/\cos[\cdot]/\sin[\cdot]$	$\odot$	Exponential/cosine/sine function
$\mathcal{H}[\cdot]$	$\odot$	Hilbert transform
$\mathrm{Arg}[\cdot]$	$\odot$	Phase spectrum

submitted to *Geophys. J. Int.*

Minimum-phase ~~properties~~ property and reconstruction of elastodynamic ~~transmission generators: Analyzing the simplest case of 1.5D media.~~ dereverberation matrix operators

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7 February 2023

SUMMARY

Minimum-phase properties are well-understood for scalar functions where they can be used as ~~a physical constraint~~ physical constraint for phase reconstruction. Existing scalar ~~minimum-phase applications~~ applications of the latter in geophysics include e.g. the reconstruction of transmission from acoustic reflection data, or multiple elimination via the ~~so-called augmented~~ augmented acoustic Marchenko method. We review scalar minimum-phase reconstruction via the conventional Kolmogorov relation, as well as a less-known factorization method. Motivated to solve practice-relevant problems beyond the scalar case, we investigate (1) the properties and (2) the reconstruction of minimum-phase ~~matrices. For simplicity, we~~ matrix functions. We consider a simple but non-trivial case of 2×2 ~~matrices representing elastic wavefields. Our analysis addresses~~ matrix response functions associated with elastodynamic wavefields. Compared to the scalar acoustic case, matrix functions possess additional freedoms. Nonetheless, the minimum-phase ~~reconstruction via the more conventional Kolmogorov relation, as well as a less-known factorization method. After a modification, the latter~~ property is still defined via a scalar function, i.e. a matrix possesses a minimum-phase property if its determinant does. We review and modify a matrix factorization method such that it can accurately reconstruct ~~the~~ a 2×2 minimum-phase matrix ~~operator~~ function related to the ~~elastic~~ elastodynamic Marchenko method. However, the reconstruction is limited to cases with sufficiently small differences between P- and S-wave travel times, which we illustrate with a synthetic example. Moreover, we show that the minimum-phase reconstruction method by factorization shares similarities with the Marchenko method in terms of the algorithm and its limitations. Our results reveal so-far unexplored matrix properties of geophysical responses that open the door towards novel data processing tools. Last but not least, it appears that minimum-phase ~~matrices~~ matrix functions possess additional, still-hidden properties that remain to be exploited e.g. for phase reconstruction.

Key words: Fourier analysis – Numerical solutions – Time-series analysis – Inverse theory – Wave propagation – Wave scattering and diffraction

1 INTRODUCTION

Phase reconstruction can be found in various fields of science and engineering (Shechtman et al. 2015). It is the process of finding a function given its Fourier amplitude spectrum or some multidimensional generalization thereof. The result is not unique but can be better constrained given some a priori knowledge of the function. The focus of this work lies on a special class called minimum-phase reconstruction. It pertains to invertible functions where the function and its inverse are characterized by energy concentrated close to the temporal origin.

In geophysics, minimum-phase is often thought to be a property of the seismic wavelet in marine acquisition (Yilmaz 2001), aside from complications resulting from band-limitation (Lamoureux & Margrave 2007). ~~More importantly,~~ However, minimum-phase is a more general property which can be a characteristic of response functions that relate wavefields measured at different spatial locations. For example, Sherwood & Trorey (1965) as well as Claerbout (1968) demonstrate that full-bandwidth 1D acoustic transmission responses and their inverses form pairs of minimum-phase signals when measured from the onset of ~~signal. To date, the properties and reconstruction of multi-dimensional (e. g. 1.5D elastodynamic, or 2D/3D acoustic wavefields)~~ the signal. The aforementioned work distinguishes transmission from reflection responses. This is often reasonable in exploration geophysics when considering a section of the subsurface embedded between top and bottom boundaries. For simplicity, we assume these boundaries are perfectly absorbing. Contrary to transmissions, reflection responses are generally not minimum-phase~~signals remain poorly understood in geophysics. This is an important problem, since next to the work by Claerbout (1998) and Fomel et al. (2003), this topic has seen a revival in the context of internal multiple elimination using the so-called Marchenko method (e.g. Dukalski et al. 2019).~~

~~The Marchenko method predicts internal multiples accurately in terms of not only kinematics but~~ ~~It does so by reconstructing the so-called dereverberation operator, which is an extrapolated inverse transmission response of the overburden (Broggini & Snieder 2012; van der Neut & Wapenaar 2~~ ~~To date, the most convincing examples apply the acoustic Marchenko method to synthetic or field data (e.g. see Ravasi et al. 2016; Staring et al. 2020; Zhang & Slob 2020). For field data, the aforementioned approach ignores elastodynamic effects, which causes artifacts (da Costa Filho et al. 2016; Reinicke et al. 2021). The elastodynamic generalization of the theory still hinges on challenges resulting from the differences in propagation velocities of the compressional and shear components (Reinicke et al. 2020). Recently, van der Neut et al. (2022) suggest~~ ~~an alternative solution that tackles these challenges using two-sided illumination, which is less common for geophysical applications. The aforementioned problems most likely extend to the elastodynamic formulation of the conventional predict-and-subtract demultiple methods (Weglein et al. 1997; Jakubowicz 1998; van Borselen 2002; Ikelle 2006; Sun & Innanen 2019) because they are the leading order of the Neumann series expansion of the Marchenko equation (Zhang et al. 2019; Dukalski & de Vos 2022)~~.

~~Recent developments combine Marchenko multiple removal with minimum-phase properties, referr~~ ~~This strategy addresses practical cases where the desired solution, i.e. the dereverberation operator, is not fully constrained by the Marchenko equation e.g. due to band-limitation~~

~~(Slob et al. 2014). The augmented Marchenko method constrains those freedoms by enforcing two physical properties of the dereverberation operator, energy conservation to obtain the so-called normal product of the dereverberation operator, and,~~ To date, the properties and the reconstruction of multi-dimensional minimum-phase ~~behavior to reconstruct the dereverberation operator from its normal product. The energy conservation constraint (step 1) has already been demonstrated for both scalar and matrix (i.e.~~ signals remain poorly understood. Here, multi-dimensional ~~) functions such as~~ signals refer to response functions that are associated with 1.5D elastodynamic or 2D~~acoustic, or~~/3D acoustic wavefields as opposed to scalar functions associated with 1.5~~D elastodynamic wavefields (e.g. see Elison et al. 2020; R~~ ~~However, minimum-phase reconstruction (step 2) is poorly understood beyond scalar functions. Therefore, the augmented Marchenko method remains limited to scalar cases. These include 1D and 1.5~~D acoustic ~~solutions (Dukalski et al. 2019; Elison et al. 2020), as well as 2D acoustic cases where a~~ wavefields. This topic remains a relevant geophysics problem which has been studied by only few authors (Claerbout 1998; Fomel et al. 2003). As a result, multi-dimensional minimum-phase ~~reconstruction under a 1.5D assumption has been shown to often provide a very good approximation (Peng et al. 2021). There are initial attempts of generalizing~~ signal reconstruction remains a barrier for numerous applications such as retrieving transmission from reflection responses (Wapenaar et al. 2003), or internal multiple elimination using the augmented Marchenko method ~~to the elastic case (Reinicke et al. 2020). However, they still depend on the understanding of, and the ability to reconstruct, matricial minimum-phase functions.~~

~~In this work, we study the minimum-phase properties and reconstruction of the dereverberation op~~ ~~The presented research~~ (e.g. Dukalski et al. 2019). The research of this paper has been motivated by ~~, but does not discuss details of,~~ the augmented Marchenko method ~~. Firstly, we define our notation. Secondly, we~~ and its generalization to elastodynamic waves (this method is not discu

In this work, we study the minimum-phase properties and reconstruction of 2×2 matrix response In Section 2, we review existing theory of minimum-phase properties and two reconstruction ~~methods~~ algorithms for the scalar case~~, including an example related to~~. Moreover, we discuss geophysical response functions and show an example of minimum-phase reconstruction for the acoustic dereverberation operator ~~. Thirdly, we explain how~~ of the Marchenko method. In Section 3, we discuss why elastodynamic response functions are matrices instead of scalars, and analyze the minimum-phase ~~properties are considerably more complex when stepping from scalar to matrix functions~~ property as well as its reconstruction for the matrix case. In Section 4, ~~and how this makes minimum-phase reconstruction more challenging. Fourthly,~~ we present two numerical examples of ~~matricial~~ minimum-phase ~~reconstruction using a modified version of the factorization method by Wilson (1972)~~ matrix reconstruction based on the factorization algorithm by Wilson (1972) with a modification inspired by the Marchenko method. The two examples ~~of matricial minimum-phase reconstruction show~~ include a case with an accurate solution as well as another case with artifacts to highlight remaining limitations. Finally, we discuss our insights ~~with a focus on similarities with~~ in Section 5 and highlight similarities between minimum-phase reconstruction and the Marchenko method~~in terms of (1) the mechanism of the algorithm as well as (2) theoretical limitations, and we present an outlook~~.

2 ~~NOTATION~~MINIMUM-PHASE PROPERTY AND RECONSTRUCTION: SCALAR CASE

~~For all numerical examples in this paper, we consider the four layer model in Figure 2 and a single ray-parameter~~ $\cancel{p_x = 2\times 10^{-4}\,\mathrm{s\,m^{-1}}}$~~. We use three models that are identical except for the S-wave velocity,~~ $\cancel{c_S}$~~, including an acoustic model (~~$\cancel{c_S = 0}$~~) and two elastic ones (~~$\cancel{c_S \neq 0}$~~).~~

3 ~~MINIMUM-PHASE PROPERTY AND RECONSTRUCTION~~

~~In this section, we,~~ In particular, we,

(2.1) ~~summarize the qualitative definition of the~~ review the scalar minimum-phase ~~property~~

~~and normal products,~~ property and how it can be used for phase reconstruction via the Kolmogorov relation,

(2.2) ~~show two existing methods for scalar~~ show a factorization method for scalar phase-reconstruction under a minimum-phase condition,

(2.3) ~~analyze the~~ introduce our notation and geophysical responses.

In part (2.3), we focus on a minimum-phase ~~property for matrices, and we explore to what extent this property can be exploited for matrix reconstruction from a normal product. Parts (3.1)-(3.2) cover existing work to prepare the discussion of the main result of this work in part (3.3).~~ function that is relevant for the Marchenko method. However, the analysis does not require in depth knowledge of the Marchenko method.

2.1 ~~Qualitative overview~~

~~Minimum-phase is a mathematical property associated with a special class of transfer functions. In general, the literature distinguishes between transfer functionswith (1) a single-input single-output (SISO), and (2) multi-inputs multi-outputs (MIMO) (Johansson 1997). An example for a SISO system in geophysics is a plane-wave response function of a layered acoustic medium. For elastic or more general acoustic media, response functions depend on multiple input and multiple output variables e.g. P- and S-wave modes (see Eq. 17), i.e. these are MIMO systems.~~

2.1 Minimum-phase in a nutshell

We start by discussing linear time-invariant (LTI) systems. Given an arbitrary input, one can obtain the output of an LTI system via temporal convolution with its impulse response. For example, seismic reflection data can be represented as a temporal convolution of the source signature with the impulse response of the subsurface. This representation assumes that the subsurface remains unchanged during the experiment. For convenience, convolutions in the time (τ) domain are often formulated as multiplications in the frequency (ω) domain, e.g.,

$$output(\omega) = g(\omega)\ input(\omega), \tag{1}$$

where $g(\omega)$ denotes an impulse response. In the following, we imply that all operations, such as products or divisions, are performed per frequency component unless explicitly mentioned. Moreover, we refer to impulse responses as responses or functions, while they may also be known as transfer functions.

The minimum-phase property is a mathematical characteristic associated with a special class of fu

Using a qualitative definition, ~~SISO or MIMO systems posses a~~ a function possesses a minimum-phase property if ~~all of the following three~~ the following conditions are satisfied ~~(Bode et al. 1945; Sherwood & Trorey 1965; Skingle et al. 1977).~~ (Bode et al. 1945; Sherwood & Trore

(i) ~~Stability:~~ The sum of all absolute ~~time-components is finite~~ time components is finite (stability).

(ii) ~~Causality: The transfer~~ The function vanishes for negative times (~~$\tau < 0$~~causality).

(iii) ~~Invertibility:~~ The inverse exists and satisfies (i) and (ii).

An important consequence ~~of the conditions (i)-(iii)~~ is that the product of minimum-phase functions produces a result with a minimum-phase property. ~~In the following sections, we will quantify the above-listed qualitative definitions for scalar functions. Moreover, we will show that the generalization to matrix functions bears challenges, especially the definition of causality for matrices~~The term "minimum-phase" suggests that some attribute is minimized, which is true for special cases, where the group delay is minimized. However, this definition is not used in our analysis.

~~Before going into further detail, we briefly discuss normal products.~~ ~~In this work, we aim to reconstruct a desired solution, $\mathbf{D}$, with a~~ We illustrate the minimum-phase property ~~from its normal product. The latter is defined as the product of a scalar, or a matrix, with its complex-conjugate transpose, e. g. $|D|^2$ or $\mathbf{D}\mathbf{D}^\dagger$. Scalar normal products may be better known as auto-correlations in the time domain and are often interpreted physically~~

~~as amplitude spectra because their phase vanishes, $\mathrm{Arg}\left[|D|^2\right] = 0$. Following this physical interpretation, retrieving the scalar solution, D, from its normal product, $|D|^2$, is often described as a phase reconstruction, while mathematically, it is a factorization problem. This generally non-unique factorization can be constrained for minimum-phase scalar functions which obey a unique phase-amplitude relationship (Smith 2007). However, the matrix case is more complicated. Matricial normal products are not necessarily phase-free on the off-diagonal elements, which are identical except for a sign-inverted phase (for 2×2 matrices). For convenience, we keep the physical interpretation from the scalar case, i. e. "amplitude spectra~~using an example. Consider the causal functions (i.e. $\tau_1 > 0$), 2

$$A(\omega) = 1 + \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1}, \tag{2}$$

$$B(\omega) = \alpha + \mathrm{e}^{-\mathrm{i}\omega\tau_1} = \left(A(\omega)\right)^* \mathrm{e}^{-\mathrm{i}\omega\tau_1}, \tag{3}$$

where α is a constant smaller than one. The variable i and the superscript "*" ~~and "phase reconstruction" refer to normal products and the retrieval of the solution, $\mathbf{D}$, from its normal product, respectively. As we will discuss in Section 3.2, it is significantly more challenging to define the minimum-phase property for matrices, and to exploit it for unique normal-product factorization. Moreover, matrices can generate two normal products, which are generally not equal, $\mathbf{D}\mathbf{D}^\dagger \neq \mathbf{D}^\dagger\mathbf{D}$. One can easily verify that the two normal products provide individually up to three, and together up to six independent equations (for 2×2 matrices) .~~ denote the imaginary unit and complex-conjugation, respectively. Hence, ~~if both normal products are known, there are more equations to constrain the reconstruction of the matrix, $\mathbf{D}$~~the functions have identical amplitude spectra, $C(\omega) = |A(\omega)| = |B(\omega)|$. Moreover, we use several common operators, which are defined in the appendix (see Table A1). The analysis of causality depends on the definition of the Fourier transform (sign choice of the exponent) which we define according to Eqs. A.1 and A.2. The phase of the functions can be visualized as an angle in the complex plane spanned between a complex number and the real axis (see Figure 1a, where $\alpha = -0.6$ and $\tau = 0.04$ s), or as a function of frequency (see Figure 1b). It can be easily seen that the functions $A(\omega)$ and $B(\omega)$ satisfy conditions

(i) and (ii) (see Figure 1c). Their inverses exist and can be found using the geometric series and Eq. 3, 1

$$(A(\omega))^{-1} = \sum_{k=0}^{\infty} (-\alpha)^k \mathrm{e}^{-\mathrm{i}\omega\tau_1 k}, \tag{4}$$

$$(B(\omega))^{-1} = \left((A(\omega))^{-1}\right)^* \mathrm{e}^{\mathrm{i}\omega\tau_1} = \sum_{k=0}^{\infty} (-\alpha)^k \mathrm{e}^{\mathrm{i}\omega\tau_1(k+1)}. \tag{5}$$

Moreover, the inverses are stable due to convergence of the geometric series in Eqs. 4 and 5. However, ~~we restrict our analysis to the situation that is relevant for the elastodynamic augmented Marchenko method where only one of the two normal products is known (this is a result from the solution is minimum-phase as we will show in Section 3.2.~~

2.2 ~~Minimum-phase reconstruction of scalar normal products~~

~~In the following, we review the well-understood scalar case of phase reconstruction based on a normal Firstly, we review the definition of the dereverberation operator and its minimum-phase property. Secondly, we will summarize two existing but very different methods for minimum-phase reconstruction.~~

~~The Marchenko method can be seen as an inversion for an operator that is minimum-phase. The literature refers to this solution as dereverberation operator,~~

$$\cancel{V^+ = T^{\downarrow -1} T^{\downarrow}_{dir} = 1 + \left(T^{\downarrow -1}\right)_{coda} T^{\downarrow}_{dir}.}$$

~~Here, the transmission $T^{\downarrow}$ is split in its direct and coda parts indicated by the subscripts *dir* and *coda*, respectively,~~

$$\cancel{T^{\downarrow} = T^{\downarrow}_{dir} + T^{\downarrow}_{coda},}$$

~~and the inverse transmission $T^{\downarrow -1}$ is often referred to as a so-called focusing function f^+ (Wapenaar et al. 2014). Since (inverse)transmissions are (advanced) delayed minimum-phase functions (Claerbout 1968),~~ only the ~~dereverberation operator V^+~~ inverse $(A(\omega))^{-1}$ is causal whereas the inverse $(B(\omega))^{-1}$ is acausal (see Figure 1c). Hence, the function $A(\omega)$ satisfies conditions (i)-(iii) and possesses a minimum-phase ~~property by definition. The augmented~~

~~Marchenko method exploits this property to reconstruct the dereverberation operator, V^+, from its normal product, V^+V^{+*}, which is obtained from energy conservation.~~ property, but the function $B(\omega)$ does not. The amplitude spectrum $C(\omega)$ has a smaller phase (zero-phase) than the function $A(\omega)$ but it violates the causality condition (ii), and hence is not minimum-phase (see Figure 1c). In the following, we omit the dependency on frequencies except for newly introduced functions.

~~In general, the reconstruction of a function from its normal product is non-unique. Scalar functions can be multiplied by an arbitrary, $\mathrm{U}(1) = \mathrm{e}^{\mathrm{i}\Phi(\omega)}$, element (Cornwell 1997) without changing their normal product, e.g.,~~ Minimum-phase reconstruction is the retrieval of a minimum-phase function from its amplitude, or power, spectrum. In general, phase reconstruction carries a degree of freedom $\mathrm{e}^{\mathrm{i}\Phi(\omega)}$,

$$\left(V^+\mathrm{U}(1)A\mathrm{e}^{\mathrm{i}\Phi(\omega)}\right)^* V^+\mathrm{U}(1)A\mathrm{e}^{\mathrm{i}\Phi(\omega)} = V^+V^+A^*A = |A|^2. \tag{6}$$

~~Next, we show two methods that constrain this freedom by exploiting~~ However, it can be shown that the aforementioned freedom vanishes under the minimum-phase ~~property.~~

2.1.1 ~~*Kolmogorov relation*~~

~~The Kolmogorov method is a well-known phase reconstruction method. It enforces the minimum-phase~~ conditions (i)-(iii) ~~(e.g. Skingle et al. 1977) and can be written~~. Thus, minimum-phase functions possess a unique amplitude-phase relationship, which can be formulated e.g. ~~for the dereverberation operator as follows,~~

$$\log\left(V^+\right) = \log\left(|V^+|\right) + \mathrm{i}\mathrm{Arg}\left[V^+\right]$$
$$= \log\left(|V^+|\right) - \mathrm{i}\mathcal{H}\left[\log\left(|V^+|\right)\right].$$

via the Kolmogorov relation (e.g. Skingle et al. 1977), 2

$$\log(A) = \log(|A|) + \mathrm{i}\mathrm{Arg}[A]$$
$$= \log(|A|) - \mathrm{i}\mathcal{H}[\log(|A|)]. \tag{7}$$

~~This equation illustrates the unique relation between phase and amplitude spectra of minimum-phase~~

~~functions~~Here, we denote the phase by $\mathrm{Arg}[A]$, the natural logarithm by $\log(\cdot)$, and the Hilbert transform by $\mathcal{H}[\cdot]$.

2.1.1 ~~*1D Minimum-phase factorization: Wilson algorithm*~~

2.2 Minimum-phase reconstruction by factorization

Wilson (1969) formulates ~~the retrieval of a~~ minimum-phase ~~function from its normal product~~ reconstruction as a recursive factorization problem, which we call the Wilson algorithm. ~~In the following, we refer to this method as the Wilson algorithm and illustrate it using the dereverberation operator, V^+. The Wilson algorithm is derived from~~ This method will be important when generalizing the minimum-phase property and reconstruction from scalars to matrices in Section 3.2. Since the Wilson method might be less-known than the Kolmogorov in Eq. 7, we summarize its scalar formulation in more detail.

Consider an arbitrary minimum-phase function $A(\omega)$. The starting point is a relation between the ~~normal product $|V^+|^2$~~amplitude spectrum $|A|$, an estimate after n iterations ~~,~~ ~~V_n^+,~~ A_n, and its update ~~, V_{n+1}^+~~ A_{n+1} (see Eq. 6 in Wilson 1969),

$$\cancel{V_n^+ V_{n+1}^{+*}} A_n A_{n+1}^* + \cancel{V_{n+1}^+ V_n^{+*}} A_{n+1} A_n^* = \cancel{V_n^+ V_n^{+*}} A_n A_n^* + \cancel{V^{+2}} AA^*. \tag{8}$$

~~Multiplication by $(V_n^+)^{-1}$ and $(V_n^{+*})^{-1}$~~ Multiplication by $(A_n)^{-1}$ and $(A_n^*)^{-1}$ leads to,

$$A_{n+1}^*\left(\cancel{V_n^+} A_n^*\right)^{-1} \cancel{V_{n+1}^{+*}} + \left(\cancel{V_n^+} A_n\right)^{-1} \cancel{V_{n+1}^+} A_{n+1} = 1 + \left(\cancel{V_n^+} A_n\right)^{-1} \cancel{V^{+2}} AA^* \left(\cancel{V_n^{+*}} A_n^*\right)^{-1} \cancel{;}. \tag{9}$$

It follows from the minimum-phase-property of the ~~dereverberation operator~~ desired solution A that Eq. 9 contains a superposition of a strictly causal term, ~~$(V_n^+)^{-1} V_{n+1}^+$~~$(A_n)^{-1} A_{n+1}$, with its time-reverse. The acausal term, ~~$\left[(V_n^+)^{-1} V_{n+1}^+\right]^*$~~$\left[(A_n)^{-1} A_{n+1}\right]^*$, can be removed by applying a ~~mute,~~ temporal mute $\Theta[\cdot]$~~, that~~. Next, the result is rearranged to obtain a recursive algorithm,

$$A_{n+1} = A_n \Theta\left[1 + (A_n)^{-1} |A|^2 (A_n^*)^{-1}\right]. \tag{10}$$

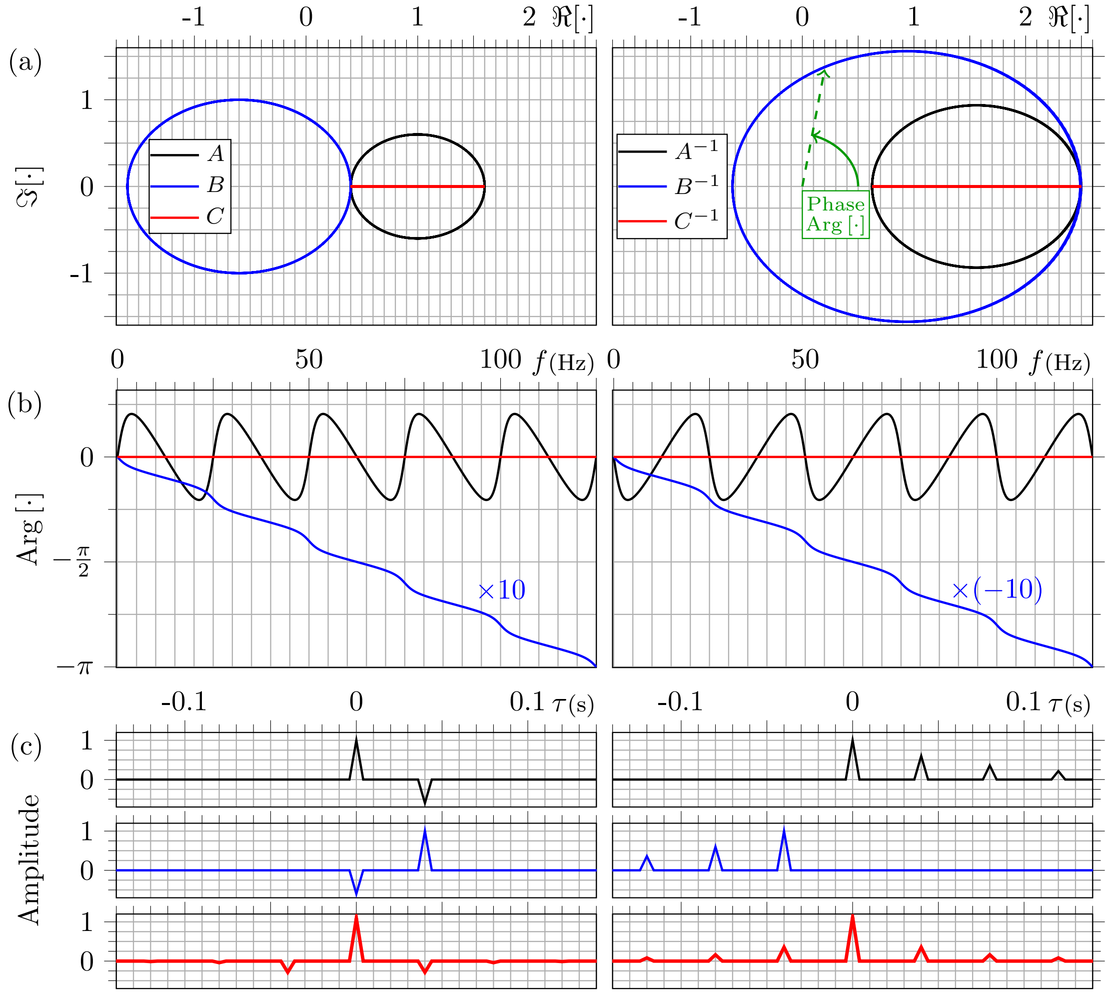

Figure 1. ~~Parameters~~ Illustration of the ~~three models used in this work. The density~~functions A, ~~ρ~~B, C (left column) and ~~the P-wave velocity, c_P, are identical for all models. An *acoustic case* is~~ their inverses (right column) defined ~~by setting the S-wave velocity to zero, $c_S = 0$~~in Eqs. 2 - 5 using $\alpha = -0.6$ and $\tau_1 = 0.04$ s. The ~~*Elastic #1* case is defined with~~ panels show (a~~non-zero S-wave velocity~~) Argand diagrams, ~~$c_S \neq 0$~~(b) phase spectra and (c) time domain representations. The ~~*Elastic #2* case is defined by reducing the S-wave velocity in one~~ axes of the ~~layers. The one-way travel times within each layer are integer-multiples of~~ Argand diagram correspond to the ~~time sampling interval, $\Delta\tau = 4$ ms, for all models~~ real ($\Re$) and ~~for P-/S-waves associated with, $p_x = 2\times10^{-4}$ s m^{-1}. This choice simplifies the interpretation~~ imaginary ($\Im$) part of the ~~medium responses~~ functions in the ~~time~~ frequency domain~~because all events perfectly coincide with a time sample, i~~. ~~e. it avoids smearing~~ The phase of ~~individual events across several time samples~~a complex number is illustrated in the top right panel. ~~In this setting~~Moreover, there is one legend per column and we ~~can accurately apply temporal mutes which allows us~~ denote $f = \frac{\omega}{2\pi}$. The minimum-phase function A and its inverse follow trajectories in the complex plane that have winding numbers around the origin equal to ~~verify~~ zero. However, the ~~accuracy~~ trajectory of the ~~discussed algorithms up to numerical noise (in~~ function B and its inverse wind five times around the ~~order~~ origin of the complex plane (deduced from the phase spectra $\frac{\pi\times10}{2\pi} = 5$, ~~$1\times10^{-15}$, for double-precision~~or $\frac{\omega_{\max}\tau_1}{2\pi} = 125$ Hz $\times$ 0.04 s $= 5$).

Here, the mute represents multiplication by the Heaviside function ~~,~~ $\mathrm{H}(\tau)$ ~~,~~ in the time domain,

$$\mathrm{H}(\tau) = \begin{cases} 1, & \tau > 0, \\ \frac{1}{2}, & \tau = 0, \\ 0, & \tau < 0. \end{cases} \tag{11}$$

~~By rearranging the result a recursive algorithm is obtained,~~

$$\cancel{V^+_{n+1} = V^+_n \Theta \left[1 + \left(V^+_n \right)^{-1} |V^+|^2 \left(V^{+*}_n \right)^{-1} \right].}$$

Since most operations in this work are formulated in the frequency domain, the mute opertator $\Theta\left[\cdot\right]$ includes Fourier transforms between the frequency and time domains. In Section 3, the mute operator will be generalized from a Heaviside function to a more general step function. Wilson (1969) shows that the ~~desired solution, V^+, is found~~ recursive algorithm in Eq. 10 converges to the desired solution A using the simplest minimum-phase function as initial estimate, ~~$V_0^+ = 1$.~~ $A_0 = 1$ (in the frequency domain). The scaling by ~~,~~ $\frac{1}{2}$ ~~, at time-zero~~ at time zero (see Eq. 11) handles the overlap of the causal and acausal terms in Eq. 9. It can also be seen as a termination condition that ensures convergence, i.e. ~~for, $V_n^+ = V^+$,~~ the solution is not updated ~~,~~ for $A_n = A$,

$$\begin{aligned} \cancel{V^+_{n+1}} A_{n+1} &= \cancel{V^+_n} A_n \Theta \left[1 + (A_n)^{-1} |A|^2 (A^*_n)^{-1} \right] \\ &= \cancel{V^+_n} A_n \Theta \left[1 + 1 \right] = \cancel{V^+_n} A_n. \end{aligned} \tag{12}$$

~~We illustrate the 1D Wilson algorithm with an example considering the acoustic model shown in F~~

~~The normal product of the~~

2.3 Geophysical scalar functions and minimum-phase

We briefly introduce our notation, define the dereverberation operator and show a numerical example of the Wilson algorithm.

In geophysics, transfer functions are often used to relate wavefields at different locations. For simplicity, we consider horizontally-layered media in the x-z space, where wavefields decouple per horizontal ray-parameter, $p_x = \frac{sin(\alpha)}{c}$ (see Eq. A.3 for definition of the domain transformation). Here, the angle α is formed by the wave front and the x-axis, and c denotes the local propagation velocity of a given wave type (P, or S which will be relevant in the elastic case).

The term response refers to a Green's function associated with a plane-wave dipole source and a monopole receiver. Hence, a response is a function that relates the wavefields at the source and receiver locations via a product per frequency. We consider an acoustic medium that is homogeneous except for a section between the depth levels z on top, and z' at the bottom. Moreover, the medium is source-free below the upper boundary at depth z. In this configuration, one can relate the wavefields on the boundaries z and z' using a scalar response $D(p_x, z', z, \omega)$ (as opposed to a matrix response) according to,

$$q(p_x, z', \omega) = D(p_x, z', z, \omega) q(p_x, z, \omega). \tag{13}$$

Here, the quantity $q(p_x, z, \omega)$ denotes an acoustic pressure wavefield. We assume all coordinates are fixed except for the frequency and use a detail-hiding notation that omits coordinates, e.g. $q_{below} = D q_{above}$ (similar to Berkhout 1982; Wapenaar 1989).

For all numerical examples in this paper, we consider the four layer model in Figure 2 and a single ray-parameter $p_x = 2 \times 10^{-4}$ s m^{-1}. We use three models that are identical except for the S-wave velocity c_S including an acoustic model ($c_S = 0$) and two elastic ones ($c_S \neq 0$).

Next, we introduce a specific transfer function namely the dereverberation operator which is the d
It can be used to remove internal multiples from seismic reflection data (e.g. van der Neut & Wapenaar 2
however, multiple elimination is not relevant for our analysis. The dereverberation operator is defined via the transmission response $T^{\downarrow}$ that relates the wavefields above and below a scattering medium ($q_{below} = T^{\downarrow} q_{above}$). In the acoustic case, it can be written as,

$$V^{+} = T^{\downarrow -1} T^{\downarrow}_{dir} = 1 + V^{+}_{coda}. \tag{14}$$

Here, the transmission $T^{\downarrow}$ is split in its direct and coda parts indicated by the subscripts "*dir*" and "*coda*", respectively,

$$T^{\downarrow} = T^{\downarrow}_{dir} + T^{\downarrow}_{coda}, \tag{15}$$

and the inverse transmission $T^{\downarrow -1}$ is often referred to as a focusing function f^{+} (Wapenaar et al. 2014). Transmissions and their inverses are minimum-phase functions, except for a positive and negative time shift, respectively (Claerbout 1968). These time shifts mutually cancel when evaluating the product in Eq. 14. Hence, the dereverberation operator possesses a minimum-phase property. For example, the function A in Eq. 2 is a dereverberation operator of an acoustic medium with two reflectors that are separated by the travel time $\frac{1}{2}\tau_1$, and the factor α represents the product of the reflection coefficients of the two interfaces.

We illustrate the scalar Wilson algorithm with an example considering the acoustic model shown i The power spectrum of the dereverberation operator $|V^{+}|^2$ (see Figure 3a) is modeled analytically (Dukalski et al. 2022) and used to evaluate Eq. 10 with $A = V^{+}$. Figures 3b-f show the solution V^{+}_{n} and its error, $V^{+}_{n} - V^{+}$, as a function of iterations (n). The convergence in Figure 4 reveals that the Wilson algorithm finds the true solution up to numerical accuracy within seven iterations.

2.4 ~~Minimum-phase property and reconstruction by normal-product factorization: Matrix case~~

3 MINIMUM-PHASE PROPERTY AND RECONSTRUCTION: MATRIX CASE

~~The concept of minimum-phase is significantly more difficult beyond scalar functions where several assumptions break.~~ In this section, we, ~~In this section, firstly, we review the definition~~

(3.1) introduce matrix functions and their link to elastodynamic wavefields,

(3.2) analyze the minimum-phase property of matrices,

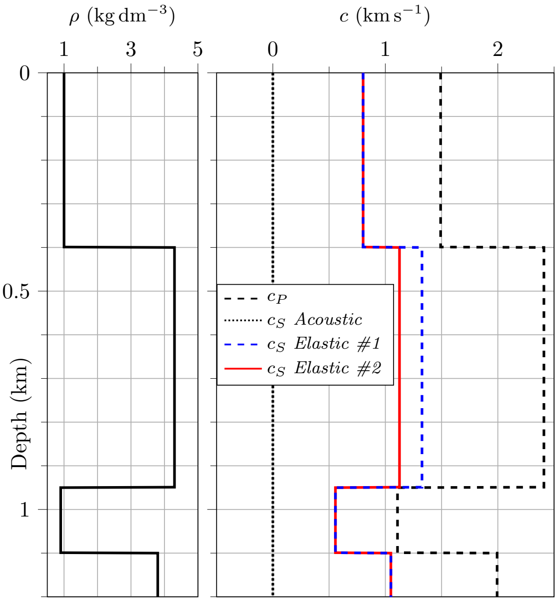

Figure 2. Parameters of the three models used in this work. The density ρ and the P-wave velocity c_P are identical for all models. An *acoustic case* is defined by setting the S-wave velocity to zero $c_S = 0$. The *Elastic #1* case is defined with a non-zero S-wave velocity $c_S \neq 0$. The *Elastic #2* case is defined by reducing the S-wave velocity in one of the layers. The one-way travel times within each layer are integer-multiples of the time sampling interval ($\Delta\tau = 4$ ms) for all models and for P-/S-waves associated with $p_x = 2 \times 10^{-4}$ s m^{-1}. This choice simplifies the interpretation of the medium responses in the time domain because all events perfectly coincide with a time sample, i.e. it avoids smearing of individual events across several time samples. In this setting, we can accurately apply temporal mutes which allows us to verify the accuracy of the discussed algorithms up to numerical noise (in the order of 1×10^{-15} for double-precision).

(3.3) review normal products and explore how minimum-phase matrices can be reconstructed from their normal products by factorization. For the reconstruction step, we focus on the special case of the elastodynamic dereverberation operator~~, $\mathbf{V}^+$, which is a minimum-phase matrix~~. ~~Secondly, we discuss minimum-phase properties for matrices. Thirdly, we analyze the additional degrees of freedom of normal-product factorization, that arise when increasing the dimensionality from scalars to matrices.Finally, we present a modified version of the matricial minimum-phase normal-product factorization method by Wilson (1972). We will illustrate our analysis numerically in Section 4.~~

~~We start by briefly introducing the elastodynamic dereverberation operator $\mathbf{V}^+$~~. ~~One can~~

~~generalize the definition of the acoustic dereverberation operator in Eqs. 14 and 15 to the elastic case by replacing scalar with matrix fields~~

3.1 Geophysical matrix functions

We briefly introduce matrix functions. The literature distinguishes between transfer functions with (1) a single input and a single output (SISO) corresponding to the scalar case discussed above, as well as (2) multi inputs and multi outputs (MIMO) (Johansson 1997). The latter can be represented by frequency-dependent matrices, where the number of rows and columns corresponds to the number of output and input variables, respectively. Hence, they are referred to as matrix functions. Compared to the scalar case, mathematical operations are generalized which can lead to previously unexplored challenges, e.g. scalar products and divisions become matrix multiplications and matrix inverses, respectively.

Elastodynamic responses can be represented by 2×2 matrix functions. Here, we consider the configuration discussed in Section 2.3 but generalize acoustic to elastic media. One can formulate the elastic extension of the wavefield-response relation in Eq. 13 as follows,

$$\mathbf{q}(p_x, z', \omega) = \mathbf{D}(p_x, z', z, \omega)\mathbf{q}(p_x, z, \omega), \tag{16}$$

with,

$$\mathbf{D} = \begin{pmatrix} D_{P,P} & D_{P,S} \\ D_{S,P} & D_{S,S} \end{pmatrix}, \text{ and, } \mathbf{q} = \begin{pmatrix} q_P \\ q_S \end{pmatrix}. \tag{17}$$

The subscripts denote P-/S-waves and we use bold font to distinguish vectors and matrices from scalars. In this context, the matrix function $\mathbf{D}$ is an elastodynamic response defined in the P-S space~~(Reinicke et al. 2020),~~

$$\mathbf{V}^{+} = \mathbf{T}^{\downarrow -1}\mathbf{T}^{\downarrow}_{dir} = \mathbf{I} + \left(\mathbf{T}^{\downarrow -1}\right)_{coda} \mathbf{T}^{\downarrow}_{dir}.$$

. The first and second subscripts of its matrix elements denote the wave type at the receiver- and source-side, respectively. For example, the element $D_{P,S}$ relates S-waves at the source location to P-waves at the receiver location. Next, we generalize the temporal mutes to matrices such that they operate, and can differ per matrix element in the P-S space,

$$\mathbf{\Theta}\left[\mathbf{D}\right] = \begin{pmatrix} \Theta_{P,P}\left[D_{P,P}\right] & \Theta_{P,S}\left[D_{P,S}\right] \\ \Theta_{S,P}\left[D_{S,P}\right] & \Theta_{S,S}\left[D_{S,S}\right] \end{pmatrix}. \tag{18}$$

~~The acoustic direct transmission, $T^{\downarrow}_{dir}$, generalizes to a so-called forward-scattered transmission, $\mathbf{T}^{\downarrow}_{dir}$, that includes all non-reflected events such as transmitted mode-converted waves (Wapenaar 2014). Analogously, the scalar identity, 1, becomes an identity matrix , $\mathbf{I}$.~~ Next, we will investigate how to define and reconstruct the minimum-phase property for matrices, e.g. per matrix element or per matrix. Moreover, we will analyze the mathematical behavior of minimum-phase matrices, e.g. whether their property is preserved by matrix products or changes of basis. Despite focusing on 2 × 2 matrices, we do not exclude generalizations to larger ones.

~~The dereverberation operator has several properties that are important for this work on minimum- Firstly, all events of the dereverberation operator arrive within a well-defined time window that only depends on the one-way travel times of P- and S-waves within each layer (Reinicke et al. 2020).S~~

3.2 Minimum-phase matrix property

The concept of minimum-phase is significantly more difficult beyond scalar functions where several ass

In the following, we discuss the ~~dereverberation operator has a finite number of events limited by the number of layers. This follows from the finite number of events of the inverse and forward-scattered transmissions (Dukalski et al. 2022). In contrast to scalars,~~ minimum-phase ~~matrix functions can contain individual acausal elements as we will discuss next.~~ property of matrices by reviewing findings from other areas (e.g. control theory).

3.2.1 ~~*Matricial minimum-phase property*~~

~~In the following, we discuss findings from other fields (e.g. control theory) on minimum-phase matrix p~~

Diagonal matrices are a trivial extension from scalars to matrices. Consider the scalar minimum-phase functions, $\cancel{g_\pm = 1 \pm \alpha e^{-i\omega\tau_1}}$~~, with ,~~ $A_\pm = 1 \pm \alpha e^{-i\omega\tau_1}$, with $|\alpha| < 1$ ~~, and ,~~ and $\tau_1 > 0$. By arranging them in a diagonal matrix ~~(~~denoted by diag$(\cdot)$ ~~), we obtain a trivial example of a~~ we obtain the minimum-phase matrix, $\cancel{\mathbf{\Lambda} = \mathrm{diag}(g_-, g_+)}$$\mathbf{\Lambda} = \mathrm{diag}(A_-, A_+)$. In contrast to this intuitive example, we will show less obvious cases of minimum-phase matrices further onwards.

Existing literature defines matrices as minimum-phase if their determinants are minimum-phase (Wiener 1955; Rosenbrock 1969; Horowitz et al. 1986). Hence, the determinant of a minimum-phase matrix satisfies the Kolmogorov relation (analogously to Eq. 7). This definition is consistent with the special case of scalar functions which are 1×1 matrices. It is also consistent with the simple matrix example above, $\mathbf{\Lambda}$, where the determinant is equal to the product of the minimum-phase diagonal elements, $\cancel{\det(\mathbf{\Lambda}) = g_- g_+}$~~, producing~~ $\det(\mathbf{\Lambda}) = A_- A_+$, producing by definition a minimum-phase result~~by definition~~.

In a general case, defining minimum-phase matrices via their determinant has several consequences:

(1) Matrix multiplications and matrix inverses preserve the minimum-phase property. This can be seen by considering the determinants of arbitrary minimum-phase matrix functions $\mathbf{A}$ and $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}), \tag{19}$$

$$\det\left(\mathbf{A}^{-1}\right) = \left(\det(\mathbf{A})\right)^{-1}. \tag{20}$$

The determinants, $\det(\mathbf{A})$ and $\det(\mathbf{B})$, are minimum-phase scalar functions. Hence, the right-hand sides of Eqs. 19 and 20 show that the matrix product $\mathbf{AB}$ and the inverse matrix $\mathbf{A}^{-1}$ possess a minimum-phase property.

(2) The minimum-phase property is basis-independent,

$$\det(\mathbf{D}) = \det\left(\mathbf{QDQ}^{-1}\right), \tag{21}$$

where $\mathbf{Q}$ is an arbitrary invertible matrix of the same size as $\mathbf{D}$. Hence, minimum-phase is a physical property that is independent of the coordinate system or domain.

(3) Minimum-phase matrices are not fully consistent with the ~~aforementioned~~ qualitative conditions (i)-(iii) in Section 2.1. The invertibility criterion (iii) is satisfied because minimum-phase determinants are non-zero. However, it is less clear how to interpret causality and stability for a matrix (criteria (i) and (ii)). In particular, minimum-phase determinants do not guarantee causality of individual matrix elements. For example, suppose the matrix,

$$\mathbf{Q} = \begin{pmatrix} 1 - 2\alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1} & 1 \\ 1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1} & 1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1} \end{pmatrix}, \tag{22}$$

is used to apply a frequency-dependent basis transformation to the ~~above-mentioned~~ minimum-phase matrix, $\cancel{\mathbf{\Lambda} = \mathrm{diag}(g_-, g_+)}\mathbf{\Lambda} = \mathrm{diag}(A_-, A_+)$. The resulting matrix,

$$\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} = \begin{pmatrix} 2 - \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1} & -\frac{1-2\alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1}}{1+\alpha \mathrm{e}^{\mathrm{i}\omega\tau_1}} \\ 1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1} & \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_1} \end{pmatrix}, \tag{23}$$

is still minimum-phase but its matrix elements are not ~~(e.g. see acausal element ,~~ such as the acausal element $1 + \alpha \mathrm{e}^{\mathrm{i}\omega\tau_1}$~~)~~.

(4) Minimum-phase matrices do not necessarily posses minimum-phase eigenvalues. A minimum-phase determinant ~~defines~~ constrains the phase spectra of the eigenvalues up to a frequency-dependent freedom, $\cancel{\zeta = \zeta(\omega)}$, $\eta = \eta(\omega)$,

$$\mathrm{Arg}\left[\lambda_1\right] = -\mathcal{H}\left[\log\left(|\lambda_1|\right)\right] + \cancel{\zeta}\eta, \tag{24}$$

$$\mathrm{Arg}\left[\lambda_2\right] = -\mathcal{H}\left[\log\left(|\lambda_2|\right)\right] - \cancel{\zeta}\eta. \tag{25}$$

There are special cases where all eigenvalues observe a minimum-phase property (i.e. $\cancel{\zeta = 0}\eta = 0$), e.g. the aforementioned matrix $\mathbf{\Lambda}$, or transmission-like responses of 2D laterally-invariant acoustic media (see examples by Wapenaar et al. 2003; Elison et al. 2020). This work focuses on ~~matricial~~ more general minimum-phase ~~normal-product factorization, and hence, goes beyond these special cases where a scalar solution per eigenvalue often suffices~~matrices, where scalar solutions per eigenvalue no longer suffice.

3.2.1 ~~*Normal-product factorization: Degrees of freedom*~~

3.3 Minimum-phase reconstruction by normal-product factorization: Matrix case

~~Compared to the scalar case, the factorization of normal products has additional degrees of freedom. Upon multiplying the matrix, $\mathbf{D}$, by an arbitrary unitary 2×2 matrix , U(2),~~ In this section, we extend minimum-phase reconstruction from scalars to matrices. Firstly, we define normal products as generalized power spectra, and we demonstrate why unique minimum-phase matrix reconstruction is significantly more challenging than its scalar version. Secondly, we modify the minimum-phase matrix reconstruction method by Wilson (1972) considering the special case of the elastodynamic dereverberation operator $\mathbf{V}^{+}$. Thirdly, we discuss similarities of this reconstruction method to the Marchenko method. We will illustrate our analysis numerically in Section 4.

3.3.1 *Normal products: Generalized power spectra*

The normal product is defined as the product of a quantity, with its complex-conjugate transpose, e.g. Scalar normal products may be better known as auto-correlations in the time domain and are often interpreted physically as power spectra in the frequency domain because their phase vanishes $\mathrm{Arg}\left[|D|^2\right] = 0$. Following this physical interpretation, retrieving the scalar solution D from its normal product ~~is preserved,~~

$$\mathbf{D}\mathrm{U}(2)\left(\mathbf{D}\mathrm{U}(2)\right)^{\dagger} = \mathbf{D}\mathbf{D}^{\dagger},$$

$|D|^2$ is often described as a phase reconstruction, while mathematically, it is a factorization problem. In Section 2, we showed that this generally non-unique factorization can be constrained for minimum-phase scalar functions (see Eqs. 6 and 7). However, the matrix case is more complicated.

There are several differences between scalar power spectra and matrix normal products.

For example, consider,

$$\mathbf{D}\mathbf{D}^{\dagger} = \begin{pmatrix} D_{P,P} & D_{P,S} \\ D_{S,P} & D_{S,S} \end{pmatrix} \begin{pmatrix} D^{*}_{P,P} & D^{*}_{S,P} \\ D^{*}_{P,S} & D^{*}_{S,S} \end{pmatrix} = \begin{pmatrix} \delta & \epsilon^{*} \\ \epsilon & \zeta \end{pmatrix}, \tag{26}$$

~~because, $\mathrm{U}(2)\left[\mathrm{U}(2)\right]^{\dagger} = \mathbf{I}$. An arbitrary U(2) element can be represented as a product of a matrix~~ with $\delta = |D_{P,P}|^2 + |D_{P,S}|^2$, $\epsilon = D^{*}_{P,P}D_{S,P} + D^{*}_{P,S}D_{S,S}$, and $\zeta = |D_{S,P}|^2 + |D_{S,S}|^2$. The off-diagonal elements of the normal product are identical except for a sign-inverted phase that is not necessarily zero $\mathrm{Arg}\left[\epsilon\right] = -\mathrm{Arg}\left[\epsilon^{*}\right]$. Nonetheless, we keep the physical interpretation from the scalar case, i.e. "power spectra" and ~~a U(1) element (Cornwell 1997),~~ "phase reconstruction" refer to normal products and the retrieval of the solution $\mathbf{D}$ from its normal product, respectively. Since matrix multiplications do not commute, there are two normal products, which are generally not equal $\mathbf{D}\mathbf{D}^{\dagger} \neq \mathbf{D}^{\dagger}\mathbf{D}$. Counting matrix elements as equations, the two normal products provide individually up to three (see Eq. 26), and together up to six independent equations (for 2×2 matrices). Hence, if both normal products are known, there are more equations to constrain the reconstruction of the matrix $\mathbf{D}$. However, we assume only one normal product is available which describes a challenge of the elastodynamic augmented Marchenko method (details are not needed here but can be found in Reinick

Compared to the scalar case, the factorization of a (single) normal product has additional degrees The normal product of the matrix $\mathbf{D}$ is preserved upon multiplication by an arbitrary unitary 2×2 matrix $\mathbf{U}_2$,

$$\mathbf{D}\mathbf{U}_2\left(\mathbf{D}\mathbf{U}_2\right)^{\dagger} = \mathbf{D}\mathbf{D}^{\dagger}, \tag{27}$$

due to the unitary property $\mathbf{U}_2\left[\mathbf{U}_2\right]^{\dagger} = \mathbf{I}$ (here $\mathbf{I}$ denotes an identity matrix). The $\mathbf{U}_2$ element can be represented as follows (the term "element" is commonly used in the relevant literature, e.g. Corn

$$\cancel{\mathrm{U}(2)}\mathbf{U}_2 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{\gamma+\alpha}{2}}\cos\left[\frac{\beta}{2}\right] & -\mathrm{e}^{\mathrm{i}\frac{\gamma-\alpha}{2}}\sin\left[\frac{\beta}{2}\right] \\ \mathrm{e}^{-\mathrm{i}\frac{\gamma-\alpha}{2}}\sin\left[\frac{\beta}{2}\right] & \mathrm{e}^{\mathrm{i}\frac{\gamma+\alpha}{2}}\cos\left[\frac{\beta}{2}\right] \end{pmatrix} \mathrm{e}^{\mathrm{i}\Phi}, \tag{28}$$

where α, β and γ are Euler angles (Hamada 2015). The ~~U(1) freedom~~ freedom $\mathrm{e}^{\mathrm{i}\Phi}$ can be constrained via the minimum-phase property of the ~~desired solution~~ determinant $\det(\mathbf{D})$ (shown in chapter 5 of Reinicke 2020),

$$\Phi = -\frac{1}{4}\mathcal{H}\left[\log\left(\left|\det(\mathbf{D}\mathbf{D}^{\dagger})\right|\right)\right]. \tag{29}$$

~~Unfortunately, the above strategy is insufficient to fully constrain the normal-product factorization. The minimum-phase constraint of the determinant provides only one equation, namely the Kolmogorov relation.Hence, it is not surprising that only~~ Unfortunately, the minimum-phase determinant only constrains Φ, i.e. one out of four free parameters~~can be determined~~. Due to this limitation, we seek for an alternative method, which is discussed next.

3.3.2 ~~2D~~ Minimum-phase matrix reconstruction by factorization~~: Wilson algorithm~~

In the following, we review a minimum-phase matrix reconstruction method, introduce the elastodynamic dereverberation operator and eventually modify the reconstruction method for the dereverberation operator.

The ~~1D~~ scalar Wilson algorithm can be generalized to matrices. Wilson (1972) proposes a matrix extension of the recursive scalar algorithm which can be written as,

$$\mathbf{D}_{n+1} = \mathbf{D}_n \mathbf{\Theta}\left[\mathbf{I} + (\mathbf{D}_n)^{-1}\mathbf{D}\mathbf{D}^{\dagger}\left(\mathbf{D}_n^{\dagger}\right)^{-1}\right], \tag{30}$$

with $\mathbf{D}_0 = \mathbf{I}$. The function ~~,~~ $\mathbf{\Theta}$ ~~,~~ element-wise mutes acausal events and scales the ~~time-zero~~ time zero components of the diagonal elements by ~~,~~ $\frac{1}{2}$ ~~(the mute,~~. Although the dereverberation operator $\mathbf{V}^{+}$ has a minimum-phase determinant (shown in the next section), it is not reconstructed correctly by the algorithm in Eq. 30 with $\mathbf{D} = \mathbf{V}^{+}$. We will show that this limitation is due to the mute $\mathbf{\Theta}\left[\cdot\right]$ ~~, involves Fourier transforms as described in Section 2.3).~~ and can be overcome using a modified mute.

~~Next, we modify the mute of the 2D Wilson algorithm to reconstruct the dereverberation operator. Although~~ For better illustration, we briefly define the elastodynamic dereverberation operator. One can generalize the acoustic definition in Eqs. 14 and 15 to the elastic case by replacing scalar with matrix responses in the P-S space (Reinicke et al. 2020),

$$\mathbf{V}^{+} = \mathbf{T}^{\downarrow -1}\mathbf{T}^{\downarrow}_{dir} = \mathbf{I} + \mathbf{V}^{+}_{coda}. \tag{31}$$

The acoustic direct transmission $T^{\downarrow}_{dir}$ generalizes to a forward-scattered transmission $\mathbf{T}^{\downarrow}_{dir}$ that includes all non-reflected events such as transmitted mode-converted waves (Wapenaar 2014). Assuming that many readers are unfamiliar with the dereverberation operator, ~~$\mathbf{V}^{+}$, has a minimum-phase determinant, the~~ we explain its properties that are important for our analysis. Firstly, the dereverberation operator has a finite number of events limited by the number of layers. This follows from the finite number of events of the inverse and forward-scattered transmissions (Dukalski et al. 2022). Secondly, all events of the dereverberation operator arrive within a well-defined time window that only depends on the one-way travel times of P- and S-waves within each layer (Reinicke et al. 2020). Lastly, and most importantly, the onset of its matrix elements in the time domain is not always at ~~time-zero~~time zero. In particular, its off-diagonal elements typically have non-zero onset times that can be acausal ~~(Reinicke et al. 2020). Hence, we~~ (shown by Reinicke et al. 2020).

Given these properties, we modify the mute of the matrix Wilson algorithm to reconstruct the der

We propose modifying the operator ~~,~~ $\mathbf{\Theta}[\cdot]$ ~~,~~ to mute all events in the time domain prior to the onset of the dereverberation operator per ~~elastic~~ matrix component. This differs from the original ~~2D~~ matrix Wilson algorithm which instead removes acausal events for all matrix elements. Using the modified mute ~~,~~ $\mathbf{\Theta}[\cdot]$ ~~,~~ in Eq. 30, it appears that the ~~2D~~ matrix Wilson algorithm can accurately factorize the normal product of the dereverberation operator (results will be shown in Section 4).

~~However, we also observe special cases where the modified 2D Wilson algorithm fails to recover the~~ ~~This coincides with the appearance of the so-called fast-multiples, which are multiples that have shorter travel times than some of the converted but non-reflected arrivals (Reinicke et al. 2020). In the presence of these fast-multiples, the diagonal elements of the dereverberation operator, V^{+}_{PP} and V^{+}_{SS}, contain acausal events that precede the spikes at time-zero associated with the identity (see Eq. 31). As a result, it is no longer clear how to define the diagonal elements of~~

~~the mute, $\boldsymbol{\Theta}[\cdot]$, which also need to scale the time-zero element by, $\frac{1}{2}$, to ensure convergence (see Eq. 12). So far, we have not been able to handle these acausal diagonal elements with another modified mute. Hence, it remains undetermined whether minimum-phase normal-product factorization of the dereverberation operator is limited to cases without fast-multiples, or whether a more general algorithm remains to be discovered. We are actively conducting further research on this topic, which may lead to follow-up publications.~~

4 NUMERICAL EXAMPLE

In this section, we show two examples of the ~~2D~~ matrix Wilson method and ~~illustrate the analysis of Section 3.2~~ analyze determinants and eigenvalues numerically. These examples are associated with the models *Elastic #1* and *Elastic #2*, which are identical except for the S-wave velocity in the second layer from the top (see Figure 2). They are designed such that ~~they generate dereverberation operators without~~ the Wilson method succeeds (*Elastic #1*) and ~~with~~ fails (*Elastic #2*) ~~fast-multiples~~to reconstruct the respective dereverberation operator correctly. In both cases, we model the dereverberation operator analytically (Dukalski et al. 2022) to calculate the normal product, and to provide a reference for the retrieved solution. For the ~~2D~~ matrix Wilson method, we define the diagonal elements of the mute ($\Theta_{PP}[\cdot]$ and $\Theta_{SS}[\cdot]$) ~~are defined~~ via the Heaviside function in Eq. 11. The off-diagonal elements ~~,~~ $\Theta_{PS}[\cdot]$ ~~, and ,~~ and $\Theta_{SP}[\cdot]$ ~~,~~ mute all events in the time domain prior to the onset of the components ~~,~~ V^+_{PS} ~~, and ,~~ and V^+_{SP}, respectively.

~~Firstly, we consider the case *Elastic #1*, which is free of fast-multiples.~~ Firstly, we consider the successful case *Elastic #1*. We use the normal product ~~,~~ $\mathbf{V}^+\mathbf{V}^{+\dagger}$ ~~,~~ shown in Figures 5a-d to evaluate eight iterations of the ~~2D~~ matrix Wilson algorithm, resulting in the solution ~~,~~ $\mathbf{V}^+_{n=8}$ ~~,~~ in Figures 5e-h. The algorithm monotonically converges to the true ~~dereverberation operator~~ solution $\mathbf{V}^+$ up to numerical noise (see Figure 4), hence, we do not show the difference plot. Figures 5e-h illustrate that the dereverberation operator has a finite number of events in the time domain that arrive within a well-defined time window as discussed in ~~Section 3.2~~Section 3.3.2. Here, the ~~dereverberation operator is~~ responses are zero outside the displayed time window, i.e. all events are shown. Figures 5e-h also show the identity term of the dereverberation operator (see Eq. 31). Moreover, the onset of the off-diagonal elements in the time domain deviates from ~~time-zero~~ time zero and is even acausal for the SP element (see Figure 5g).

~~Secondly, we modified the model till fast-multiples emerged (*Elastic #2* case), at which point the~~ Secondly, we modify the model until the proposed method for normal-product factorization becomes inaccurate (case *Elastic #2*). Compared to the previous example, the travel-time difference between P- and S-waves increased, leading to ~~fast-multiples, and hence,~~ acausal events in the diagonal elements ~~,~~ V^+_{PP} ~~, and ,~~ and V^+_{SS}. ~~We repeat the previous experiment using the normal product of the dereverberation operator shown in Figures 6a-d. Moreover, we~~ As a result, it is no longer clear how to define the diagonal elements of the mute $\Theta_{PP}[\cdot]$ and $\Theta_{SS}[\cdot]$, which also need to scale the time zero element by $\frac{1}{2}$ to ensure convergence (see Eq. 12). Here, we only adjust the off-diagonal elements of the mute, $\Theta_{PS}[\cdot]$ ~~, and ,~~ and $\Theta_{SP}[\cdot]$, to account for the changed onset of the dereverberation operator in the time domain. Then we repeat the previous experiment using the normal product of the dereverberation operator shown in Figures 6a-d. Figures 6e-h show the retrieved dereverberation operator after evaluating eight iterations of Eq. 30 ~~,~~ $\mathbf{V}^+_{n=8}$, and the difference with respect to the modeled reference ~~,~~ $\mathbf{V}^+$. The convergence (see Figure 4) indicates that the relative error of the retrieved solution is in the order of 10 %.

Lastly, we analyze the determinants and eigenvalues of the dereverberation ~~operator~~operators. We ~~verified for both, the *Elastic #1* and the *Elastic #2*, cases that the determinant~~ verify that the determinants of the modeled dereverberation ~~operator satisfies~~ operators $\det(\mathbf{V}^+)$ satisfy the Kolmogorov relation up to numerical noise (relative error in the order of ~~,~~ 1×10^{-14}) ~~. Figure 7 shows~~ for both cases, *Elastic #1* and *Elastic #2*. Next, we inspect the determinants of the retrieved dereverberation operators ~~. In the *Elastic #1* case, the retrieved solution, $\mathbf{V}^+_{n=8}$, has a~~ after eight iterations $\det(\mathbf{V}^+_{n=8})$ (see Figure 7). We observe that it satisfies the minimum-phase ~~determinant. However, in the presence of fast-multiples (~~conditions in the case *Elastic #1* but it violates them in the case *Elastic #2*~~), the determinant~~

~~of the retrieved solution, det $(\mathbf{V}^{+}_{n=8})$, contains acausal events~~. This violation can be easily verified by the acausal events of the determinant (see close-up in Figure 7c)~~, and therefore, does not satisfy the Kolmogorov relation and is not minimum-phase. Even after enforcing the minimum-phase property~~. The phase error of the determinant can be corrected using Eq. 7. However, the retrieved ~~solution differs from the true one~~response $\mathbf{V}^{+}_{n=8}$ carries an additional error represented by the Euler angles (see Eq. 28) that cannot be removed. The eigenvalues of the dereverberation operators do not satisfy the Kolmogorov relation for any of the tested cases. Even in the ~~absence of fast-multiples~~ successful case (*Elastic #1*), the phase spectra of the eigenvalues differ severely from their minimum-phase spectra defined via the Kolmogorov relation in Eq. 7. This can be illustrated via the phase-freedom ~~, ζ,~~ η defined in Eqs. 24 and 25, which is far from trivial (see Figure 8).

5 DISCUSSION

Our analysis has shown that the causality condition of minimum-phase functions can be less intuitive The minimum-phase property does not necessarily hold for individual matrix elements but it does for the determinant. Hence, minimum-phase matrix functions can contain acausal matrix elements. Our numerical examples indicate that the matrix Wilson algorithm can accurately handle acausal off-diagonal elements, while acausal diagonal elements appear to be an obstacle. This limitation is not obvious from the algorithm in Eq. 30. In the presented examples, the temporal mute suppresses acausal events on the diagonal, but not on the off-diagonal, elements. Hence, the subsequent matrix multiplication by $\mathbf{D}_n$ could still introduce acausal events on the diagonals (see Eq. 30). It remains undetermined whether normal-product factorization of minimum-phase matrices is limited to cases with strictly causal diagonal elements, or, whether a more general algorithm remains to be discovered.

Our interest in minimum-phase matrices is motivated by the Marchenko method. The latter formulates internal multiple elimination for seismic reflection data as an inverse problem~~that~~. It aims to retrieve the dereverberation operator and it is often underconstrained in practice. Existing work demonstrates for the scalar case how two additional constraints can be used to accurately reconstruct the ~~desired solution, i.e. the~~ dereverberation operator. Firstly, the normal product ~~(=amplitude spectrum)~~ of the dereverberation operator is retrieved via energy conservation. Secondly, the dereverberation operator is reconstructed from its normal product by exploiting its minimum-phase property ~~(Dukalski et al. 2019; Elison et al. 2020; Peng et al. 2021)~~(Dukalski et al. 2019; Elison et al. 2020; Pen

In previous work, we tried to generalize this strategy to the elastic case where the dereverberation operator is no longer a scalar but a 2×2 matrix, and identified two challenges (Reinicke et al. 2020):

(1) Once the normal product of the elastodynamic dereverberation operator is retrieved, it remains unclear how to reconstruct the operator uniquely from its normal product using its minimum-phase property.

(2) Energy conservation provides the normal product of the inverse transmission. The dereverberation operator ~~,~~ $\mathbf{V}^{+}$ ~~,~~ is minimum-phase but the inverse transmission ~~,~~ $\mathbf{T}^{\downarrow -1}$ (also known as $\mathbf{F}^{+}$) ~~,~~ is not. This is not an issue for scalars, because the scalar normal-products of the inverse transmission and the dereverberation operator are identical up to a frequency-independent constant. This holds because the acoustic direct transmission is a single pulse, $T^{\downarrow}_{dir} = \alpha \mathrm{e}^{-\mathrm{i}\omega\tau_{dir}}$, with travel time ~~,~~ τ_{dir},

$$V^{+}V^{+*} = T^{\downarrow -1}T^{\downarrow}_{dir}T^{\downarrow -1*}T^{\downarrow *}_{dir} = T^{\downarrow -1}T^{\downarrow -1*}|\alpha|^{2}. \tag{32}$$

However, this relation is more complicated for the elastic case ~~because~~ where the direct transmission generalizes to a forward-scattered transmission including mode conversions ~~,~~ $\mathbf{T}^{\downarrow}_{dir}$. Moreover, Eq. 32 cannot be extended from the scalar to the matrix case because matrix multiplications do not commute.

In this paper, we focused on the first challenge. Addressing the second one is beyond the scope of this work.

We notice similarities between the Marchenko method and the here-discussed ~~2D~~ matrix Wilson method. Both methods use the same ingredients including temporal convolutions and correlations as well as temporal mutes. The modified mute of the ~~2D Wilson method ,~~ matrix

Wilson method $\mathbf{\Theta}[\cdot]$ ~~,~~ is inspired by, and is nearly identical to, one of the two mutes of the Marchenko method ~~,~~ $\mathbf{P}_B[\cdot]$ (see Eq. 16 in Reinicke et al. 2020). The two mutes only differ at ~~time-zero~~ time zero of the diagonal elements, where the Wilson mute scales its argument by ~~,~~ $\frac{1}{2}$ ~~, instead of ,~~ instead of 1 ~~,~~ to ensure convergence. Moreover, both methods face limitations related to the mutes. ~~Fast-multiples~~ It has been shown that the Marchenko method fails to reconstruct the desired solution in the presence of fast-multiples. The latter are multiples that have shorter travel times than some of the converted but non-reflected arrivals. As a result, fast-multiples introduce temporal overlaps between signals that the ~~mute operators~~ Marchenko method ought to separate ~~. As a result, the Wilson and~~ with the mute. These temporal overlaps are due to acausal events in the diagonal elements of the dereverberation operator V^+_{PP} and V^+_{SS}. This limitation of the Marchenko method ~~fail to reconstruct the desired solutionin the presence of fast-multiples~~coincides with the cases where the matrix Wilson method fails to retrieve the correct solution. The question is whether ~~this is~~ fast multiples pose a fundamental limitation, or whether there is another, more robust solution strategy for the Marchenko and matrix Wilson methods. Despite the remaining challenges, the ~~2D~~ matrix Wilson algorithm could potentially help to retrieve a better estimate of the desired dereverberation operator. For example, ~~Peng et al. (2021)~~ Peng et al. (2022) show that the 2D acoustic augmented Marchenko method can reconstruct the correct dereverberation operator, even though they apply a scalar, instead of a ~~matricial,~~ matrix minimum-phase reconstruction. They propose a recursive application of the 2D Marchenko method and a scalar minimum-phase correction. Similarly, one could attempt to recursively apply the elastodynamic Marchenko method and the ~~2D Wilson algorithm.~~ matrix Wilson algorithm ignoring the challenge of fast multiples.

~~Our analysis has shown that the causality condition of minimum-phase functions can be less intuit~~ ~~The minimum-phase property holds not necessarily for individual matrix elements but for the determinant. Hence, minimum-phase matrix functions can contain acausal matrix elements. Our numerical examples indicate that the 2D Wilson algorithm can accurately handle acausal off-diagonal elements, while acausal diagonal elements appear to be an obstacle.~~

~~This limitation is not obvious from the algorithm in Eq. 30. In the presented examples, the temporal mute,~~ $\cancel{\mathbf{\Theta}[\cdot]}$~~, suppresses acausal events on the diagonal, but not on the off-diagonal, elements. Hence, the subsequent matrix multiplication by,~~ $\cancel{\mathbf{D}_n}$~~, could still introduce acausal events on the diagonals.~~

Minimum-phase matrices and normal-product factorization provide physical relationships that remain mostly unexplored, especially in geophysics. For example, the results of this work could bring new momentum to the research on reconstructing transmission from reflection data in the multi-dimensional acoustic or elastic case (i.e. beyond the work of Wapenaar et al. 2003). Moreover, we illustrated that normal-product factorization has four (real-valued) unknown parameters (for 2×2 matrices) but the determinant provides a single ~~, complex-valued equation~~phase. Despite the mismatch in number of unknowns and equations, we demonstrated that the modified ~~2D~~ matrix Wilson algorithm can reconstruct a special class of minimum-phase matrices. This raises the question whether ~~minimum-phase matrices have~~ there are additional, so-far unexplored fundamental properties ~~other than a~~ of minimum-phase ~~determinant~~matrices. If so, the follow up question is whether these properties allow for a unique factorization of normal products in more general cases, e.g. including fast-multiples. Answering these questions is beyond the scope of this paper but it is a matter of ongoing research. Last but not least, we investigated the simplest non-trivial matrix case, i.e. 2×2 matrices, but generalizations are not excluded. It would be particularly interesting to analyze multi-dimensional acoustic cases which will be subject of future work.

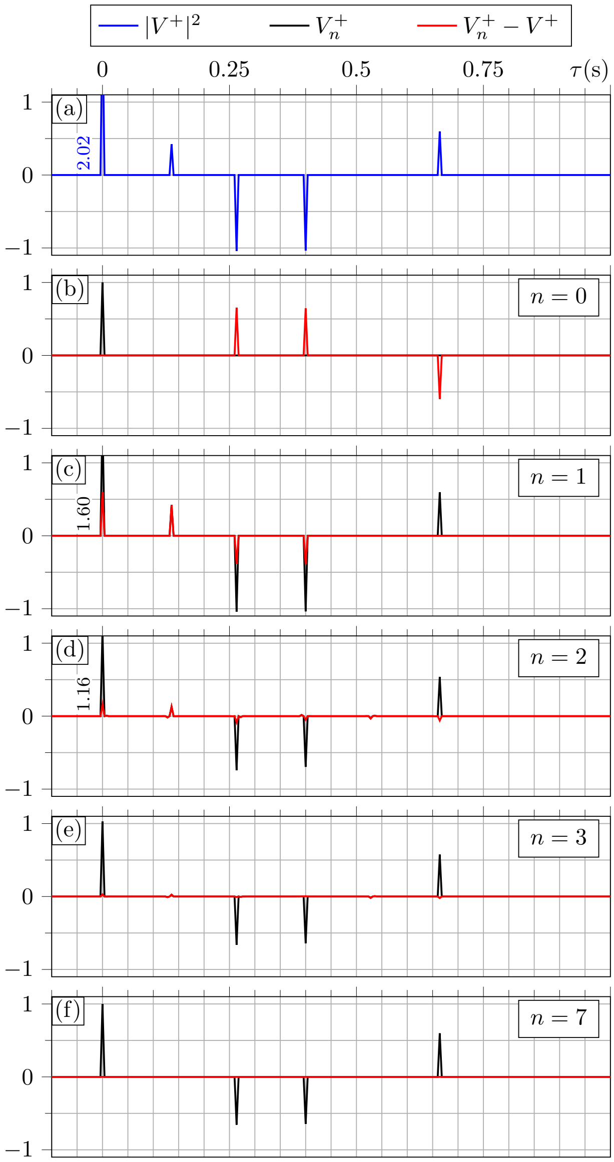

Figure 3. All responses are shown in the time domain. (a) ~~Normal product~~ Autocorrelation of the dereverberation operator associated with the acoustic model shown in Figure 2. Negative times are not shown because (scalar~~normal products~~) autocorrelations are symmetric in time, $(|V^+|^2)^* = |V^+|^2$. Panels (b)-(f) show the dereverberation operator as it is recursively reconstructed via the Wilson algorithm in Eq. 10 (V_n^+ in black) and its error ($V_n^+ - V^+$ in red). The initial estimate ($n = 0$) is an identity, i.e. a single spike at ~~time-zero~~time zero. After seven iterations the true solution is retrieved up to numerical noise (see Figure 4). For better illustration, strong events are clipped and their amplitudes are indicated with labels.

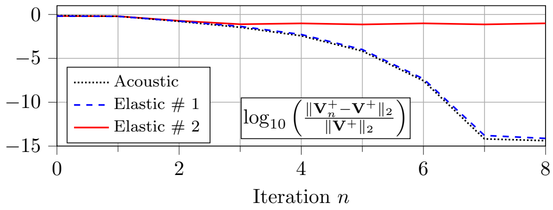

Figure 4. Convergence of the ~~1D~~ scalar and ~~2D~~ matrix Wilson algorithms in Eqs. 10 and 30 associated with the dereverberation operators (V^+ and $\mathbf{V}^+$) of the acoustic and elastic models in Figure 2, respectively. The convergence is defined as the relative error with respect to the true solution as indicated by the legend. For the *acoustic* and the *Elastic #1* case, the Wilson algorithm converges up to numerical noise within seven iterations. For the *Elastic #2* case, the relative error converges to approximately 10 %.

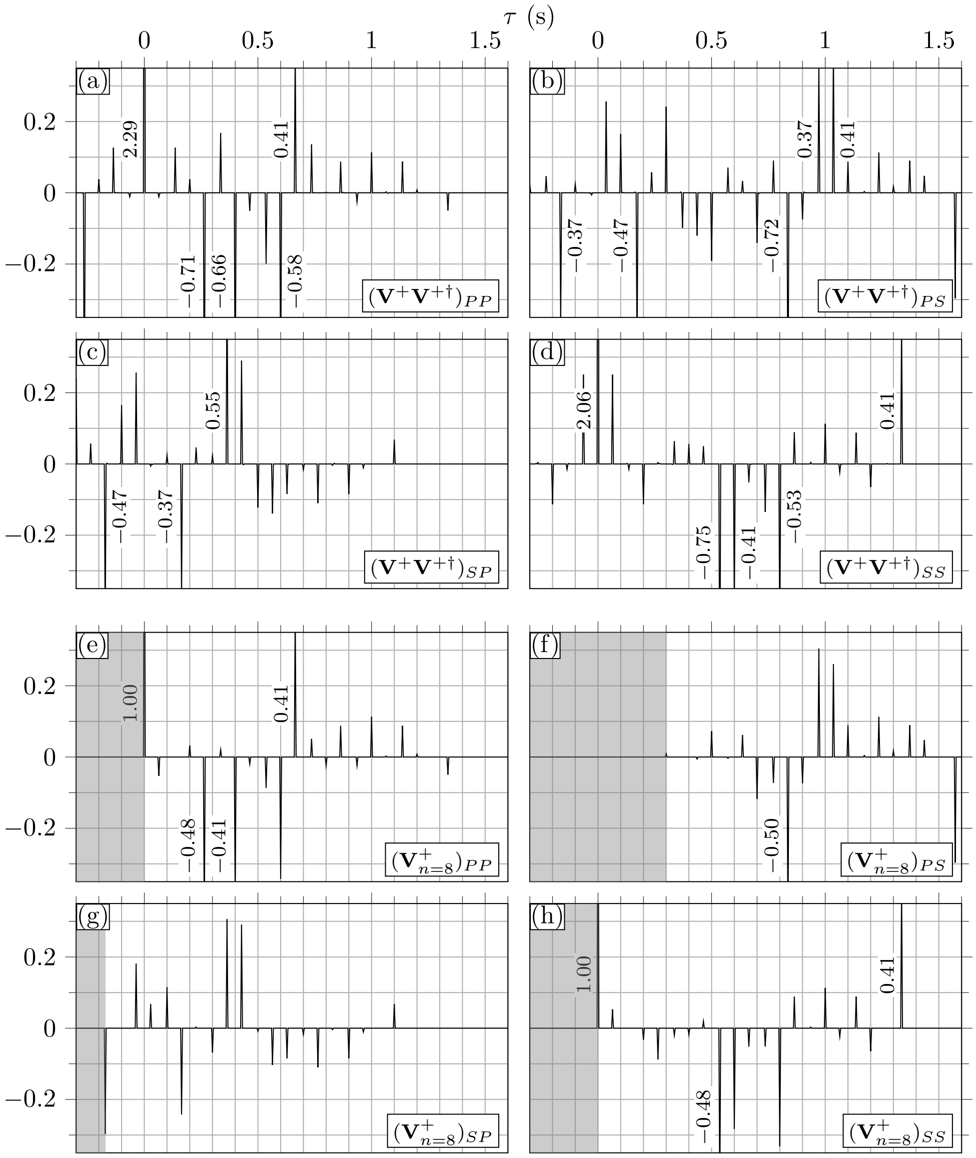

Figure 5. (a)-(d) Normal product ~~,~~ $\mathbf{V}^+\mathbf{V}^{+\dagger}$ ~~,~~ of the dereverberation operator associated with the model *Elastic #1* (see Figure 2). The panels show the four elastic components analogously to the 2×2 matrix in Eq. 17. (e)-(h) Retrieved dereverberation operator after eight iterations. The grey areas indicate the time samples that are muted by the modified operator ~~,~~ $\boldsymbol{\Theta}\,[\cdot]$ ~~,~~ in Eq. 30. We do not show a difference or reference plot because the retrieved and modeled dereverberation operators are identical up to numerical noise (see convergence in Figure 4). All panels show responses in the time domain to facilitate the interpretation.

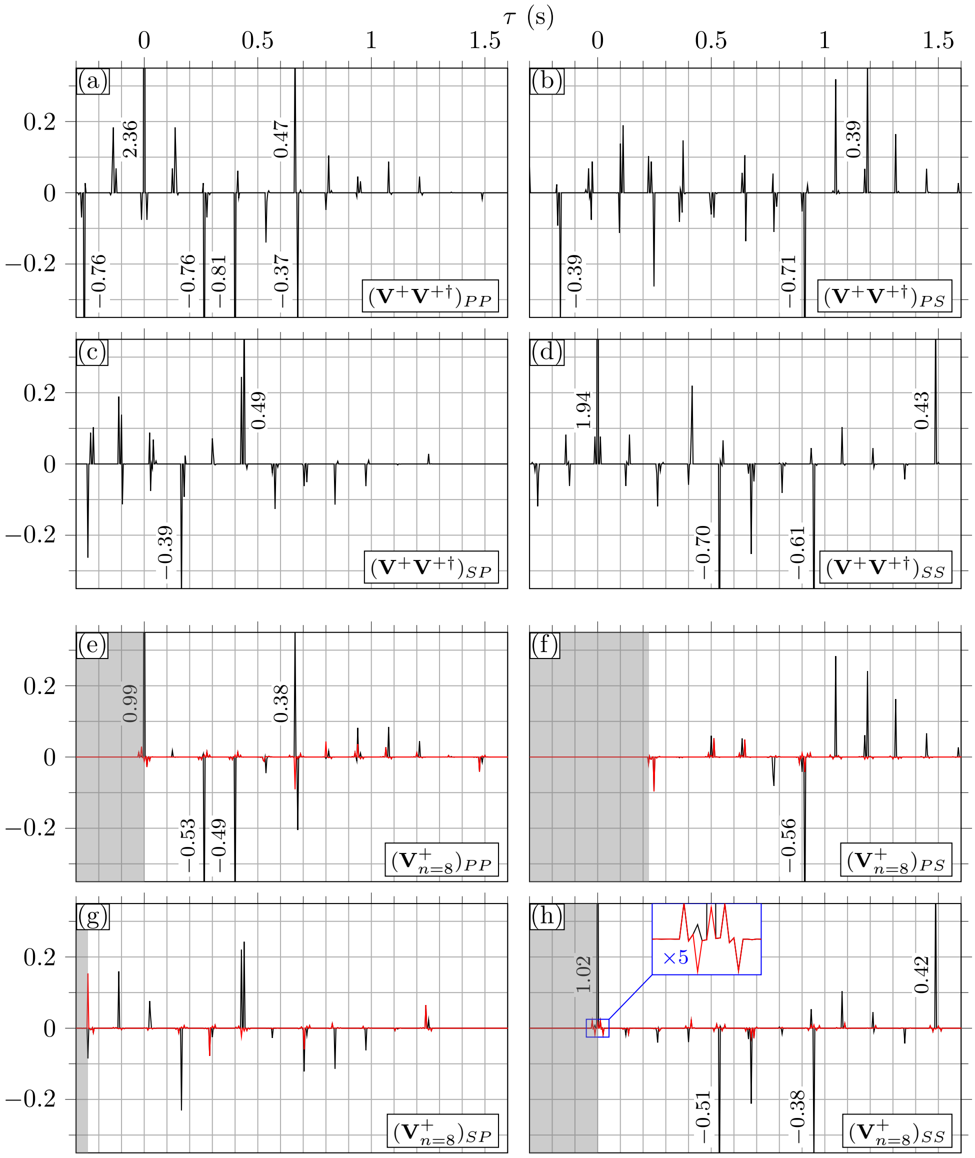

Figure 6. Idem as Figure 5 but associated with the model *Elastic #2* (see Figure 2). In this case, the dereverberation operator has acausal events on the diagonals (PP and SS components)~~due to so-called fast-multiples (these are multiples that share a temporal overlap with the forward-scattered events, for details see Reinicke e~~ The acausal events on the diagonals appear to be an issue for the Wilson algorithm. The dereverberation operator is reconstructed only up to a relative error in the order of 10 % (see Figure 4), instead of numerical noise as in the previous example in Figure 5.

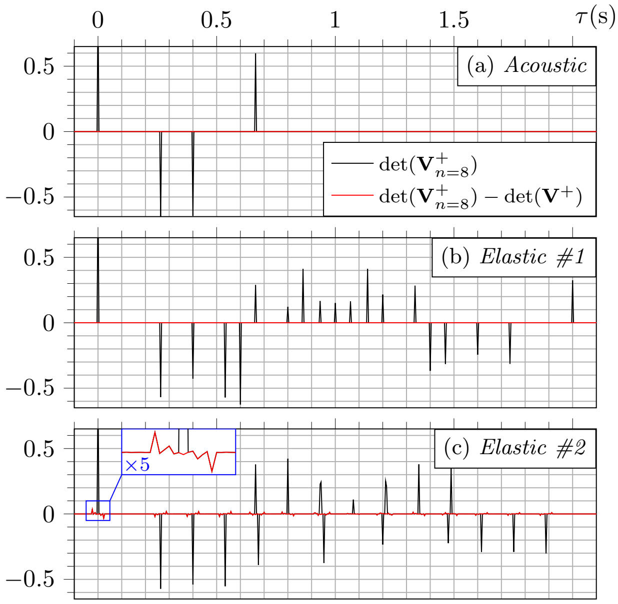

Figure 7. Determinants of the retrieved dereverberation operators (black) and the difference with respect to the modeled solutions (red). The panels are associated with the (a) *Acoustic*, (b) *Elastic #1*, and (c) *Elastic #2*, cases shown in Figures 3, 5 and 6, respectively. In the *Acoustic* case, the dereverberation operator is a scalar function, and hence, identical to its determinant. Nonetheless, it is shown for completeness. For the *Elastic #2* case, the determinant of the retrieved dereverberation operator is not minimum-phase, which can be easily seen via the acausal events shown in the magnified box in blue. The difference plot indicates that acausal events are absent in the determinant of the true solution, which possesses a minimum-phase property.

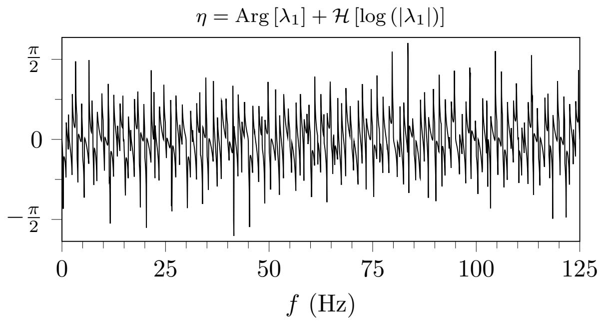

Figure 8. Phase-freedom $\cancel{, \zeta,}$ η of the eigenvalues of the dereverberation operator shown in Figures 5e-h, which is associated with the model *Elastic #1* (also see Eq. 24). The horizontal axis denotes the temporal frequency $\cancel{,}$ $f = \frac{\omega}{2\pi}$.

6 CONCLUSION

Minimum-phase properties become significantly more complicated when stepping from scalar to matrix functions. Since the minimum-phase ~~matrices are defined via the determinant, they can contain non-minimum-phase~~ property of a matrix only imposes conditions on its determinant, there are no constraints on individual matrix elements, e.g. ~~acausal ones as we demonstrated for the~~ they can be acausal.

Our analysis has been motivated by challenges of the Marchenko method. Hence, we focused on the minimum-phase properties of the elastodynamic dereverberation operator~~of the 1.5D~~, which is a solution of the Marchenko method. We showed that this 2×2 minimum-phase ~~operator~~ matrix function can be uniquely reconstructed from its normal product using a modified version of the ~~2D~~ matrix Wilson algorithm. Compared to the original Wilson method, we modified the temporal mute that curiously is identical to one of the two mute operators of the Marchenko method, except for the time zero element.

However, the proposed solution ~~is limited to cases without so-called~~ appears to be limited to dereverberation operators with causal diagonal elements. Thus, the method excludes cases with fast-multiples that can occur in the presence of large P- and S-wave velocity differences. Moreover, the dereverberation operator can be seen as a special class of minimum-phase matrices, i.e. the proposed factorization method does not necessarily generalize for other minimum-phase matrices. ~~Our results are consistent with existing scalar solutions for minimum-phase reconstruction. Compared to the original Wilson method, we modified the mute operator that curiously is identical to one of the two mute operators of the Marchenko method, except for the time-zero element.~~

The presented results suggest that ~~matricial~~ the minimum-phase ~~properties~~ property of matrices could play an important role in physics-driven data processing. ~~In particular, the matricial minimum-phase normal-product factorization could be key in overcoming theoretical challenges of the elastodynamic Marchenko method, which remains an underconstrained problem.~~ This work scratches the surface of minimum-phase matrices in the context of geophysics and indicates interesting directions for future research.

7 ~~DATA AVAILABILITY~~

DATA AVAILABILITY

The data underlying this article cannot be shared publicly due to company regulations. The data will be shared on reasonable request to the corresponding author.

ACKNOWLEDGMENT

The research of Kees Wapenaar has received funding from the European Research Council (grant no. 742703).

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APPENDIX A: NOTATION

We use the following Fourier transforms (per ray-parameter) where the real-part is denoted by $\Re$,

$$\mathbf{q}(p_x, z, \omega) = \int_{-\infty}^{\infty} \mathbf{q}(p_x, z, \tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau, \tag{A.1}$$

$$\mathbf{q}(p_x, z, \tau) = \frac{1}{\pi}\Re\left[\int_{0}^{\infty} \mathbf{q}(p_x, z, \omega)\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\omega\right]. \tag{A.2}$$

In this work, all equations are formulated for plane waves, i.e. per the ray-parameter p_x. We define the transformation from the offset-time domain $\mathbf{q}(x, z, t)$ to the ray-parameter

Table A1. Definition of additional operators used in this paper. All operators are applied per ray-parameter, p_x, and per frequency, ω, except for the Hilbert transform and the L_2 norm which take into account all frequencies. When applied to matrices, the operators act in the P-S space, except for the operations marked with "$\odot$" which act per matrix element. The L_2 norm is calculated using all frequencies and all wavefield components, i.e. a single and four components for acoustic and elastodynamic waves, respectively.

Symbol		**Operation**
Superscript "$*$"		Complex-conjugate
Superscript "$\dagger$"		Complex-conjugate transpose
Superscript "-1"		Inverse
$\log(\cdot)$		Natural logarithm
$\det(\cdot)$		Determinant
$\lVert\cdot\rVert_2$		L_2 norm
$\lvert\cdot\rvert$	$\odot$	Absolute value
$e^{[\cdot]}/\cos[\cdot]/\sin[\cdot]$	$\odot$	Exponential/cosine/sine function
$\mathcal{H}[\cdot]$	$\odot$	Hilbert transform
$\mathrm{Arg}[\cdot]$	$\odot$	Phase spectrum

intercept-time domain $\mathbf{q}(p_x, z, \tau)$ as,

$$\mathbf{q}(p_x, z, \tau) = \int_{-\infty}^{\infty} \mathbf{q}(x, z, \tau + p_x x)\mathrm{d}x. \tag{A.3}$$