On the Velocity of a Small Rigid Body in a Viscous Incompressible Fluid in Dimension Two and Three

Bravin, Marco; Nečasová, Šárka

DOI
10.1007/s10884-023-10263-5

Publication date
2023

Document Version
Final published version

Published in
Journal of Dynamics and Differential Equations

Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-023-10263-5

On the Velocity of a Small Rigid Body in a Viscous Incompressible Fluid in Dimension Two and Three

Marco Bravin[1] · Šárka Nečasová[2]

Received: 25 August 2022 / Revised: 13 January 2023 / Accepted: 25 March 2023

Abstract
In this paper we study the evolution of a small rigid body in a viscous incompressible fluid, in particular we show that a small particle is not accelerated by the fluid in the limit when its size converges to zero under a lower bound on its mass. This result is based on a new a priori estimate on the velocities of the centers of mass of rigid bodies that holds in the case when their masses are also allowed to decrease to zero.

Keywords Fluid-structure · Asymptotic analysis · Navier-Stokes

1 Introduction

In the recent works [13] and [14], the authors showed that the presence of a small rigid body is negligible in a viscous incompressible fluid. In this paper we study the trajectory of this small object. We show that a small rigid body is not influenced by the fluid under some constrain on its mass, in particular the rigid body is not accelerated by the fluid and it moves with its initial velocity.

Let start by introducing the equations that describe the dynamic of the system. For $d = 2, 3$, let us denote by $\mathcal{S}(t) \subset \mathbb{R}^d$ the position of the rigid body at time $t \in \mathbb{R}^+ = [0, +\infty)$. The fluid occupies the domain $\mathcal{F}(t) = \mathbb{R}^d \setminus \mathcal{S}(t)$ and it is modelled by the Navier–Stokes equations

$$\begin{aligned} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p &= 0 && \text{for } x \in \mathcal{F}(t), \\ \operatorname{div}(u) &= 0 && \text{for } x \in \mathcal{F}(t), \\ u &= u_{\mathcal{S}} && \text{for } x \in \partial \mathcal{S}(t), \\ u &\longrightarrow 0 && \text{as } |x| \longrightarrow +\infty, \end{aligned} \tag{1}$$

where $u : \mathbb{R}^+ \times \mathcal{F}(t) \longrightarrow \mathbb{R}^d$ is the velocity field and $p : \mathbb{R}^+ \times \mathcal{F}(t) \longrightarrow \mathbb{R}$ is the pressure which is a scalar quantity. The real number $\nu > 0$ is the viscosity coefficient. Finally $u_{\mathcal{S}}$ is the velocity of the rigid body.

✉ Marco Bravin
bravinmarco92@gmail.com

1 Delft Institute of Applied Mathematics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

2 Institute of Mathematics, Czech Academy of Sciences Žitná 25, 2628 CD Praha 1, Czech Republic

Published online: 18 April 2023

Regarding the rigid body we assume that it occupies $\mathcal{S}(0) = \mathcal{S}^{in}$ a closed, connected, simply connected subset of $\mathbb{R}^d$ with no empty interior and smooth boundary and that it has density $\rho^{in} : \mathcal{S}^{in} \longrightarrow \mathbb{R}$ such that $\rho^{in} > 0$. The dynamic of $\mathcal{S}(t)$ is completely determined by the evolution of the center of mass $h(t)$ and the angular rotation $Q(t)$ around the center of mass. More precisely

$$\mathcal{S}(t) = \left\{ x \in \mathbb{R}^d \text{ such that } Q^T(t)(x - h(t)) \in \mathcal{S}^{in} \right\}.$$

Here for a matrix A we denote by A^T its transpose. The density of the rigid body $\mathcal{S}(t)$ is given by $\rho(t,x) = \rho^{in}(Q^T(t)(x - h(t)))$ and its velocity $u_{\mathcal{S}} : \mathbb{R}^+ \times \mathcal{S}(t) \longrightarrow \mathbb{R}^d$ is

$$u_{\mathcal{S}}(t,x) = \frac{d}{dt}(h(t) + Q(t)y)\Big|_{y=Q^T(t)(x-h(t))} = h'(t) + Q'(t)Q^T(t)(x - h(t)).$$

The matrix $Q(t)$ is a rotation, we deduce that $Q'(t)Q^T(t)$ is skew-symmetric and can be identify in dimension three with a vector $\omega(t)$ in the following way

$$Q'(t)Q^T(t)x = \omega(t) \times x$$

for any $x \in \mathbb{R}^3$. We call ω the angular velocity. If we denote by $\ell(t) = h'(t)$ the velocity of the center of mass, the solid velocity is

$$u_{\mathcal{S}}(t,x) = \ell(t) + \omega(t) \times (x - h(t)).$$

Let recall that the mass m and the center of mass h is defined as

$$m = \int_{\mathcal{S}(t)} \rho(t,x)\,dx \quad \text{and} \quad h(t) = \frac{1}{m}\int_{\mathcal{S}(t)} \rho(t,x)x\,dx$$

and without loss of generality we assume $h(0) = 0$ and $Q(0) = 0$. The evolution of the center of mass $h(t)$ and $Q(t)$ follows the Newton's laws that write

$$m\ell'(t) = -\oint_{\partial\mathcal{S}(t)} \Sigma(u,p)n ds, \tag{2}$$
$$\mathcal{J}(t)\omega'(t) = \mathcal{J}(t)\omega(t) \times \omega(t) - \oint_{\partial\mathcal{S}(t)} (x - h(t)) \times \Sigma(u,p)n ds,$$

where n is the normal component to $\partial\mathcal{F}(t)$ exiting from the fluid domain, the inertia matrix $\mathcal{J}(t)$ is defined as

$$\mathcal{J}(t) = \int_{\mathcal{S}(t)} \rho(t,x)\left[|x - h(t)|^2\mathbb{I} - (x - h(t)) \otimes (x - h(t))\right] dx$$

where $\mathbb{I}$ is the identity matrix of d dimensions. Finally the stress tensor

$$\Sigma(u,p) = 2\nu D(u) - p\mathbb{I} \quad \text{where} \quad D(u) = \frac{\nabla u + (\nabla u)^T}{2}$$

is the symmetric gradient.

For the system (1)-(2) the initial conditions are

$$u(0,.) = u^{in} \quad \text{in } \mathcal{F}(t), \quad \ell(0) = \ell^{in} \quad \text{and} \quad \omega(0) = \omega^{in}. \tag{3}$$

where u^{in} is the initial fluid velocity, $\ell^{in} \in \mathbb{R}^d$ and $\omega^{in} \in \mathbb{R}^{2d-3}$. Moreover they satisfy the compatibility conditions

$$\operatorname{div}(u^{in}) = 0 \quad \text{in } \mathcal{F}(0) \quad \text{and} \quad u^{in} \cdot n = \ell^{in} + \omega^{in} \times x \quad \text{on } \partial\mathcal{S}(0). \tag{4}$$

Notice that in the case of dimension two the matrix $Q'(t)Q^T(t)$ can be identify with a scalar quantity that we denote again by ω and $u_{\mathcal{S}}(t,x) = \ell(t) + \omega(t)(x-h(t))^\perp$, where for $x \in \mathbb{R}^2$ we denote by $x^\perp = (-x_2, x_1)^T$. Moreover the inertia matrix $\mathcal{J}$ becomes a scalar independent of time and the second equation of (2) simplifies.

Let us recall that the system (1)-(2)-(3) has been widely studied in the literature. In fact the first works on the existence of Hopf-Leray type weak solutions are [27] and [33] where they consider the case $\Omega = \mathbb{R}^3$. These results were then extended in [26]- [7]- [8]- [9]- [10]. Uniqueness was shown in [24] in dimension two and in [30] in dimension three under Prodi-Serrin conditions. Regularity was studied in dimension three under Prodi-Serrin conditions in [31]. Well-posedness of strong solutions in Hilbert space setting was proved in [25]- [15]- [34]- [35] and in the Banach space setting in [18]- [29]. Notice that similar results hold in the case the Navier-Slip boundary conditions are prescribed on $\partial\mathcal{S}$, see [32]- [19]- [2]- [1]- [6]. Existence of time-periodic solutions to the Navier-Stokes equations around a moving body was studied in [17] while the steady case was tackled in [17].

Among all different types of solutions, in this work we consider Hopf-Leray type weak solutions for the system (1)-(2)-(3). To recall this definition let introduce some notations from [14]. Let denote by

$$\tilde{\rho} = \chi_{\mathcal{F}(t)} + \rho\chi_{\mathcal{S}(t)} \tag{5}$$

which is the extension by 1 of the density of the rigid body. Here for a set $A \subset \mathbb{R}^d$, we denote by χ_A the indicator function of A, more precisely $\chi_A(x) = 1$ for $x \in A$ and 0 elsewhere. Similarly we define the global velocity

$$\tilde{u} = u\chi_{\mathcal{F}(t)} + u_{\mathcal{S}}\chi_{\mathcal{S}(t)} = u\chi_{\mathcal{F}(t)} + (\ell(t) + \omega \times (x - h(t)))\,\chi_{\mathcal{S}(t)}.$$

Notice that if $\tilde{u}^{in} \in L^2(\mathcal{F}(0))$, then the compatibility conditions (4) on the initial data imply that div $(\tilde{u}^{in}) = 0$ in an appropriate weak sense. After all these preliminaries we introduce the definition of Hopf-Leray type weak solution for the system (1)-(2)-(3).

Definition 1 Let $\mathcal{S}^{in}$ and ρ^{in} be the initial position and density of the rigid body, let $(u^{in}, \ell^{in}, \omega^{in})$ be initial velocities satisfying hypothesis (4) and such that $\tilde{u}^{in} \in L^2(\mathbb{R}^d)$. Then a triple (u, ℓ, ω) is a Hopf-Leray weak solution for system (1)-(2)-(3) associated with initial data $\mathcal{S}^{in}$, ρ^{in}, u^{in}, ℓ^{in} and ω^{in}, if

- the functions u, ℓ and ω satisfy

$$\ell \in L^\infty(\mathbb{R}^+;\mathbb{R}^d), \quad \omega \in L^\infty(\mathbb{R}^+;\mathbb{R}^{2d-3})$$
$$u \in L^\infty(\mathbb{R}^+; L^2(\mathcal{F}(t))) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathcal{F}(t))), \quad \text{and} \quad \tilde{u} \in C_w(\mathbb{R}^+; L^2(\mathbb{R}^d));$$

- the vector field $\tilde{u}$ is divergence free in $\mathbb{R}^d$ with $D(\tilde{u}) = 0$ in $\mathcal{S}(t)$;
- the vector field $\tilde{u}$ satisfies the system in the following sense:

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \tilde{\rho}\tilde{u}\cdot(\partial_t\varphi + \tilde{u}\cdot\nabla\varphi) - 2\nu D(\tilde{u}) : D(\varphi)\,dx\,dt = \int_{\mathbb{R}^d} \tilde{\rho}^{in}\tilde{u}^{in}\cdot\varphi(0,.)\,dt, \tag{6}$$

  for any test function $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that div $(\varphi) = 0$ and $D(\varphi) = 0$ in $\mathcal{S}(t)$.
- The following energy inequality holds

$$\int_{\mathbb{R}^d} \tilde{\rho}(t,.)|\tilde{u}(t,.)|^2\,dx + 4\nu\int_0^t\int_{\mathbb{R}^d} |D(\tilde{u})|^2\,dxdt \le \int_{\mathbb{R}^d} \tilde{\rho}|\tilde{u}^{in}|^2, \tag{7}$$

  for almost any time $t \in \mathbb{R}^+$.

The existence of weak solutions for the system (1)-(2)-(3) is now classical and can be found for example in [15]- [35]- [32]- [9].

Theorem 1 *For initial data $\mathcal{S}^{in}$, ρ^{in}, u^{in}, ℓ^{in} and ω^{in} satisfying the hypothesis* (4) and such that $\tilde{u}^{in} \in L^2(\mathbb{R}^d)$, there exist a Hopf-Leray weak solution (u, ℓ, ω) of the system (1)-(2)-(3).

Let now introduce a small parameter $\varepsilon > 0$ that control the size of the rigid body. We will consider initial rigid body of the size $\mathcal{S}^{in}_\varepsilon \subset B_0(\varepsilon)$. In [13] and [14] the authors studied the limit as ε goes to zero for solutions of the system (1)-(2)-(3) under some mild assumptions on the initial data ρ^{in}_ε, u^{in}_ε, ℓ^{in}_ε and ω^{in}_ε and in particular they show that in the limit the the presence of the small rigid body does not influence the limit dynamics. These results can be resume as follows.

Theorem 2 (Th. 3 of [13] and Th. 2 of [14].) *Let $(u_\varepsilon, \ell_\varepsilon, \omega_\varepsilon)$ be a sequence of Hopf-Leray solutions associated with the initial data $\mathcal{S}^{in}_\varepsilon$, ρ^{in}_ε, u^{in}_ε, ℓ^{in}_ε and ω^{in}_ε satisfying the hypothesis* (4) and such that $\tilde{u}^{in}_\varepsilon \in L^2(\mathbb{R}^d)$. If we assume that

- The rigid body $\mathcal{S}^{in}_\varepsilon \subset B_0(\varepsilon)$;
- The mass of the rigid body $m_\varepsilon/\varepsilon^d \longrightarrow +\infty$;
- The initial velocity $\tilde{u}^{in}_\varepsilon \longrightarrow u^{in}$ in $L^2(\mathbb{R}^d)$ and $m_\varepsilon|\ell_\varepsilon|^2 + (\mathcal{J}_\varepsilon \omega^{in}_\varepsilon) \cdot \omega^{in}_\varepsilon \longrightarrow 2E$;

Then up to subsequence

$$\tilde{u}_\varepsilon \xrightarrow{w} u \quad \text{weak-}\star \text{ in } L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d)) \text{ and weak in } L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^d))$$

where u is a distributional solution to the Navier–Stokes equations that satisfies the energy inequality

$$\int_{\mathbb{R}^d} |u(t,.)|^2\, dx + 4\nu \int_0^t \int_{\mathbb{R}^d} |D(u)|^2\, dxdt \leq \int_{\mathbb{R}^d} |u^{in}|^2 + 2E. \tag{8}$$

Notice that there are many results in this direction. For example in [28] the authors studied the vanishing object problem in dimension two under the assumption $m_\varepsilon = \varepsilon^2 m$. For viscous compressible fluid the three dimensional case was tackled in [4] and improved in [12] and more recently in [11] was studied the two dimensional case for a weakly compressible fluid. Notice that for viscous compressible fluid the vanishing object problem in two dimensions is still an open problem due to the lack of integrability of the pressure.

In the case of dimension two and for an inviscid incompressible fluid modelled by the Euler equations, i.e. system (1)-(2)-(3) with $\nu = 0$, the vanishing object problem was studied in [21]-[23]. In this case the presence of the small rigid body creates a vortex point in the limiting dynamics and the intensity is associated with the initial circulation around the object. In this case the authors were able to determine the position of the rigid body in the limit and it coincides with the center of the vortex.

The goal of this paper is to study the evolution of the small rigid body in the limit as $\varepsilon \longrightarrow 0$ in the case the fluid is viscous and incompressible. In particular we will show that its dynamics is not influenced by the fluid.

Theorem 3 *Let $(u_\varepsilon, \ell_\varepsilon, \omega_\varepsilon)$ be a sequence of Hopf-Leray solutions associated with the initial data $\mathcal{S}^{in}_\varepsilon$, ρ^{in}_ε, u^{in}_ε, ℓ^{in}_ε and ω^{in}_ε satisfying the hypothesis* (4) and such that $\tilde{u}^{in}_\varepsilon \in L^2(\mathbb{R}^d)$. If to the assumptions of Theorem 2 we add

- $m_\varepsilon/\varepsilon^{1/2} \longrightarrow +\infty$ for $d = 3$ and $m_\varepsilon \geq C > 0$ for $d = 2$;
- $\ell^{in}_\varepsilon \longrightarrow \ell^{in}$.

Then

$$\ell_\varepsilon \longrightarrow \ell^{in} \quad \text{in } L^\infty_{loc}(\mathbb{R}^+).$$

Notice that in the above theorem $\ell_\varepsilon \longrightarrow \ell^{in}$ and ℓ^{in} is independent of time. This means that the small rigid body in not accelerated by the fluid.

In the special case the rigid bodies $\mathcal{S}^{in}_\varepsilon = B_\varepsilon(0)$ and they have densities ρ_ε constant in the space variables, the assumption on the masses in Theorem 3 can be rewritten in the form $\rho_\varepsilon \varepsilon^{5/2} \longrightarrow +\infty$ for $d = 3$ and $\rho_\varepsilon \varepsilon^2 \geq C > 0$ for $d = 2$. This implies that the densities ρ_ε of the rigid ball diverge to infinity.

The difficulty of the result is the fact that we allow the mass of the rigid body to go to zero, in particular it is not enough to show that in the limit $m\ell' = 0$ because $m = 0$. Moreover from the energy estimate we only deduce $m_\varepsilon |\ell_\varepsilon|^2$ uniformly bounded and this is not enough to have an *a priori* bound on ℓ_ε.

Finally notice that the evolution of the small rigid body seems richer in the case of a two dimensional inviscid incompressible fluid but this is due to the fact that in this setting it is possible to consider initial data with non-zero circulation around the object. In the case of zero circulation, the limit velocity of the center of mass of the small rigid body is trivial in the sense that $\ell(t) = \ell^{in}$, see Section 1.4 of [20]. Notice that a velocity field that has non-zero circulation around the body behaves as $1/|x|$ when $|x|$ goes to $+\infty$, in particular it is not L^2 so it does not enter in the theory of Hopf-Leray weak solutions, see also the comments in section 2 of [28]. Let recall that an existence result in this direction is available in [3] where the author considers initial data for the velocity field of the type $u^{in} + x^\perp/|x^2|$ with $u^{in} \in L^2(\mathcal{F}^{in})$ for the system (1)-(2)-(3). The vanishing rigid body problem is still open in this setting.

Let now move to the proof of Theorem 3. The remaining part of the paper is divided in two main sections. First of all we recall some useful cut off. In the second one we show the L^∞ convergence for the velocity of the center of mass.

2 Some Appropriate Cut-Off

In this section we introduce some cut-off functions that have been considered also in [13]. They have the property that they optimized the L^d norm of the gradient and we denote them by $\eta_{\varepsilon,\alpha_\varepsilon}$. The parameter $\varepsilon > 0$ indicates that $\eta_{\varepsilon,\alpha_\varepsilon} = 1$ in the ball $B_\varepsilon(0)$ and α_ε indicates that the support of the $\eta_{\varepsilon,\alpha_\varepsilon}$ is contained in the ball of size $\varepsilon\alpha_\varepsilon$.

Proposition 1 *For any $\varepsilon > 0$ and $\alpha_\varepsilon \geq 2$, there exists a cut-off function $\eta_{\varepsilon,\alpha_\varepsilon} \in C^\infty_c(B_{\varepsilon\alpha_\varepsilon}(0))$ such that $\eta_{\varepsilon,\alpha_\varepsilon}(x) = 1$ for $x \in B_\varepsilon(0)$, $\|\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty} \leq 1$ and the following bounds hold with constant C independent of ε and α_ε.*

1. *For $1 \leq q < +\infty$*

$$\|\eta_{\varepsilon,\alpha}\|_{L^q} + \||x|\nabla\eta_{\varepsilon,\alpha}\|_{L^q(\mathbb{R}^d)} \leq C(\varepsilon\alpha_\varepsilon)^{d/q}.$$

2. *We have*

$$\big\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\big\|^d_{L^d(\mathbb{R}^d)} + \big\||x|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}\big\|^d_{L^d(\mathbb{R}^d)} \leq \frac{C}{(\log\alpha_\varepsilon)^{d-1}}.$$

3. *For $1 \leq q < d$,*

$$\big\|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\big\|^q_{L^q(\mathbb{R}^d)} + \big\||x|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}\big\|^q_{L^q(\mathbb{R}^d)} \leq \frac{C}{d-q}\frac{(\varepsilon\alpha_\varepsilon)^{d-q}}{(\log\alpha_\varepsilon)^q}.$$

The proof of the above proposition is a straight-forward extension of Lemma 3 of [13], so let us postpone the proof in Appendix A.

For $\varepsilon > 0$ and $\alpha_\varepsilon \geq 2$, let us introduce a family of linear operators $\Psi_{\varepsilon,\alpha_\varepsilon}$ that associate to any vector $z \in \mathbb{R}^d$ a divergence free vector field $\Psi_{\varepsilon,\alpha_\varepsilon}[z] \in C_c^\infty(B_{\varepsilon,\alpha_\varepsilon}(0))$ such that it is equal to z in $B_\varepsilon(0)$. In dimension three

$$\Psi_{\varepsilon,\alpha_\varepsilon} : \mathbb{R}^3 \longrightarrow C_c^\infty(\mathbb{R}^3)$$

is the map $z \mapsto \Psi_{\varepsilon,\alpha_\varepsilon}[z]$ defined by

$$\Psi_{\varepsilon,\alpha_\varepsilon}[z](x) = \text{curl}\left(\eta_{\varepsilon,\alpha_\varepsilon}(x)\begin{pmatrix} z_2x_3 \\ z_3x_1 \\ z_1x_2 \end{pmatrix}\right). \tag{9}$$

for any x in $\mathbb{R}^3$

Similarly in dimension two the operator $\Psi_{\varepsilon,\alpha_\varepsilon} : \mathbb{R}^2 \longrightarrow C_c^\infty(\mathbb{R}^2)$ is defined by

$$\Psi_{\varepsilon,\alpha_\varepsilon}[z](x) = \nabla^\perp\left(\eta_{\varepsilon,\alpha_\varepsilon}(x)(z_2x_1 - z_1x_2)\right), \tag{10}$$

for any x in $\mathbb{R}^2$.

In the following we use square brackets only to denote to which vector $z \in \mathbb{R}^d$ we apply the operator $\Psi_{\varepsilon,\alpha_\varepsilon}$.

Lemma 1 *The family of operators $\Psi_{\varepsilon,\alpha_\varepsilon}$ satisfies the following bounds where the constant C are independent of ε and α_ε.*

1. *For $1 \leq q < +\infty$, we have*

$$\|\Psi_{\varepsilon,\alpha_\varepsilon}[z](x)\|_{L^q(R^d)} \leq C(\varepsilon\alpha_\varepsilon)^{d/q}|z|.$$

2. *We have*

$$\left\|\nabla\Psi_{\varepsilon,\alpha_\varepsilon}[z](x)\right\|^d_{L^d(\mathbb{R}^d)} \leq C\frac{|z|^d}{(\log\alpha_\varepsilon)^{d-1}}.$$

3. *For $1 \leq q < d$,*

$$\left\|\nabla\Psi_{\varepsilon,\alpha_\varepsilon}[z](x)\right\|^q_{L^q(\mathbb{R}^d)} = C\frac{|z|^q}{d-q}\frac{(\varepsilon\alpha_\varepsilon)^{d-q}}{(\log\alpha_\varepsilon)^q}.$$

Proof Let us show the lemma in dimension two for simplicity. Notice that

$$\Psi_{\varepsilon,\alpha_\varepsilon}[z](x) = \nabla^\perp\eta_{\varepsilon,\alpha_\varepsilon}(x)(z_2x_1 - z_1x_2) + \eta_{\varepsilon,\alpha_\varepsilon}(x)z$$

We estimate

$$\begin{aligned}\|\Psi_{\varepsilon,\alpha_\varepsilon}[z]\|_{L^q(\mathbb{R}^d)} &\leq C\left(\||x|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} + \|\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)}\right)|z| \\ &\leq C(\varepsilon\alpha_\varepsilon)^{d/q}.\end{aligned}$$

from point 1 of Proposition 1. To estimate the gradient of $\Psi_{\varepsilon,\alpha_\varepsilon}[z]$, we compute

$$\nabla\Psi_{\varepsilon,\alpha_\varepsilon}[z](x) = \nabla^\perp \otimes \nabla\eta_{\varepsilon,\alpha_\varepsilon}(x)(z_2x_1 - z_1x_2) + \nabla^\perp\eta_{\varepsilon,\alpha_\varepsilon}(x) \otimes z^\perp + z \otimes \nabla\eta_{\varepsilon,\alpha_\varepsilon}(x).$$

where for two vector in $u, v \in \mathbb{R}^d$ we use the notation $u \otimes v \in \mathbb{R}^{d\times d}$ to denote the matrix with entries $(u \otimes v)_{ij} = u_iv_j$. For $1 \leq q \leq d$, we bound

$$\left\|\nabla\Psi_{\varepsilon,\alpha_\varepsilon}[z](x)\right\|^q_{L^q(\mathbb{R}^d)} \leq C\left(\||x|\nabla^2\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} + \|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)}\right)|z|.$$

We then deduce points 2 and 3 of Lemma 1 from points 2 and 3 of Proposition 1 respectively. □

We use the special test functions (9) and (10) in equation (6) to prove the main result.

3 Proof of Theorem 3

In this section we will prove Theorem 3 in dimension two and three. From the definition of Leray-Hopf weak solutions and in particular from (7), we have

$$m_\varepsilon|\ell_\varepsilon(t)|^2 + \big(\mathcal{J}_{\mathcal{S}_\varepsilon(t)}\omega_\varepsilon(t)\big)\cdot\omega_\varepsilon(t) + \int_{\mathcal{F}_\varepsilon(t)} |u_\varepsilon(t,.)|^2\,dx + 4\nu\int_0^t\int_{\mathbb{R}^d} |D(u_\varepsilon)|^2\,dxdt$$
$$\leq \int_{\mathbb{R}^d} \tilde{\rho}_\varepsilon^{in}|\tilde{u}_\varepsilon^{in}|^2 \leq C, \tag{11}$$

with C independent of ε. The last inequality is a consequence of the convergence $\tilde{u}_\varepsilon^{in} \longrightarrow \tilde{u}^{in}$ in $L^2(\mathbb{R}^2)$ and $m_\varepsilon|\ell_\varepsilon|^2 + (\mathcal{J}_\varepsilon\omega_\varepsilon^{in})\cdot\omega_\varepsilon^{in} \longrightarrow 2E$ that is stated in Theorem 2. The above bound was used in [13] and [14] to show Theorem 2 but it does not give information on ℓ_ε in the case the mass of the rigid body converges to zero. We will now present a new estimate that gives us control of the L^∞-norm of ℓ_ε. Recall that that the L^∞-norm in the interval $(0,T)$ can be defined by

$$\|f\|_{L^\infty(0,T)} = \sup_{\gamma\in C_c^\infty(0,T)} \frac{\left|\int_0^T f\gamma\,d\tau\right|}{\|\gamma\|_{L^1(0,T)}}.$$

We will use this form to show our results.

Given $\gamma \in C_c^\infty([0,T];\mathbb{R}^d)$, we consider the divergence free vector fields

$$\psi_\varepsilon(t,x) = \frac{1}{m_\varepsilon}\Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_t^T \gamma(\tau)\,d\tau\right](x - h_\varepsilon(t))$$

for $t \in [0,T]$ and ψ_ε identically zero for $t > T$. Using (9), the function ψ_ε in dimension three is

$$\psi_\varepsilon(t,x) = \frac{1}{m_\varepsilon}\operatorname{curl}\left(\eta_{\varepsilon,\alpha_\varepsilon}(x - h_\varepsilon(t))\begin{pmatrix}\int_t^T \gamma_2(\tau)\,d\tau(x_3 - h_{3,\varepsilon}(t))\\ \int_t^T \gamma_3(\tau)\,d\tau(x_1 - h_{1,\varepsilon}(t))\\ \int_t^T \gamma_1(\tau)\,d\tau(x_2 - h_{2,\varepsilon}(t))\end{pmatrix}\right),$$

for $t \leq T$. First of all notice that for $x \in \mathcal{S}_\varepsilon(t)$

$$\psi_\varepsilon(t,x) = \frac{1}{m_\varepsilon}\Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_t^T \gamma(\tau)\,d\tau\right](x - h_\varepsilon(t)) = \frac{1}{m_\varepsilon}\int_t^T \gamma(\tau)\,d\tau.$$

We deduce that

$$-\int_{\mathbb{R}^+}\int_{\mathcal{S}_\varepsilon(t)} \rho_\varepsilon u_{\mathcal{S},\varepsilon}\cdot\partial_t\psi_\varepsilon\,dxdt = \int_0^T \ell_\varepsilon\cdot\gamma\,dt,$$

and similarly

$$\int_{\mathcal{S}_\varepsilon(0)} \rho_\varepsilon u_{\mathcal{S},\varepsilon}^{in}\cdot\psi_\varepsilon(0,.)\,dx = \ell_\varepsilon^{in}\cdot\int_0^T \gamma(\tau)\,d\tau.$$

To show the L^∞ convergence of ℓ_ε, we rewrite equation (6) with $\varphi = \psi_\varepsilon$ in the form

$$\int_0^T (\ell_\varepsilon - \ell^{in}) \cdot \gamma \, dt = \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot (\partial_t \psi_\varepsilon + u_\varepsilon \cdot \nabla \psi_\varepsilon) - 2\nu D(u_\varepsilon) : D(\psi_\varepsilon) \, dx \, dt + \int_{\mathcal{F}_\varepsilon(0)} u_\varepsilon^{in} \cdot \psi_\varepsilon(0, .) \, dx + (\ell_\varepsilon^{in} - \ell^{in}) \cdot \int_0^T \gamma \, dt. \tag{12}$$

In particular, if we show that the absolute value of the right hand side of (12) is bounded by

$$c(\varepsilon) \|\gamma\|_{L^1(0,T)}$$

with $c(\varepsilon) \longrightarrow 0$, then the convergence $\ell_\varepsilon \longrightarrow \ell^{in}$ follows from (12).

It remains to estimate the right hand side of (12). Let start by computing for $x \in \mathcal{F}_\varepsilon(t)$

$$\begin{aligned} \partial_t \psi_\varepsilon &= \partial_t \left(\frac{1}{m_\varepsilon} \Psi_{\varepsilon,\alpha_\varepsilon} \left[\int_t^T \gamma(\tau) \, d\tau \right] (x - h_\varepsilon(t)) \right) \\ &= - \frac{1}{m_\varepsilon} \Psi_{\varepsilon,\alpha_\varepsilon} \left[\gamma(t) \right] (x - h_\varepsilon(t)) - \ell_\varepsilon \cdot \nabla \left(\frac{1}{m_\varepsilon} \Psi_\varepsilon \left[\int_t^T \gamma(\tau) \, d\tau \right] \right) (x - h_\varepsilon(t)) \end{aligned} \tag{13}$$

At this step the proof in dimension two and three start to differ so let start by consider the case of dimension three.

Proof of Theorem 3 for d = 3. In this case, we choose $\alpha_\varepsilon = 2$. For simplicity we write $\Psi_{\varepsilon,2} = \Psi_\varepsilon$. Let estimate any term of the right hand side of (12) separately. Using the computation (13), we deduce that

$$\begin{aligned} \left| \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot \partial_t \psi_\varepsilon \, dx dt \right| &\le \left| \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot \frac{1}{m_\varepsilon} \Psi_\varepsilon \left[\gamma(t) \right] (x - h_\varepsilon(t)) \, dx dt \right| \\ &\quad + \left| \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot \left[\ell_\varepsilon \cdot \nabla \left(\frac{1}{m_\varepsilon} \Psi_\varepsilon \left[\int_t^T \gamma(\tau) \, d\tau \right] \right) (x - h_\varepsilon(t)) \right] dx dt \right| \\ &\le \|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \left\| \frac{\Psi_\varepsilon}{m_\varepsilon} \left[\gamma(t) \right] \right\|_{L^1(0,T;L^2(\mathbb{R}^3))} \\ &\quad + \|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \left\| \frac{1}{m_\varepsilon} \nabla \Psi_\varepsilon \left[\int_t^T \gamma(\tau) \, d\tau \right] \right\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \|\ell_\varepsilon\|_{L^1(0,T)} \\ &\le C \|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \frac{\varepsilon^{3/2}}{m_\varepsilon} \|\gamma\|_{L^1(0,T)} \\ &\quad + C \|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \frac{\varepsilon^{1/2}}{m_\varepsilon} \left\| \int_t^T \gamma(\tau) \, d\tau \right\|_{L^\infty(0,T)} \|\ell_\varepsilon\|_{L^1(0,T)} \\ &\le C \|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \frac{\varepsilon^{1/2}}{m_\varepsilon} \left(\varepsilon + T \|\ell_\varepsilon\|_{L^\infty(0,T)} \right) \|\gamma\|_{L^1(0,T)}, \end{aligned} \tag{14}$$

where in the third inequality we used points 1 and 3 of Lemma 1 for $q = 2$. Moreover notice that $\|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))}$ is uniformly bounded in ε from (11).

To estimate the convective term

$$\left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot (u_\varepsilon \cdot \nabla \psi_\varepsilon)\, dx dt\right|$$
$$\leq \|u_\varepsilon\|^2_{L^2(0,T;L^6(\mathcal{F}_\varepsilon(t)))} \left\| \frac{1}{m_\varepsilon} \nabla \Psi_\varepsilon \left[\int_t^T \gamma(\tau)\, d\tau\right]\right\|_{L^\infty(0,T;L^{3/2}(\mathcal{F}_\varepsilon(t)))}$$
$$\leq C\|\tilde{u}_\varepsilon\|^2_{L^2(0,T;L^6(\mathbb{R}^3))} \frac{\varepsilon}{m_\varepsilon}\|\gamma\|_{L^1(0,T)}, \tag{15}$$

where we used point 3 of Lemma 1 for $q = 3/2$. Moreover $\|\tilde{u}_\varepsilon\|_{L^2(0,T;L^6(\mathbb{R}^3))}$ is uniformly bounded in ε, in fact by Sobolev embedding and Korn inequality we have

$$\|\tilde{u}_\varepsilon\|_{L^6(\mathbb{R}^3)} \leq \|\nabla \tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} \leq C\|D\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^3)}.$$

Inequality (11) implies that

$$\|\tilde{u}_\varepsilon\|_{L^2(0,T;L^6(\mathbb{R}^3))} \leq C\|D\tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq C.$$

Similarly

$$\left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} Du_\varepsilon \cdot D\psi_\varepsilon\, dx dt\right|$$
$$\leq \|Du_\varepsilon\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \left\| \frac{1}{m_\varepsilon} \nabla \Psi_\varepsilon \left[\int_t^T \gamma(\tau)\, d\tau\right]\right\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))}$$
$$\leq C\|Du_\varepsilon\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \frac{\varepsilon^{1/2}}{m_\varepsilon}\sqrt{T}\|\gamma\|_{L^1(0,T)}. \tag{16}$$

For the term involving the initial fluid velocity, we estimate

$$\left|\int_{\mathcal{F}_\varepsilon(0)} u^{in} \cdot \psi_\varepsilon(0,.)\, dx\right| \leq \left|\int_{\mathcal{F}(0)} \frac{1}{m_\varepsilon} u^{in}_\varepsilon \cdot \Psi_\varepsilon \left[\int_0^T \gamma(\tau)\, d\tau\right] dx\right|$$
$$\leq C\|u^{in}_\varepsilon\|_{L^2(\mathcal{F}(0))} \frac{\varepsilon^{3/2}}{m_\varepsilon}\|\gamma\|_{L^1(0,T)}, \tag{17}$$

where we used point 1 of Lemma 1 for $q = 2$, Finally

$$\left|(\ell^{in}_\varepsilon - \ell^{in}) \cdot \int_0^T \gamma\, d\tau\right| \leq \left|\ell^{in}_\varepsilon - \ell^{in}\right| \|\gamma\|_{L^1(0,T)}. \tag{18}$$

Putting estimates (14)- (15)-(16)-(17)-(18) together and using the uniform estimates (11), the convergence of the initial data and the hypothesis that $m_\varepsilon/\varepsilon^{1/2} \longrightarrow +\infty$, we deduce from (12) that

$$\left|\int_0^T (\ell_\varepsilon - \ell^{in}) \cdot \gamma\, dt\right| \leq \tilde{c}(\varepsilon)\|\gamma\|_{L^1} + \bar{c}(\varepsilon)\|\ell_\varepsilon\|_{L^\infty(0,T)}\|\gamma\|_{L^1(0,T)},$$

where $\tilde{c}(\varepsilon), \bar{c}(\varepsilon) \longrightarrow 0$ and

$$\bar{c}(\varepsilon) = C\|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \frac{\varepsilon^{1/2}}{m_\varepsilon} T. \tag{19}$$

If we divide the right and the left hand side by the L^1 norm of γ and we take the sup for $\gamma \in C^\infty_c(0,T)$, we deduce

$$\|\ell_\varepsilon - \ell^{in}\|_{L^\infty} \leq \tilde{c}(\varepsilon) + \bar{c}(\varepsilon)\|\ell_\varepsilon - \ell^{in}\|_{L^\infty(0,T)} + \bar{c}(\varepsilon)|\ell^{in}|.$$

We can now absorb the second term of the right hand side and deduce that $\ell_\varepsilon \longrightarrow \ell^{in}$ in L^∞. □

Let now move to the case of dimension two. In this case the the energy inequality (11) implies that ℓ_ε is uniformly bounded in $L^\infty(0,T)$ thanks to the hypothesis $m_\varepsilon \geq C > 0$. It remains to show the strong convergence.

Proof of Theorem 3 for d = 2. Choose α_ε such that $\varepsilon\alpha_\varepsilon \longrightarrow 0$ and $\alpha_\varepsilon \longrightarrow +\infty$. Consider test functions

$$\psi_\varepsilon(x,t) = \frac{1}{m_\varepsilon}\Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_t^T \gamma(\tau)\,d\tau\right](x - h_\varepsilon(t)),$$

for $\gamma : [0,T] \longrightarrow \mathbb{R}^2$ with $\gamma \in C_c^\infty((0,T))$. Notice that (12) holds also in the case of dimension two, in fact the same computations are true. We now show that the right hand side of (12) converges to zero.

Let us estimate

$$\begin{aligned}
\left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot \partial_t \psi_\varepsilon\,dxdt\right| &\leq \left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot \frac{1}{m_\varepsilon}\Psi_{\varepsilon,\alpha_\varepsilon}\left[\gamma(t)\right](x - h_\varepsilon(t))\,dxdt\right| \\
&\quad + \left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot \left[\ell_\varepsilon \cdot \nabla\left(\frac{1}{m_\varepsilon}\Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_t^T \gamma\,d\tau\right](x - h_\varepsilon(t))\right)\right]dxdt\right| \\
&\leq \|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))}\left\|\frac{\Psi_{\varepsilon,\alpha_\varepsilon}[\gamma(t)]}{m_\varepsilon}\right\|_{L^1(0,T;L^2(\mathbb{R}^2))} \\
&\quad + \|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))}\left\|\frac{1}{m_\varepsilon}\nabla\Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_t^T \gamma(\tau)\,d\tau\right]\right\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}\|\ell_\varepsilon\|_{L^1(0,T)} \\
&\leq C\|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))}\frac{1}{m_\varepsilon}\left(\varepsilon\alpha_\varepsilon + \frac{1}{\sqrt{\log(\alpha_\varepsilon)}}T\|\ell_\varepsilon\|_{L^\infty(0,T)}\right)\|\gamma\|_{L^1(0,T)}.
\end{aligned}$$

In the third inequality we used points 1 and 2 of Lemma 1 for $q = 2$ and $d = 2$. Then we consider

$$\begin{aligned}
&\left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot (u_\varepsilon \cdot \nabla\psi_\varepsilon)\,dxdt\right| \\
&\leq \|u_\varepsilon\|^2_{L^2(0,T;L^4(\mathcal{F}_\varepsilon(t)))}\left\|\frac{1}{m_\varepsilon}\nabla\Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_t^T \gamma(\tau)\,d\tau\right]\right\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \\
&\leq C\|\tilde{u}_\varepsilon\|^2_{L^2(0,T;L^4(\mathbb{R}^2))}\frac{1}{m_\varepsilon}\frac{1}{\sqrt{\log(\alpha_\varepsilon)}}\|\gamma\|_{L^1(0,T)},
\end{aligned}$$

where we used point 2 of Lemma 1 for $q = 2$. The term

$$\begin{aligned}
&\left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} Du_\varepsilon \cdot D\psi_\varepsilon\,dxdt\right| \\
&\leq \|Du_\varepsilon\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))}\left\|\frac{1}{m_\varepsilon}\nabla\Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_t^T \gamma(\tau)\,d\tau\right]\right\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))} \\
&\leq C\|Du_\varepsilon\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))}\frac{1}{m_\varepsilon}\frac{1}{\sqrt{\log(\alpha_\varepsilon)}}\sqrt{T}\|\gamma\|_{L^1(0,T)}.
\end{aligned}$$

Finally for the terms associated with the initial data,

$$\left|\int_{\mathcal{F}_\varepsilon(0)} u^{in} \cdot \psi_\varepsilon(0,.)\, dx\right| \le \left|\int_{\mathcal{F}(0)} \frac{1}{m_\varepsilon} u_\varepsilon^{in} \cdot \Psi_{\varepsilon,\alpha_\varepsilon}\left[\int_0^T \gamma(\tau)\, d\tau\right] dx\right|$$
$$\le C\|u_\varepsilon^{in}\|_{L^2(\mathcal{F}_\varepsilon(0))}\frac{\varepsilon\alpha_\varepsilon}{m_\varepsilon}\|\gamma\|_{L^1(0,T)},$$

where we used point 1 of Lemma 1 for $q = 2$, and

$$\left|(\ell_\varepsilon^{in} - \ell^{in}) \cdot \int_0^T \gamma\, d\tau\right| \le \left|\ell_\varepsilon^{in} - \ell^{in}\right| \|\gamma\|_{L^1(0,T)}. \tag{20}$$

Putting all the above estimates together, we deduce from (12) that

$$\left|\int_0^T (\ell_\varepsilon - \ell^{in}) \cdot \gamma\, dt\right| \le \tilde{c}(\varepsilon)\|\gamma\|_{L^1},$$

where $\tilde{c}(\varepsilon) \longrightarrow 0$. In fact we have that $\|u_\varepsilon\|_{L^\infty(0,T;L^2(\mathcal{F}_\varepsilon(t)))}$, $\|\ell_\varepsilon\|_{L^\infty(0,T)}$, $\|Du_\varepsilon\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))}$ are uniformly bounded in ε from (11) and Korn inequality, $1/m_\varepsilon$ and $\|\tilde{u}_\varepsilon^{in}\|_{L^2(\mathcal{F}_\varepsilon(0))}$ are uniformly bounded by hypothesis, $1/\sqrt{\log(\alpha_\varepsilon)} \longrightarrow 0$ from the choice $\alpha_\varepsilon \longrightarrow +\infty$ and $|\ell_\varepsilon^{in} - \ell^{in}| \longrightarrow 0$ by hypothesis. It only remains to show a uniform bound for $\|\tilde{u}_\varepsilon\|_{L^2(0,T;L^4(\mathbb{R}^2))}$. By Sobolev embedding see for instance Lemma 4 of [5], we have

$$\|\tilde{u}_\varepsilon\|_{L^2(0,T;L^4(\mathbb{R}^2))} \le C\|\tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))} + C\|\nabla\tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^2))}$$
$$\le C\|\tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\mathcal{F}_\varepsilon(t)))} + C\|D\tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^2))},$$

where in the last step we use the Korn inequality. Notice that the right hand side of the above inequality is uniformly bounded due to (11).

If we divide the right and the left hand side of (20) by the L^1 norm of γ and we take the sup for $\gamma \in C_c^\infty(0,T)$, we deduce

$$\|\ell_\varepsilon - \ell^{in}\|_{L^\infty(0,T)} \le \tilde{c}(\varepsilon) \longrightarrow 0$$

as ε converges to zero. □

Author Contributions All authors contributed equally to this work.

Funding M.B. is supported by the NWO grant OCENW.M20.194. Š. N. has been supported by Praemium Academiæ of Š. Nečasová and by the Czech Science Foundation (GAČR) project 22-01591 S. The Institute of Mathematics, CAS is supported by RVO:67985840.

Availability of Data and Materials The authors of this work claim that the manuscript has no associate data.

Declarations

Conflict of Interest Not applicable.

Ethics Approval and Consent to Participate The submitted work is original and have not been published elsewhere in any form or language (partially or in full). All authors have approved the participation and this submission.

Consent for Publication This manuscript has not been published and is not under consideration for publication elsewhere. All the authors have approved the manuscript and agree with this submission.

A Proof of Proposition 1

In this section we prove Proposition 1 which is a straight-forward extension of Lemma 3 of [13]. First of all for $A, B \in \mathbb{R}$ with $0 < A < B$, we denote by $\alpha = B/A > 1$ and we define the functions

$$f_{A,B}(z) = \begin{cases} 1 & \text{for } 0 \le z < A, \\ \frac{\log z - \log B}{\log A - \log B} & \text{for } A \le z \le B, \\ 0 & \text{for } z > B. \end{cases}$$

It holds that $f_{A,B} \in W^{1,\infty}(\mathbb{R}^+)$. We define the d-dimensional cut-off

$$\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x) = f_{\varepsilon,\alpha_\varepsilon\varepsilon}(|x|),$$

for $x \in \mathbb{R}^d$ and $\alpha_\varepsilon > 1$.

Proposition 2 *The functions $\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}$ have the following properties*

1. $\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty(\mathbb{R}^d) \le 1}$.
2. *For* $1 \le q < +\infty$
$$\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} + \||x|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} \le C(\varepsilon\alpha_\varepsilon)^{d/q}.$$
3. *We have*
$$\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|^d_{L^d(\mathbb{R}^d)} = \frac{2^{d-1}\pi}{(\log\alpha_\varepsilon)^{d-1}}.$$
4. *For* $1 \le q < d$,
$$\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|^q_{L^q(\mathbb{R}^3)} = \frac{2^{d-1}\pi}{d-q}\frac{\alpha_\varepsilon^{d-q}-1}{(\log\alpha_\varepsilon)^q}\varepsilon^{d-q}.$$
5. *For* $1 \le q < d$,
$$\||x|\nabla^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|^q_{L^q(B_{\varepsilon\alpha_\varepsilon}(0)\setminus B_\varepsilon)} \le \frac{2^{d-1}\pi}{d-q}\frac{\alpha_\varepsilon^{d-q}-1}{(\log\alpha_\varepsilon)^q}\varepsilon^{d-q}.$$

Proof After passing to spherical coordinates the proof is straight-forward. For example to show part 2. in dimension three, we compute

$$\begin{aligned}\left\|\tilde{\nabla}\eta_{\varepsilon,\alpha_\varepsilon}\right\|^3_{L^3(\mathbb{R}^3)} &= \int_0^{2\pi}\int_0^\pi\int_\varepsilon^{\varepsilon\alpha_\varepsilon}\left|\frac{1}{r}\frac{1}{\log(\varepsilon)-\log(\varepsilon\alpha_\varepsilon)}\right|^3 \sin(\varphi)r^2\,drd\varphi d\theta \\ &= \frac{4\pi}{(\log(\alpha_\varepsilon))^3.}\left[\log(r)\right]_\varepsilon^{\varepsilon\alpha_\varepsilon} \\ &= \frac{4\pi}{(\log(\alpha_\varepsilon))^2.}\end{aligned}$$

See Lemma 2 of [13] for the proof in dimension two. □

As noticed in [13] the functions $\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}$ satisfy all the bounds of Proposition 1 but they are not smooth, in fact they are not even C^2 on $\partial B_\varepsilon(0) \cup \partial B_{\varepsilon\alpha_\varepsilon}(0)$. To tackle this issue, the authors of [13] modify the functions $\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}$ as follows. They consider $g \in C_c^\infty([0, 12/10))$ such that $0 \le g \le 1$ and $g(y) = 1$ for $y \in [0, 11/10]$. Then they define

$$\eta_{\varepsilon,\alpha_\varepsilon}(x) = 1 + \left(1 - g\left(\frac{|x|}{\varepsilon}\right)\right)\left(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x) g\left(\frac{13}{10}\frac{|x|}{\alpha_\varepsilon \varepsilon}\right) - 1\right), \tag{21}$$

which rewrites

$$\eta_{\varepsilon,\alpha_\varepsilon}(x) = \begin{cases} 1 & \text{for} \quad |x| < \frac{11}{10}\varepsilon, \\ 1 + \left(1 - g\left(\frac{|x|}{\varepsilon}\right)\right)\left(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x) - 1\right) & \text{for} \quad \frac{11}{10}\varepsilon \le |x| < \frac{12}{10}\varepsilon, \\ \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x) & \text{for} \quad \frac{12}{10}\varepsilon \le |x| < \frac{11}{13}\varepsilon\alpha_\varepsilon, \\ \tilde{\eta}_{\varepsilon,\alpha_\varepsilon}(x) g\left(\frac{13}{10}\frac{|x|}{\alpha_\varepsilon\varepsilon}\right) & \text{for} \quad \frac{11}{13}\varepsilon\alpha_\varepsilon \le |x| < \frac{12}{13}\varepsilon\alpha_\varepsilon, \\ 0 & \text{for} \quad |x| \ge \frac{12}{13}\varepsilon\alpha_\varepsilon. \end{cases}$$

The functions $\eta_{\varepsilon,\alpha_\varepsilon}$ are smooth. It remains to show that they satisfy all the properties stated in Proposition 1

Proof of Proposition 1 We verify that the family $\eta_{\varepsilon,\alpha_\varepsilon}$ defined in (21) satisfies all the properties stated in Proposition 1. First of all by definition $\eta_{\varepsilon,\alpha_\varepsilon} \in C_c^\infty(B_{\varepsilon\alpha_\varepsilon}(0))$, $\eta_{\varepsilon,\alpha_\varepsilon}(x) = 1$ for $x \in B_\varepsilon(0)$ and $\|\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty} \le 1$. Let now bound the L^q norm of $\nabla\eta_{\varepsilon,\alpha_\varepsilon}$. As in [13], we denote by

$$g_\varepsilon^1(x) = \left(1 - g\left(\frac{|x|}{\varepsilon}\right)\right), \quad g_\varepsilon^2(x) = g\left(\frac{13}{10}\frac{|x|}{\alpha_\varepsilon\varepsilon}\right)$$

and by $A_{r,R} = B_R(0) \setminus B_r(0)$ the annulus for $0 < r < R$. Finally we notice that

$$\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} = \left\| \frac{\log(|x|/\varepsilon)}{\log(\alpha_\varepsilon)} \right\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} \le \frac{C}{\log(\alpha_\varepsilon)}$$

and similarly

$$\|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{13}\varepsilon\alpha_\varepsilon}\right)} = \left\| \frac{\log(|x|/(\varepsilon\alpha_\varepsilon))}{\log(\alpha_\varepsilon)} \right\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{10}\varepsilon\alpha_\varepsilon}\right)} \le \frac{C}{\log(\alpha_\varepsilon)}.$$

For $1 \le q \le p$, we estimate

$$\begin{aligned} \|\nabla\eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} &\le \|\nabla(g_\varepsilon^1(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1))\|_{L^q\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} + \|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q\left(A_{\frac{12}{10}\varepsilon, \frac{11}{13}\varepsilon\alpha_\varepsilon}\right)} \\ &\quad + \|\nabla(g_\varepsilon^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon})\|_{L^q\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{10}\varepsilon\alpha_\varepsilon}\right)} \\ &\le \|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} \left(\|g_\varepsilon^1\|_{L^\infty(\mathbb{R}^d)} + 1 + \|g_\varepsilon^2\|_{L^\infty(\mathbb{R}^d)}\right) \\ &\quad \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon, \frac{12}{10}\varepsilon}\right)} \|\nabla g_\varepsilon^1\|_{L^q(\mathbb{R}^d)} + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon, \frac{12}{13}\varepsilon\alpha_\varepsilon}\right)} \|\nabla g_\varepsilon^2\|_{L^q(\mathbb{R}^d)} \\ &\le C\|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} + \frac{C}{\log(\alpha_\varepsilon)}((\alpha_\varepsilon\varepsilon)^{(d-q)/q} + \varepsilon^{(d-q)/q}), \end{aligned}$$

where we use that $1 - g_\varepsilon^1$ and g_ε^2 are appropriate rescaling of g to estimate the L^q norm of ∇g_ε^1 and ∇g_ε^2. The bounds of the L^q norm of $\nabla \eta_{\varepsilon,\alpha_\varepsilon}$ follows from the above estimate and Proposition 2.

Similarly we estimate

$$\begin{aligned}
\||x|\nabla \eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} \leq & \||x|\nabla(g_\varepsilon^1(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}-1))\|_{L^q\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)} + \||x|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q\left(A_{\frac{12}{10}\varepsilon,\frac{11}{13}\varepsilon\alpha_\varepsilon}\right)} \\
& + \||x|\nabla(g_\varepsilon^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon})\|_{L^q\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{10}\varepsilon\alpha_\varepsilon}\right)} \\
\leq & \||x|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)}\left(\|g_\varepsilon^1\|_{L^\infty(\mathbb{R}^d)} + 1 + \|g_\varepsilon^2\|_{L^\infty(\mathbb{R}^d)}\right) \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)}\||x|\nabla g_\varepsilon^1\|_{L^q(\mathbb{R}^d)} \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{13}\varepsilon\alpha_\varepsilon}\right)}\||x|\nabla g_\varepsilon^2\|_{L^q(\mathbb{R}^d)} \\
\leq & C\left(\||x|\nabla^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} + \|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)}\right) + \frac{C}{\log(\alpha_\varepsilon)}((\alpha_\varepsilon\varepsilon)^{d/q} + \varepsilon^{d/q}).
\end{aligned}$$

where we use that $1 - g_\varepsilon^1$ and g_ε^2 are appropriate rescaling of g to estimate the L^∞ norm of $|x|\nabla g_\varepsilon^1$ and $|x|\nabla g_\varepsilon^2$. Finally we estimate

$$\begin{aligned}
\||x|\nabla^2 \eta_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} \leq & \||x|\nabla^2(g_\varepsilon^1(\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}-1))\|_{L^q\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)} + \||x|\nabla^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q\left(A_{\frac{12}{10}\varepsilon,\frac{11}{13}\varepsilon\alpha_\varepsilon}\right)} \\
& + \||x|\nabla^2(g_\varepsilon^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon})\|_{L^q\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{10}\varepsilon\alpha_\varepsilon}\right)} \\
\leq & \||x|\nabla^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)}\left(\|g_\varepsilon^1\|_{L^\infty(\mathbb{R}^d)} + 1 + \|g_\varepsilon^2\|_{L^\infty(\mathbb{R}^d)}\right) \\
& + \|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)}\left(\||x|\nabla g_\varepsilon^1\|_{L^\infty(\mathbb{R}^d)} + \||x|\nabla g_\varepsilon^2\|_{L^\infty(\mathbb{R}^d)}\right) \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon} - 1\|_{L^\infty\left(A_{\frac{11}{10}\varepsilon,\frac{12}{10}\varepsilon}\right)}\||x|\nabla^2 g_\varepsilon^1\|_{L^q(\mathbb{R}^d)} \\
& + \|\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^\infty\left(A_{\frac{11}{13}\varepsilon\alpha_\varepsilon,\frac{12}{13}\varepsilon\alpha_\varepsilon}\right)}\||x|\nabla^2 g_\varepsilon^2\|_{L^q(\mathbb{R}^d)} \\
\leq & C\left(\||x|\nabla^2\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)} + \|\nabla\tilde{\eta}_{\varepsilon,\alpha_\varepsilon}\|_{L^q(\mathbb{R}^d)}\right) \\
& + \frac{C}{\log(\alpha_\varepsilon)}((\alpha_\varepsilon\varepsilon)^{(d-q)/q} + \varepsilon^{(d-q)/q}).
\end{aligned}$$

where as before we use that $1 - g_\varepsilon^1$ and g_ε^2 are appropriate rescaling of g to estimate the L^q norm of $|x|\nabla^2\nabla g_\varepsilon^1$ and $|x|\nabla^2 g_\varepsilon^2$. □

References

1. Al Baba, H., Ghosh, A., Muha, B., Nečasová, Š: $L^p - L^q$-strong solution to fluid-rigid body interaction system with Navier slip boundary condition. J. Elliptic Parabolic Equ. **7**(2), 439–489 (2021)
2. Bravin, M.: Energy equality and uniqueness of weak solutions of a "viscous incompressible fluid+ rigid body" system with Navier slip-with-friction conditions in a 2D bounded domain. J. Math. Fluid Mech. **21**(2), 1–31 (2019)
3. Bravin, M.: On the "2D viscous incompressible fluid+ rigid body" system with Navier conditions and unbounded energy. Comptes Rendus. Mathématique **358**(3), 303–319 (2020)
4. Bravin, M., Nečasová, Š: On the vanishing rigid body problem in a viscous compressible fluid. J. Diff. Equ. **345**, 45–77 (2023)

5. Bravin, M., Nečasová, Š.: On the Trajectory of a Light Small Rigid Body in an Incompressible Viscous Fluid. (2022) arXiv preprint arXiv:2211.15610
6. Chemetov, N.V., Nečasová, Š, Muha, B.: Weak-strong uniqueness for fluid-rigid body interaction problem with slip boundary condition. J. Math. Phys. **60**, 13 (2019)
7. Conca, C., Martín, Jorge San, H., Tucsnak, M.: Motion of a rigid body in a viscous fluid. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics **328**(6), 473–478 (1999)
8. Desjardins, B., Esteban, M.J.: Existence of weak solutions for the motion of rigid bodies in a viscous fluid. Arch. Ration. Mech. Anal. **146**(1), 59–71 (1999)
9. Feireisl, E.: On the motion of rigid bodies in a viscous fluid. Appl. Mathemat. **47**(6), 463–484 (2002)
10. Feireisl, E.: On the motion of rigid bodies in a viscous incompressible fluid. J. Evol. Equ. **3**, 419–441 (2003)
11. Feireisl, E., Roy, A., Zarnescu, A.: On the Motion of a Nearly Incompressible Viscous Fluid Containing a Small Rigid Body. (2022) arXiv preprint arXiv:2206.02931
12. Feireisl, E., Roy, A., Zarnescu, A.: On the Motion of a Small Rigid Body in a Viscous Compressible Fluid. (2022) arXiv preprint arXiv:2208.07933
13. He, J., Iftimie, D.: A small solid body with large density in a planar fluid is negligible. J. Dyn. Diff. Equat. **31**(3), 1671–1688 (2019)
14. He, J., Iftimie, D.: On the small rigid body limit in 3D incompressible flows. J. Lond. Math. Soc. **104**(2), 668–687 (2021)
15. Galdi, G. P., Silvestre, A. L.: Strong Solutions to the Problem of Motion of a Rigid Body in a Navier-Stokes Liquid under the Action of Prescribed Forces and Torques. In Nonlinear problems in mathematical physics and related topics I (pp. 121-144). Springer, Boston, MA (2002)
16. Galdi, G.P., Silvestre, A.L.: The steady motion of a Navier-Stokes liquid around a rigid body. Arch. Ration. Mech. Anal. **184**(3), 371–400 (2007)
17. Galdi, G.P., Silvestre, A.: Existence of time-periodic solutions to the Navier-Stokes equations around a moving body. Pac. J. Math. **223**(2), 251–267 (2006)
18. Geissert, M., Götze, K., Hieber, M.: L^p-theory for strong solutions to fluid-rigid body interaction in Newtonian and generalized Newtonian fluids. Trans. Am. Math. Soc. **365**(3), 1393–1439 (2013)
19. Gérard-Varet, D., Hillairet, M.: Existence of weak solutions up to collision for viscous fluid-solid systems with slip. Commun. Pure Appl. Math. **67**(12), 2022–2076 (2014)
20. Glass, O., Munnier, A., Sueur, F.: Dynamics of a Point Vortex as Limits of a Shrinking Solid in an Irrotational Fluid. (2014) arXiv preprint arXiv:1402.5387
21. Glass, O., Munnier, A., Sueur, F.: Point vortex dynamics as zero-radius limit of the motion of a rigid body in an irrotational fluid. Invent. Math. **214**(1), 171–287 (2018)
22. Glass, O., Lacave, C., Sueur, F.: On the motion of a small body immersed in a two-dimensional incompressible perfect fluid. Bull. Soc. Math. France **142**(3), 489–536 (2014)
23. Glass, O., Lacave, C., Sueur, F.: On the motion of a small light body immersed in a two dimensional incompressible perfect fluid with vorticity. Commun. Math. Phys. **341**(3), 1015–1065 (2016)
24. Glass, O., Sueur, F.: Uniqueness results for weak solutions of two-dimensional fluid-solid systems. Arch. Ration. Mech. Anal. **218**(2), 907–944 (2015)
25. Grandmont, C., Maday, Y.: Existence for an unsteady fluid-structure interaction problem. ESAIM: Mathemat. Modell. Numer. Anal. **34**(3), 609–636 (2000)
26. Gunzburger, M.D., Lee, H.C., Seregin, G.A.: Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. J. Math. Fluid Mech. **2**(3), 219–266 (2000)
27. Judakov, N.V.: The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. Dinamika Splošn. Sredy **18**, 249–253 (1974)
28. Lacave, C., Takahashi, T.: Small moving rigid body into a viscous incompressible fluid. Arch. Ration. Mech. Anal. **223**(3), 1307–1335 (2017)
29. Maity, D., Tucsnak, M.: Lp-Lq maximal regularity for some operators associated with linearized incompressible fluid-rigid body problems. Mathemat. Anal. Fluid Mech.-Selec. Recent Res **710**, 175–201 (2018)
30. Muha, B., Nečasová, Š, Radošević, A.: A uniqueness result for 3D incompressible fluid-rigid body interaction problem. J. Math. Fluid Mech. **23**(1), 1–39 (2021)
31. Muha, B., Nečasová, Š., Radošević, A.: On the Regularity of Weak Solutions to the Fluid-Rigid Body Interaction P[roblem. (2022) arXiv preprint arXiv:2211.03080
32. Planas, G., Sueur, F.: On the "viscous incompressible fluid+ rigid body" system with Navier conditions. Annales de l'Institut Henri Poincaré C **31**(1), 55–80 (2014)
33. Serre, D.: Chute libre d'un solide dans un fluide visqueux incompressible existence. Japan J. Appl. Mathemat. **4**(1), 99–110 (1987)

34. Takahashi, T.: Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. Adv. Diff. Equ. **8**(12), 1499–1532 (2003)
35. Takahashi, T., Tucsnak, M.: Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. J. Math. Fluid Mech. **6**(1), 53–77 (2004)