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DOI
10.1007/s00605-022-01771-5

Publication date
2022

Document Version
Final published version

Published in
Monatshefte fur Mathematik

Citation (APA)
Marynets, V., Marynets, K., & Kohutych, O. (2022). On a novel approach for the investigation and approximation of solutions to the systems of higher order nonlinear PDEs. *Monatshefte fur Mathematik*, *200 (2023)*(4), 835-848. https://doi.org/10.1007/s00605-022-01771-5

Monatshefte für Mathematik (2023) 200:835–848
https://doi.org/10.1007/s00605-022-01771-5

On a novel approach for the investigation and approximation of solutions to the systems of higher order nonlinear PDEs

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Received: 7 March 2022 / Accepted: 3 September 2022 / Published online: 16 September 2022

Abstract
We study a boundary value problem for a system of the third order semi-linear partial differential equations with nonlocal boundary conditions. We establish sufficient conditions of existence, uniqueness, regularity and sign-preserving property of solutions of the studied problem and construct an iterative method for its approximation.

Keywords Vector-functions · Functional matrices · Non-local boundary conditions · Comparison functions · Integro-differential equations · Differential inequalities

Mathematics Subject Classification 35G30 · 35C15 · 35B05

1 Introduction

Mathematical modeling of the processes of water filtration through the double-layered porous media [1], heat distribution in the heterogeneous environment [2], dampness distribution in the soil [11] lead to a scalar linear differential equation (DE) of the

Communicated by Adrian Constantin.

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form:

$$
\begin{aligned}
&m(t,x)D^{(1.2)}u(t,x)+\alpha(t,x)D^{(1.1)}u(t,x)+d(t,x)D^{(1.0)}u(t,x)\\
&+\ \ \eta(t,x)D^{(0.2)}u(t,x)+a(t,x)D^{(0.1)}u(t,x)+b(t,x)u(t,x)=g(t,x), \quad (1)
\end{aligned}
$$

where $D^{(i,j)}u(t,x)=\frac{\partial^{i+j}u(t,x)}{\partial^i t\partial^j x}$ – denotes a mixed partial derivative of the function $u(t,x)$ of the order i with resprect to t and of the order j with respect to x, $m(t,x)$, $\alpha(t,x)$, $d(t,x)$, $\eta(t,x)$, $a(t,x)$, $b(t,x)$ and $g(t,x)$ are given continuous functions in the domain of consideration.

Questions of existence and uniqueness of solutions to the mixed problems for the DE (1) under different local and nonlocal boundary conditions are studied in [5, 13, 14]. In [8, 9] the authors investigate and construct approximate solutions to the boundary value problems (BVPs) in the case of systems of the third order semi-linear DEs under local and nonlocal boundary constraints. Authors also obtain sufficient conditions of existence and uniqueness of solutions to the studied BVPs, their sign-preserving property and prove theorems about the differential inequalities.

The current paper is an extention of the results obtained in [8–10]. In particular, we study a BVP for a system of the third order semilinear partial differential equations (PDEs) coupled with the nonlocal boundary condition of the Nakhushev type. We construct a modification of the two-sided method to approximate a solution of the studied problem. In addition, we essentially improve the sufficient existence and uniqueness conditions for the solution, obtained earlier in [8, 9].

2 Problem setting and auxiliary statements

Let us study the following problem: in the space of functions $C^*(\overline{D}) := C^{(1.2)}(D)\cap C^{(1.1)}(\overline{D})$, with $D=\{(t,x): t\in(0,b), x\in(0,a)\}$ find a solution to the BVP

$$
\mathcal{L}_3 U(t,x)=f\left(t,x,U(t,x),D^{(0.1)}U(t,x)\right):=f\left[U(t,x)\right], \quad (2)
$$

where $\mathcal{L}_3$ is a differential operator defined by the differential expression

$$
l_3U(t,x):=D^{(1.2)}U(t,x)+A_1(t,x)D^{(0.2)}U(t,x)+A_2(t,x)D^{(1.1)}(t,x),
$$

$U(t,x):=(u_i(t,x))$, $f\left[U(t,x)\right]:=(f_i\left[U(t,x)\right])$, $i=\overline{1,n}$ are vector-functions, $A_2(t,x):=\left(\delta_{ij}a_{ij}^{(r)}(t,x)\right)$, $r=1,2$, $j=\overline{1,n}$, are given matrices, δ_{ij} is the Kronecker symbol, and the boundary conditions

$$
\begin{aligned}
&U(0,x)=T(x),\ \ x\in[0,a],\\
&D^{(0.1)}U(t,a)=\Psi(t),\ \ t\in[0,b], \qquad (3)\\
&\int_{x_0}^{a} D^{(1.0)}U(t,x)dx=\Omega(t),\ \ t\in[0,b],\ \ 0\le x_0\le x\le a,
\end{aligned}
$$

and $T(x) := (\tau_i(x))$, $\Psi(t) := (\psi_i(t))$, $\Omega(t) := (\omega_i(t))$ are given vector-functions, $D^k U : D \to D_k \subset \mathbb{R}^n$, $k = (k_1, k_2)$, with D_k being some bounded domains, $f : \overline{B} \to \mathbb{R}^n$, $B = D \times \prod_{k_1,k_2} D_{(k_1,k_2)} \subset \mathbb{R}^{2(n+1)}$, $k_1 = 0, 1$, $k_2 = 0, 1, 2$.

From now on we assume that $T(x) \in C^2[0, a]$, $\Psi(t) \in C^1[0, b]$, $A_2(t, x) \in C(D)$, $A_1(t, x) \in C^{(0.1)}(D)$, $\Omega(t) \in C[0, b]$, the right hand-side of the DE (2) $f\,[U(t, x)] \in C(\overline{B})$ and the condition

$$T'(a) = \Psi(0) \tag{4}$$

holds.

Lemma 1 *If $f\,[U(t, x)] \in C(\overline{B})$, $T(x) \in C^2[0, a]$, $\Psi(t) \in C^1[0, b]$, $A_2(t, x) \in C(D)$, $A_1(t, x) \in C^{(0.1)}(D)$, $\Omega(t) \in C[0, b]$, then the BVP (2) and the system of integro-differential equations*

$$U(t, x) = S(t, x) + \int_0^t \left\{ LF[U(\eta, \zeta)] - \frac{1}{a - x_0} \int_{x_0}^a LF[U(\eta, \zeta)]dx \right\} d\eta \tag{5}$$

are equivalent.

Here

$$\begin{aligned}
&S(t, x) := \frac{1}{a - x_0} \left\{ \int_0^t \Omega(\eta)d\eta + \int_{x_0}^a [T(x) - \Phi(t, x)]\, dx \right\} + \Phi(t, x),\\
&\Phi(t, x) := (\phi_i(t, x)),\\
&\phi_i(t, x) := \int_a^x \tau_i'(\xi) exp \left(\int_t^0 a_{ii}^{(1)}(\eta, \xi)d\eta \right) d\xi +\\
&\int_a^x \int_0^t \left[a_{ii}^{(1)}(\eta, a)\psi_i(\eta) + \psi_i'(\eta) \right] k_{ii}(\xi, t; a, \eta)d\eta\, d\xi,\\
&k_{ii}(x, t; \xi, \eta) := exp \left(\int_x^\xi a_{ii}^{(2)}(\eta, \tau)d\tau + \int_t^\eta a_{ii}^{(1)}(\tau, x)d\tau \right),\\
&F\,[U(t, x)] := \left(f_i[U(t, x)] + [a_{ii}^{(1)}(t, x)a_{ii}^{(2)}(t, x) + D^{(0.1)}a_{ii}^{(1)}(t, x)]D^{(0.1)}u_i(t, x) \right),\\
&LF\,[U(\eta, \zeta)] := \int_x^a \int_\xi^a K(\xi, t; \zeta, \eta)F[U(\eta, \zeta)]d\zeta d\xi,
\end{aligned}$$

and

$$K(\xi, t; \zeta, \eta) := \left(\delta_{ij} k_{ij}(\xi, t; \zeta, \eta)\right) \tag{6}$$

is a matrix.

Obviously, $S(t, x) \in C^{(2.1)}(D) \cap C^{(1.1)}(\overline{D})$ and it satisfies all of the boundary conditions (3). Moreover, using the Ansatz $Z(t, x) := U(t, x) - S(t, x)$ in the BVP

(2) we obtain a problem with already homogeneous boundary conditions (3). Hence, without loss of generality we let $T(x) = \Psi(t) = \Omega(t) = 0$, or in other words that $S(t,x) = 0$.

Definition 1 We say, that a vector-function $F[U(t,x)] \in C_3(\overline{B})$, if it satisfies the following conditions:

1. $F[U(t,x)] \in C(\overline{B})$;
2. in the space of vector-functions $C^{(0.1)}(\overline{B}_1)$, $\overline{B}_1 \in \mathbb{R}^{2(2n+1)}$, $proj_{xOt}\overline{B}_1 = \overline{D}$ there exists a vector-function

   $$H(t,x,U(t,x),D^{(0.1)}U(t,x);V(t,x),D^{(0.1)}V(t,x)) := H[U(t,x);V(t,x)] := (h_i[U(t,x);V(t,x)]),\ i=\overline{1,n}$$

   such that

   - $H[U(t,x);V(t,x)] \equiv F[U(t,x)]$;
   - for arbitrary in $C^*(\overline{D})$ pairs of functions $U(t,x), V(t,x) \in \overline{B}_1$ satisfying conditions

     $$D^{(0.k_2)}[U(t,x)-V(t,x)] \geq (\leq)\ 0,\ k_2 = 0\ (k_2 = 1),\ (t,x) \in \overline{D}_1$$

     in the domain $\overline{B}_1$ the inequality holds

     $$H[U(t,x);V(t,x)] \geq H[V(t,x);U(t,x)]; \tag{7}$$

3. vector-function $H[U(t,x);V(t,x)]$ satisfies the Lipschitz condition, i.e. for arbitrary in $C^*(\overline{D})$ vector-functions $U_r(t,x), V_r(t,x) \in \overline{B}_1$, $r = 1,2$ an inequality holds:

   $$|\ H[U_1(t,x);U_2(t,x)] - H[V_1(t,x);V_2(t,x)]\ | \leq \overline{L}\sum_{r=1}^{2}\left(|\ W_r(t,x) + D^{(0.1)W_r(t,x)}\ |\right),$$

where $W_r(t,x) := U_r(t,x) - V_r(t,x)$, $r = 1,2$ and $\overline{L}$ is the Lipschitz matrix.

Remark 1 It is straightforward that if the vector-function $F[U(t,x)] \in C(\overline{B})$ and its first order partial derivatives with respect to all of its arguments starting from the third one are bounded, then $F[U(t,x)]$ is always in the space of functions $C_3(\overline{B})$. The inverse statement is false.

3 Constructive method of investigation and approximation of solutions to the BVP (2)

Let us first introduce the following notations:

$$
\begin{aligned}
&W_p(t,x) := Z_p(t,x) - V_p(t,x),\ (t,x)\in\overline{D},\ p\in\mathbb{N}_0;\\
&f^p(t,x) := H[Z_p(t,x); V_p(t,x)];\\
&f_p(t,x) := H[V_p(t,x); Z_p(t,x)];\\
&A_p(t,x) := Z_p(t,x) - T_1 f^p(\eta,\zeta) - T_2 f_p(\eta,\zeta);\\
&B_p(t,x) := V_p(t,x) - T_1 f_p(\eta,\zeta) - T_2 f^p(\eta,\zeta);\\
&T_1 f^p(\eta,\zeta) := \int_0^t Lf^p(\eta,\zeta)d\eta;\\
&T_2 f^p(\eta,\zeta) := -\frac{1}{a-x_0}\int_0^t\int_{x_0}^a Lf^p(\eta,\zeta)dxd\eta,\\
&F^p(t,x) := (F_i^p(t,x)),\ F_p(t,x) := (F_{i,p}(t,x)) \text{ — are vector-functions;}\\
&D^{(0.k_2)}Z_p^*(t,x) := D^{(0.k_2)}Z_p(t,x) - C_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x);\\
&D^{(0.k_2)}V_p^*(t,x) := D^{(0.k_2)}V_p(t,x) + Q_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x);\\
&C_{p,k_2}(t,x) := (\delta_{i,j}c_{i,p,k_2}(t,x));\\
&Q_{p,k_2}(t,x) := (\delta_{i,j}q_{i,p,k_2}(t,x));
\end{aligned} \tag{8}
$$

— are functional matrices with non-negative coefficients satisfying the estimates:

$$
\begin{matrix} 0\le c_{i,p,k_2}(t,x)\le 0.5;\\ 0\le q_{i,p,k_2}(t,x)\le 0.5,\end{matrix}\quad (t,x)\in\overline{D},\ k_2=0,1, i=\overline{1,n}; \tag{9}
$$

$$
\begin{aligned}
F_{i,p}(t,x) &:= h_i[v_{1,p+1}(t,x),\dots,v_{i-1,p+1}(t,x), v_{i,p}^*(t,x),\dots,v_{n,p}^*(t,x);\\
&\qquad z_{1,p+1}(t,x),\dots,z_{i-1,p+1}(t,x), z_{i,p}^*(t,x),\dots,z_{n,p}^*(t,x)];\\
F_i^p(t,x) &:= h_i[z_{1,p+1}(t,x),\dots,z_{i-1,p+1}(t,x), z_{i,p}^*(t,x),\dots,z_{n,p}^*(t,x);\\
&\qquad v_{1,p+1}(t,x),\dots,v_{i-1,p+1}(t,x), v_{i,p}^*(t,x),\dots,v_{n,p}^*(t,x)];\\
R^p(t,x) &:= T_1F^p(\eta,\zeta) + T_2F_p(\eta,\zeta);\\
R_p(t,x) &:= T_1F_p(\eta,\zeta) + T_2F^p(\eta,\zeta);
\end{aligned}
$$

Let us construct sequences of vector-functions according to formulas:

$$
\begin{aligned} Z_{p+1}(t,x) &= R^p(t,x);\\ V_{p+1}(t,x) &= R_p(t,x),\end{aligned} \tag{10}
$$

where for a zero approximation we take arbitrary in the space $C^{(0.1)}(\overline{D})$ vector-functions $Z_0(t,x), V_0(t,x)\in\overline{B}_1$ satisfying conditions:

$$
\begin{aligned} D^{(0.k_2)}W_0(t,x)\ge(\le)\,0,\ D^{(0.k_2)}A_0(t,x)\ge(\le)\,0,\\ D^{(0.k_2)}B_0(t,x)\le(\ge)\,0,\ (t,x)\in\overline{D},\ k_2=0\ (k_2=1).\end{aligned} \tag{11}
$$

Definition 2 Arbitrary from $C^{(0.1)}(\overline{D})$ vector-functions $Z_0(t,x)$, $V_0(t,x) \in \overline{B}$ satisfying conditions (11) are called the comparison functions to the BVP (2).

Note, that due to (9), (11) we have

$$D^{(0.k_2)}V_0(t,x) \le (\ge) D^{(0.k_2)}V_0^*(t,x) \le (\ge) D^{(0.k_2)}Z_0^*(t,x) \le (\ge)$$
$$D^{(0.k_2)}Z_0(t,x),\ (t,x) \in \overline{D},\ k_2 = 0\ (k_2 = 1),$$

and thus, $D^{(0.k_2)}V_0^*(t,x)$, $D^{(0.k_2)}Z_0^*(t,x) \in \overline{B}_1$.

From (8), (10) we obtain:

$$\begin{aligned} &D^{(0.k_2)}[Z_p(t,x) - Z_{p+1}(t,x)] \\ &\quad = D^{(0.k_2)}\{A_p(t,x) + T_1[f^p(\eta,\zeta) - F^p(\eta,\zeta)] + T_2[f_p(\eta,\zeta) - F_p(\eta,\zeta)],\} \end{aligned} \tag{12}$$

$$\begin{aligned} &D^{(0.k_2)}[V_p(t,x) - V_{p+1}(t,x)] \\ &\quad = D^{(0.k_2)}\{B_p(t,x) + T_1[f_p(\eta,\zeta) - F_p(\eta,\zeta)] + T_2[f^p(\eta,\zeta) - F^p(\eta,\zeta)],\} \\ &D^{(0.k_2)}W_{p+1}(t,x) = D^{(0.k_2)}[R^p(t,x) - R_p(t,x)] \\ &\quad = D^{(0.k_2)}\{T_1[F^p(\eta,\zeta) - F_p(\eta,\zeta)] + T_2[F_p(\eta,\zeta) - F^p(\eta,\zeta)]\}. \end{aligned} \tag{13}$$

$$\begin{aligned} &D^{(0.k_2)}A_{p+1}(t,x) = D^{(0.k_2)}\{T_1[F^p(\eta,\zeta) - f^{p+1}(\eta,\zeta)] \\ &\quad + T_2[F_p(\eta,\zeta) - f_{p+1}(\eta,\zeta)]\}, \\ &D^{(0.k_2)}B_{p+1}(t,x) = D^{(0.k_2)}\{T_1[F_p(\eta,\zeta) - f_{p+1}(\eta,\zeta)] \\ &\quad + T_2[F^p(\eta,\zeta) - f^{p+1}(\eta,\zeta)]\} \end{aligned} \tag{14}$$

Taking into account inequalities (7), (9), (11), from (12)–(14) in virtue of the method of mathematical induction it is easy to check that if on every iteration step (10), (11) we pick components of the matrices $C_{p,k_2}(t,x)$ and $Q_{p,k_2}(t,x)$ such, that the conditions

$$\begin{aligned} &D^{(0.k_2)}[Z_p(\eta,\zeta) - Z_{p+1}(\eta,\zeta)] - C_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x) \ge (\le) 0, \\ &D^{(0.k_2)}[V_p(\eta,\zeta) - V_{p+1}(\eta,\zeta)] + Q_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x) \le (\ge) 0, \\ &\qquad (t,x) \in \overline{D},\ k_2 = 0\ (k_2 = 1) \end{aligned} \tag{15}$$

hold, then the constructed vector-functions $D^{(0.k_2)}Z_p(t,x)$, $D^{(0.k_2)}V_p(t,x)$ satisfy the inequalities:

$$\begin{aligned} &D^{(0.k_2)}V_p(t,x) \le (\ge) D^{(0.k_2)}V_{p+1}(t,x) \le (\ge) \\ &\qquad D^{(0.k_2)}Z_{p+1}(t,x) \le (\ge) D^{(0.k_2)}Z_p(t,x), \\ &D^{(0.k_2)}A_p(t,x) \ge (\le)\ 0,\ D^{(0.k_2)}B_p(t,x) \le (\ge)\ 0, \\ &\qquad (t,x) \in \overline{D},\ p \in \mathbb{N},\ k_2 = 0\ (k_2 = 1). \end{aligned} \tag{16}$$

Lemma 2 *If the vector-function $F[U(t,x)] \in C_3(\overline{B})$, $A_1(t,x) \in C^{(0.1)}(D)$, $A_2(t,x) \in C(D)$, and in the domain $\overline{B}_1$ there exist comparison functions $Z_0(t,x)$, $V_0(t,x)$ to the BVP (2), then the set of functional matrices $C_{p,k_2}(t,x)$ and $Q_{p,k_2}(t,x)$, satisfying conditions (15), is non-empty.*

Proof Let us pick on every iteration step of (10), (11), (14) elements of the matrices $C_{p,k_2}(t,x)$, $Q_{p,k_2}(t,x)$ in the form

$$c_{i,p,k_2}(t,x) = \begin{cases} D^{(0.k_2)}\alpha_{i,p}(t,x)\rho^{-1}_{i,p,k_2}(t,x), & D^{(0.k_2)}w_{i,p}(t,x) \neq 0, \\ 0, & D^{(0.k_2)}w_{i,p}(t,x) = 0, \end{cases} \tag{17}$$

$$q_{i,p,k_2}(t,x) = \begin{cases} -D^{(0.k_2)}\beta_{i,p}(t,x)\rho^{-1}_{i,p,k_2}(t,x), & D^{(0.k_2)}w_{i,p}(t,x) \neq 0, \\ 0, & D^{(0.k_2)}w_{i,p}(t,x) = 0, \end{cases} \tag{18}$$

$$\rho_{i,p,k_2}(t,x) := D^{(0.k_2)}[\alpha_{i,p}(t,x) - \beta_{i,p}(t,x) + w_{i,p}(t,x)],$$
$$(t,x) \in \overline{D},\ k_2 = 0\ (k_2 = 1),\ p \in \mathbb{N}.$$

Obviously, such non-negative functions $c_{i,p,k_2}(t,x)$, $q_{i,p,k_2}(t,x)$ satisfy conditions (9), and, due to (16), also the inequalities

$$\begin{aligned} &D^{(0.k_2)}[Z_p(t,x) - Z_{p+1}(t,x)] - C_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x) \\ &\quad = D^{(0.k_2)}\{A_p(t,x) + T_1[f^p(\eta,\zeta) - F^p(\eta,\zeta)] + T_2[f_p(\eta,\zeta) - F_p(\eta,\zeta)]\} \\ &\qquad - C_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x) \geq (\leq) D^{(0.k_2)}A_p(t,x) - C_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x) \\ &\quad = (E - P_{p,k_2}(t,x))D^{(0.k_2)}A_p(t,x) \geq (\leq)\ 0, \end{aligned}$$

where $P_{p,k_2}(t,x) := \left(\delta_{ij}D^{(0.k_2)}w_{i,p}(t,x)\rho^{-1}_{i,p,k_2}(t,x)\right)$ is a matrix, and

$$\begin{aligned} D^{(0.k_2)}[V_p(t,x) - V_{p+1}(t,x)] + Q_{p,k_2}(t,x)D^{(0.k_2)}W_p(t,x) \leq (\geq) \\ (E - P_{p,k_2}(t,x))D^{(0.k_2)}B_p(t,x) \leq (\geq)\ 0. \end{aligned}$$

The obtained inequalities prove the lemma. □

Theorem 1 *Let $F[U(t,x)] \in C_3(\overline{B})$, $A_1(t,x) \in C^{(0.1)}(D)$, $A_2(t,x) \in C(D)$ and in the domain $\overline{B}_1$ there exist comparison functions $Z_0(t,x)$, $V_0(t,x)$ to the BVP (2).*

Then the vector-functions $D^{(0.k_2)}Z_p(t,x)$, $D^{(0.k_2)}V_p(t,x)$, constructed according to the iteration scheme (10), (11), (14), satisfy in the domain $\overline{B}_1$ the inequalities (16), for all $(t,x) \in \overline{D}$ and $p \in \mathbb{N}$.

Let us show that the constructed sequences of vector-functions $\{D^{(0.k_2)}Z_p(t,x)\}$, $\{D^{(0.k_2)}V_p(t,x)\}$ uniformly converge to the same limit, that is a solution to the system of integro-differential equations (5). In virtue of (16) it is sufficient to show that

$$\lim_{p\to\infty} D^{(0.k_2)}W_p(t,x) = 0.$$

Proof Denote by

$$\|W_0(t,x)\|_{C^{(0.1)}(\overline{D})} := \max_{i=\overline{1,n}} \sup_{\overline{D}} \Big(|\ w_{i,0}(t,x)\ | + |\ D^{(0.1)} w_{i,0}(t,x)\ | \Big) \le d;$$

$$\|\overline{L}\| := 0.5l;$$

$$\max_{i=\overline{1,n}} \sup_{\overline{D}\times\overline{D}} k_{i,i}(x,t;\xi,\eta) \le K;$$

$$\|E - C_{p,k_2}(t,x) - Q_{p,k_2}(t,x)\| := \gamma_{p,k_2};$$

$$\max_{p,k_2} := \gamma \le 1.$$

From (13) follows that

$$\begin{aligned}
&|\ \omega_{i,p+1}(t,x)\ | \\
&\le Kl\gamma \int_0^t \Bigg\{ \int_x^a \int_\xi^a \sum_{i=1}^n \Big[|\ w_{i,p}(\eta,\zeta)\ | + |\ D^{(0.1)} w_{i,p}(\eta,\zeta)\ | \Big] d\zeta d\xi \\
&\quad + \frac{1}{a-x_0} \int_{x_0}^a \int_x^a \int_\xi^a \sum_{i=1}^n \Big[|\ w_{i,p}(\eta,\zeta)\ | + |\ D^{(0.1)} w_{i,p}(\eta,\zeta)\ | \Big] d\zeta d\xi dx \Bigg\} d\eta, \\
&|\ D^{(0.1)} w_{i,p+1}(\eta,\zeta)\ | \\
&\le \int_0^t \int_x^a Kl\gamma \sum_{i=1}^n \Big[|\ w_{i,p}(\eta,\zeta)\ | + |\ D^{(0.1)} w_{i,p}(\eta,\zeta)\ | \Big] d\zeta d\eta.
\end{aligned} \tag{19}$$

From (19) using the mathematical induction method we obtain the estimates:

$$\begin{aligned}
|\ D^{(0.1)} w_{i,p}(t,x)\ | &\le \frac{(At)^p}{p!} 0.5d; \\
A &= Kl\gamma\rho n; \\
\frac{1}{2}\rho &= sup\left(a, \frac{a^2}{2}\left(1 + \frac{a}{3}\right)\right),
\end{aligned}$$

for all $p \in \mathbb{N}$, $i = \overline{1,n}$, $(t,x) \in \overline{D}$.

Thus,

$$\|D^{(0.1)} W_p(t,x)\|_{C^{(0.k_2)}(\overline{D})} \le \frac{(Ab)^p}{p!} 0.5d. \tag{20}$$

From the estimates (20) it follows that

$$\lim_{p\to\infty} D^{(0.k_2)} W_p(t,x) = 0,$$

i.e.,

$$\lim_{p\to\infty} D^{(0.k_2)} Z_p(t,x) = \lim_{p\to\infty} D^{(0.k_2)} V_p(t,x) = D^{(0.k_2)} U(t,x).$$

It is easy to check that the limit vector-function $U(t, x)$ is the solution to the integro-differential system (5) and hence, to the BVP (2).

Theorem 2 *Let conditions of the Theorem 1 to be hold. Then the sequences of vector-functions $\{Z_p(t, x)\}$, $\{V_p(t, x)\}$ constructed by (10), (11), (14) in the domain $\overline{B}_1$:*

1. *uniformly converge to the unique regular solution of the BVP (2) for $(t, x) \in \overline{D}$;*
2. *estimates (20) hold;*
3. *in the domain $\overline{B}_1$ inequalities*

$$\begin{aligned} D^{(0.k_2)}V_p(t, x) \le (\ge) D^{(0.k_2)}V_{p+1}(t, x) \le (\ge) D^{(0.k_2)}U(t, x) \le (\ge) \\ D^{(0.k_2)}Z_{p+1}(t, x) \le (\ge) D^{(0.k_2)}Z_p(t, x), \;\; (t, x) \in \overline{D}, \;\; k_2 = 0 \; (k_2 = 1); \end{aligned} \tag{21}$$

   hold;

4. *convergence of the method (10), (11), (14) is not slower than the convergence of the Picard method.*

Proof Let

$$\begin{aligned} \overline{Z}_{p+1}(t, x) &= T_1 f^p(\eta, \zeta) + T_2 f_p(\eta, \zeta); \\ \overline{V}_{p+1}(t, x) &= T_1 f_p(\eta, \zeta) + T_2 f^p(\eta, \zeta). \end{aligned}$$

One can prove the uniqueness of solution to the BVP (2) and the inequality (21) by contradiction. For a detailed proof we refer to (Marynets et. al. 2019).

Let us prove statement 4 of the theorem. For this purpose assume, that $Z_p(t, x)$ and $V_p(t, x)$ are the comparison vector-functions of the problem (2). Then

$$\overline{Z}_{p+1}(t, x) - Z_{p+1}(t, x) = T_1\left[f^p(\eta, \zeta) - F^p(\eta, \zeta)\right] + T_2\left[f_p(\eta, \zeta) - F_p(\eta, \zeta)\right].$$

In virtue of the inequalities (7) and (9)

$$\begin{aligned} f^p(t, x) - F^p(t, x) \ge 0, \\ f_p(t, x) - F_p(t, x) \le 0 \end{aligned}$$

and thus,

$$\overline{Z}_{p+1}(t, x) - Z_{p+1}(t, x) \ge 0.$$

Analogically we obtain that

$$\overline{V}_{p+1}(t, x) - V_{p+1}(t, x) \le 0.$$

Hence,

$$\overline{V}_{p+1}(t, x) \le V_{p+1}(t, x) \le Z_{p+1}(t, x) \le \overline{Z}_{p+1}(t, x).$$

The last inequality finishes the proof.

Remark 2 1. Functions $Z_p(t,x)$ and $V_p(t,x)$ satisty the first two boundary conditions in (3) and

$$\int_{x_0}^{a} D^{(1.0)} Z_p(t,x)dx = D^{(1.0)} \int_0^t \int_{x_0}^{a} L[F^{p-1}(\eta,\zeta) - F_{p-1}(\eta,\zeta)]dxd\eta = -\int_{x_0}^{a} D^{(1.0)} V_p(t,x)dx.$$

2. Since for the p-th approximation to the exact solution we take the vector-function

$$\tilde{U}_p(t,x) = \frac{1}{2}[Z_p(t,x) + V_p(t,x)],$$

then $\tilde{U}_p(t,x)$ will satisfy all boundary conditions in (3).
3. It is worth mentioning that some approches for construction of the iterative methods with the improved convergence in the case of the operator equations were studied in [6, 7]. Similar results for different classes of problems in the theory of differential equations were also obtained in [3, 4, 12].

Corollary 1 *If the vector-function* $F[U(t,x)] \in C_3(\overline{B})$*, matrices* $A_1(t,x) \in C^{(0.1)}(D)$*,* $A_2(t,x) \in C(D)$*, and in the space* $C^*(\overline{D})$ *there exists such vector-function* $V_0(t,x)$ $(Z_0(t,x)) \in \overline{B}_1$ *that*

$$\begin{aligned} &D^{(0.k_2)}\{-T_1H[0; V_0(\eta,\zeta)] - T_2H[V_0(\eta,\zeta); 0]\} \geq (\leq)\ 0;\\ &\quad D^{(0.k_2)}V_0(t,x) \leq (\geq)\ 0;\\ &D^{(0.k_2)}\{-T_1H[V_0(\eta,\zeta); 0] - T_2H[0; V_0(\eta,\zeta) + V_0(t,x)]\} \leq (\geq)\ 0;\\ &\quad k_2 = 0\ (k_2 = 1) \end{aligned}$$

$$\left(\begin{array}{c} D^{(0.k_2)}\{Z_0(t,x) - T_1H[Z_0(\eta,\zeta); 0] - T_2H[0; Z_0(\eta,\zeta)]\} \geq (\leq)\ 0;\\ D^{(0.k_2)}\{-T_1H[0; Z_0(\eta,\zeta)] - T_2H[Z_0(\eta,\zeta); 0]\} \leq (\geq)\ 0;\\ D^{(0.k_2)}Z_0(t,x) \geq (\leq)\ 0;\\ k_2 = 0\ (k_2 = 1), \end{array}\right),$$

then solution to the BVP (2) *with the homogeneous boundary conditions* (3) *satisfies the inequalities:*

$$\begin{aligned} &D^{(0.k_2)}U(t,x) \leq (\geq)\ 0\\ &(D^{(0.k_2)}U(t,x) \geq (\leq)\ 0),\\ &k_2 = 0\ (k_2 = 1)\ (t,x) \in \overline{D}. \end{aligned}$$

Together with the BVP (2) we consider the following problem:

$$\mathcal{L}_3 Z(t,x) = f_1(t,x,Z(t,x),D^{(0.1)}(t,x)) := f_1[Z(t,x)]. \tag{22}$$

From now on we assume, that the right hand-sides of the problems (2) and (22) satisfy conditions below:

1. $f[U(t,x)] \in C_3(\overline{B})$;
2. vector-function $f_1[Z(t,x)] \in C(\overline{B})$, and in the domain $\overline{B}$ it has bounded first order partial derivatives with respect to $Z(t,x)$ and $D^{(0.1)}Z(t,x)$, i.e.,

$$\frac{\partial f_{1,i}[Z(t,x)]}{\partial z_j(t,x)} := b_{i,j}^{(0)}(t,x) < \infty;$$
$$\frac{\partial f_{1,i}[Z(t,x)]}{\partial D^{(0.1)} z_j(t,x)} := b_{i,j}^{(1)}(t,x) < \infty,$$

satistying conditions:

$$\begin{aligned} & b_{i,j}^{(0)}(t,x) \geq 0, \\ & b_{i,j}^{(1)}(t,x) + \delta_{i,j}\left[D^{(0.1)}a_{i,j}^{(1)}(t,x) + a_{i,j}^{(1)}(t,x)a_{i,j}^{(2)}(t,x)\right] \leq 0; \end{aligned} \tag{23}$$

3. for an arbitrary vector-function $V(t,x) \in \overline{B}$ from the space $C^{(0.1)}(\overline{D})$ it holds that

$$f_1[V(t,x)] \geq (\leq)\; f[V(t,x)]. \tag{24}$$

Theorem 3 *Assume, that the matrices $A_1(t,x) \in C^{(0.1)}(D)$, $A_2(t,x) \in C(D)$, the right hand-sides of the problems (2), (22) satisfy conditions (1)–(3) above, and in the domain $\overline{B}_1$ there exist the comparison vector-functions to the BVP (2), (22).*

Then for the solutions of these problems the inequalities

$$U(t,x) \leq (\geq) Z(t,x).$$

hold, where $(t,x) \in \overline{D}$.

Proof According to the Theorem 2 solutions to the BVP (2), (22) exist, are unique and regular. Thus, by putting $W(t,x) := Z(t,x) - U(t,x)$ and applying the Mean Value Theorem, we get [10]

$$\mathcal{L}_3 W(t,x) = A_3(t,x)W(t,x) + A_4(t,x)D^{(0.1)}W(t,x) + A_5(t,x), \tag{25}$$

where $A_3(t,x) := \left(\tilde{b}_{i,j}^{(0)}(t,x)\right)$, $A_3(t,x) := \left(\tilde{b}_{i,j}^{(1)}(t,x)\right)$, $i,j = \overline{1,n}$ are matrices, $\tilde{b}_{i,j}^{(k_2)}(t,x)$ are derivatives of $b_{i,j}^{(k_2)}(t,x)$ for some fixed $D^{(0.k_2)}Z(t,x) \in B$, $k_2 = 0, 1$, and due to (24)

$$A_5(t,x) := f_1[U(t,x)] - f[U(t,x)] \geq (\leq)\; 0. \tag{26}$$

It is straightforward that the vector-function satisfies the homogeneous boundary conditions (3) and

$$F[W(t,x)] := \Big[A_4(t,x) + D^{(0.1)}A_1(t,x) + A_1(t,x)A_2(t,x)\Big] D^{(0.1)}W(t,x) + \\ A_3(t,x)W(t,x) + A_5(t,x), \tag{27}$$

i.e., in virtue of (23) $F[W(t,x)] \equiv H[W(t,x);0]$ and

$$F[0] \geq (\leq)\ 0,\ (t,x) \in \overline{D}. \tag{28}$$

Taking into account (26)–(28) and due to the Corollary 1 solution of the system (25) satisfies the inequalities:

$$D^{(0.k_2)}W(t,x) \geq (\leq)\ 0\ (D^{(0.k_2)}W(t,x) \leq (\geq)\ 0),\ k_2 = 0\ (k_2 = 1),\ (t,x) \in D.$$

This completes the proof.

4 Example

Let us consider an illustrative example: *in the space of functions* $C^*(D_0)$,

$$D_0 = \{(t,x) \mid t \in (0,1), x \in (0,1)\},$$

find a solution to a scalar differential equation

$$\begin{aligned} &D^{(1.2)}U(t,x) - t(1+0,5t^2)^{-1}D^{(0.2)}U(t,x) - (1+x)^{-1}D^{(1.1)}U(t,x) \\ &\quad = (1+x)(1+0,5t^2)[U^3(t,x) + 0,1tx] \\ &\quad\quad - t[(1+x)(1+0,5t^2)]^{-1}D^{(0.1)}U(t,x), \end{aligned} \tag{29}$$

coupled with the boundary conditions of the form:

$$\begin{aligned} U(0,x) = 0,\quad x \in [0,1], \\ D^{(0.1)}U(t,1) = 0,\ \int_{0,5}^{1} D^{(1.0)}U(t,\xi)d\xi = 0,\ t \in [0,1]. \end{aligned} \tag{30}$$

Note, that in the case of non-homogeneous boundary conditions they can always be reduced to the homogeneous ones.

For the BVP (29), (30) the kernel $K(x,t;\xi,\eta)$, defined in (6), is given by

$$K(x,t;\xi,\eta) = \frac{(1+x)(1+0,5t^2)}{(1+\xi)(1+0,5\eta^2)}.$$

Table 1 Comparison characteristics $C_{p,k_2}(t,x)$, $Q_{p,k_2}(t,x)$ of the iterative method (10)

N/p	$C_{p,k_2}(t,x)=0$, $Q_{p,k_2}(t,x)=0$, $\sup\limits_{\overline{D}_0}\lvert W_p(t,x)\rvert$	$C_{p,k_2}(t,x)\neq 0$, $Q_{p,k_2}(t,x)\neq 0$, $\sup\limits_{\overline{D}_0}\lvert W_p(t,x)\rvert$
0	$1,4$	$0,75$
1	$1,8\cdot 10^{-2}$	$8,5\cdot 10^{-3}$
2	$0,1\cdot 10^{-6}$	$0,6\cdot 10^{-7}$

For the comparison functions of the studied problem (29), (30), we take the following:

$$Z_0(t,x) = (1+0,5t^2)t\left(\frac{95}{192} - 0,5x + \frac{1}{6}x^3\right),$$

$$V_0(t,x) = (1+0,5t^2)t\left(-\frac{95}{192} + 0,5x - \frac{1}{6}x^3\right).$$

Obviously, $W_0(t,x) \geq 0$, $D^{(0.1)}W_0(t,x) \leq 0$, $(t,x) \in \overline{D}_0$.

Let us now implement the iterative method (10) for the BVP (29), (30) in the case of particular values of $C_{p,k_2}(t,x)$, $Q_{p,k_2}(t,x)$, defined by (17), (18), which are here just scalar functions.

The comparison characteristics of our computations are given in the Table 1.

From the results, presented in the table, follows that the convergence of the iterative method (10) in governed by $C_{p,k_2}(t,x)$ and $Q_{p,k_2}(t,x)$. Depending on their choice we can obtain different modifications to the considered method.

As one can see, already on the second iteration step we are able to obtain an approximate solution to the BVP (29), (30) with a very high precision. This solution is given by

$$\begin{aligned}\tilde{U}_2(t,x) &= \frac{1}{2}\left[Z_2(t,x) + V_2(t,x)\right] \\ &= 0,5\cdot 10^{-2}t^2(1+0,5t^2)(1,25x^4 + 1,67x^3 - 2,5x^2 - 5x + 3,9) + \mathcal{O}(10^{-7}).\end{aligned}$$

If necessary, one can continue the iteration process and construct further approximations to the exact solution with an even higher precision than those, obtained on the second iteration step.

Acknowledgements Authors are thankful for valuable suggestions and comments of the reviewers that helped to improve the paper.

Author Contributions The authors have contributed equally to the manuscript.

Funding There is no funding to declare.

Data availability Not applicable.

Code availability Not applicable.

Declarations

Conflict of interest The authors declare no conflict of interests.

Ethics approval Not applicable.

Consent to participate Not applicable.

Consent to publication Not applicable.

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