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Model-Free Bounds for Multi-Asset Options Using Option-Implied Information and Their Exact Computation

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Abstract. We consider derivatives written on multiple underlyings in a one-period financial market, and we are interested in the computation of model-free upper and lower bounds for their arbitrage-free prices. We work in a completely realistic setting, in that we only assume the knowledge of traded prices for other single- and multi-asset derivatives and even allow for the presence of bid–ask spread in these prices. We provide a fundamental theorem of asset pricing for this market model, as well as a superhedging duality result, that allows to transform the abstract maximization problem over probability measures into a more tractable minimization problem over vectors, subject to certain constraints. Then, we recast this problem into a linear semi-infinite optimization problem and provide two algorithms for its solution. These algorithms provide upper and lower bounds for the prices that are ε-optimal, as well as a characterization of the optimal pricing measures. These algorithms are efficient and allow the computation of bounds in high-dimensional scenarios (e.g., when $d = 60$). Moreover, these algorithms can be used to detect arbitrage opportunities and identify the corresponding arbitrage strategies. Numerical experiments using both synthetic and real market data showcase the efficiency of these algorithms, and they also allow understanding of the reduction of model risk by including additional information in the form of known derivative prices.

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1. Introduction

The classical paradigm in finance and theoretical economics assumes the existence of a model that provides an accurate description of the evolution of asset prices, and all subsequent computations about hedging strategies, exotic derivatives, risk measures, and so forth, are based on this model. However, academics, practitioners, and regulators have realized that all models provide only a partially accurate description of this reality; thus, either methods need to be developed to aggregate the results of many models or approaches must be devised that allow for computations in the absence of a specific model. The first approach led to the introduction of robust methods in asset pricing and no-arbitrage theory (Rigotti and Shannon 2005, Maruhn 2009, Dana and Riedel 2013, Epstein and Ji 2013, Neufeld and Nutz 2013, Possamaï et al. 2013, Bayraktar et al. 2015, Bouchard and Nutz 2015, Beissner 2017, Yan et al. 2022, Beissner and Riedel 2019, Bouchard et al. 2019), whereas the second one led to model-free methods in asset pricing and no-arbitrage theory (Hobson 1998; Bertsimas and Bushueva 2006; Beiglböck et al. 2013; Henry-Labordère 2013; Davis et al. 2014; Dolinsky and Soner 2014a; Galichon et al. 2014; Riedel 2015; Acciaio et al. 2016; Burzoni et al. 2016, 2019, 2021; Cheridito et al. 2017; Dolinsky and Neufeld 2018; Bartl et al. 2019, 2020; Hu et al. 2019; Lütkebohmert and Sester 2019).

In this work, we consider derivatives written on multiple underlyings in a one-period financial market, and we are interested in the computation of upper and lower bounds for their arbitrage-free prices. We work in a completely realistic setting, in that we only assume the knowledge of traded prices for other single- and multi-asset derivatives and even allow for the presence of bid–ask spread in these prices. In other words, we work in a model-free setting in the presence of option-implied information, and make no assumption about the probabilistic evolution of asset prices (i.e., their marginal distributions) or their dependence structure.

The computation of bounds for the prices of multi-asset options, most often basket options, is a classical problem in the mathematical finance literature and has connections with several other branches of mathematics, such as probability theory, optimal transport, operations research, and optimization. In the most classical setting, one assumes that the marginal distributions are known, and the joint law is unknown; this framework is known as dependence uncertainty. In this framework, several authors have derived bounds for multi-asset options using tools from probability theory, such as copulas and Fréchet–Hoeffding bounds (Dhaene et al. 2002a, b; Hobson et al. 2005a, b; Chen et al. 2008). These bounds turned out to be very wide for practical applications; hence, recently there was an interest in methods that allow for the inclusion of additional information on the dependence structure to reduce this gap. This led to the creation of improved Fréchet–Hoeffding bounds and the pricing of multi-asset options in the presence of additional information on the dependence structure (Tankov 2011, Puccetti et al. 2016, Lux and Papapantoleon 2017).

The setting of dependence uncertainty is intimately linked with optimal transport theory, and its tools have also been used to derive bounds for multi-asset option prices (see Bartl et al. 2022 for a formulation in the presence of additional information on the joint distribution). More recently, De Gennaro Aquino and Bernard (2020), Eckstein and Kupper (2021), and Eckstein et al. (2021) have translated the model-free superhedging problem into an optimization problem over classes of functions by extending results in optimal transport, and used neural networks and the stochastic gradient descent algorithm for the computation of the bounds.

Ideas from operations research and optimization have also been applied for the computation of model-free bounds in settings that are closer to ours and do not necessarily assume knowledge of the marginal distributions (or, equivalently, knowledge of call option prices for a continuum of strikes). Bertsimas and Popescu (2002) consider the computation of the model-free bounds on a single-asset call option given the moments of the underlying asset price and the model-free bounds on a single-asset call option given other single-asset call and put option prices. In addition, they also consider specific conditions under which the model-free bounds on a multi-asset option can be theoretically computed in polynomial time. d'Aspremont and El Ghaoui (2006) consider a framework where the prices of forwards and single-asset call options are known and compute upper and lower bounds on basket options prices using linear programming. In the more general case where the prices of other basket options are also known, they derive a relaxation to the problem that can be solved using linear programming. This work was later extended by various authors. Peña et al. (2010b) improve the results of d'Aspremont and El Ghaoui (2006) when computing the lower bounds on basket options prices in two special cases: (i) when the number of assets is limited to two and prices of basket options are known and (ii) when the prices of only a forward and a single-asset call option per asset are known. Peña et al. (2012) develop a linear programming-based approach for the problem of computing the upper price bound of a basket option given bid and ask prices of vanilla call options. Peña et al. (2010a) study the problem of computing the upper and lower bounds on basket and spread option prices when the prices of other basket and spread option prices are known. Their approach involves solving a large linear programming problem via the Dantzig–Wolfe decomposition in which the corresponding subproblem is solved using mixed-integer programming. Compared with d'Aspremont and El Ghaoui (2006), Peña et al. (2010a), and Peña et al. (2010b, 2012), the numerical methods we develop in Section 3 apply to settings that are much more general, where the derivative being priced and the traded derivatives with known prices can be any continuous piece-wise affine function (including, but not limited to, vanilla, basket, spread, and rainbow options, as well as any linear combination of these options). Moreover, as we demonstrate in Section 4, these methods are able to efficiently compute the price bounds in high-dimensional scenarios, for example, when 60 assets are considered. This is considerably higher compared with existing studies. Daum and Werner (2011) develop a discretization-based algorithm for solving linear semi-infinite programming problems that returns a feasible solution and apply the algorithm to compute the upper bounds on basket or spread options prices when single-asset call, put, and exotic options prices are known. Cho et al. (2016) develop methods similar to Daum and Werner (2011) but for lower bounds on basket or spread options prices. The algorithm we introduce in Section 3.1 takes a similar approach but is able to solve the problem when the prices of multi-asset options with a more general class of payoff functions are known. Kahalé (2017) uses a central cutting plane algorithm to compute the super- and subreplicating prices of financial derivatives using hedging portfolios that consist of other financial derivatives in the multiperiod discrete-time setting. The algorithm only works under the assumption that the underlying state space (i.e., the space of asset prices) is finite. When the state space is infinite, it is discretized before applying the central cutting plane algorithm, and the discretization error is analyzed. However, the approach of discretizing the state space has limited applicability to the multidimensional settings (i.e., with multiple underlying assets) because of the curse of dimensionality. The algorithm we develop in Section 3.2 is also based on a central cutting plane algorithm, but it allows us to efficiently compute model-free price bounds in high-dimensional state spaces for financial derivatives that depend on multiple assets.

Our contributions are three-fold: First, we provide a fundamental theorem of asset pricing for the market model described previously, as well as a superhedging duality, that allows to transform the abstract maximization problem over probability measures into a more tractable problem over vectors, subject to certain constraints. Second, we recast this problem into a linear semi-infinite optimization problem and provide two algorithms for its solution. These algorithms provide upper and lower bounds for the prices of multiasset derivatives that are ε-optimal, as well as a characterization of the optimal pricing measures. These algorithms are efficient and allow the computation of bounds in high-dimensional scenarios (e.g., when $d = 60$) that were not possible by previous methods. Moreover, these algorithms can be used to detect arbitrage opportunities in multi-asset financial markets and to identify the corresponding arbitrage strategies. Third, we perform numerical experiments using synthetic data and real market data to showcase the efficiency of these algorithms. These experiments allow us to understand the reduction of the no-arbitrage gap, that is, the difference between the upper and lower no-arbitrage bounds, by including additional information in the form of known derivative prices. The no-arbitrage gap directly reflects the model-risk associated to a particular derivative and the information available in the market. The numerical experiments show a decrease of the model-risk by the inclusion of additional information, although this decrease is not uniform and depends on the form of information and the specific structure of the payoff functions.

This paper is organized as follows: In Section 2, we present the modeling framework, state the no-arbitrage theorem and the superhedging duality, and discuss a setting that is relevant for practical applications. In Section 3, we present the algorithms that have been developed for the computation of model-free bounds and state theorems that show the validity of these algorithms. In Section 4, we discuss various numerical experiments using both synthetic and real market data that show the efficiency of the algorithms and the reduction of model-risk by the inclusion of additional information in the form of known derivative prices. We also perform a numerical experiment to show the ability of the algorithms to detect arbitrage opportunities. Appendix A contains the proofs of the main results of this paper. The online appendices contain additional remarks and discussions about the theoretical results, the numerical methods, and the numerical experiments, as well as the proofs of the results in Section 2.

2. Duality in the Presence of Option-Implied Information

In this section, we introduce a general framework for a model-free, one-period, financial market where multiple assets and several single- and multi-asset derivatives written on these assets are traded simultaneously. Model-free means that we will not make any assumption about the probabilistic model that governs the evolution of asset prices. Instead, we will use information available in the financial market and implied by the prices of single- and multi-asset derivatives. We will provide both a fundamental theorem and a superhedging duality in this setting, where our results and proofs are inspired by Bouchard and Nutz (2015). Moreover, we will describe concrete examples of this framework that are of practical interest.

Throughout this work, all vectors are column vectors unless otherwise stated. We denote vectors and vector-valued functions by boldface symbols. For a vector $\boldsymbol{x}$ in a Euclidean space, let $[\boldsymbol{x}]_j$ denote the jth component of $\boldsymbol{x}$. For simplicity, we also use x_j to denote $[\boldsymbol{x}]_j$ when there is no ambiguity. Let $\|\boldsymbol{x}\|$ denote the Euclidean norm of $\boldsymbol{x}$. Let $\langle \boldsymbol{x}, \boldsymbol{x}' \rangle$ denote the Euclidean inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$. We denote by $\boldsymbol{e}_i$ the ith standard basis vector of a Euclidean space, by $\boldsymbol{0}$ the vector with all entries equal to zero, that is, $\boldsymbol{0} = (0, \dots, 0)^{\mathrm{T}}$, and by $\boldsymbol{1}$ the vector with all entries equal to one, that is, $\boldsymbol{1} = (1, \dots, 1)^{\mathrm{T}}$. We call a subset of a Euclidean space a polyhedron or a polyhedral set if it is the intersection of finitely many closed half-spaces. We call a subset of a Euclidean space a polytope if it is a bounded polyhedron.

Let Ω be a Polish space equipped with its Borel σ-algebra denoted by $\mathcal{B}(\Omega)$. Let $\mathcal{P}(\Omega)$ denote the set of Borel probability measures on Ω. Let $g_j : \Omega \to \mathbb{R}$ be Borel measurable for $j = 1, \dots, m$, for some fixed $m \in \mathbb{N}$, and let $\boldsymbol{g} : \Omega \to \mathbb{R}^m$ denote the vector-valued Borel measurable function where the jth component corresponds to g_j. Let $\underline{\pi}_j, \overline{\pi}_j \in \mathbb{R}$ be such that $\underline{\pi}_j \le \overline{\pi}_j$ for $j = 1, \dots, m$. Let $\boldsymbol{y} = (y_1, \dots, y_m)^{\mathrm{T}} \in \mathbb{R}^m$ and define $\pi : \mathbb{R}^m \to \mathbb{R}$ as

$$\pi(\boldsymbol{y}) := \sum_{j=1}^{m} y_j^+ \overline{\pi}_j - y_j^- \underline{\pi}_j, \tag{2.1}$$

where $y_j^+ := \max\{y_j, 0\}, y_j^- := \max\{-y_j, 0\}$. Let $\langle \boldsymbol{y}, \boldsymbol{g} \rangle$ denote the function $\sum_{j=1}^{m} y_j g_j : \Omega \to \mathbb{R}$.

We make the following *no-arbitrage* assumption.

Assumption 2.1 (No-Arbitrage). *The following implication holds for any* $\boldsymbol{y} \in \mathbb{R}^m$:

$$\langle \boldsymbol{y}, \boldsymbol{g} \rangle - \pi(\boldsymbol{y}) \ge 0 \Rightarrow \langle \boldsymbol{y}, \boldsymbol{g} \rangle - \pi(\boldsymbol{y}) = 0,$$

where the inequality and the equality are both understood as pointwise.

Remark 2.2. Assumption 2.1 is inspired by the no-arbitrage assumption introduced in definition 1.1 of Bouchard and Nutz (2015), where the set of possible models for the market is $\mathcal{P}(\Omega)$, that is, all Borel probability measures, and a single time step is considered. The difference between Assumption 2.1 and the no-arbitrage assumption in Bouchard and Nutz (2015) is that the price of a financial derivative in the present work is not a singleton but can lie anywhere between the corresponding bid and ask prices. There are other notions of no-arbitrage that are weaker than Assumption 2.1, for example, the "no uniform strong arbitrage" assumption in definition 2.1 of Bartl et al. (2022).

Let $f:\Omega\to\mathbb{R}$ be a Borel measurable function and define the functional $\phi(f)$ as follows:

$$\phi(f) := \inf\{c+\pi(\boldsymbol{y}) : c\in\mathbb{R}, \boldsymbol{y}\in\mathbb{R}^m, c+\langle \boldsymbol{y},\boldsymbol{g}\rangle \ge f\}. \tag{2.2}$$

Let $\mathcal{Q}$ be defined as follows:

$$\mathcal{Q} := \left\{\mu\in\mathcal{P}(\Omega) : \underline{\pi}_j \le \int_\Omega g_j \,\mathrm{d}\mu \le \overline{\pi}_j, \text{ for } j=1,\dots,m\right\}.$$

The main results of this section are the following fundamental theorem and superhedging duality, whose proofs are provided in Appendix EC.4.

Theorem 2.3 (Fundamental Theorem). *The following are equivalent:*

(i) *Assumption 2.1 holds.*

(ii) *For all $\nu\in\mathcal{P}(\Omega)$, there exists $\mu\in\mathcal{Q}$ such that $\nu\ll\mu$.*

Theorem 2.4 (Superhedging Duality). *Under Assumption 2.1, the following statements hold.*

(i) *We have that $\phi(f) > -\infty$.*

(ii) *There exists $\boldsymbol{y}\in\mathbb{R}^m$ such that $\phi(f)+\langle \boldsymbol{y},\boldsymbol{g}\rangle-\pi(\boldsymbol{y})\ge f$. Hence, the infimum in (2.2) is attained when $\phi(f)<\infty$.*

(iii) *We have the following superhedging duality result:*

$$\phi(f) = \sup_{\mu\in\mathcal{Q}} \int_\Omega f\,\mathrm{d}\mu. \tag{2.3}$$

Remark 2.5. No-arbitrage conditions and fundamental theorems of asset pricing are essential tools to understand and characterize the viability of a model in a financial market. They must be tailored to the modeling assumptions and the specific applications in mind; hence, a multitude of comparable statements exist in the literature. Analogously to our no-arbitrage condition, the fundamental theorem presented in Theorem 2.3 is closely related to the first fundamental theorem in Bouchard and Nutz (2015). The main difference is the presence of a bid–ask spread, which means that we cannot exactly reduce our results to their theorem and another proof is needed; this proof is motivated by the results in Bouchard and Nutz (2015). There are several other versions of a fundamental theorem in the presence of model uncertainty in the mathematical finance literature (discrete time models: Bayraktar et al. 2014, Acciaio et al. 2016, Bayraktar and Zhang 2016, Burzoni et al. 2021; continuous time models: Dolinsky and Soner 2014b, Biagini et al. 2017).

Let us point out that the fundamental theorem presented here plays a particular role in conjunction with the numerical methods developed in the next section. More specifically, it provides a sufficient condition for the detection of arbitrage opportunities by numerically testing the violation of the no-arbitrage condition. Moreover, it provides a sufficient condition for repairing derivative prices by removing arbitrage opportunities from the market, in the same spirit as Cohen et al. (2020). These results are novel in the related literature on multi-asset model-free price bounds and are facilitated by the tailor-made fundamental theorem.

Remark 2.6. Superhedging dualities are also classical and essential tools in mathematical finance, typically tailored to specific modeling assumptions and applications. The superhedging duality presented in Theorem 2.4 is motivated by the superhedging theorem in Bouchard and Nutz (2015), with the main difference being once again that we are considering an interval of bid and ask prices instead of a single price. There are multiple comparable duality results or superhedging theorems in various areas of mathematics. In the mathematical finance literature, these results are known as superhedging theorems or (martingale) optimal transport dualities (Beiglböck et al. 2013, Bayraktar et al. 2014, Dolinsky and Soner 2014a, Acciaio et al. 2016, Cheridito et al. 2017). In the operations research literature, these results are known as perfect or strong dualities (Bertsimas and Popescu 2002, d'Aspremont and El Ghaoui 2006, Nishihara et al. 2007, Peña et al. 2010b). These latter dualities are typically based on classical results in mathematical programming (Karlin and Studden 1966, Hettich and Kortanek 1993).

Let us point out that the superhedging duality (2.3) is crucial when verifying the ε-optimality of a measure in the numerical algorithms introduced in Section 3 (see Theorem 3.7 and Corollary 3.8).

The canonical way to interpret the framework developed earlier is as follows: when $\Omega=\mathbb{R}^d_+$, then there exist d underlying risky assets that are traded in the financial market, and Ω represents the (nonnegative) prices of the assets at a fixed future date. Investing into a unit of the asset i then corresponds to the payoff function $g(\boldsymbol{x})\equiv \mathrm{proj}_i(\boldsymbol{x}) := x_i$ for $\boldsymbol{x}\in\mathbb{R}^d_+$. Moreover, there exist m traded derivatives (typically $m\gg d$) with known bid and ask prices $(\underline{\pi}_j,\overline{\pi}_j)_{j=1:m}$, written either on single or on multiple assets. The payoff function g_j of a single-asset derivative depends on the price of only a single asset, that is, $g_j=\widetilde{g}_j\circ\mathrm{proj}_i$ for some $i\in\{1,\dots,m\}$ and $\widetilde{g}_j:\mathbb{R}_+\to\mathbb{R}$. For example, $g_j(\boldsymbol{x}) = (x_i-\kappa_j)^+$ corresponds to a call option with strike κ_j. The payoff function g_j of a multi-asset derivative depends on the prices of multiple assets. For example, $g_j(\boldsymbol{x}) = (\langle \boldsymbol{w}, \boldsymbol{x}\rangle - \kappa_j)^+$ corresponds to a basket call option with weight $\boldsymbol{w}$ and strike κ_j. These derivatives encode all the information available in this market. Specifically, information about the marginals of the probability measures in $\mathcal{Q}$ is *implied* by the bid and ask prices of single-asset derivatives, whereas *partial* information on the joint distribution is *implied* by the bid and ask prices of multi-asset derivatives.

In this setting, the right-hand side of (2.3) is the model-free upper bound for the price of a derivative with payoff function f written on these d assets. The optimization takes place over all probability measures that are compatible with the option-implied information, that is, all probability measures that produce prices for a given option within its respective bid and ask prices. The duality result in (2.3) states that this model-free upper bound equals the least superhedging price achieved by trading in the m derivatives according to the strategy $(c, \boldsymbol{y})$, that is, holding c units of cash and y_j units of derivative j for $j = 1, \ldots, m$, where the minimization takes place over all $(c, \boldsymbol{y})$ such that the payoff f is dominated, that is, $c + \langle \boldsymbol{y}, \boldsymbol{g}\rangle \geq f$.

Section EC.1 in the online appendices contains additional discussions about the duality result, including Example EC.1.2 that demonstrates that the supremum on the right-hand side of (2.3) is not necessarily attained, as well as Proposition EC.1.3, which provides a specific setting in which Assumption 2.1 holds.

3. Numerical Methods for the Computation of Bounds

The superhedging duality in Theorem 2.4 allows transformation of the abstract maximization problem over probability measures into a more tractable minimization problem over vectors that satisfy certain constraints. The aim of this section is to develop novel numerical methods for the exact and efficient computation of upper and lower bounds on $\phi(f)$. More specifically, we will develop methods for the computation of upper and lower bounds $\phi(f)^{\mathrm{UB}}$ and $\phi(f)^{\mathrm{LB}}$, which are ε-optimal, that is,

$$\phi(f)^{\mathrm{LB}} \leq \phi(f) \leq \phi(f)^{\mathrm{UB}} \quad \text{and} \quad \phi(f)^{\mathrm{UB}} - \phi(f)^{\mathrm{LB}} \leq \varepsilon,$$

for $\varepsilon > 0$. Our methods allow us to also characterize the optimal pricing measure associated with the primal maximization problem. Therefore, we provide a complete solution to both optimization problems, and can characterize the solution both in terms of ε-optimal hedging strategies and in terms of the optimal pricing measure.

Let $\overline{\boldsymbol{\pi}} := (\overline{\pi}_1, \ldots, \overline{\pi}_m)^{\mathrm{T}}$ and $\underline{\boldsymbol{\pi}} := (\underline{\pi}_1, \ldots, \underline{\pi}_m)^{\mathrm{T}}$. The minimization problem $\phi(f)$ in (2.2) can be equivalently formulated as a linear semi-infinite programming (LSIP) problem, that is, as an optimization problem with a linear objective and an infinite number of linear constraints, one for each $\omega \in \Omega$,

$$\begin{aligned} \phi(f) = \text{minimize} \quad & c + \langle \boldsymbol{y}^+, \overline{\boldsymbol{\pi}}\rangle - \langle \boldsymbol{y}^-, \underline{\boldsymbol{\pi}}\rangle \\ \text{subject to} \quad & c + \langle \boldsymbol{y}^+ - \boldsymbol{y}^-, \boldsymbol{g}(\omega)\rangle \geq f(\omega) \quad \forall \omega \in \Omega, \\ & c \in \mathbb{R}, \boldsymbol{y}^+ \geq \boldsymbol{0}, \boldsymbol{y}^- \geq \boldsymbol{0}. \end{aligned} \tag{3.1}$$

LSIP problems are classical optimization problems that have been thoroughly studied in the related literature (Goberna and López 1998, 2018). More general semi-infinite programming problems, including nonlinear semi-infinite programming problems and *generalized* semi-infinite programming problems (where the index set can depend on the decision variable), have also been studied in the literature (Reemtsen and Rückmann 1998, López and Still 2007, Stein 2012). In this section, we develop novel algorithms tailored to solving (3.1) under different assumptions on the space Ω and the functions $\boldsymbol{g}$ and f.

Let us first introduce the notion of continuous piecewise affine (CPWA) functions and their radial functions.

Definition 3.1 (Continuous Piece-Wise Affine Function and Its Radial Function). We call a function $h : \mathbb{R}^d \to \mathbb{R}$ a CPWA function if it can be represented as

$$h(\boldsymbol{x}) = \sum_{k=1}^{K} \xi_k \max\{\langle \boldsymbol{a}_{k,i}, \boldsymbol{x}\rangle + b_{k,i} : 1 \leq i \leq I_k\}, \tag{3.2}$$

where $K \in \mathbb{N}$, $I_k \in \mathbb{N}$ for $k = 1, \ldots, K$, and $\boldsymbol{a}_{k,i} \in \mathbb{R}^d$, $b_{k,i} \in \mathbb{R}$, $\xi_k \in \{-1, 1\}$ for $i = 1, \ldots, I_k, k = 1, \ldots, K$. The *radial function* of h, denoted by $\widetilde{h} : \mathbb{R}^d_+ \to \mathbb{R}$, is defined as

$$\widetilde{h}(\boldsymbol{z}) := \sum_{k=1}^{K} \xi_k \max\{\langle \boldsymbol{a}_{k,i}, \boldsymbol{z}\rangle : 1 \leq i \leq I_k\}.$$

The class of CPWA functions contains many popular payoff functions in finance, including vanilla call and put options, basket options, spread options, call/put-on-max options, call/put-on-min options, best-of-call options, and so on. We refer the reader to Section EC.2.1 in the online appendices for the CPWA representations of these payoff functions and some properties of CPWA functions.

3.1. CPWA Payoff Functions on Unbounded Domains

In the first setting, we work under the following assumptions.

Assumption 3.2 (Setting 1). *We assume the following:*

(i) *The space Ω is given by $\Omega = \mathbb{R}^d_+$;*

(ii) *The payoff functions f and $(g_j)_{j=1:m}$ are CPWA functions on Ω;*

(iii) *It holds that $\phi(f) < \infty$ and $\phi(-f) < \infty$.*

In the sequel, for notational reasons, we use x in place of ω when Ω is a subset of the Euclidean space. Let us introduce the notion of the slack function for the LSIP problem in (3.1).

Definition 3.3 (Slack Function). Let $y \in \mathbb{R}^m$ be fixed, and denote the slack function of the LSIP problem in (3.1) by $s_y : \mathbb{R}^d_+ \to \mathbb{R}$, which is defined as $s_y(x) := \langle y, g(x)\rangle - f(x)$.

Algorithm 1 (Exterior Cutting Plane (ECP) Algorithm (Under Assumption 3.2))

Input: $\overline{\pi}$, $\underline{\pi}$, $(g_j)_{j=1:m}$, f, $X^{(0)} \subset \mathbb{R}^d_+$, $\underline{\phi}$, $\overline{x} > \mathbf{0}$, $\varepsilon > 0$, $\tau > 0$, $0 < \delta \le 1$

Output: $\phi(f)^{\mathrm{UB}}$, $\phi(f)^{\mathrm{LB}}$, $c^\star$, $y^\star$, X

1 Formulate the function $\widetilde{s}_y(\cdot)$ and use Algorithm 0 (see Section EC.2.2) to generate radial constraints, denoted by a system of linear inequalities with auxiliary variables $\widetilde{\sigma}$.
2 Construct a system of linear inequalities σ that contains all variables and inequalities in $\widetilde{\sigma}$, additional variables $c \in \mathbb{R}$, $y^+ \ge \mathbf{0}$ and $y^- \ge \mathbf{0}$, and an additional equality $y = y^+ - y^-$.
3 Add the linear inequality $c + \langle y^+, \overline{\pi}\rangle - \langle y^-, \underline{\pi}\rangle \ge \underline{\phi} - \tau$ to σ.
4 $r \leftarrow 0$.
5 **repeat**
6 **for** *each* $x \in X^{(r)}$ **do**
7 Add the linear inequality $c + \langle y^+ - y^-, g(x)\rangle \ge f(x)$ to σ.
8 Solve the linear programming (LP) problem: $\underline{\varphi}^{(r)} \leftarrow$ minimize $c + \langle y^+, \overline{\pi}\rangle - \langle y^-, \underline{\pi}\rangle$ subject to linear constraints σ, and denote the computed minimizer as $(c^{(r)}, y^{+(r)}, y^{-(r)})$.
9 $y^{(r)} \leftarrow y^{+(r)} - y^{-(r)}$.
10 Formulate the global minimization problem: $\min_{\mathbf{0}\le x\le \overline{x}} s_{y^{(r)}}(x)$ into a mixed-integer linear programming (MILP) problem (see (EC.2.8)).
11 $s^{(r)} \leftarrow c^{(r)} + \min_{\mathbf{0}\le x\le \overline{x}} s_{y^{(r)}}(x)$ (solve the MILP problem via the BnB algorithm).
12 $X^{(r+1)} \leftarrow \{x : (x, (\lambda_k), (\zeta_k), (\delta_{k,i}), (\iota_{k,i}))$ is an integer feasible solution found by the BnB algorithm while solving the MILP problem such that $c^{(r)} + s_{y^{(r)}}(x) \le \delta s^{(r)}\}$.
13 $r \leftarrow r + 1$.
until $s^{(r-1)} \ge -\varepsilon$;
14 $\phi(f)^{\mathrm{LB}} \leftarrow \underline{\varphi}^{(r-1)}$, $\phi(f)^{\mathrm{UB}} \leftarrow \underline{\varphi}^{(r-1)} - s^{(r-1)}$, $c^\star \leftarrow c^{(r-1)} - s^{(r-1)}$, $y^\star \leftarrow y^{(r-1)}$, $X \leftarrow \bigcup_{l=0}^{r-1} X^{(l)}$.
15 **if** $\phi(f)^{\mathrm{UB}} < \underline{\phi}$ **then**
16 **return** the problem (3.1) is unbounded.
17 **else**
18 **return** $\phi(f)^{\mathrm{UB}}$, $\phi(f)^{\mathrm{LB}}$, $c^\star$, $y^\star$, X.

To numerically solve the LSIP problem (3.1), let us now introduce the cutting plane discretization method, detailed in Algorithm 1, which is inspired by the conceptual algorithm 11.4.1 in Goberna and López (1998). In Line 1 of Algorithm 1, the so-called "radial constraints" are generated using Algorithm 0 in Section EC.2.2 of the online appendices. The purpose of this step is to generate sufficient and necessary constraints on y to guarantee that the slack function $s_y(\cdot)$ is bounded from below. We refer the reader to Section EC.2.2 for a detailed explanation about this step. In Lines 10 and 11, the global minimization problem $\min_{\mathbf{0}\le x\le\overline{x}} s_{y^{(r)}}(x)$ is solved by formulating it into a mixed-integer linear programming (MILP) problem in (EC.2.8), as discussed in Lemma EC.2.8. The MILP problem can be solved efficiently by state-of-the-art solvers such as Gurobi (Gurobi Optimization 2020) that uses the so-called branch-and-bound (BnB) algorithm. We refer the reader to Remark EC.2.9 for a brief description of the BnB algorithm.

We name Algorithm 1 the ECP method because every constraint (also known as cut) generated in Line 7 does not restrict the feasible set of (3.1) and hence is exterior to the feasible set. We refer the reader to Section EC.2.2 for detailed discussions about various aspects of Algorithm 1. Specifically, Remark EC.2.10 explains the inputs of Algorithm 1, and Remark EC.2.11 discusses the differences between Algorithm 1 and the conceptual algorithm 11.4.1 in Goberna and López (1998). Under the assumption that the inputs of Algorithm 1 are specified according to Remark EC.2.10, Theorem 3.4 shows the properties of Algorithm 1, whose proof is provided in Appendix A.1.

Theorem 3.4 (Properties of Algorithm 1). *Let Assumption 3.2 hold. Assume that $\overline{x}$ and $\underline{\phi}$ are specified as stated in Remark EC.2.10. Then, the following statements hold.*

(i) *If Assumption 2.1 holds, then $\underline{\varphi}^{(r)}$ is nondecreasing in r. At any stage of Algorithm 1, $s^{(r)} \le 0$ and $\underline{\varphi}^{(r)} \le \phi(f) \le \underline{\varphi}^{(r)} - s^{(r)}$.*

(ii) *If Assumption 2.1 holds, then Algorithm 1 terminates after finitely many iterations with an ε-optimal solution $(c^\star, y^\star)$ of (2.2) and $\phi(f)^{\mathrm{LB}} \le \phi(f) \le \phi(f)^{\mathrm{UB}}$ with $\phi(f)^{\mathrm{UB}} - \phi(f)^{\mathrm{LB}} \le \varepsilon$.*

(iii) *If Line 16 of Algorithm 1 is reached, then Assumption 2.1 is violated, and Problem (3.1) is unbounded.*

Remark 3.5. In Section 3.2, under the more restrictive assumption that $\Omega = \{x \in \mathbb{R}^d : \mathbf{0} \le x \le \overline{x}\}$ for some $\overline{x} > \mathbf{0}$[1] (see Assumption 3.6), we show that Algorithm 1 also produces an ε-optimal solution to the right-hand side of (2.3), which corresponds to the most extreme pricing measure in the original model-free superhedging problem. This will be explained in detail in Corollary 3.8.

3.2. CPWA Payoff Functions on Bounded Domains

In the second setting, we adopt similar but more restrictive assumptions than in the first one.

Assumption 3.6 (Setting 2). *We assume the following:*

(i) *The space* Ω *is given by* $\Omega = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{0} \le \boldsymbol{x} \le \overline{\boldsymbol{x}}\}$ *for* $\overline{\boldsymbol{x}} = (\overline{x}_1, \ldots, \overline{x}_d)^{\mathrm{T}} > \boldsymbol{0}$;

(ii) *The payoff functions* f *and* $(g_j)_{j=1:m}$ *are CPWA functions on* Ω;

(iii) *It holds that* $\phi(f) < \infty$ *and* $\phi(-f) < \infty$.

Algorithm 2 (Accelerated Central Cutting Plane (ACCP) Algorithm (Under Assumption 3.6))

Input: $\overline{\boldsymbol{\pi}}, \underline{\boldsymbol{\pi}}, (g_j)_{j=1:m}, f, X^{(0)} \subset \Omega, \overline{\boldsymbol{x}}, \underline{\phi}, \overline{\phi}, c^{\star(0)}, \boldsymbol{y}^{\star(0)}$, $\overline{c} > 0, \overline{\boldsymbol{y}} > \boldsymbol{0}, \varepsilon > 0, \tau > \varepsilon, 0 \le \gamma < 1, 0 < \zeta < 1, 0 < \delta \le 1$

Output: $\phi(f)^{\mathrm{UB}}, \phi(f)^{\mathrm{LB}}, c^\star, \boldsymbol{y}^\star, c^\dagger, \boldsymbol{y}^\dagger, X^\dagger, X$

1 $r \leftarrow 0$, $\underline{\varphi}^{(0)} \leftarrow \underline{\phi} - \tau$, $\overline{\varphi}^{(0)} \leftarrow \overline{\phi}$, flag $\leftarrow$ false.

2 Mark all elements of $X^{(0)}$ as active and removable.

3 **while** $\overline{\varphi}^{(r)} - \underline{\varphi}^{(r)} > \varepsilon$

4 $\quad r \leftarrow r+1$, $\underline{\varphi}^{(r)} \leftarrow \underline{\varphi}^{(r-1)}$, $\overline{\varphi}^{(r)} \leftarrow \overline{\varphi}^{(r-1)}$, $\varphi^{(r)} \leftarrow \frac{\underline{\varphi}^{(r)} + \overline{\varphi}^{(r)}}{2}$, $c^{\star(r)} \leftarrow c^{\star(r-1)}$, $\boldsymbol{y}^{\star(r)} \leftarrow \boldsymbol{y}^{\star(r-1)}$.

5 $\quad$ **if** *flag is true* **then**

6 $\quad\quad \varphi^{(r)} \leftarrow (\underline{\varphi}^{(r)} + \varphi^{(r)})/2$.

7 $\quad X \leftarrow \bigcup_{0 \le l \le r-1} \{\boldsymbol{x} \in X^{(l)} : \boldsymbol{x}$ is marked active$\}$.

8 $\quad$ Compute the Chebyshev center of $\sigma(\overline{c}, \overline{\boldsymbol{y}}, \underline{\varphi}^{(r)}, \varphi^{(r)}, X)$ by solving an LP problem.

9 $\quad$ **if** *the LP problem in Line 8 is infeasible* **then**

10 $\quad\quad \underline{\varphi}^{(r)} \leftarrow \min\{c + \langle \boldsymbol{y}^+, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^-, \underline{\boldsymbol{\pi}} \rangle : (c, \boldsymbol{y}^+, \boldsymbol{y}^-)$ satisfies $\sigma(\overline{c}, \overline{\boldsymbol{y}}, -\infty, \infty, X)\}$ (which is an LP problem). Let $(c^\dagger, \boldsymbol{y}^{+\dagger}, \boldsymbol{y}^{-\dagger})$ be a minimizer of this LP problem.

11 $\quad\quad \boldsymbol{y}^\dagger \leftarrow \boldsymbol{y}^{+\dagger} - \boldsymbol{y}^{-\dagger}$, $X^\dagger \leftarrow X$.

12 $\quad\quad$ Mark all elements of $\bigcup_{1 \le l \le r-1} X^{(r)}$ as removable.

13 $\quad\quad \rho^{(r)} \leftarrow -1$, $X^{(r)} \leftarrow \emptyset$.

14 $\quad\quad$ Skip to the next iteration.

15 $\quad$ Let $(c^{(r)}, \boldsymbol{y}^{+(r)}, \boldsymbol{y}^{-(r)})$ be the Chebyshev center and let $\rho^{(r)}$ be the radius of the largest inscribed ball of $\sigma(\overline{c}, \overline{\boldsymbol{y}}, \underline{\varphi}^{(r)}, \varphi^{(r)}, X)$.

16 $\quad \boldsymbol{y}^{(r)} \leftarrow \boldsymbol{y}^{+(r)} - \boldsymbol{y}^{-(r)}$.

17 $\quad$ Formulate the global minimization problem: minimize $c^{(r)} + s_{\boldsymbol{y}^{(r)}}(\boldsymbol{x})$ subject to $\boldsymbol{0} \le \boldsymbol{x} \le \overline{\boldsymbol{x}}$ into an MILP problem (see (EC.2.8)). Solve it with relative gap tolerance ζ. Let $\overline{s}^{(r)}$ be its approximate optimal value. Let $\underline{s}^{(r)}$ be its lower bound at termination.

18 $\quad X^{(r)} \leftarrow \{\boldsymbol{x} : (\boldsymbol{x}, (\lambda_k), (\zeta_k), (\delta_{k,i}), (\iota_{k,i}))$ is an integer feasible solution found by the BnB algorithm while solving (EC.2.8) such that $c^{(r)} + s_{\boldsymbol{y}^{(r)}}(\boldsymbol{x}) \le \delta \overline{s}^{(r)}\}$. Mark all elements of $X^{(r)}$ as active and removable.

19 $\quad$ **if** $c^{(r)} + \pi(\boldsymbol{y}^{(r)}) - \underline{s}^{(r)} < \overline{\varphi}^{(r)} - \varepsilon$ **then**

20 $\quad\quad \overline{\varphi}^{(r)} \leftarrow c^{(r)} + \pi(\boldsymbol{y}^{(r)}) - \underline{s}^{(r)}$, $c^{\star(r)} \leftarrow c^{(r)} - \underline{s}^{(r)}$, $\boldsymbol{y}^{\star(r)} \leftarrow \boldsymbol{y}^{(r)}$.

21 $\quad\quad$ **if** $\underline{s}^{(r)} \ge 0$ **then**

22 $\quad\quad\quad$ Mark all elements of $\bigcup_{1 \le l \le r} X^{(l)}$ as removable.

23 $\quad\quad\quad$ Skip to the next iteration (Line 3).

24 $\quad\quad$ **if** *flag is true* **then**

25 $\quad\quad\quad$ flag $\leftarrow$ false.

26 $\quad\quad\quad$ Mark all elements of $X^{(r)}$ as nonremovable, and skip to the next iteration (Line 3).

27 $\quad\quad$ flag $\leftarrow$ true.

28 $\quad\quad$ **for** *each* $0 \le l \le r$ *such that* $\rho^{(r)} < \gamma \rho^{(l)}$ **do**

29 $\quad\quad\quad$ **for** *each* $\boldsymbol{x} \in X^{(l)}$ *marked as removable* **do**

30 $\quad\quad\quad\quad$ **if** $c^{(r)} + \langle \boldsymbol{y}^{(r)}, \boldsymbol{g}(\boldsymbol{x}) \rangle - (1 + \|\boldsymbol{g}(\boldsymbol{x})\|_2^2)^{\frac{1}{2}} \rho^{(r)} > f(\boldsymbol{x})$ **then**

31 $\quad\quad\quad\quad\quad$ Set $\boldsymbol{x}$ as inactive.

32 $\phi(f)^{\mathrm{UB}} \leftarrow \overline{\varphi}^{(r)}$, $\phi(f)^{\mathrm{LB}} \leftarrow \underline{\varphi}^{(r)}$, $c^\star \leftarrow c^{\star(r)}$, $\boldsymbol{y}^\star \leftarrow \boldsymbol{y}^{\star(r)}$, $X \leftarrow \bigcup_{0 \le l \le r} \{\boldsymbol{x} \in X^{(l)} : \boldsymbol{x}$ is marked active$\}$.

33 **if** $\phi(f)^{\mathrm{UB}} < \underline{\phi}$ **then**

34 $\quad$ **return** the problem (3.1) is unbounded.

35 **else**

36 $\quad$ **return** $\phi(f)^{\mathrm{UB}}, \phi(f)^{\mathrm{LB}}, c^\star, \boldsymbol{y}^\star, c^\dagger, \boldsymbol{y}^\dagger, X^\dagger, X$.

Let us introduce a version of the ACCP method inspired by Betrò (2004), detailed in Algorithm 2. In Algorithm 2, we maintain and update a sequence of lower bounds $(\underline{\varphi}^{(r)})_{r \ge 0}$ of $\phi(f)$, a sequence of upper bounds $(\overline{\varphi}^{(r)})_{r \ge 0}$ of $\phi(f)$, and polytopes in $\mathbb{R}^{2m+1}$ that are denoted by $\sigma(\overline{c}, \overline{\boldsymbol{y}}, \underline{\varphi}^{(r)}, \varphi^{(r)}, X)$, which have the form

$$
\begin{aligned}
&\sigma(\overline{c}, \overline{\boldsymbol{y}}, \underline{\varphi}^{(r)}, \varphi^{(r)}, X) \\
&\quad := \{(c, \boldsymbol{y}^+, \boldsymbol{y}^-) : |c| \le \overline{c}, \boldsymbol{0} \le \boldsymbol{y}^+ \le \overline{\boldsymbol{y}}, \boldsymbol{0} \le \boldsymbol{y}^- \le \overline{\boldsymbol{y}}, \\
&\qquad \underline{\varphi}^{(r)} \le c + \langle \boldsymbol{y}^+, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^-, \underline{\boldsymbol{\pi}} \rangle \le \varphi^{(r)}, \\
&\qquad c + \langle \boldsymbol{y}^+ - \boldsymbol{y}^-, \boldsymbol{g}(\boldsymbol{x}) \rangle \ge f(\boldsymbol{x}), \ \forall \boldsymbol{x} \in X\}, \qquad (3.3)
\end{aligned}
$$

where $\overline{c}, \overline{\boldsymbol{y}}$ specify a bounding box, $\underline{\varphi}^{(r)}$ and $\overline{\varphi}^{(r)}$ specify the lower and upper bounds on $c + \pi(\boldsymbol{y}^+ - \boldsymbol{y}^-)$, and $X \subset \Omega$ specifies a collection of feasibility constraints. In Algorithm 2, $\varphi^{(r)}$ is between the lower bound $\underline{\varphi}^{(r)}$ and the upper bound $\overline{\varphi}^{(r)}$ and is used as a speculative upper objective cut. The idea of Algorithm 2 is that $(\underline{\varphi}^{(r)})_{r \ge 0}$ is a nondecreasing sequence of lower bounds that approaches $\phi(f)$ from below, whereas $(\overline{\varphi}^{(r)})_{r \ge 0}$ is a nonincreasing sequence of upper bounds that approaches $\phi(f)$ from above. These facts will be made clear later in Theorem 3.7. Algorithm 2 has various advantages over Algorithm 1. Most importantly, the MILP problem in Line 17 of Algorithm 2 only needs to be solved approximately with a large error tolerance, and some linear constraints are removed in Line 31 to make solving the LP problem in Line 8 faster. We refer the reader to Betrò (2004) and section 11.4 of Goberna and López (1998) for further discussions.

A crucial step of Algorithm 2 is to compute the Chebyshev center, that is, the center of the largest inscribed ball, of the polytope $\sigma(\overline{c}, \overline{\boldsymbol{y}}, \underline{\varphi}^{(r)}, \varphi^{(r)}, X)$ in Line 8. It is well known that the Chebyshev center of a polytope can be computed by solving an LP problem (e.g., the problem (Q_r) before the conceptual algorithm 11.4.2 of Goberna and López 1998).

We refer the reader to Section EC.2.3 for detailed discussions about various aspects of Algorithm 2. Specifically, Remark EC.2.13 explains its inputs, and Remark EC.2.14 discusses the differences between Algorithm 2 and the ACCP algorithm by Betrò (2004). Assuming that the inputs of Algorithm 2 are specified according to Remark EC.2.13, Theorem 3.7 shows the properties of Algorithm 2, whose proof is provided in Appendix A.2.

Theorem 3.7 (Properties of Algorithm 2). *Let Assumption 3.6 hold. Assume that* $\underline{\phi}, \overline{\phi}, c^{\star(0)}, \ \boldsymbol{y}^{\star(0)}, \overline{c}, \overline{\boldsymbol{y}}$ *are specified as stated in Remark EC.2.13. Then, the following statements hold.*

(i) *If Assumption 2.1 holds, then* $\underline{\varphi}^{(r)}$ *is nondecreasing in* r, $\overline{\varphi}^{(r)}$ *is nonincreasing in* r. *Moreover, at any stage of Algorithm 2,* $\underline{\varphi}^{(r)} \le \phi(f) \le \overline{\varphi}^{(r)}$, *and* $c^{\star(r)} + \langle \boldsymbol{y}^{\star(r)}, \boldsymbol{g}\rangle \ge f$ *holds.*

(ii) *If Assumption 2.1 holds, then Algorithm 2 terminates after finitely many iterations with an* ε*-optimal solution* $(c^\star, \boldsymbol{y}^\star)$ *of* (2.2) *and* $\phi(f)^{\mathrm{LB}} \le \phi(f) \le \phi(f)^{\mathrm{UB}}$ *with* $\phi(f)^{\mathrm{UB}} - \phi(f)^{\mathrm{LB}} \le \varepsilon$.

(iii) *If Assumption 2.1 holds, then* $c^\dagger$, $\boldsymbol{y}^\dagger$ *and* $X^\dagger$ *are defined when Algorithm 2 terminates. If* $|c^\dagger| < \overline{c}$ *and* $-\overline{\boldsymbol{y}} < \boldsymbol{y}^\dagger < \overline{\boldsymbol{y}}$, *then the following LP problem with decision variables* $(\mu_{\boldsymbol{x}})_{\boldsymbol{x}\in X^\dagger}$:

$$\begin{aligned} &\textit{maximize} && \sum_{\boldsymbol{x}\in X^\dagger} \mu_{\boldsymbol{x}} f(\boldsymbol{x}) \\ &\textit{subject to} && \sum_{\boldsymbol{x}\in X^\dagger} \mu_{\boldsymbol{x}} = 1, \quad \underline{\boldsymbol{\pi}} \le \sum_{\boldsymbol{x}\in X^\dagger} \mu_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}) \le \overline{\boldsymbol{\pi}}, \\ &&& \mu_{\boldsymbol{x}} \ge 0 \quad \forall \boldsymbol{x} \in X^\dagger \end{aligned} \tag{3.4}$$

has an optimal solution $(\mu^\star_{\boldsymbol{x}})_{\boldsymbol{x}\in X^\dagger}$. *Let* $\mu^\star$ *be a finitely supported measure defined by* $\mu^\star := \sum_{\boldsymbol{x}\in X^\dagger} \mu^\star_{\boldsymbol{x}} \delta_{\boldsymbol{x}}$. *Then,* $\mu^\star$ *is* ε*-optimal for the right-hand side of* (2.3).

(iv) *If Line 34 of Algorithm 2 is reached, then Assumption 2.1 is violated, and Problem* (3.1) *is unbounded.*

Theorem 3.7(iii) explicitly provides a pricing measure that is an ε-optimal solution to the original model-free superhedging problem. The ECP method (Algorithm 1) is also applicable in Setting 2 with Line 1 removed. It has the same property as Theorem 3.7(iii), which is detailed in the next corollary and proved in Appendix A.2.

Corollary 3.8. *Under Assumption 2.1 and Assumption 3.6, Algorithm 1 (with Line 1 removed) terminates after finitely many iterations, and the following LP problem with decision variables* $(\mu_{\boldsymbol{x}})_{\boldsymbol{x}\in X}$:

$$\begin{aligned} &\textit{maximize} && \sum_{\boldsymbol{x}\in X} \mu_{\boldsymbol{x}} f(\boldsymbol{x}) \\ &\textit{subject to} && \sum_{\boldsymbol{x}\in X} \mu_{\boldsymbol{x}} = 1, \quad \underline{\boldsymbol{\pi}} \le \sum_{\boldsymbol{x}\in X} \mu_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}) \le \overline{\boldsymbol{\pi}}, \\ &&& \mu_{\boldsymbol{x}} \ge 0 \quad \forall \boldsymbol{x} \in X \end{aligned} \tag{3.5}$$

has an optimal solution $(\mu^\star_{\boldsymbol{x}})_{\boldsymbol{x}\in X}$. *Define the finitely supported measure* $\mu^\star$ *by* $\mu^\star := \sum_{\boldsymbol{x}\in X} \mu^\star_{\boldsymbol{x}} \delta_{\boldsymbol{x}}$. *Then,* $\mu^\star$ *is* ε*-optimal for the right-hand side of* (2.3).

Remark 3.9. Under Assumptions 3.2 and 3.6, the payoff functions of the traded derivatives and the target derivative f must be CPWA functions. Thus, the proposed Algorithms 1 and 2 are unable to directly treat derivatives with non-CPWA payoff functions, such as digital and power options. Hence, it would be necessary to first approximate these payoff functions by CPWA functions to treat such derivatives using the proposed algorithms. Existing studies have proposed methods to treat non-CPWA payoff functions under more restrictive assumptions. Bertsimas and Popescu (2002) develop a method to compute the price bounds on a single vanilla call option given moments of the underlying asset price. Moreover, in the multi-asset setting, they show that the theoretical time complexity to compute the price bounds is polynomial when all payoff functions are the sum of a CPWA function and a quadratic function, under the assumption that all of the CPWA functions share the same partition of $\mathbb{R}^d$ and that the number of polyhedra in the partition is polynomial in the number of assets and the number of traded derivatives. However, this assumption is rather restrictive because the presence of a fixed number of traded vanilla options written on each asset incurs a partition of $\mathbb{R}^d$ in which the number of polyhedra is exponential in the number of assets. In Daum and Werner (2011), a method is developed to compute the price bounds on a basket call option given the prices of vanilla call and put options and the prices of single-asset digital and power options. In this setting, because of the structure of the basket call option, the global optimization problem associated with the LSIP problem can be reduced to a sequence of one-dimensional global optimization problems that can then be efficiently solved.

On the computational side, our assumption that all derivatives have CPWA payoff functions makes it possible to formulate the global minimization problem associated with the LSIP problem (3.1) into an MILP problem, which then allows us to efficiently solve the LSIP problem in high-dimensional situations (e.g., when $d = 60$) using state-of-the-art solvers. On the practical side, the exclusion of derivatives with non-CPWA payoff functions does not harm the applicability of the methods we have developed because most relevant financial derivatives have CPWA payoff functions (see Example EC.2.1). Let us point out that many trading platforms of digital options are unregulated, some adopt ethically questionable practices, whereas the involved risk is anyhow difficult to manage and hedge. Because of these, the trading of digital options has been banned in many countries including Australia,[2] Canada,[3] Israel,[4] and all European Union (EU) countries.[5] Power options, on the other hand, are conceptual financial derivatives that are not traded in real markets.

Remark 3.10. Algorithms 1 and 2 both solve a sequence of constrained optimization problems which are relaxations of the LSIP problem (3.1), where the semi-infinite constraint is reduced to finitely many constraints. This approach is known as the discretization method. Instead of enforcing these constraints strictly, there is an alternative discretization method in which the constraints are replaced by penalty functions (Coope and Price 1998, Auslender et al. 2009). In this method, a sequence of unconstrained optimization problems is solved. We refer the reader to Auslender et al. (2015) for a comparison of different variants of this method. Another approach that is based on penalization is the integral-type penalization method (Borwein and Lewis 1991a, b; Lin et al. 1998; Auslender et al. 2009). This approach transforms the original LSIP problem into an unconstrained convex optimization problem with an integral-type penalty term, which can be subsequently solved by the stochastic (sub)gradient descent (SGD) algorithm. This is similar to a recent approach adopted for solving optimal transport, martingale optimal transport, and related problems (De Gennaro Aquino and Bernard 2020; Eckstein et al. 2020, 2021; Eckstein and Kupper 2021). We have empirically tested the *integral-type penalization plus SGD* approach in our problem and found that it is unstable and highly sensitive to the initialization and the hyperparameter settings of the SGD algorithm.

4. Numerical Experiments and Results

In this section, we perform three experiments using synthetically generated derivative prices and one experiment using real market data (i) to demonstrate the performance of the proposed approaches under these settings, (ii) to quantify the effect of the additional information from traded multi-asset options on the width of the no-arbitrage gap, that is, the difference between the upper and lower model-free bounds, and (iii) to show that the proposed algorithms are capable of detecting arbitrage opportunities in a financial market.

We refer the reader to Section EC.2.4 in the online appendices for details about the implementation of the proposed numerical algorithms. The code used in this work is available on GitHub.[6] In the subsequent numerical experiments, we consider financial derivatives with the following CPWA payoff functions, whose representations are discussed in Example EC.2.1.

(i) Trading in the ith asset: $g(\boldsymbol{x}) = x_i$.

(ii) Vanilla call option on the ith asset with strike $\kappa > 0$: $g(\boldsymbol{x}) = (x_i - \kappa)^+$.

(iii) Basket call option with weights $\boldsymbol{w} \in \mathbb{R}_+^d$ and strike $\kappa > 0$: $g(\boldsymbol{x}) = (\sum_i w_i x_i - \kappa)^+$.

(iv) Spread call option with weights $\boldsymbol{w} \in \mathbb{R}^d \backslash \mathbb{R}_+^d$ (e.g., $\boldsymbol{w} = \boldsymbol{e}_i - \boldsymbol{e}_j$) and strike $\kappa \in \mathbb{R}$, e.g., $g(\boldsymbol{x}) = (x_i - x_j - \kappa)^+$.

(v) Call-on-max (rainbow) option on assets $i_1, \ldots, i_l$ with strike $\kappa \geq 0$: $g(\boldsymbol{x}) = [(x_{i_1} \vee x_{i_2} \vee \ldots \vee x_{i_l}) - \kappa]^+$.

(vi) Call-on-min (rainbow) option of assets $i_1, \ldots, i_l$ with strike $\kappa \geq 0$: $g(\boldsymbol{x}) = [(x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_l}) - \kappa]^+$.

(vii) Best-of-calls option of assets $i_1, \ldots, i_l$ with strikes $\kappa_1, \ldots, \kappa_l \geq 0$: $g(\boldsymbol{x}) = (x_{i_1} - \kappa_1)^+ \vee (x_{i_2} - \kappa_2)^+ \vee \ldots \vee (x_{i_l} - \kappa_l)^+$.

Moreover, in the numerical experiments with synthetically generated prices, we consider market models of the following type:

- The marginal distribution of the price of an asset at terminal time is a log-normal distribution. Under Setting 2, the ith marginal distribution is truncated to $[0, \overline{x}_i]$ for $i = 1, \ldots, d$.
- The dependence structure among the marginals of the d assets at terminal time is a t-copula with a positive definite correlation matrix $\mathbf{C}$ and ν degrees of freedom.

Given these market models, the prices of the single-asset derivatives listed previously can be computed in closed-form by taking the discounted expectations of the corresponding payoff functions (with respect to a pricing measure). We have assumed that the interest rate is equal to zero for the sake of simplicity. For the multi-asset derivatives listed previously, we approximate their prices via Monte Carlo integration by randomly generating one million independent samples from the copula model and subsequently using these samples to approximate the expectations of the payoff functions. The markets models are also used to compute reference prices for the target derivatives (with payoff f). However, they are not used in the computation of the model-free bounds. In the computation of the bounds, the only information used are the prices of single- and multi-asset derivatives that are synthetically generated from these market models. To simulate an incomplete market with the presence of bid–ask spread, we specify multiple market models with different parameters and subsequently take the minimum (respectively, maximum) price of a derivative among its prices under all models as the bid (respectively, ask) price of the derivative.

Under the market models described previously, the pricing measure $\hat{\mu}$ has strictly positive density with respect to the Lebesgue measure on Ω. Moreover, the way the prices of derivatives are generated guarantees that $\hat{\mu} \in \mathcal{Q}$. Therefore, Assumption 2.1 holds by Proposition EC.1.3.

4.1. Experiment 1

In this experiment, we consider a financial market with five assets ($d = 5$). We consider Setting 2 (i.e., Assumption 3.6), where $\Omega = [0, 100]^5$. Our goal is to compute the model-free lower and upper price bounds for a call-on-max option with payoff function

$f(\mathbf{x}) = (x_2 \vee x_3 \vee x_4 - \kappa)^+$, where the strike price κ ranges from 0 to 10 with an increment of 0.2. We assume that a total of 439 financial derivatives are traded in the market ($m = 439$). These include the following:

- The five assets x_1, x_2, x_3, x_4, x_5.
- Vanilla call options on the five assets with strikes $1, 2, \ldots, 10$.
- Basket call options with the following weights and strikes $1, 2, \ldots, 10$: $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)^{\mathrm{T}}$, $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)^{\mathrm{T}}$, $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}\right)^{\mathrm{T}}$, $\left(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}\right)^{\mathrm{T}}$, $\left(\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^{\mathrm{T}}$, $\left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^{\mathrm{T}}$, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right)^{\mathrm{T}}$, $\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^{\mathrm{T}}$, $\left(0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\mathrm{T}}$, $\left(0, \frac{1}{2}, \frac{1}{2}, 0, 0\right)^{\mathrm{T}}$, $\left(0, \frac{1}{2}, 0, \frac{1}{2}, 0\right)^{\mathrm{T}}$, $\left(0, 0, \frac{1}{2}, \frac{1}{2}, 0\right)^{\mathrm{T}}$.
- Spread call options with the following weights and strikes $-5, -4, \ldots, 0, 1, \ldots, 5$: $\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_1 - \mathbf{e}_3$, $\mathbf{e}_1 - \mathbf{e}_4$, $\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_2 - \mathbf{e}_4$, $\mathbf{e}_2 - \mathbf{e}_5$, $\mathbf{e}_3 - \mathbf{e}_4$, $\mathbf{e}_3 - \mathbf{e}_5$, $\mathbf{e}_4 - \mathbf{e}_5$, $\mathbf{e}_2 - \mathbf{e}_1$, $\mathbf{e}_3 - \mathbf{e}_1$, $\mathbf{e}_4 - \mathbf{e}_1$, $\mathbf{e}_3 - \mathbf{e}_2$, $\mathbf{e}_4 - \mathbf{e}_2$, $\mathbf{e}_5 - \mathbf{e}_2$, $\mathbf{e}_4 - \mathbf{e}_3$, $\mathbf{e}_5 - \mathbf{e}_3$, $\mathbf{e}_5 - \mathbf{e}_4$.
- Call-on-max (rainbow) options on the following 6 groups of assets and strikes $0, 1, \ldots, 10$: $\{x_1, x_2, x_3, x_4, x_5\}$, $\{x_1, x_2, x_3, x_4\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_2, x_3\}$, $\{x_2, x_4\}$, $\{x_3, x_4\}$.

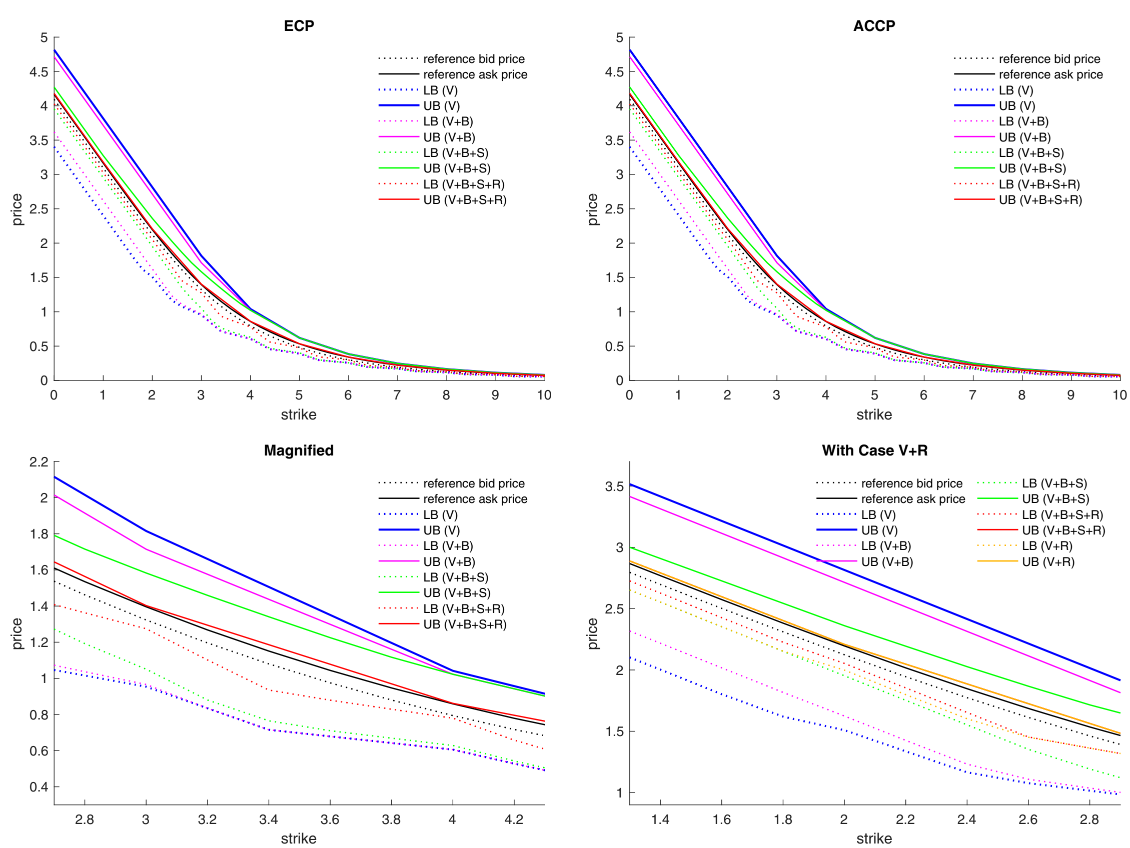
The bid and ask prices of the assets and derivatives are synthetically generated from the market models specified in Section EC.3.1. We consider five cases, where we use certain subsets of the traded derivatives to compute the model-free lower and upper price bounds for the target derivative:

Case 1 (denoted as V): we use only vanilla options;
Case 2 (V+B): we use vanilla and basket options;
Case 3 (V+B+S): we use vanilla, basket, and spread options;
Case 4 (V+B+S+R): we use vanilla, basket, spread and call-on-max (rainbow) options;
Case 5 (V+R): we use vanilla and call-on-max options.

We compute the lower and upper bounds of the call-on-max option with payoff function f using the ECP method (Algorithm 1 with Line 1 removed) and the ACCP method (Algorithm 2). The inputs of the two algorithms for this experiment are specified in Section EC.3.1.

Figure 1 shows the computed lower and upper price bounds of the call-on-max option with different strikes, along with their reference bid and ask prices. Let us point out that the price bounds computed by

Figure 1. (Color online) Experiment 1: Model-Free Lower and Upper Price Bounds of Call-on-Max Options with Strikes Between 0 and 10



Notes. (Bottom left) Magnified version of a part of the top right. (Bottom right) Magnified version of a part of the top right with the Case V+R included.

the two algorithms are almost identical. Indeed, we have checked that all the absolute differences between the bounds computed by the two algorithms are below $\varepsilon = 0.001$. This is a consequence of Theorem 3.4(ii) and Theorem 3.7(ii) and confirms the correctness of the computed price bounds.

The following observations ensue from the price bounds computed in this example (Figure 1). (i) The price bounds in Cases 1–4 are distinct, and the gap between the lower and upper bounds shrinks when the prices of more traded derivatives are added. This means that observing the market prices of more traded derivatives substantially restricts the class of possible pricing measures $\mathcal{Q}$ and reduces the no-arbitrage gap between the bounds. On the dual side (2.2), this can be equivalently interpreted as having the information about more traded derivatives provides more ways to subreplicate and superreplicate the given payoff function and thus makes the gap between the subreplication price and the superreplication price smaller. (ii) The addition of rainbow options (R) in Case 4 results in a significant reduction of the no-arbitrage gap. For example, in Case 4, when the strike is 3.2, the upper bound is 2.07% higher than the reference ask price and the lower bound is 7.88% lower than the reference bid price. The respective percentages in Case 3 are 15.09% and 26.39% for the upper and lower bounds. The reason is that the traded call-on-max options provide more information to determine the price of the target derivative because they are similar in structure to the target derivative. This becomes concrete when one considers the dual optimization problem (2.2), where these call-on-max options offer direct ways to subreplicate and superreplicate the target payoff, for example, $(x_2 \vee x_3 - \kappa)^+ \le (x_2 \vee x_3 \vee x_4 - \kappa)^+ \le (x_1 \vee x_2 \vee x_3 \vee x_4 - \kappa)^+$. (iii) Conversely, the addition of spread options (Case 3) to vanilla and basket options (Case 2) only yields a significant improvement to the bounds for small strikes (≤ 4), because spread options with large strikes are not traded in the market. (iv) In all cases, we observe that the no-arbitrage gap is significantly smaller for the (integer) strikes, where traded derivative prices are present. In the bottom left panel of Figure 1, for example, the upper bounds in Case V+B+S+R almost touches the reference ask prices at strikes 3 and 4, being only 0.47% and 0.30% higher than the respective reference ask prices, whereas for the lower bound, the gap is very small at the same strikes, being 3.67% and 1.82% lower than the respective reference bid prices. Conversely, for the intermediate strikes between 3 and 4, for example, when the strike is 3.4, the gap grows to 3.05% for the upper bound and to 13.42% for the lower bound. This is because all the traded options in the synthetic financial market have integer strike prices. In the dual optimization problem (2.2), one needs to interpolate traded options with integer strike prices to subreplicate and superreplicate the call-on-max option with noninteger strike prices. Therefore, whenever possible, practitioners should include in the sub- and superreplicating portfolios derivatives with the same strike price as that of the target derivative to reduce the no-arbitrage gap. (v) As observed from the bottom right panel of Figure 1, the upper bounds in Case 5 (V+R) coincide with the upper bounds in Case 4, whereas the lower bounds in Case 5 coincide with Case 3 for strikes between 0 and 1.8, coincide with Case 4 for strikes between 2.6 and 10, and fall between Case 3 and Case 4 for strikes between 2 and 2.4. This shows that, although the inclusion of more derivatives produces tighter bounds than fewer derivatives, derivatives with similar payoff structure provide more improvement than other derivatives. Therefore, because the computation of price bounds is faster when fewer derivatives are included in the sub- and superreplicating portfolios, practitioners should prioritize including derivatives with payoff structure similar to that of the target derivative when the computation time is limited.

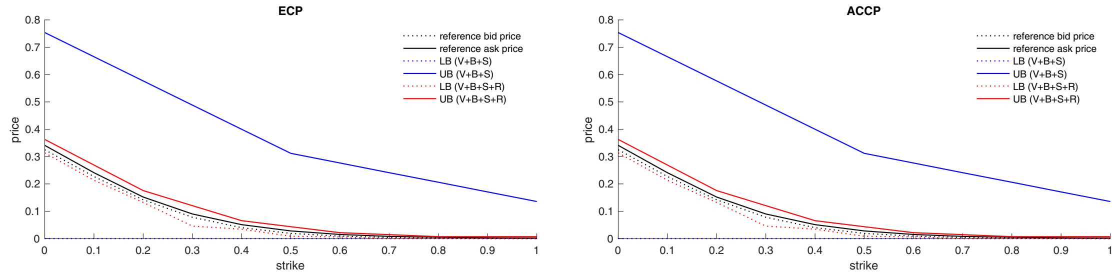
4.2. Experiment 2

In this experiment, we consider a financial market with 60 assets ($d = 60$). We consider Setting 2 (i.e., Assumption 3.6) where $\Omega = [0, 100]^{60}$. We use Algorithms 1 and 2 to compute the model-free lower and upper price bounds for a call-on-min option on the first 50 of 60 assets, with the strike price ranging from zero to one, with an increment of 0.1. The purpose of this experiment is to demonstrate that Algorithms 1 and 2 work even when the number of assets is large.

A total of 400 financial derivatives are traded in the market ($m = 400$). These financial derivatives include the following: the 60 assets, 180 vanilla call options (V), three basket call options (B), 147 spread call options (S), and 10 call-on-min options (R). The bid and ask prices of the assets and derivatives are synthetically generated using the market models specified in Section EC.3.2. For simplicity, we only consider Cases V+B+S and V+B+S+R in this experiment. The inputs of the two algorithms for this experiment are detailed in Section EC.3.2.

Figure 2 shows the computed lower and upper price bounds for the call-on-min option with different strikes, along with the reference bid and ask prices. Once again, the price bounds computed by the two algorithms are almost identical, and we checked that all the absolute differences between the bounds computed by the two algorithms are below $\varepsilon = 0.001$. The following observations ensue from this example, which are mostly in line with the observations from the previous one. (i) The price bounds in the two cases are distinct, and the addition of more information

Figure 2. (Color online) Experiment 2: Model-Free Lower and Upper Price Bounds for a Call-on-Min Option with Strikes Between Zero and One



improves the bounds and reduces the no-arbitrage gap. (ii) The addition of traded prices of call-on-min options results in a significant improvement of the bounds because the payoffs used for sub- and superreplicating and the target payoff are of the same type. (iii) However, in this high-dimensional example, we notice that the lower price bounds in Case $V{+}B{+}S$ are identically zero, showing that the traded vanilla, basket, and spread options do not provide enough information for a nontrivial lower price bound of the call-on-min options and that it is not possible to subreplicate the payoff of a call-on-min option with these traded options. Therefore, we conclude once again that, whenever possible, practitioners should include in their sub- and superreplicating portfolios not only as many derivatives as possible but also as many derivatives with similar payoff structure as possible.

Table 1 shows the total number of LP and MILP problems solved throughout this experiment by the two algorithms. The ACCP algorithm achieved convergence faster than the ECP algorithm in this experiment. Moreover, in the ACCP algorithm, the MILP problems were only approximately solved with relative gap tolerance $\zeta = 0.8$, as explained in Remark EC.2.13. As a result, the ACCP algorithm was much faster than the ECP algorithm in this experiment.

4.3. Experiment 3

In this experiment, we want to demonstrate how the fundamental theorem can be combined with the numerical algorithms developed to detect arbitrage opportunities in the financial market. We consider the case where the no-arbitrage assumption (Assumption 2.1) does not hold and use Algorithm 2 to detect the presence of arbitrage opportunities in the market. We can actually detect a very delicate form of arbitrage because we consider a financial market with several single-asset options and two multi-asset options; the multi-asset options are priced within their own no-arbitrage intervals, that is, when considered separately from each, there is no arbitrage in the market. However, when they are considered together, an arbitrage opportunity arises, and this is detected by the numerical algorithm. This experiment is inspired by similar examples in (Tavin 2015, section 4) and (Papapantoleon and Yanez Sarmiento 2021, section 5.2); in their setting, the marginals of the pricing measure are given.

Table 1. Experiment 2: Total Number of LP and MILP Problems Solved by the Two Algorithms

Algorithm	Problem	$V{+}B{+}S$	$V{+}B{+}S{+}R$
ECP	LP	4,789	3,339
	MILP	4,789	3,339
ACCP	LP	1,639	1,714
	MILP	1,461	1,574

We consider Setting 2 (Assumption 3.6) where $\Omega = [0, 100]^5$, and consider the five assets x_1, x_2, x_3, x_4, and x_5 and vanilla call options on the five assets with strikes $1, 2, \ldots, 10$ as the traded financial derivatives. In addition, we include a call-on-min option on the five assets with strike 1 and a put-on-min option on the five assets with strike 4. The bid and ask prices of the single-asset derivatives are synthetically generated using the method specified at the beginning of Section 4.

We set the bid and ask prices of the call-on-min option as 0.83 and 0.85, respectively. As for the put-on-min option, we set its bid and ask prices as 3.18 and 3.20, respectively. Subsequently, we let $f = 0$ and run Algorithm 2 with $\varepsilon = 0.001$, $\tau{=}1$, $\gamma = 0.1$, $\zeta = 0.8$, $\delta = 0.7$, $\bar{c} = 100$, $\overline{\boldsymbol{y}} = 100 \cdot \mathbf{1}$, $\underline{\phi} = \overline{\phi} = 0$. When only the call-on-min option is considered a traded multi-asset option, Algorithm 2 terminates without reaching Line 34, and the outputs satisfy $\phi(f)^{\mathrm{LB}} > -0.001$, $\phi(f)^{\mathrm{UB}} = 0$. Similarly, when only the put-on-min option is considered as traded multi-asset option, Algorithm 2 terminates without reaching Line 34, and the outputs again satisfy $\phi(f)^{\mathrm{LB}} > -0.001$, $\phi(f)^{\mathrm{UB}} = 0$. These numerical results imply that there is no arbitrage opportunity in the market with the single-asset derivatives and the call-on-min option, as well as in the market with the single-asset derivatives and the put-on-min option. However, when the single-asset derivatives together with both the call-on-min option and the put-on-min option are considered as traded options, Algorithm 2 reaches Line 34 before termination, indicating the violation of Assumption 2.1 and the presence of an arbitrage opportunity, as stated by Theorem 3.7(iv). The detected arbitrage strategy is given by $c^\star$ and $\boldsymbol{y}^\star$, which specify a portfolio with nonnegative payoff such that $c^\star + \pi(\boldsymbol{y}) < 0$.

In Experiment 4 (see Section 4.4), a similar procedure for detecting arbitrage opportunities using Algorithm 1 is applied to Setting 1 (Assumption 3.2) where bid and ask prices of traded derivatives are obtained from *real market data*. This demonstrates the real-world applicability of the proposed algorithms for arbitrage detection.

4.4. Experiment 4

In this experiment, we use *real market prices* of European call and put options written on the Dow Jones Industrial Average (DJIA) index and European call and put options written on the 30 constituent stocks of the DJIA index. This type of market data has been considered by Hobson et al. (2005b), d'Aspremont and El Ghaoui (2006), and Peña et al. (2010b, 2012) for illustration.

4.4.1. Data Collection. The following market prices (corresponding to the closing prices on April 5, 2021, at 1600 hours EDT) were collected from MarketWatch[7] on April 6, 2021.

- The prices of the 30 constituent stocks of the DJIA.
- The bid and ask prices of the call and put options written on the 30 constituent stocks of the DJIA with expiration date May 21, 2021.
- The bid and ask prices of the call and put options written on the SPDR Dow Jones Industrial Average ETF Trust (symbol: DIA), which is an exchange traded fund (ETF) that tracks the DJIA index. These DIA options are regarded as basket options written on the 30 constituent stocks with equal weights $w_{\mathrm{DIA}} := 0.0658$. The weight w_{DIA} is equal to $\frac{1}{100}$ of the inverse of the Dow divisor[8] calculated based on the stock prices on April 5, 2021.

Remark 4.1. Although Experiments 1 and 2 have demonstrated that one should gather the market prices of as many derivatives as possible to obtain tight price bounds, exotic options such as spread options, call-on-max options, and call-on-min options are usually only traded in over-the-counter (OTC) markets. Therefore, the prices of these exotic options are not publicly available, and we are thus unable to collect real market data of this type.

4.4.2. Data Preprocessing. We apply a procedure using Algorithm 1 to detect whether arbitrage opportunities are present with the bid and ask prices of call and put options written on each of the stocks and on DIA, similar to Experiment 3. We found that the prices of DIA options are arbitrage-free, whereas arbitrage opportunities are present among the prices of options written on 5 of the 30 underlying stocks. Before proceeding to the next step of the experiment, we adjust the bid and ask prices slightly to remove these arbitrage opportunities. In this process, the l_1-norm of the price adjustment is minimized to encourage sparsity in the same spirit as Cohen et al. (2020). We refer the reader to Section EC.3.3 in the online appendices for details of this arbitrage removal process. After this process, 27 of the 4,304 prices have been adjusted, and the largest change (in absolute value) is \$0.38. This shows that only very few options were mis-priced, and the market was close to being arbitrage-free.

4.4.3. Experimental Setting. We consider Setting 1 (Assumption 3.2) and let $\Omega = \mathbb{R}_+^{30}$. We rank the 30 constituent stocks of the DJIA index based on the market capitalization of the respective companies; that is, in $(x_1, \dots, x_{30})^{\mathrm{T}} \in \Omega$, x_1 corresponds to the stock price of the company with the highest market capitalization, and x_{30} corresponds to the stock price of the company with the lowest market capitalization. Our goal is to use Algorithm 1 to compute the model-free lower and upper price bounds of two basket call options with the following payoff functions:

$$
\begin{aligned}
f_1(\boldsymbol{x}) &= \left[\left(\sum_{i=6}^{30} w_{\mathrm{DIA}} x_i\right) - \kappa\right]^+, \\
f_2(\boldsymbol{x}) &= \left[\left(\sum_{i=1}^{10} 1.2 w_{\mathrm{DIA}} x_i\right) + \left(\sum_{i=11}^{20} w_{\mathrm{DIA}} x_i\right) + \left(\sum_{i=21}^{30} 0.8 w_{\mathrm{DIA}} x_i\right) - \kappa\right]^+,
\end{aligned}
\tag{4.1}
$$

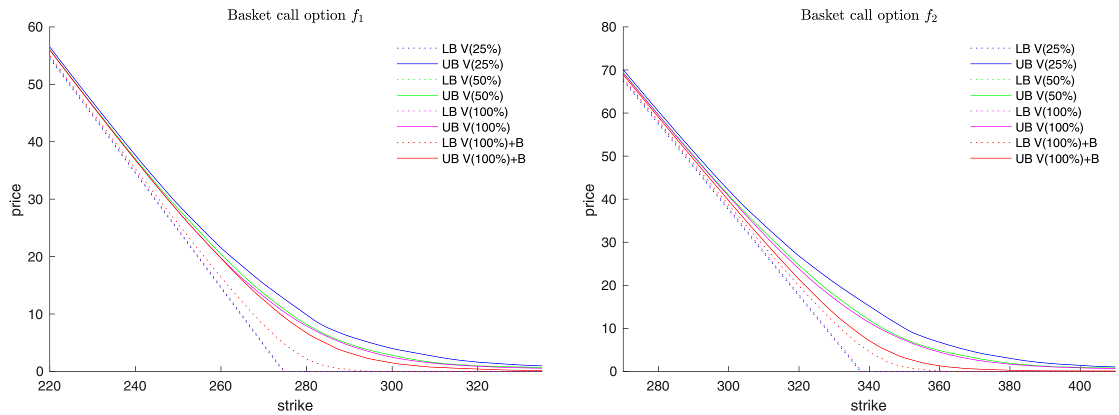
where κ is the strike price that is varied in this experiment. Therefore, f_1 is the payoff of a basket call option written on DIA with the five largest companies in terms of market capitalization excluded (i.e., a basket option written on a subset of the 30 constituent stocks of the DJIA index); f_2 is the payoff of a basket call option written on a weight-adjusted version of DIA, where the weights of the 10 largest companies are increased by 20% and the weights of the 10 smallest companies are decreased by 20%.

After preprocessing, we are left with the following 2,152 traded financial derivatives ($m = 2{,}152$):

- 980 vanilla call options and 980 vanilla put options written on the 30 stocks.
- 96 basket call options and 96 basket put options written on DIA.

We consider four different cases when computing the model-free lower and upper bounds. In the first three cases, denoted by *V(25%)*, *V(50%)*, and *V(100%)*, we randomly select 25%, 50%, and 100% of the vanilla options, respectively. In the fourth case, denoted by *V(100%)+B*, we use all vanilla and basket options. The

Figure 3. (Color online) Experiment 4: Model-Free Lower and Upper Price Bounds for Basket Call Options



inputs of the ECP algorithm for this experiment are similar to those in Experiment 1.

4.4.4. Experimental Results. Figure 3 shows the computed model-free upper and lower price bounds of the two basket call options with payoff functions f_1 and f_2 defined in (4.1). When pricing the first basket call option f_1, the results from the first three cases show that, when more vanilla options are considered, the uncertainty about the marginals of the pricing measure decreases and the model-free upper and lower price bounds are improved, where the improvements of the upper bounds are much more noticeable compared with the lower bounds. In Case *V(100%)+B*, when other basket options are considered in addition to the vanilla options, the upper price bounds are further improved for large strikes, and the lower price bounds are improved substantially, especially for strikes between \$260 and \$280. Compared with Case *V(100%)*, the inclusion of basket options results in a maximum reduction of \$5.59 in the no-arbitrage gap when the strike is \$275, which reduces the gap by 54.7%. The results when pricing the second basket call option f_2 are in line with the results from pricing f_1, and the improvements of the upper and lower price bounds in Case *V(100%)+B* are more significant across all strikes. Compared with Case *V(100%)*, the inclusion of basket options results in a maximum reduction of \$9.79 in the no-arbitrage gap when the strike is \$338, which corresponds to a reduction of 79.4%. We conclude from these observations that when pricing a derivative using real market data, the inclusion of more traded derivatives in the sub- and superreplicating portfolios, especially derivatives whose payoff structures are similar to the target derivative, can produce substantial improvements to the arbitrage-free price bounds, and that the proposed algorithms are applicable to real market data.

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Appendix A. Proofs of the Main Results

A.1. Proof of Theorem 3.4

Proof of Theorem 3.4. By the assumption about $\underline{\phi}$ in Remark EC.2.10, there exist c_0 and y_0 such that $c_0 + \langle y_0, g\rangle \geq -f$ and $\underline{\phi} = -c_0 - \pi(y_0)$. For any c and y such that $c + \langle y, g\rangle \geq f$, it holds that $c_0 + c + \langle y_0 + y, g\rangle \geq 0$ and thus $\langle y_0 + y, g\rangle - \pi(y_0 + y) \geq -c_0 - c - \pi(y_0 + y)$. By Assumption 2.1, $-c_0 - c - \pi(y_0 + y) \leq 0$, and therefore $-c_0 - c \leq \pi(y_0 + y) \leq \pi(y_0) + \pi(y)$; thus, $\underline{\phi} = -c_0 - \pi(y_0) \leq c + \pi(y)$. This implies that $\underline{\phi} \leq \phi(f)$. Let $(\hat{c}, \hat{y}^+, \hat{y}^-)$ be an optimizer of (3.1), which exists because of Theorem 2.4(ii). We have $\hat{c} + \pi(\hat{y}^+ - \hat{y}^-) = \phi(f) > \underline{\phi} - \tau$. Let $\sigma^{(r)}$ denote the system of linear inequalities σ at iteration r. Proposition EC.2.6 states that $\inf_{x \in \mathbb{R}^d_+} s_y(x) > -\infty$ if and only if $y \in \mathbb{R}^m$ satisfies all constraints in $\widetilde{\sigma}$. It hence holds by Line 2 that $(\hat{c}, \hat{y}^+, \hat{y}^-)$ satisfies all constraints in $\sigma^{(r)}$ for all r. Consequently, we have

$$\underline{\varphi}^{(r)} = \inf_{(c, y^+, y^-) \text{ satisfies } \sigma^{(r)}} \{c + \pi(y^+ - y^-)\} \leq \hat{c} + \pi(\hat{y}^+ - \hat{y}^-) = \phi(f). \tag{A.1}$$

The variable $\underline{\varphi}^{(r)}$ is nondecreasing in r because more constraints are added to σ. Moreover, by Proposition EC.2.4(iv) and the assumption on $\overline{x}$, for any $y \in \mathbb{R}^m$ that satisfies all constraints in $\widetilde{\sigma}$, it holds that $\inf_{x\in\mathbb{R}^d_+} s_y(x) = \inf_{0\le x\le\overline{x}} s_y(x)$. By Definition 3.3, for all r and for any $x\in\Omega$,

$$\begin{aligned}
&c^{(r)} - s^{(r)} + \langle y^{(r)}, g(x)\rangle - f(x)\\
&= c^{(r)} - c^{(r)} - \inf_{0\le x'\le\overline{x}}\{s_{y^{(r)}}(x')\} + \langle y^{(r)}, g(x)\rangle - f(x)\\
&= \langle y^{(r)}, g(x)\rangle - f(x) - \inf_{x'\in\mathbb{R}^d_+} s_{y^{(r)}}(x')\\
&= \langle y^{(r)}, g(x)\rangle - f(x) - \inf_{x'\in\mathbb{R}^d_+}\left\{\langle y^{(r)}, g(x')\rangle - f(x')\right\}\\
&\ge 0,
\end{aligned} \tag{A.2}$$

and thus by Line 8, $\underline{\varphi}^{(r)} - s^{(r)} = c^{(r)} - s^{(r)} + \pi(y^{(r)}) \ge \phi(f)$. This and (A.1) also show that $s^{(r)}\le 0$. We have proved statement (i).

If Algorithm 1 terminates, then by (A.2), it holds for all $x\in\mathbb{R}^d_+$ that $c^\star + \langle y^\star, g(x)\rangle - f(x) = c^{(r-1)} - s^{(r-1)} + \langle y^{(r-1)}, g(x)\rangle - f(x) \ge 0$. Therefore, $(c^\star, y^\star)$ is feasible for (2.2) and $\phi(f)^{\mathrm{LB}} \le \phi(f) \le \phi(f)^{\mathrm{UB}}$ follows directly from statement (i). We have $s^{(r-1)} \ge -\varepsilon$ at termination, and thus, $\phi(f)^{\mathrm{UB}} - \phi(f)^{\mathrm{LB}} \le \varepsilon$ and $(c^\star, y^\star)$ is ε-optimal. We now show that Algorithm 1 terminates. Proposition EC.2.4(iv) states that there exists a partition $\mathfrak{C}$ of $\{x\in\mathbb{R}^d: 0\le x\le\overline{x}\}$, such that each $C\in\mathfrak{C}$ is a polytope, $\bigcup_{C\in\mathfrak{C}} C = \{x\in\mathbb{R}^d: 0\le x\le\overline{x}\}$, and that for all $y\in\mathbb{R}^m$, $s_y(\cdot)$ is an affine function when restricted to each $C\in\mathfrak{C}$. Let[9] $\mathfrak{F} := \{F\neq\emptyset \text{ is a face of some } C\in\mathfrak{C}\}$. By theorem 18.2 of Rockafellar (1970),

$$\bigcup_{F\in\mathfrak{F}} \operatorname{relint}(F) = \bigcup_{C\in\mathfrak{C}} C = \{x\in\mathbb{R}^d_+ : 0\le x\le\overline{x}\}. \tag{A.3}$$

Moreover, by theorem 19.1 of Rockafellar (1970), $|\mathfrak{F}|<\infty$. Let $x^{(r)}$ be a minimizer of the MILP problem in Line 11, that is, $c^{(r)} + s_{y^{(r)}}(x^{(r)}) = s^{(r)}$. We prove that either Algorithm 1 terminates, or for each $F\in\mathfrak{F}$, there exists at most one $r\in\mathbb{N}$ such that $x^{(r)}\in\operatorname{relint}(F)$. Suppose, for the sake of contradiction, that Algorithm 1 does not terminate and that there exists $r, l\in\mathbb{N}, r<l$, and $x^{(r)}, x^{(l)}\in\operatorname{relint}(F)$ for some $F\in\mathfrak{F}$. Because $x^{(r)}\in X^{(r)}$, we have $c^{(l)} + s_{y^{(l)}}(x^{(r)})\ge 0$ by Line 7. We also have $c^{(l)} + s_{y^{(l)}}(x^{(l)}) = s^{(l)} < 0$, because otherwise, Algorithm 1 will terminate at the lth iteration. For every $\lambda\in\mathbb{R}$, let $x_\lambda := (1-\lambda)x^{(r)} + \lambda x^{(l)}$. Because $x^{(r)}, x^{(l)}\in\operatorname{relint}(F)$, there exists $\hat{\lambda}>1$ such that $x_{\hat{\lambda}}\in F\subset C$. Because $c^{(l)} + s_{y^{(l)}}(\cdot)$ is an affine function when restricted to the set C, we have by $c^{(l)} + s_{y^{(l)}}(x^{(r)})\ge 0$ and $c^{(l)} + s_{y^{(l)}}(x^{(l)}) < 0$ that

$$\begin{aligned}
c^{(l)} + s_{y^{(l)}}(x_{\hat{\lambda}}) &= (1-\hat{\lambda})(c^{(l)} + s_{y^{(l)}}(x^{(r)})) + \hat{\lambda}(c^{(l)} + s_{y^{(l)}}(x^{(l)}))\\
&\le \hat{\lambda}(c^{(l)} + s_{y^{(l)}}(x^{(l)})) < c^{(l)} + s_{y^{(l)}}(x^{(l)}),
\end{aligned}$$

contradicting the fact that $x^{(l)}$ is a minimizer of the MILP problem in Line 11. Because for each r, $x^{(r)}\in\operatorname{relint}(F)$ for some $F\in\mathfrak{F}$ as a consequence of (A.3), and because $|\mathfrak{F}|<\infty$, Algorithm 1 terminates eventually. The proof of statement (ii) is now complete.

Finally, if Line 16 of Algorithm 1 is reached, then $\underline{\phi} > \underline{\varphi}^{(r-1)} - s^{(r-1)}$. By (A.2), $c^{(r-1)} - s^{(r-1)} + \langle y^{(r-1)}, g\rangle \ge f$. By the assumption about $\underline{\phi}$ in Remark EC.2.10, there exist c_0 and y_0 such that $c_0 + \langle y_0, g\rangle \ge -f$ and $\underline{\phi} = -c_0 - \pi(y_0)$. Hence, we have $c_0 + c^{(r-1)} - s^{(r-1)} + \langle y_0 + y^{(r-1)}, g\rangle \ge 0$, and thus

$$\begin{aligned}
\langle y_0 + y^{(r-1)}, g\rangle - \pi(y_0 + y^{(r-1)}) &\ge -c_0 - c^{(r-1)} + s^{(r-1)} - \pi(y_0 + y^{(r-1)})\\
&\ge -c_0 - \pi(y_0) - c^{(r-1)} + s^{(r-1)} - \pi(y^{(r-1)})\\
&= \underline{\phi} - \underline{\varphi}^{(r-1)} + s^{(r-1)} > 0,
\end{aligned}$$

which is a violation of Assumption 2.1. The proof is now complete. □

A.2. Proofs of Theorem 3.7 and Corollary 3.8

Proof of Theorem 3.7. If Assumption 2.1 holds, then by the same argument as in the proof of Theorem 3.4(i), $\underline{\phi}\le\phi(f)$. Hence, $\underline{\varphi}^{(0)} \le \phi(f) \le \overline{\varphi}^{(0)}$ and $c^{\star(0)} + \langle y^{\star(0)}, g\rangle \ge f$ follow from our assumptions. For $r\ge 1$, suppose that $\underline{\varphi}^{(r-1)} \le \phi(f)$. Then, $\underline{\varphi}^{(r)} \neq \underline{\varphi}^{(r-1)}$ only when Line 10 is reached. This implies that $\sigma(\overline{c}, \overline{y}, \underline{\varphi}^{(r-1)}, \varphi^{(r)}, X) = \emptyset$. By the assumption in Remark EC.2.13, there exists an optimizer $(\hat{c}, \hat{y}^+, \hat{y}^-)$ of (3.1) that satisfies $|\hat{c}|\le\overline{c}-1$, $0\le\hat{y}^+\le\overline{y}-\mathbf{1}$, $0\le\hat{y}^-\le\overline{y}-\mathbf{1}$. In particular, $\hat{c} + \langle\hat{y}^+ - \hat{y}^-, g(x)\rangle \ge f(x)$ for all $x\in X$ and

$$\hat{c} + \langle\hat{y}^+, \overline{\pi}\rangle - \langle\hat{y}^-, \underline{\pi}\rangle = \hat{c} + \pi(\hat{y}) = \phi(f). \tag{A.4}$$

By (A.4) and by the assumption that $\underline{\varphi}^{(r-1)} \le \phi(f)$, it holds that $\sigma(\overline{c}, \overline{y}, \underline{\varphi}^{(r-1)}, \varphi^{(r)}, X) = \emptyset$ implies $\varphi^{(r)} < \phi(f)$. Therefore, by Line 10, $\underline{\varphi}^{(r)} > \varphi^{(r)} > \underline{\varphi}^{(r-1)}$. Because $(\hat{c}, \hat{y}^+, \hat{y}^-)$ also satisfies all constraints in the LP problem in Line 10, we have, again by (A.4), that $\underline{\varphi}^{(r)} \le \phi(f)$. By induction, we have proved that $\underline{\varphi}^{(r)}$ is nondecreasing in r and $\underline{\varphi}^{(r)} \le \phi(f)$ for all r.

For $r\ge 1$, $\overline{\varphi}^{(r)} \neq \overline{\varphi}^{(r-1)}$, $c^{\star(r)} \neq c^{\star(r-1)}$, or $y^{\star(r)} \neq y^{\star(r-1)}$ only if Line 20 is reached. By Line 19 and Line 20, $\overline{\varphi}^{(r)} < \overline{\varphi}^{(r-1)}$. By the same reasoning as in the proof of Theorem 3.4(i) in Equation (A.2), we have $c^{\star(r)} + \langle y^{\star(r)}, g\rangle \ge f$ and $\overline{\varphi}^{(r)} \ge \phi(f)$. We have thus proved statement (i).

If Assumption 2.1 holds and Algorithm 2 terminates, then $\phi(f)^{\mathrm{LB}} \le \phi(f) \le \phi(f)^{\mathrm{UB}}$ and the feasibility and ε-optimality of $(c^\star, y^\star)$ follow directly from statement (i) and Line 3. Thus, we only need to show that Algorithm 2 terminates. Notice that the strong Slater condition in theorem 1 of Betrò (2004) holds because one may take $(\hat{c}, \hat{y}^+, \hat{y}^-)$ defined earlier and choose any $0<\eta<\frac{1}{2}$. Subsequently, one checks that

$$\begin{aligned}
&|\hat{c}+\eta| \le \overline{c} - \eta, && \eta\mathbf{1} \le \hat{y}^+ + \eta\mathbf{1} \le \overline{y} - \eta\mathbf{1},\\
&\eta\mathbf{1} \le \hat{y}^- + \eta\mathbf{1} \le \overline{y} - \eta\mathbf{1}, && (\hat{c}+\eta) + \langle(\hat{y}^+ + \eta\mathbf{1}) - (\hat{y}^- + \eta\mathbf{1}), g\rangle \ge f + \eta.
\end{aligned}$$

Thus, $(\hat{c}+\eta, \hat{y}^+ + \eta\mathbf{1}, \hat{y}^- + \eta\mathbf{1})$ satisfies the strong Slater condition. Moreover, under Assumption 3.6, $\sup_{x\in\Omega}\|g(x)\|<\infty$. Suppose, for the sake of contradiction, that Algorithm 2 loops infinitely and does not terminate. Then, one can deduce that after finitely many iterations, Line 10 is never reached, because each time Line 10 is reached, $\overline{\varphi}^{(r)} - \underline{\varphi}^{(r)} \le \frac{3}{4}(\overline{\varphi}^{(r-1)} - \underline{\varphi}^{(r-1)})$ by Line 4 and Line 6. Similarly, Line 20 is never reached again after finitely many iterations since each time Line 20 is reached $\overline{\varphi}^{(r)} - \underline{\varphi}^{(r)} \le \overline{\varphi}^{(r-1)} - \underline{\varphi}^{(r-1)} - \varepsilon$. The rest of the proof of statement (ii) follows exactly as the proof of theorem 1 in Betrò (2004).

For statement (iii), notice that because $\underline{\varphi}^{(0)} = \underline{\phi} - \tau \leq \phi(f) - \tau < \phi(f)^{\mathrm{LB}}$, Line 10 is reached at least once before termination. Thus, $c^{\dagger}$ and $\boldsymbol{y}^{\dagger}$ are defined. Let $\boldsymbol{y}^{+\dagger}$ and $\boldsymbol{y}^{-\dagger}$ be defined in Line 10. Then, by Line 10, $(c^{\dagger}, \boldsymbol{y}^{+\dagger}, \boldsymbol{y}^{-\dagger})$ is an optimal solution of the LP problem:

$$\begin{aligned} &\text{minimize} && c + \langle \boldsymbol{y}^{+}, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^{-}, \underline{\boldsymbol{\pi}} \rangle \\ &\text{subject to} && c + \langle \boldsymbol{y}^{+} - \boldsymbol{y}^{-}, \boldsymbol{g}(\boldsymbol{x}) \rangle \geq f(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in X^{\dagger}, \\ &&& -\overline{c} \leq c \leq \overline{c}, \mathbf{0} \leq \boldsymbol{y}^{+} \leq \overline{\boldsymbol{y}}, \mathbf{0} \leq \boldsymbol{y}^{-} \leq \overline{\boldsymbol{y}}. \end{aligned} \tag{A.5}$$

Thus, because $\underline{\varphi}^{(r)}$ is updated whenever $(c^{\dagger}, \boldsymbol{y}^{\dagger})$ are updated, we have $\phi(f)^{\mathrm{LB}} = c^{\dagger} + \langle \boldsymbol{y}^{+\dagger}, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^{-\dagger}, \underline{\boldsymbol{\pi}} \rangle$, and $c^{\dagger} + \langle \boldsymbol{y}^{+\dagger} - \boldsymbol{y}^{-\dagger}, \boldsymbol{g}(\boldsymbol{x}) \rangle \geq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in X^{\dagger}$. Let $\widetilde{B} := \{(c, \boldsymbol{y}^{+}, \boldsymbol{y}^{-}) : -\overline{c} \leq c \leq \overline{c}, \mathbf{0} \leq \boldsymbol{y}^{+} \leq \overline{\boldsymbol{y}}, \mathbf{0} \leq \boldsymbol{y}^{-} \leq \overline{\boldsymbol{y}}\} \subset \mathbb{R}^{2m+1}$. By the assumption of statement (iii), $-\overline{c} < c^{\dagger} < \overline{c}, \mathbf{0} \leq \boldsymbol{y}^{+\dagger} < \overline{\boldsymbol{y}}, \mathbf{0} \leq \boldsymbol{y}^{-\dagger} < \overline{\boldsymbol{y}}$, and we claim that $(c^{\dagger}, \boldsymbol{y}^{+\dagger}, \boldsymbol{y}^{-\dagger})$ is also optimal for the following LP problem:

$$\begin{aligned} &\text{minimize} && c + \langle \boldsymbol{y}^{+}, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^{-}, \underline{\boldsymbol{\pi}} \rangle \\ &\text{subject to} && c + \langle \boldsymbol{y}^{+} - \boldsymbol{y}^{-}, \boldsymbol{g}(\boldsymbol{x}) \rangle \geq f(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in X^{\dagger}, \\ &&& \boldsymbol{y}^{+} \geq \mathbf{0}, \boldsymbol{y}^{-} \geq \mathbf{0}. \end{aligned} \tag{A.6}$$

Suppose, for the sake of contradiction that (A.6) has optimal solution $(\widetilde{c}, \widetilde{\boldsymbol{y}}^{+}, \widetilde{\boldsymbol{y}}^{-})$ with $\kappa := \widetilde{c} + \langle \widetilde{\boldsymbol{y}}^{+}, \overline{\boldsymbol{\pi}} \rangle - \langle \widetilde{\boldsymbol{y}}^{-}, \underline{\boldsymbol{\pi}} \rangle < \phi(f)^{\mathrm{LB}}$. Then, because $\phi(f)^{\mathrm{LB}}$ is the optimal value of (A.5), we have $(\widetilde{c}, \widetilde{\boldsymbol{y}}^{+}, \widetilde{\boldsymbol{y}}^{-}) \notin \widetilde{B}, \widetilde{\boldsymbol{y}}^{+} \geq \mathbf{0}, \widetilde{\boldsymbol{y}}^{-} \geq \mathbf{0}$. Let $c_{\lambda} := \lambda c^{\dagger} + (1-\lambda)\widetilde{c}$, $\boldsymbol{y}_{\lambda}^{+} := \lambda \boldsymbol{y}^{+\dagger} + (1-\lambda)\widetilde{\boldsymbol{y}}^{+}$, $\boldsymbol{y}_{\lambda}^{-} := \lambda \boldsymbol{y}^{-\dagger} + (1-\lambda)\widetilde{\boldsymbol{y}}^{-}$. Then, there exists some $\lambda \in (0,1)$, such that $(c_{\lambda}, \boldsymbol{y}_{\lambda}^{+}, \boldsymbol{y}_{\lambda}^{-}) = \lambda(c^{\dagger}, \boldsymbol{y}^{+\dagger}, \boldsymbol{y}^{-\dagger}) + (1-\lambda)(\widetilde{c}, \widetilde{\boldsymbol{y}}^{+}, \widetilde{\boldsymbol{y}}^{-}) \in \widetilde{B}$, $c_{\lambda} + \langle \boldsymbol{y}_{\lambda}^{+} - \boldsymbol{y}_{\lambda}^{-}, \boldsymbol{g}(\boldsymbol{x}) \rangle \geq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in X^{\dagger}$, and $c_{\lambda} + \langle \boldsymbol{y}_{\lambda}^{+}, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}_{\lambda}^{-}, \underline{\boldsymbol{\pi}} \rangle = \lambda \phi(f)^{\mathrm{LB}} + (1-\lambda)\kappa < \phi(f)^{\mathrm{LB}}$, contradicting the optimality of $(c^{\dagger}, \boldsymbol{y}^{+\dagger}, \boldsymbol{y}^{-\dagger})$ for (A.5). Therefore, $(c^{\dagger}, \boldsymbol{y}^{+\dagger}, \boldsymbol{y}^{-\dagger})$ is also optimal for (A.6), whose corresponding dual LP problem is exactly (3.4). Then, an optimal solution $(\mu_{\boldsymbol{x}}^{\star})_{\boldsymbol{x} \in X^{\dagger}}$ of (3.4) exists, its corresponding finitely supported measure $\mu^{\star}$ is a probability measure that satisfies $\underline{\pi}_j \leq \int_{\Omega} g_j \mathrm{d}\mu^{\star} \leq \overline{\pi}_j$ for $j = 1, \ldots, m$, and thus, $\mu^{\star} \in \mathcal{Q}$. Moreover, because of the strong duality of LP problems, $\int_{\Omega} f \mathrm{d}\mu^{\star} = \phi(f)^{\mathrm{LB}} \geq \phi(f) - \varepsilon$ by statement (ii), and $\mu^{\star}$ is ε-optimal for the right-hand side of (2.3) by Theorem 2.4(iii). We have completed the proof of statement (iii).

The proof of statement (iv) is exactly the same as the proof of Theorem 3.4(iii). The proof is now complete. □

Proof of Corollary 3.8. The proof that Algorithm 1 terminates is identical to the proof of Theorem 3.4(ii). Hence, as in Theorem 3.4(ii), we have $\phi(f)^{\mathrm{LB}} \geq \phi(f) - \varepsilon$. Because Line 1 is not used and $\phi(f)^{\mathrm{LB}} \geq \underline{\phi} - \varepsilon > \underline{\phi} - \tau$, we have that $\phi(f)^{\mathrm{LB}}$ is the optimal value of the following LP problem:

$$\begin{aligned} &\text{minimize} && c + \langle \boldsymbol{y}^{+}, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^{-}, \underline{\boldsymbol{\pi}} \rangle \\ &\text{subject to} && c + \langle \boldsymbol{y}^{+} - \boldsymbol{y}^{-}, \boldsymbol{g}(\boldsymbol{x}) \rangle \geq f(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in X = \bigcup_{l=0}^{r-1} X^{(l)}, \end{aligned}$$

whose dual LP problem is exactly (3.5). Consequently, by the argument in the proof of Theorem 3.7(iii), $\int_{\Omega} f \mathrm{d}\mu^{\star} = \phi(f)^{\mathrm{LB}} \geq \phi(f) - \varepsilon$, and $\mu^{\star}$ is ε-optimal for the right-hand side of (2.3) by Theorem 2.4(iii). □

Endnotes

[1] We adopt the notation $\boldsymbol{x} = (x_1, \ldots, x_d)^{\mathrm{T}} > \mathbf{0} \Longleftrightarrow x_1 > 0, \ldots, x_d > 0$.

[2] See https://asic.gov.au/about-asic/news-centre/find-a-media-release/2021-releases/21-064mr-asic-bans-the-sale-of-binary-options-to-retail-clients/, accessed April 18, 2021.

[3] See https://www.investmentexecutive.com/news/from-the-regulators/binary-options-ban-takes-effect/, accessed April 18, 2021.

[4] See https://www.reuters.com/article/us-israel-binaryoptions-lawmaking/israel-ban-on-binary-options-gets-final-parliamentary-approval-idUSKBN1CS2L1, accessed April 18, 2021.

[5] See https://www.esma.europa.eu/press-news/esma-news/esma-agrees-prohibit-binary-options-and-restrict-cfds-protect-retail-investors, accessed April 18, 2021.

[6] See https://github.com/qikunxiang/ModelFreePriceBounds.

[7] See http://marketwatch.com.

[8] See https://www.investopedia.com/terms/d/dowdivisor.asp, accessed April 30, 2021.

[9] A convex subset C' of a convex set $C \subseteq \mathbb{R}^d$ is called a face of C if for all $\lambda \in (0,1)$ and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in C$, $\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in C'$ implies that $\boldsymbol{x}_1 \in C', \boldsymbol{x}_2 \in C'$ (Rockafellar 1970, section 18).

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