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Countably compact group topologies on arbitrarily large free Abelian groups

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ABSTRACT

We prove that if there are $\mathfrak{c}$ incomparable selective ultrafilters then, for every infinite cardinal κ such that $\kappa^\omega = \kappa$, there exists a group topology on the free Abelian group of cardinality κ without nontrivial convergent sequences and such that every finite power is countably compact. In particular, there are arbitrarily large countably compact groups. This answers a 1992 question of D. Dikranjan and D. Shakhmatov.

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1. Introduction

1.1. Some history

It is well known that a non-trivial free Abelian group does not admit a compact Hausdorff group topology. Tomita [21] showed that it does not admit even a group topology whose countable power is countably compact.

Tkachenko [19] showed in 1990 that the free Abelian group generated by $\mathfrak{c}$ elements can be endowed with a countably compact Hausdorff group topology under CH. Tomita [21], Koszmider, Tomita and Watson [14], and Madariaga-Garcia and Tomita [16] obtained such examples using weaker assumptions. Boero, Castro Pereira and Tomita obtained such an example using a single selective ultrafilter [2]. Using $2^{\mathfrak{c}}$ selective ultrafilters, the example in [16] showed the consistency of a countably compact group topology on the free Abelian group of cardinality $2^{\mathfrak{c}}$. All forcing examples obtained so far had their cardinalities bounded by $2^{\mathfrak{c}}$.

Boero and Tomita [3] showed from the existence of $\mathfrak{c}$ selective ultrafilters that there exists a free Abelian group of cardinality $\mathfrak{c}$ whose square is countably compact. Tomita [25] showed that there exists a group topology on the free Abelian group of cardinality $\mathfrak{c}$ that makes all its finite powers countably compact.

E. van Douwen showed in [7] that the cardinality of a countably compact group cannot be a strong limit of countable cofinality.

Using the result in the abstract, we obtain the following:

Theorem 1.1. *Assume* GCH. *Then a free Abelian group of infinite cardinality κ can be endowed with a countably compact group topology (without non-trivial convergent sequences) if and only if $\kappa = \kappa^\omega$.*

The result above answers a question of Dikranjan and Shakhmatov that was posed in the survey by Comfort, Hoffman and Remus [5].

Because of the way our examples are constructed we can raise their weights in the same way as in the papers [22] or [4] and obtain the following result – the examples in these references are Boolean but the trick is similar.

Theorem 1.2. *It is consistent that there is a proper class of cardinals of countable cofinality that can occur as the weight of a countably compact free Abelian group.*

1.2. Basic results, notation and terminology

We recall that a topological space is *countably compact* if, and only if, every countable open cover of it has a finite subcover.

Definition 1.3. Let $\mathcal{U}$ be a filter on ω and let $(x_n : n \in \omega)$ be a sequence in a topological space X. We say that $x \in X$ is a *$\mathcal{U}$-limit point* of $(x_n : n \in \omega)$ if, for every neighborhood U of x, the set $\{n \in \omega : x_n \in U\}$ belongs to $\mathcal{U}$.

If X is Hausdorff, every sequence has at most one $\mathcal{U}$-limit and we write $x = \mathcal{U}\text{-lim}\,(x_n : n \in \omega)$ in that case.

The set of all free ultrafilters on ω is denoted by ω^*. The following proposition is a well known result on ultrafilter limits.

Proposition 1.4. *A topological space is countably compact if and only if each sequence in it has a $\mathcal{U}$-limit point for some $\mathcal{U} \in \omega^*$.*

The concept of almost disjoint families will be useful in our construction.

Definition 1.5. An almost disjoint family is an infinite family $\mathcal{A}$ of infinite subsets of ω such that distinct elements of $\mathcal{A}$ have a finite intersection.

It is well known that there exists an almost disjoint family of size continuum (see [15]).

Definition 1.6. The unit circle group $\mathbb{T}$ will be the metric group $(\mathbb{R}/\mathbb{Z}, \delta)$ where the metric δ is given by

$$\delta(x + \mathbb{Z}, y + \mathbb{Z}) = \min\{|x - y + a| : a \in \mathbb{Z}\}$$

for every $x, y \in \mathbb{R}$.

Given an open interval (a, b) of $\mathbb{R}$ with $a < b$, we let $\delta((a, b)) = b - a$.

An arc of $\mathbb{T}$ is a set of the form $I + \mathbb{Z} = \{a + \mathbb{Z} : a \in I\}$, where I is an open interval of $\mathbb{R}$. An arc is said to be proper if it is distinct from $\mathbb{T}$.

If U is a proper arc and $U = \{a + \mathbb{Z} : a \in I\} = \{b + \mathbb{Z} : a \in J\}$, then the Euclidean length of I equals the Euclidean length of J, and we define the length of U as $\delta(U) = \delta(I)$. We also let $\delta(\mathbb{T}) = 1$.

Given an arc U such that $\delta(U) \leq \frac{1}{2}$, it follows that $\operatorname{diam}_\delta U = \delta(U)$.

Our free Abelian groups will all be represented as directs sums of copies of the group of integers $\mathbb{Z}$; we fix some notation. The additive group of rationals will also be used, so in the following definition one should read $\mathbb{Z}$ or $\mathbb{Q}$ for G.

Definition 1.7. If f is a map from a set X to a group G then the *support* of f, denotes $\operatorname{supp} f$ is defined to be the set $\{x \in X : f(x) \neq 0\}$.

We define $G^{(X)} = \{f \in G^X : |\operatorname{supp} f| < \omega\}$.

If Y is a subset of X then, as an abuse of notation, we often write $G^{(Y)} = \{x \in G^{(X)} : \operatorname{supp} x \subseteq Y\}$.

Given $x \in X$, we denote by χ_x the characteristic function of $\{x\}$, whose support is $\{x\}$ and which value $\chi_x(x) = 1$.

For a sequence $\zeta : \omega \to X$ in X we define $\chi_\zeta : \omega \to G^X$ by $\chi_\zeta(n) = \chi_{\zeta(n)}$.

Finally, for $x \in X$, we let $\vec{x} : \omega \to X$ be the constant sequence with value x.

Note that then $\chi_{\vec{x}}$ is also constant, with value χ_x.

Definition 1.8. Let $\mathcal{U}$ be a filter on ω and X a set. We say that the sequences $f, g \in X^\omega$ are $\mathcal{U}$*-equivalent* and write $f \equiv_{\mathcal{U}} g$ iff $\{n \in \omega : f(n) = g(n)\} \in \mathcal{U}$.

It is easy to verify that $\equiv_{\mathcal{U}}$ is an equivalence relation. We denote the equivalence class of $f \in X^\omega$ by $[f]_{\mathcal{U}}$. We also denote the set of all equivalence classes by $X^\omega/\mathcal{U}$.

If R is a ring and X is an R-module, then $X^\omega/\mathcal{U}$ has a natural R-module structure given by $[f]_{\mathcal{U}} + [g]_{\mathcal{U}} = [f + g]_{\mathcal{U}}$, $[-f]_{\mathcal{U}} = -[f]_{\mathcal{U}}$, $r \cdot [f]_{\mathcal{U}} = [r \cdot f]_{\mathcal{U}}$ and the class of the zero function as its zero element.

If p is a free ultrafilter, then the *ultrapower* of the R-module X by p is the R-module X^ω/p.

For the rest of this paper we will fix a cardinal number κ that satisfies $\kappa^\omega = \kappa$.

Throughout this article, we will work inside ultrapowers of $\mathbb{Q}^{(\kappa)}$. These ultrapowers contain copies of ultrapowers of $\mathbb{Z}^{(\kappa)}$, which will be useful for the construction. So it is useful to define some notation.

Definition 1.9. Let p be a free ultrafilter on ω. We define $\operatorname{Ult}(\mathbb{Q}, p)$ as the $\mathbb{Q}$-vector space $(\mathbb{Q}^{(\kappa)})^\omega/p$ and $\operatorname{Ult}(\mathbb{Z}, p) = \{[g]_p : g \in \mathbb{Z}^\omega\}$ with the subgroup structure.

Notice that each $[g]_p$ in $\mathrm{Ult}(\mathbb{Z}, p)$ is formally an element of $(\mathbb{Q}^{(\kappa)})^\omega/p$, not of $(\mathbb{Z}^{(\kappa)})^\omega/p$. Nevertheless it is clear that $(\mathbb{Z}^{(\kappa)})^\omega/p$ is isomorphic to $\mathrm{Ult}(\mathbb{Z}, \kappa)$ via the obvious isomorphism that carries the equivalence class of a sequence $g \in (\mathbb{Z}^\kappa)^\omega$ in $(\mathbb{Z}^{(\kappa)})^\omega/p$ to its class in $(\mathbb{Q}^{(\kappa)})^\omega/p$.

2. Selective ultrafilters

In this section we review some basic facts about selective ultrafilters, the Rudin-Keisler order and some lemmas we will use in the next sections.

Definition 2.1. A selective ultrafilter (on ω), also called Ramsey ultrafilter, is a free ultrafilter p on ω with the property that for every partition $(A_n : n \in \omega)$ of ω, either there exists n such that $A_n \in p$ or there exists $B \in p$ such that $|B \cap A_n| = 1$ for every $n \in \omega$.

The following proposition is well known. We provide [13] as a reference.

Proposition 2.2. *Let p be a free ultrafilter on ω. Then the following are equivalent:*

a) p is a selective ultrafilter,
b) for every $f \in \omega^\omega$, there exists $A \in p$ such that f is either constant or one-to-one on A,
c) for every function $f : [\omega]^2 \to 2$ there exists $A \in p$ such that f is constant on $[A]^2$.

The Rudin-Keisler order is defined as follows:

Definition 2.3. Let $\mathcal{U}$ be a filter on ω and $f : \omega \to \omega$. We define $f_*(\mathcal{U}) = \{A \subseteq \omega : f^{-1}[A] \in \mathcal{U}\}$.

It is easy to verify that $f_*(\mathcal{U})$ is a filter; if $\mathcal{U}$ is an ultrafilter then so is $f_*(\mathcal{U})$; if $f, g : \omega \to \omega$, then $(f \circ g)_* = f_* \circ g_*$; and $(\mathrm{id}_\omega)_*$ is the identity over the set of all filters. This implies that if f is bijective, then $(f^{-1})_* = (f_*)^{-1}$.

Definition 2.4. Let $\mathcal{U}$ and $\mathcal{V}$ be filters. We say that $\mathcal{U} \leq \mathcal{V}$ (or $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$, if we need to be clear) iff there exists $f \in \omega$ such that $f_*(\mathcal{V}) = \mathcal{U}$.

The *Rudin-Keisler order* is the set of all free ultrafilters over ω ordered by $\leq_{\mathrm{RK}}$. We say that two ultrafilters p and q are equivalent iff $p \leq q$ and $q \leq p$.

It is easy to verify that $\leq$ is a preorder and that the equivalence defined above is indeed an equivalence relation. Moreover, the equivalence class of a fixed ultrafilter is the set of all fixed ultrafilters, so the relation restricts to ω^* without modifying the equivalence classes. We refer to [13] for the following proposition:

Proposition 2.5. *The following are true:*

(1) If p and q are ultrafilters, then $p \leq q$ and $q \leq p$ is equivalent to the existence of a bijection $f : \omega \to \omega$ such that $f_*(p) = q$.
(2) The selective ultrafilters are exactly the minimal elements of the Rudin-Keisler order.

This implies that if $f : \omega \to \omega$ and p is a selective ultrafilter, then $f_*(p)$ is either a fixed ultrafilter or a selective ultrafilter. If $f_*(p)$ is the ultrafilter generated by n, then $f^{-1}[\{n\}] \in p$, so, in particular, if f is finite to one and p is selective, then $f_*(p)$ is a selective ultrafilter equivalent to p.

The existence of selective ultrafilters is independent from ZFC. Martin's Axiom for countable orders implies the existence of $2^{\mathfrak{c}}$ pairwise incomparable selective ultrafilters in the Rudin-Keisler order.

The lemma below appears in [23].

Lemma 2.6. *Let $(p_k : k \in \omega)$ be a family of pairwise incomparable selective ultrafilters. For each k let $(a_{k,i} : i \in \omega)$ be a strictly increasing sequence in ω such that $\{a_{k,i} : i \in \omega\} \in p_k$ and $i < a_{k,i}$ for all $i \in \omega$. Then there exists $\{I_k : k \in \omega\}$ such that:*

a) $\{a_{k,i} : i \in I_k\} \in p_k$, for each $k \in \omega$.
b) $I_i \cap I_j = \emptyset$ whenever $i, j \in \omega$ and $i \neq j$, and
c) $\{[i, a_{k,i}] : i \in I_k$ and $k \in \omega\}$ is a pairwise disjoint family.

In the course of the construction we will often use families of ultrafilters indexed by ω and finite sequences of infinite subsets of ω. The following definition fixes some convenient notation.

Definition 2.7. A *finite tower* in ω is a finite sequence $(A_0, \dots, A_{k-1})$ of infinite subsets of ω such that $A_{t+1} \subseteq A_t$ for every $t < k-1$. The set of all finite towers in ω is called $\mathcal{T}$. If $T = (A_0, \dots, A_{k-1})$ then $l(T) = A_{k-1}$, the last term of the sequence T. For the empty sequence we write $l(\emptyset) = \omega$.

Lemma 2.8. *Assume there are $\mathfrak{c}$ incomparable selective ultrafilters. Then there is a family of incomparable selective ultrafilters $(p_{T,n} : T \in \mathcal{T}, n \in \omega)$ such that $l(T) \in p_{T,n}$ whenever $T \in \mathcal{T}$ and $n \in \omega$.*

Proof. Index the $\mathfrak{c}$ incomparable selective ultrafilters faithfully as $\{q_{T,n} : T \in \mathcal{T}, n \in \omega\}$. For each T, let $f_T : \omega \to l(T)$ be a bijection and define $p_{T,n} = f_{T*}(q_{T,n})$. Since f is one-to-one, it follows that $p_{T,n}$ is a selective ultrafilter equivalent to $q_{T,n}$. The family $(p_{T,n} : T \in \mathcal{T}, n \in \omega)$ is as required. $\square$

3. Main ideas

From now on we fix a family $(p_{T,n} : n \in \omega, T \in \mathcal{T})$ of selective ultrafilters as provided by Lemma 2.8.

The idea will be to use these ultrafilters to assign p-limits to enough injective sequences in $\mathbb{Z}^{(\kappa)}$ to ensure countable compactness of the resulting topology. We take some inspiration from [2] where a large independent family was used such that, up to a permutation every injective sequence in $\mathbb{Z}^{(\mathfrak{c})}$ was part of this family. Since this group has cardinality $\mathfrak{c}$, there were indeed enough permutations to accomplish this. For an arbitrarily large group, we shall consider large linearly independent pieces to make sure every sequence has an accumulation point.

The following definition will be used to construct a witness for linearly independence in an ultraproduct that does not depend on the free ultrafilter.

Definition 3.1. Let $\mathcal{F}$ be a subset of $(\mathbb{Z}^{(\kappa)})^\omega$ and $A \in [\omega]^\omega$. We shall call $\mathcal{F}$ *linearly independent mod A^** iff for every free ultrafilter p with $A \in p$ the set

$$\{[f]_p : f \in \mathcal{F}\} \,\dot\cup\, \{[\chi_{\vec{\xi}}]_p : \xi < \kappa\}$$

is linearly independent in the $\mathbb{Q}$-vector space $\mathrm{Ult}(\mathbb{Q}, p)$, and if $[f]_p \neq [g]_p$ whenever f and g are distinct elements of $\mathcal{F}$.

Notice that it is implicit in our definition that $\{[f]_p : f \in \mathcal{F}\}$ and $\{[\chi_{\vec{\xi}}]_p : \xi < \kappa\}$ are disjoint. We will abbreviate "linearly independent mod A^*" to l.i. mod A^*.

An application of Zorn's Lemma will establish the following lemma.

Lemma 3.2. *Every set of sequences that is l.i. mod A^* can be extended to a maximal linearly independent set mod A^*.* $\square$

It should be clear that $A \subseteq B \subseteq \omega$ and A and B are infinite, then a set that is l.i. mod B^* is also l.i. mod A^*. By using recursion, this easily implies the following corollary:

Corollary 3.3. *There exists a family* $(\mathcal{E}_T : T \in \mathcal{T})$ *such that:*

(1) For every $T \in \mathcal{T}$ *the set* $\mathcal{E}_T$ *is maximal l.i. mod* $l(T)^*$*, and*
(2) For every $T \in \mathcal{T}$*, if* $n \leq |T|$ *then* $\mathcal{E}_{T|n} \subseteq \mathcal{E}_T$*.*

We note explicitly that even though $\mathcal{E}_T$ is only demanded to be maximal l.i. mod $l(T)^*$ it will, because of item (2), depend on all of T, not just on $l(T)$.

Lemma 3.4. *Let* g *be an element of* $(\mathbb{Z}^{(\kappa)})^\omega$ *and let* $\mathcal{E} \subseteq (\mathbb{Z}^{(\kappa)})^\omega$ *be maximal l.i. mod* B^**. Then there exist an infinite subset* A *of* B*, a finite subset* E *of* $\mathcal{E}$*, a finite subset* D *of* κ*, and sets* $\{r_f : f \in E\}$ *and* $\{s_\nu : \nu \in D\}$ *of rational numbers such that*

$$g|_A = \sum_{f \in E} r_f \cdot f|_A + \sum_{\nu \in D} s_\nu \cdot \chi_{\vec{\nu}}|_A.$$

Proof. If $g \in \mathcal{E}$ or $g = \chi_{\vec{\nu}}$ for some $\nu < \kappa$, then we are done. Otherwise, by the maximality of $\mathcal{E}$, there exists a free ultrafilter p with $B \in p$ such that the set

$$\{[g]_p\} \cup \{[h]_p : h \in \mathcal{E}\} \cup \{[\chi_{\vec{\xi}}]_p : \xi < \kappa\}$$

is not linearly independent.

This means that we can find finite subsets E and D of $\mathcal{E}$ and κ respectively and finite sets $\{r_f : f \in E\}$ and $\{s_\nu : \nu \in D\}$ of rational numbers such that

$$[g]_p = \sum_{f \in E} r_f \cdot [f]_p + \sum_{\nu \in D} s_\nu \cdot [\chi_{\vec{\nu}}]_p.$$

Now choose $A \in p$ with $A \subseteq B$ that witnesses this equality. $\square$

Corollary 3.5. *If* $\mathcal{E} \subseteq (\mathbb{Z}^{(\kappa)})^\omega$ *is maximal l.i. mod* B^**, then* $|\mathcal{E}| = \kappa$*.*

Proof. First notice that $|\mathcal{E}| \leq |(\mathbb{Z}^{(\kappa)})^\omega| = \kappa^\omega = \kappa$. Assume $|\mathcal{E}| < \kappa$. Then the set $C = \bigcup\{\operatorname{supp} f(n) : n \in \omega, f \in \mathcal{E}\}$ has cardinality less than κ.

Take some injective sequence $\langle \xi_n : n \in \omega \rangle$ in $\kappa \setminus C$ and define $g : \omega \to \mathbb{Z}^{(\kappa)}$ by $g(n) = \chi_{\xi_n}$ for all n. Clearly then $\bigcup\{\operatorname{supp} g(n) : n \in \omega\}$ is disjoint from C, all values of g are non-zero and the values have disjoint supports.

Apply Lemma 3.4 to obtain sets A, E, D, $\{r_f : f \in E\}$, and $\{s_\nu : \nu \in D\}$ such that

$$g|_A = \sum_{f \in E} r_f \cdot f|_A + \sum_{\nu \in D} s_\nu \cdot \chi_{\vec{\nu}}|_A. \tag{$*$}$$

Since A is infinite and D is finite, there is a $k \in A$ such that $\xi_k \notin D$. Now $f(k)(\xi_k) = 0$ when $f \in E$ because $\xi_k \notin C$, and $\chi_{\vec{\nu}}(k)(\xi_k) = 0$ when $\nu \in D$ because $\xi_k \notin D$, and also $g(k)(\xi_k) = 1$, which contradicts $(*)$. $\square$

Henceforth we fix a family $(\mathcal{E}_T : T \in \mathcal{T})$ as in Corollary 3.3 and enumerate each $\mathcal{E}_T$ faithfully as $\mathcal{E}_T = \{f^T_\xi : \kappa \leq \xi < \kappa + \kappa\}$.

Definition 3.6. For each $T \in \mathcal{T}$ and $n \in \omega$, we denote by $G_{T,n}$ the intersection of $\mathrm{Ult}(\mathbb{Z}, p_{T,n})$ and the free Abelian group generated by $\{\frac{1}{n!}[f^T_\xi]_{p_{T,n}} : \kappa \leq \xi < \kappa + \kappa\} \cup \{\frac{1}{n!}[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\}$.

For the next lemma, we are going to use the following proposition:

Proposition 3.7. *If G is an abelian group and H is a subgroup of G such that G/H is an infinite cyclic group, then there exists $a \in G$ such that $G = H \oplus \langle a \rangle$.*

A proof may be found in [8, 14.4]. This is not the statement of the theorem but it is exactly what is proved by the author.

The main idea of the proof of the following lemma is to mimic the well known proof of the fact that every subgroup of a free abelian group is free.

Lemma 3.8. *The group $G_{T,n}$ has a basis of the form $\{[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\} \dot\cup \{[f]_{p_{T,n}} : f \in \mathcal{F}_{T,n}\}$ for some subset $\mathcal{F}_{T,n}$ of $(\mathbb{Z}^{(\kappa)})^\omega$.*

Proof. Let H_μ the group generated by $\{\frac{1}{n!}[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \mu\}$ if $\mu \leq \kappa$ and by the union of $\{\frac{1}{n!}[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\}$ and $\{\frac{1}{n!}[f^T_\xi]_{p_{T,n}} : \kappa \leq \xi < \mu\}$ when $\kappa < \mu \leq \kappa + \kappa$.

Let $G_\mu = H_\mu \cap \mathrm{Ult}(\mathbb{Z}, p_{T,n})$ for all μ.

For every $\mu < \kappa + \kappa$ we shall find h_μ so that $G_{\mu+1} = G_\mu \oplus \big\langle\{[h_\mu]_{p_{T,n}}\}\big\rangle$, as follows.

For $\mu < \kappa$ the group G_μ is generated by $\{[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \mu\}$, so $G_{\mu+1} = G_\mu \oplus \langle\{[\chi_{\vec{\mu}}]\}\rangle$ and we have $h_\mu = \chi_{\vec{\mu}}$.

For $\mu \geq \kappa$ observe that $G_{\mu+1} \cap H_\mu = G_\mu$, so:

$$\frac{G_{\mu+1}}{G_\mu} = \frac{G_{\mu+1}}{G_{\mu+1} \cap H_\mu} \approx \frac{G_{\mu+1} + H_\mu}{H_\mu} \leq \frac{H_{\mu+1}}{H_\mu}.$$

The group $\frac{H_{\mu+1}}{H_\mu}$ is cyclic infinite, so either $\frac{G_{\mu+1}}{G_\mu}$ is infinite and cyclic or $G_{\mu+1} = G_\mu$. By Proposition 3.7 there exists $a_\mu \in G_{\mu+1}$ such that $G_{\mu+1} = G_\mu \oplus \langle\{a_\mu\}\rangle$ (and $a_\mu = 0$ in case $G_{\mu+1} = G_\mu$). Take h_μ such that $[h_\mu]_{p_{T,n}} = a_\mu$.

For every $\mu < \kappa + \kappa$, it follows that $G_{\mu+1} = G_\mu \oplus \big\langle\{[h_\mu]_{p_{T,n}}\}\big\rangle$. Since $G_{T,n} = \bigcup_{\mu<\kappa+\kappa} G_\mu$, it follows that $G_{T,n} = \bigoplus_{\mu<\kappa+\kappa}\big\langle\{[h_\mu]_{p_{T,n}}\}\big\rangle$.

The set $\mathcal{F}_{T,n} = \{h_\mu : \kappa \leq \mu < \kappa + \kappa, [h_\mu]_{p_{T,n}} \neq 0\}$ is as required. $\square$

For the rest of this article we fix such a set $\mathcal{F}_{T,n}$ as above for each pair (T, n) in $\mathcal{T} \times \omega$.

The next lemma makes good on the promise from the beginning of this section as it shows how to make our topology countably compact.

Lemma 3.9. *Assume that for every pair (T, n) in $\mathcal{T} \times \omega$ every sequence f in $\mathcal{F}_{T,n}$ has a $p_{T,n}$-limit in $\mathbb{Z}^{(\kappa)}$. Then every finite power of $\mathbb{Z}^{(\kappa)}$ is countably compact.*

Proof. A sequence in some finite power of $\mathbb{Z}^{(\kappa)}$ is represented by finitely many members of $(\mathbb{Z}^{(\kappa)})^\omega$, say $g_0, \dots, g_m$. We show that there is one ultrafilter p such that p-$\lim g_i$ exists for all i, namely $p_{T,n}$ for a suitable T and n.

Recursively, we define a tower $T = (A_0, \dots, A_m)$ and for $i \leq m$ finite subsets E_i and D_i of $\mathcal{E}_{T|_i}$ and κ respectively together with finite sets $(r^i_f : f \in E_i)$ and $(s^i_\nu : \nu \in D_i)$ of rational numbers such that

$$g_i|_{A_i} = \sum_{f\in E_i} r^i_f \cdot f|_{A_i} + \sum_{\nu\in D_i} s^i_\nu \cdot \chi_{\vec{\nu}}|_{A_i} \tag{$*$}$$

For $i = 0$, use Lemma 3.4 applied to $\mathcal{E}_\emptyset$ to obtain A_0, E_0, D_0, $(r^0_f : f \in E_0)$ and $(s^0_\nu : \nu \in D_0)$ such that $(*)$ holds with $i = 0$.

To go from i to $i+1$ apply Lemma 3.4 to $\mathcal{E}_{(A_0,\ldots,A_i)}$ to obtain A_{i+1}, E_{i+1}, D_{i+1}, $(r^{i+1}_f : f \in E_{i+1})$, and $(s^{i+1}_\nu : \nu \in D_{i+1})$ so that $(*)$ holds for $i+1$.

Let $A = A_m$ and let n be sufficiently large so that $n!r^i_f$ and $n!s^i_\nu$ are integers, for all $i \leq m$, $f \in E_i$, and $\nu \in D_i$. Then $g_i|_A = \sum_{f\in E_i} n! \cdot r^i_f \cdot (\frac{1}{n!} \cdot f)|_A + \sum_{\nu\in D_i} n! \cdot s^i_\nu \cdot (\frac{1}{n!} \cdot \chi_{\vec{\nu}})|_A$ for all i.

As $l(T) = A \in p_{T,n}$ and for each E_i is a subset of $\mathcal{E}_T$, it follows that $[g_i]_{p_{T,n}} \in G_{T,n}$. Therefore, each $[g_i]_{p_{T,n}}$ is an integer combination of $\{[f]_{p_{T,n}} : f \in \mathcal{F}_{T,n}\} \cup \{[\chi_\xi]_{p_{T,n}} : \xi < \kappa\}$. Then, by hypothesis, it follows that each g_i has a $p_{T,n}$-limit. This completes the proof. □

4. Constructing homomorphisms

Through this section, we let $G = \mathbb{Z}^{(\kappa)}$ and we let $\{h_\xi : \omega \leq \xi < \kappa\}$ be an enumeration of G^ω such that $\operatorname{supp} h_\xi(n) \subseteq \xi$ whenever $n \in \omega$ and $\omega \leq \xi < \kappa$, and so that each element of G^ω appears at least $\mathfrak{c}$ many times.

Lemma 4.1. *There exists a family $(J_{T,n} : T \in \mathcal{T}, n \in \omega)$ of pairwise disjoint subsets of κ such that $\{h_\xi : \xi \in J_{T,n}\} = \mathcal{F}_{T,n}$.*

Proof. For each $f \in G^\omega$ there is an injective map $\phi_f : \mathcal{T} \times \omega \to \{\xi \in \kappa : f = h_\xi\}$. Let $J_{T,n} = \{\phi_f(T,n) : f \in \mathcal{F}_{T,n}\}$ and we are done. □

For the rest of this section, we fix a family $(J_{T,n} : T \in \mathcal{T}, n \in \omega)$ as above.

The following lemma is the key to the main result.

Lemma 4.2. *Assume we have a non-zero element d of G, an injective sequence r in G, and a countably infinite subset D of κ such that*

(1) $\omega \cup \operatorname{supp} d \cup \bigcup_{n\in\omega} \operatorname{supp} r(n) \subseteq D$,
(2) $\operatorname{supp} h_\xi(n) \subseteq D$ *for all* $n \in \omega$ *and* $\xi \in D \setminus \omega$.

Then there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that:

(1) $\phi(d) \neq 0$
(2) $p_{T,n}$-$\lim_k \phi(h_\xi(k)) = \phi(\chi_\xi)$, *whenever* $T \in \mathcal{T}$, $n \in \omega$, *and* $\xi \in D \cap J_{T,n}$.
(3) $\phi \circ r$ *does not converge.*

Before proving this lemma, we show how to use it to prove the main result. First, we use it to prove another lemma:

Lemma 4.3. *Assume d is a non-zero element of G and r is an injective sequence in G. Then there exists a homomorphism $\phi : \mathbb{Z}^{(\kappa)} \to \mathbb{T}$ such that*

(1) $\phi(d) \neq 0$
(2) $p_{T,n}$-$\lim_k \phi(h_\xi(k)) = \phi(\chi_\xi)$, *whenever* $T \in \mathcal{T}$, $n \in \omega$ *and* $\xi \in J_{T,n}$.
(3) $\phi \circ r$ *does not converge.*

Proof. Using a closing-off argument construct a countable subset D of κ that intersects infinitely many sets $J_{T,n}$, and that contains ω, $\operatorname{supp} d$, $\operatorname{supp} r(n)$ for all n as well as $\operatorname{supp} h_\xi(n)$ whenever $\xi \in D \setminus \omega$ and $n \in \omega$.

By the previous Lemma, there exists a homomorphism $\phi_0 : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that $\phi_0(d) \neq 0$, $\phi \circ r$ does not converge, and $p_{T,n}\text{-}\lim_k \phi_0(h_\xi(k)) = \phi_0(\chi_\xi)$ whenever $T \in \mathcal{T}$, $n \in \omega$ and $\xi \in D \cap J_{T,n}$.

We let $\langle \alpha_\delta : \delta < \kappa \rangle$ be the monotone enumeration of $\kappa \setminus D$. For $\gamma \leq \kappa$, let $D_\gamma = D \cup \{\alpha_\delta : \delta < \gamma\}$. So $D_0 = D$ and $D_\kappa = \kappa$.

Recursively, we construct, for $\gamma \leq \kappa$, an increasing sequence of homomorphisms $\phi_\gamma : \mathbb{Z}^{(D_\gamma)} \to \mathbb{T}$ such that $p_{T,n}\text{-}\lim_k \phi_\gamma(h_\xi(k)) = \phi_\gamma(\chi_\xi)$ whenever $T \in \mathcal{T}$, $n \in \omega$ and $\xi \in D_\gamma \cap J_{T,n}$. Our homomorphism ϕ will be ϕ_κ. The basis step 0 is already done, and for limit steps, we just unite all previous homomorphisms.

To define $\phi_{\gamma+1}$ given ϕ_γ it suffices to specify the value $\phi_{\gamma+1}(\chi_{\alpha_\gamma})$.

If $\alpha_\gamma \in J_{T,n}$ for some $T \in \mathcal{T}$ and $n \in \omega$ then we put $\phi_{\gamma+1}(\chi_{\alpha_\gamma}) = p_{T,n}\text{-}\lim_n \phi_\gamma(h_\gamma(n))$. This is well defined because $\operatorname{supp} h_\gamma(n) \subseteq \gamma \subseteq D_\gamma$ for all n and because $\mathbb{T}$ is compact. In the other case let $\phi_{\gamma+1}(\chi_{\alpha_\gamma}) = 0$. $\square$

We can now prove our main result.

Theorem 4.4. *Assume the existence of pairwise incompatible $\mathfrak{c}$ selective ultrafilters and that κ is an infinite cardinal such that $\kappa^\omega = \kappa$. Then the free abelian group of cardinality κ has a Hausdorff group topology without nontrivial converging sequences such that all of its finite powers are countably compact.*

Proof. Following the notation of the rest of the article, given $d \in G \setminus \{0\}$ and an injective sequence r in G, Lemma 4.3 provides a homomorphism $\phi_{d,r} : G \to \mathbb{T}$ such that $\phi_d(d) \neq 0$, such that $\phi_{d,r} \circ r$ does not converge, and such that $p_{T,n}\text{-}\lim_k \phi_{d,r}(h_\xi(k)) = \phi_{d,r}(\chi_\xi)$ whenever $T \in \mathcal{T}$, $n \in \omega$ and $\xi \in J_{T,n}$. We give G the initial topology generated by the collection of homomorphisms $\{\phi_{d,r} : d \in G \setminus \{0\},\ r \in G^\omega \text{ is injective}\}$ thus obtained and the natural topology of $\mathbb{T}$.

Since the initial topology generated by any collection of group homomorphisms is a group topology we do indeed obtain a group topology. Since $\mathbb{T}$ is Hausdorff and for every $d \neq 0$ there are many $\phi_{d,r}$ with $\phi_{d,r}(d) \neq 0$ it follows at once that our topology is Hausdorff.

To see that every finite power of G is countably compact we use Lemma 3.9.

Given $T \in \mathcal{T}$, $n \in \omega$ and $f \in \mathcal{F}_{T,n}$, there exist $\xi \in J_{T,n}$ such that $h_\xi = f$. For every $d \in G \setminus \{0\}$ and injective $r \in G^\omega$, we have $p_{T,n}\text{-}\lim_n \phi_{d,r}(h_\xi(n)) = \phi_{d,r}(\chi_\xi)$. So $p_{T,n}\text{-}\lim f(n) = \chi_\xi$ and we are done.

Since for a given injective sequence r and any $d \in G^\omega$ the sequence $\phi_{d,r} \circ r$ does not converge and $\phi_{d,r}$ is continuous, it follows that r does not converge. So G has no nontrivial convergent sequences. $\square$

Towards the proof of Lemma 4.2 we formulate a definition and a (very) technical lemma.

Definition 4.5. Let $\epsilon > 0$. An ϵ-*arc function* (for $\mathbb{Z}^{(\kappa)}$) is a function ψ from κ into the set of open arcs of $\mathbb{T}$ (including $\mathbb{T}$ itself) such that for all α either $\psi(\alpha) = \mathbb{T}$ or the length of $\psi(\alpha)$ is equal to ϵ, and the set $\{\alpha \in \kappa : \psi(\alpha) \neq \mathbb{T}\}$ is finite. We will call this finite set the support of ψ and denote it by $\operatorname{supp} \psi$.

Given two arc functions ψ and ϱ we write $\psi \leq \varrho$ if $\overline{\psi(\alpha)} \subseteq \varrho(\alpha)$ or $\psi(\alpha) = \varrho(\alpha)$ for each $\alpha \in \kappa$.

We shall obtain our homomorphisms using limits of such arc functions. The following lemmas are instrumental in its construction.

The following result follows from an argument implicit in the construction of [2], but it may be difficult to extract it from that paper. We postpone its rather technical proof to the next section.

Lemma 4.6. *Let p be a selective ultrafilter and $\mathcal{F}$ a finite subset of G^ω such that the set $\{[f]_p : f \in \mathcal{F}\} \cup \{[\chi_{\vec{\alpha}}]_p : \alpha < \kappa\}$ is linearly independent.*

Then for a given $\epsilon > 0$ and a finite subset E of κ there exist $A \in p$ and a sequence $(\delta_n : n \in A)$ of positive real numbers such that

($\star$) *whenever* $\{U_f : f \in \mathcal{F}\}$ *is a family of arcs of length* ϵ *and* ϱ *is an arc function of length at least* ϵ *with* $\operatorname{supp}\varrho \subseteq E$ *there exist for each* $n \in A$ *a* δ_n*-arc function* $\psi_n \leq \varrho$ *such that* $\operatorname{supp}\psi_n = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f(n) \cup E$, *and* $\sum_{\mu\in\operatorname{supp} f(n)} f(n)(\mu) \cdot \psi_n(\mu) \subseteq U_f$ *for each* $f \in \mathcal{F}$.

Now we proceed to prove Lemma 4.2. We will use the following lemma:

Lemma 4.7. *Let* $(\mathcal{F}^k : k \in \omega)$ *be a sequence of countable subsets of* G^ω *and let* $(p_k : k \in \omega)$ *be a sequence of pairwise incomparable selective ultrafilters such that for each* $k \in \omega$ *the set* $\{[f]_{p_k} : f \in \mathcal{F}^k\} \dot\cup \{[\chi_{\vec{\xi}}]_{p_k} : \xi \in \kappa\}$ *is linearly independent and* $[f]_{p_k} \neq [g]_{p_k}$ *whenever* $f \neq g$ *in* $\mathcal{F}^k$. *Furthermore let for every* $k \in \omega$ *and* $f \in \mathcal{F}^k$ *an ordinal* $\xi_{f,k}$ *in* κ *be given. In addition let* d *and* d' *be non-zero in* G *and with disjoint supports. Finally, let* D *be a countable subset of* κ *that contains* $\omega \cup \operatorname{supp} d \cup \operatorname{supp} d'$ *and* $\bigcup_n \operatorname{supp} f(n)$ *for every* $f \in \bigcup_k \mathcal{F}^k$.

Then there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ *such that* $\phi(d) \neq 0$, $\phi(d') \neq 0$ *and* $p_k\text{-}\lim_n \phi(f(n)) = \phi(\chi_{\xi_{f,k}})$, *whenever* $k \in \omega$ *and* $f \in \mathcal{F}^k$.

Proof. Write D as the union of an increasing sequence $(D_n : n \in \omega)$ of finite nonempty subsets, and take a similar sequence $(\mathcal{F}^k_n : n \in \omega)$ for each $\mathcal{F}^k$.

Take a sufficiently small positive number ϵ_0 and an ϵ_0-arc function ϱ_* such that $\operatorname{supp} d \cup \operatorname{supp} d' \subseteq \operatorname{supp}\varrho_*$ and $0 \notin \overline{\sum_{\mu\in\operatorname{supp} d} d(\mu)\varrho_*(\mu)} \cup \overline{\sum_{\mu\in\operatorname{supp} d'} d'(\mu)\varrho_*(\mu)}$.

Let $E_0 = \operatorname{supp}\varrho_* \cup D_0$ and $B^k_0 = \omega$ for each $k \in \omega$.

We will define, by recursion, for $m \in \omega$: finite sequences $(B^k_m : 0 \leq k \leq m)$, finite sets $E_m \subseteq \kappa$, and real numbers $\epsilon_m > 0$ satisfying:

(1) For all k and m in ω we have $B^k_m \in p_k$,
(2) For each $m \geq 1$ and $k \leq m$, we have a sequence $(\delta^k_{m,n} : n \in \omega)$ of positive real numbers such that: if $(U_f : f \in \mathcal{F}^k_m)$ is a family of arcs of length ϵ_{m-1} and ϱ is an arc function of length ϵ_{m-1} and $\operatorname{supp}\varrho \subseteq E_{m-1}$ then for each $n \in \omega$ there exists a $\delta^k_{m,n}$-arc function ψ with $\psi \leq \varrho$, and $\operatorname{supp}\psi = \bigcup_{f\in\mathcal{F}^k_m} \operatorname{supp} f(n) \cup E_{m-1}$, and $\sum_{\mu\in\operatorname{supp} f(n)} f(n)(\mu)\psi(\mu) \subseteq U_f$ for each $f \in \mathcal{F}^k_m$.
(3) For all k and m we have $B^k_{m+1} \subseteq B^k_m$.
(4) $\epsilon_{m+1} = \frac{1}{2}\min(\{\delta^k_{l,n} : k \leq l \leq m+1 \text{ and } n \in (m+2) \cap B^k_l\} \cup \{\epsilon_m\})$.

Suppose we have defined B^k_l for all k as well as E_l and ϵ_l for all $l \leq m$. As will be clear from the step below the set B^k_m is only non-trivial whenever $k \leq m$. Therefore we let $B^k_{m+1} = B^k_m = \omega$ for $k > m+1$ and we concentrate on the case $k \leq m+1$.

Let $k \leq m+1$. By Lemma 4.6, there exist $B^k_{m+1} \in p_k$ and $(\delta^k_{m+1,n} : n \in \omega)$ that satisfy (2) for $m+1$. Without loss of generality we can assume that $B^k_{m+1} \subseteq B^k_m$.

Condition (4) now specifies ϵ_{m+1}.

Setting $E_{m+1} = E_m \cup \bigcup\{\operatorname{supp} f(k) : k \leq m, f \in \bigcup_{k\leq m+1} \mathcal{F}^k_{m+1}\} \cup D_{m+1}$ completes the recursion.

For each $k \in \omega$, apply the selectivity of p_k, to choose an increasing sequence $(a_{k,i} : i \in \omega)$ with $\{a_{k,i} : i \in \omega\} \in p_k$ and such that $a_{k,i} \in B^k_i$ and $a_{k,i} > i$ for all i.

Next apply Lemma 2.6 and let $(I_k : k \in \omega)$ be a sequence of pairwise disjoint subsets of ω such that $\{a_{k,i} : i \in I_k\} \in p_k$ and the family of intervals $\{[i, a_{k,i}] : k \in \omega, i \in I_k\}$ is pairwise disjoint. Without loss of generality we can assume that $k < \min I_k$.

Enumerate $\bigcup_{k\in\omega} I_k$ in increasing order as $(i_t : t \in \omega)$. For each $t \in \omega$, let k_t be such that $i_t \in I_{k_t}$. Thus, for each t we have $i_t \in I_{k_t}$, and hence $i_t \geq \min I_{k_t} > k_t$ and $a_{k_t,i_t} > i_t$.

By recursion we define a sequence of arc functions, $(\varrho_{i_t} : t \in \omega)$, such that $\varrho_{i_0} \leq \varrho_*$ and $\varrho_{i_{t+1}} \leq \varrho_{i_t}$.

We start with $t = 0$. Then we have $k_0 < i_0 < a_{k_0,i_0}$, and $a_{k_0,i_0} \in B^{k_0}_{i_0}$, and $\epsilon_{i_0-1} \leq \epsilon_0$.

Since ϱ_* has length at least ϵ_{i_0-1}, there exists an arc function ϱ_{i_0} of length $\delta^{k_0}_{i_0,a_{k_0,i_0}}$ such that $\sum_{\mu\in\operatorname{supp} f} f(a_{k_0,i_0})(\mu)\varrho_{i_0}(\mu) \subseteq \varrho_*(\xi_{f,k_0})$, for each $f \in \mathcal{F}^{k_0}_{i_0}$. We have by definition that $\delta^{k_0}_{i_0,a_{k_0,i_0}} > \epsilon_{i_1-1}$.

Suppose $t > 0$ and that $\varrho_{i_{t-1}}$ has been defined with length at least $\epsilon_{i_{t-1}}$.

Apply item (2) to the arc function $\varrho_{i_{t-1}}$, the finite set $\mathcal{F} = \mathcal{F}^{k_t}_{i_t}$, the number $\epsilon_{i_{t-1}}$, the finite set $E_{i_{t-1}}$, the arcs $U_f = \varrho_{i_{t-1}}(\xi_{f,k_t})$ for $f \in \mathcal{F}^{k_t}_{i_t}$, and $n = a_{k_t,i_t} \in B^{k_t}_{i_t}$ to obtain an arc function $\varrho_{i_t} \leq \varrho_{i_{t-1}}$ such that $\sum_{\mu\in\operatorname{supp} f(a_{k_t,i_t})} f(a_{k_t,i_t})(\mu)\varrho_{i_t}(\mu) \subseteq \varrho_{i_{t-1}}(\xi_{f,k_t})$ for all $f \in \mathcal{F}^{k_t}_{i_t}$, and ϱ_{i_t} has length $\delta^{k_t}_{i_t,a_{k_t,i_t}}$.

Because $k_t < i_t < a_{k_t,i_t} \leq i_{t+1} - 1$ and $a_{k_t,i_t} \in B^{k_t}_{i_t}$ we get $\delta^{k_t}_{i_t,a_{k_t,i_t}} > \epsilon_{i_{t+1}-1}$.

If $\xi \in D_{i_t}$ then $\xi \in \operatorname{supp} \varrho_{i_t}$ and the length of $\varrho_{i_t}(\xi)$ is not greater than ϵ_{i_t-1} which in turn is not larger than $\frac{1}{2^{i_t-1}} \leq \frac{1}{2^t}$.

It follows that for all $\xi \in D$ the intersection $\bigcap_{t\in\omega} \varrho_{i_t}(\xi)$ consists of a unique element; we define $\phi(\chi_\xi)$ to be that element and extend ϕ to a group homomorphism.

By construction $\phi(f(a_{k_t,i_t}))$ is in $\sum_{\mu\in\operatorname{supp} f(a_{k_t,i_t})} f(a_{k_t,i_t})(\mu)\varrho_{i_t}(\mu)$ which is a subset of $\varrho_{i_{t-1}}(\xi_{f,k_t})$ whenever $f \in \mathcal{F}^{k_t}_{i_t}$. Therefore, the sequence $(\phi(f(a_{k,i})))_{i\in I_k}$ converges to $\phi(\chi_{\xi_{f,k}})$, for each $k \in \omega$ and $f \in \mathcal{F}^k$.

Furthermore $\phi(d) \in \sum_{\mu\in\operatorname{supp} d} d(\mu)\varrho_*(\mu)$, therefore, $\phi(d) \neq 0$; and likewise $\phi(d') \neq 0$.

It is clear that this implies the conclusion of Lemma 4.7. $\square$

Now we are ready to prove Lemma 4.2.

Proof of Lemma 4.2. There are only countably many (and wlog infinitely many since we may increase it) pairs $(T, n) \in \mathcal{T} \times \omega$ such that $J_{T,n} \cap D \neq \emptyset$. We enumerate them faithfully as $((T_m, n_m) : m \geq 2)$.

For $m \geq 2$ let $\mathcal{F}^m = \{h_\xi : \xi \in D \cap J_{T_m,n_m}\}$ and $p_m = p_{T_m,n_m}$. Let p_0 and p_1 be two selective ultrafilters that were not listed incompatible with the ones listed and with each other and let $\mathcal{F}^0 = \mathcal{F}^1 = \{r\}$. For each $m \geq 2$ and $\xi \in J_{T_m,n_m} \cap D$, let $\xi_{h_\xi,n_m} = \xi$. Let $k, k' \in \omega$ be distinct elements of $\omega \setminus \operatorname{supp} d$. Then, by applying Lemma 4.7 with $d' = \chi_k - \chi_{k'}$, there exist $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ satisfying (1) and (2). To see it also satisfies (3), notice that p_0-$\lim \phi \circ r \neq p_1$-$\lim \phi \circ r$. $\square$

5. Proof of Lemma 4.6

In this section we present a proof of Lemma 4.6. We will need the notion of integer stack, which was defined in [25].

The integer stacks are collections of sequences in $\mathbb{Z}^{(\mathfrak{c})}$ that are usually associated to a selective ultrafilter. Given a finite set of sequences $\mathcal{F}$ it is possible to associate it to an integer stack which generates the same $\mathbb{Q}$-vector space as $\mathcal{F}$. The sequences in the stack have some nice properties that help us to construct well behaved arcs when constructing homomorphisms, and the linear relations between $\mathcal{F}$ and the sequences of the stack help us to transform these arcs into arcs that work for the functions of $\mathcal{F}$. Below, we give the definition of integer stack.

Definition 5.1. An *integer stack $\mathcal{S}$ on A* consists of

(i) an infinite subset A of ω;
(ii) natural numbers s, t, and M; positive integers r_i for $0 \leq i < s$ and positive integers $r_{i,j}$ for $0 \leq i < s$ and $0 \leq j < r_i$;
(iii) functions $f_{i,j,k} \in (\mathbb{Z}^{(\mathfrak{c})})^A$ for $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$ and elements $g_l \in (\mathbb{Z}^{(\mathfrak{c})})^A$ for $0 \leq l < t$;
(iv) sequences $\xi_i \in \mathfrak{c}^A$ for $0 \leq i < s$ and $\mu_l \in \mathfrak{c}^A$ for $0 \leq l < t$ and
(v) real numbers $\theta_{i,j,k}$ for $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$.

These are required to satisfy the following conditions:

(1) $\mu_l(n) \in \operatorname{supp} g_l(n)$ for each $n \in A$;

(2) $\mu_{l^*}(n) \notin \operatorname{supp} g_l(n)$ for each $n \in A$ and $0 \leq l^* < l < t$;
(3) the elements of $\{\mu_l(n) : 0 \leq l < t \text{ and } n \in A\}$ are pairwise distinct;
(4) $|g_l(n)| \leq M$ for each $n \in A$ and $0 \leq l < t$;
(5) $(\theta_{i,j,k} : 0 \leq k < r_{i,j})$ is a linearly independent subset of $\mathbb{R}$ as a $\mathbb{Q}$-vector space for each $0 \leq i < s$ and $0 \leq j < r_i$;
(6) $\lim_{n\in A} \frac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j,0}(n)(\xi_i(n))} = \theta_{i,j,k}$ for each $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$;
(7) the sequence $\big(|f_{i,j,k}(n)(\xi_i(n))| : n \in A\big)$ diverges monotonically to ∞, for each $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$;
(8) $|f_{i,j,k}(n)(\xi_i(n))| > |f_{i,j,k^*}(n)(\xi_i(n))|$ for each $n \in A$, $i < s$, $j < r_i$ and $0 \leq k < k^* < r_{i,j}$;
(9) $\left(\frac{|f_{i,j,k}(n)(\xi_i(n))|}{|f_{i,j^*,k^*}(n)(\xi_i(n))|} : n \in A\right)$ converges monotonically to 0 for each $0 \leq i < s$, $0 \leq j^* < j < r_i$, $0 \leq k < r_{i,j}$, and $0 \leq k^* < r_{i,j^*}$; and
(10) $\{f_{i,j,k}(n)(\xi_{i^*}(n)) : n \in A\} \subseteq [-M, M]$ for each $0 \leq i^* < i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$.

It is not difficult to show that the sequences of the stack are linearly independent. Moreover, if p is a free ultrafilter, $\mathcal{S}$ is a stack over A, and $A \in p$, then it is not difficult to see that $([g_l]_p : l < t) \cup ([f_{i,j,k}]_p : i < s, j < r_i, k < r_{i,j})$ is linearly independent in the $\mathbb{Q}$-vector space $\mathbb{Q}^{(\mathfrak{c})}/p$. We leave the details as an exercise to the reader.

As the reader might notice, integer stacks are defined for the group $\mathbb{Z}^{(\mathfrak{c})}$. However, we are working with $\mathbb{Z}^{(\kappa)}$. Thus, before we begin our proof, we show that it suffices to prove Lemma 4.6 for sequences of $\mathbb{Z}^{(\mathfrak{c})}$. Formally, we state:

Lemma 5.2. *Let p be a selective ultrafilter and $\mathcal{F}$ be a finite subset of $(\mathbb{Z}^{(\mathfrak{c})})^\omega$ such that the set $\{[f]_p : f \in \mathcal{F}\} \cup \{[\chi_{\vec{\alpha}}]_p : \alpha < \mathfrak{c}\}$ is linearly independent.*

Then for a given $\epsilon > 0$ and a finite subset E of $\mathfrak{c}$ there exist $A \in p$ and a sequence $(\delta_n : n \in A)$ of positive real numbers such that

($\star$) *whenever $\{U_f : f \in \mathcal{F}\}$ is a family of arcs of length ϵ and ϱ is an arc function for $\mathbb{Z}^{(\mathfrak{c})}$ of length at least ϵ with $\operatorname{supp}\varrho \subseteq E$ there exists, for each $n \in A$, a δ_n-arc function for $\mathbb{Z}^{(\mathfrak{c})}$ $\psi_n \leq \varrho$ such that $\operatorname{supp}\psi_n = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f(n) \cup E$, and $\sum_{\mu\in\operatorname{supp} f(n)} f(n)(\mu) \cdot \psi_n(\mu) \subseteq U_f$ for each $f \in \mathcal{F}$.*

In what follows we prove that Lemma 4.6 follows easily from Lemma 5.2. The rest of this section will be dedicated to proving Lemma 5.2.

Proof of Lemma 4.6 from Lemma 5.2. Fix a selective ultrafilter p and a finite subset $\mathcal{F}$ of G^ω such that $\{[f]_p : f \in \mathcal{F}\} \cup \{[\chi_{\vec{\alpha}}]_p : \alpha < \mathfrak{c}\}$ is linearly independent. Also, let $\epsilon > 0$ and $E \in [\kappa]^{<\omega}$ be given. Let $R = \bigcup\{\operatorname{supp} f(n) : f \in \mathcal{F}, n \in \omega\} \cup E$. R is countable, so let $\phi : R \to \mathfrak{c}$ be injective.

For each $f \in \mathcal{F}$, define $f' \in (\mathbb{Z}^{(\mathfrak{c})})^\omega$ so that for each $n \in \omega$, $\operatorname{supp} f'(n) = \phi[\operatorname{supp} f(n)]$ and $(f'(n))(\phi(\xi)) = (f(n))(\xi)$ for every $\xi \in \operatorname{supp} f(n)$. Also, let $E' = \phi[E]$ and $R' = \phi[R]$.

Claim: $\{[f']_p : f \in \mathcal{F}\} \cup \{[\chi_{\vec{\alpha}}]_p : \alpha < \kappa\}$ is linearly independent. To see that, fix $C \in [\kappa]^{<\omega}$ and assume that for some rational numbers $(a_f : f \in \mathcal{F})$ and $(b_\alpha : \alpha \in C)$ we have:

$$\sum_{f\in\mathcal{F}} a_f[f'] + \sum_{\alpha\in C} b_\alpha[\chi_{\vec{\alpha}}]_p = 0.$$

This means that there exists $Z \in p$ such that for every $n \in Z$, we have:

$$0 = \sum_{f\in\mathcal{F}} a_f f'(n) + \sum_{\alpha\in C} b_\alpha \chi_\alpha.$$

Since for all $f \in \mathcal{F}$, $n \in \omega$ and $\alpha \notin R'$ we have $f'(n)(\alpha) = 0$, by fixing any $n \in Z$ and calculating the expression above in any such α we obtain that $b_\alpha = 0$.

Now fix $n \in Z$ and $\alpha_0 \in R$. We have that:

$$\sum_{f\in\mathcal{F}} a_f(f(n))(\alpha_0) + \sum_{\alpha\in\phi^{-1}[C]} b_{\phi(\alpha)}\chi_\alpha(\alpha_0) = \sum_{f\in\mathcal{F}} a_f(f(n))(\alpha_0) + \sum_{\alpha\in\phi^{-1}[C]} b_{\phi(\alpha)}\chi_\alpha(\alpha_0).$$

$$= \sum_{f\in\mathcal{F}} a_f(f'(n))(\phi(\alpha_0)) + \sum_{\alpha\in\phi^{-1}[C]} b_{\phi(\alpha)}\chi_{\phi(\alpha)}(\phi(\alpha_0))$$

$$= \left(\sum_{f\in\mathcal{F}} a_f f'(n) + \sum_{\alpha\in C\cap R'} b_\alpha\chi_\alpha\right)(\phi(\alpha_0)) = \left(\sum_{f\in\mathcal{F}} a_f f'(n) + \sum_{\alpha\in C} b_\alpha\chi_\alpha\right)(\phi(\alpha_0)) = 0.$$

Moreover, for Now fix $n \in Z$ and $\alpha_0 \in \kappa \setminus R$, it easily follows that $\sum_{f\in\mathcal{F}} a_f(f(n))(\alpha_0) + \sum_{\alpha\in\phi^{-1}[C]} b_{\phi(\alpha)}\chi_\alpha(\alpha_0) = 0$. Thus, we have that for all $n \in Z$, $\sum_{f\in\mathcal{F}} a_f f(n) + \sum_{\alpha\in\phi^{-1}[C]} b_{\phi(\alpha)}\chi_\alpha = 0$, which implies by hypothesis that $a_f = 0$ for every $f \in \mathcal{F}$ and $b_\alpha = 0$ for every $\alpha \in C \cap R'$. This proves the claim.

Thus, by hypothesis, there exist $A \in p$ and a sequence $(\delta_n : n \in A)$ of positive real numbers such that

($\star$) whenever $(U_f : f \in \mathcal{F})$ is a family of arcs of length ϵ and ϱ is an arc function for $\mathbb{Z}^{(\mathfrak{c})}$ of length at least ϵ with $\operatorname{supp}\rho' \subseteq E'$ there exist for each $n \in A$ a δ_n-arc function for $\mathbb{Z}^{(\mathfrak{c})}$ $\psi_n \leq \rho'$ such that $\operatorname{supp}\psi'_n = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f'(n) \cup E'$, and $\sum_{\mu\in\operatorname{supp} f'(n)} f'(n)(\mu)\cdot\psi'_n(\mu) \subseteq U_f$ for each $f \in \mathcal{F}$.

We show that A and $(\delta_n : n \in A)$ also work for $\mathcal{F}$ and E. Let $(U_f : f \in \mathcal{F})$ be a family of arcs of length ϵ and ρ be an arc function for G of length at least ϵ with $\operatorname{supp}\rho \subseteq E$. Define ρ' an arc function for $\mathbb{Z}^{(\mathfrak{c})}$ so that $\operatorname{supp}\rho' = \Phi[\operatorname{supp}\rho]$ and for $\mu \in \operatorname{supp}\rho$, $\rho'(\phi(\mu)) = \rho(\mu)$. Then $\operatorname{supp}\rho' \subseteq E'$ and ρ' is an arc function of length at least ϵ, so there exist $n \in A$ and a δ_n-arc function for $\mathbb{Z}^{(c)}$ $\psi'_n \leq \rho'$ such that $\operatorname{supp}\psi'_n = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f'(n) \cup E'$ and $\sum_{\mu\in\operatorname{supp} f'(n)} f'(n)(\mu)\cdot\psi'_n(\mu) \subseteq U_f$ for each $f \in \mathcal{F}$. Define an arc function ψ_n for G whose support is $\bigcup_{f\in\mathcal{F}} \operatorname{supp} f(n) \cup E$ and for $\mu \in \operatorname{supp}\psi_n$, $\psi_n(\mu) = \psi'_n(\phi(\mu))$. Then clearly $\psi_n(\mu) \leq \rho(\mu)$ and:

$$\sum_{\mu\in\operatorname{supp} f(n)} f(n)(\mu)\cdot\psi_n(\mu) = \sum_{\mu\in\operatorname{supp} f(n)} f'(n)(\phi(\mu))\cdot\psi'_n(\phi(\mu)) = \sum_{\mu\in\operatorname{supp} f'(n)} f'(n)(\mu)\cdot\psi'_n(\mu) \subseteq U_f$$

This completes the proof. □

Now we work toward the proof of Lemma 5.2.

Definition 5.3. Given an integer stack $\mathcal{S}$ and a natural number N, the *Nth root of* $\mathcal{S}$, written $\frac{1}{N}\mathcal{S}$, is obtained by keeping all the structure in $\mathcal{S}$ with the exception of the functions; these are divided by N. Thus a function $f_{i,j,k} \in \mathcal{S}$ is replaced by $\frac{1}{N}f_{i,j,k}$ in $\frac{1}{N}\mathcal{S}$ for each $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$ and a function $g_l \in \mathcal{S}$ is replaced by $\frac{1}{N}g_l$ in $\frac{1}{N}\mathcal{S}$ for each $0 \leq l < t$.

A *stack* is then defined to be the Nth root of an integer stack for some positive integer N.

The lemma below gives the relation between a finite sequence of sequences in $\mathbb{Z}^{(\mathfrak{c})}$ and a stack $\mathcal{S}$ that is associated to it. The first part of this lemma is proved in [25]. The second part, which we called (2) was stated in [2] with no proof presented there, since it follows directly from statements of several lemmas and constructions from [25]. In order to see that (2) holds one has to read the whole proof of Lemma 7.1. of [25] keeping an eye on the relevant properties that imply it. The construction does not need to be changed at all. Since the construction is long and complicated and a few typos are present on that paper, we decided to reproduce it on this paper keeping track of the important things needed to observe that (2) holds for the sake of the completeness. As the construction is long and no new mathematical content is present, we decided to put in an appendix at the end of this paper.

Lemma 5.4 *(A version of Lemma 7.1. of [25]). Let $h_0, \ldots, h_{m-1}$ be sequences in $\mathbb{Z}^{(\mathfrak{c})}$ and $\mathcal{U} \in \omega^*$ be a selective ultrafilter so that $\{[h_0]_{\mathcal{U}}, \ldots, [h_{m-1}]_{\mathcal{U}}\} \cup \{[\chi_{\check{\alpha}}]_{\mathcal{U}} : \alpha < \mathfrak{c}\}$ is linearly independent in the vector space $\mathbb{Q}^{(\mathfrak{c})}/\mathcal{U}$. Then there exists $A \in \mathcal{U}$, $N \in \omega \setminus \{0\}$ and a stack $\frac{1}{N}\mathcal{S}$ on A such that:*

(1) If $\mathbb{Z}^{(\mathfrak{c})}$ is given a group topology and the elements of the stack $\frac{1}{N}\mathcal{S}$ have a $\mathcal{U}$-limit in $\mathbb{Z}^{(\mathfrak{c})}$ then h_i has a $\mathcal{U}$-limit in $\mathbb{Z}^{(\mathfrak{c})}$ for each $0 \le i < m$.
(2) For each $i < m$, $h_i|_A$ is an integer combination of the elements of the sequences of the stack $\frac{1}{N}\mathcal{S}$ restricted to A. On the other hand, each sequence of the integer stack $\mathcal{S}$ restricted to A is an integer combination of $\{h_0, \ldots, h_{m-1}\}$ restricted to A.

We will say in this case that the finite sequence $\{h_0, \ldots, h_{m-1}\}$ is associated to $(\frac{1}{N}\mathcal{S}, A, \mathcal{U})$.

Now we define some integers related to Kronecker's Theorem that will be useful in our proof. The existence of these integers are a direct consequence of Kronecker's Theorem and may be an exercise to the reader, however, the details may be found on Lemma 4.3. of [25]. These integers were defined and used in that paper.

Definition 5.5. If $\{\theta_0, \ldots, \theta_{r-1}\}$ is a linearly independent subset of the $\mathbb{Q}$-vector space $\mathbb{R}$ and $\epsilon > 0$ then $L(\theta_0, \ldots, \theta_{r-1}, \epsilon)$ denotes a positive integer, L, such that $\{(\theta_0 x + \mathbb{Z}, \ldots, \theta_{r-1}x + \mathbb{Z}) : x \in I\}$ is ϵ-dense in $\mathbb{T}^r$ in the usual Euclidean product metric, for any interval I of length at least L.

The last lemma we are going to need is Lemma 8.3 from [25], stated below.

Lemma 5.6. *Let ϵ, γ and ρ be positive reals, N a positive integer and ψ be an arc function. Let $\mathcal{S}$ be an integer stack on $A \in [\omega]^\omega$ and s, t, r_i, $r_{i,j}$, M, $f_{i,j,k}$, g_l, ξ_i, μ_j and $\theta_{i,j,k}$ be as in Definition 5.1.*

Let L be an integer greater or equal to $\max\{L(\theta_{i,j,0}, \ldots, \theta_{i,j,r_{i,j}-1}, \frac{\epsilon}{24}) : 0 \le i < s \text{ and } 0 \le j < r_i\}$ and let $r = \max\{r_{i,j} : 0 \le i < s \text{ and } 0 \le j < r_i\}$.

Suppose that $n \in A$ is such that

(a) $\{V_{i,j,k} : 0 \le i < s, 0 \le j < r_i \text{ and } 0 \le k < r_{i,j}\} \cup \{W_l : 0 \le l < t\}$ is a family of open arcs of length ϵ;
(b) $\delta(\psi(\beta)) \ge \epsilon$ for each $\beta \in \operatorname{supp}\psi$;
(c) $\epsilon > 3N \cdot \rho \cdot \max\big(\{\|g_l(n)\| : 0 \le l < t\} \cup \bigcup\{\|f_{i,j,k}(n)\| : 0 \le i < s, 0 \le j < r_i, 0 \le k < r_{i,j}\}\big)$;
(d) $3MNs\gamma < \epsilon$;
(e) $|f_{i,r_i-1,0}(n)(\xi_i(n))| \cdot \gamma > 3L$ for each $0 \le i < s$;
(f) $|f_{i,j-1,0}(n)(\xi_i(n))| \cdot \frac{\epsilon}{6\sqrt{r_{i,j}}|f_{i,j,0}(n)|} > 3L$ for each $0 \le i < s$ and $0 < j < r_i$;
(g) $\left|\theta_{i,j,k} - \frac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j,0}(n)(\xi_i(n))}\right| < \frac{\epsilon}{24\sqrt{r}L}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$ and
(h) $\operatorname{supp}\psi \cap \{\mu_0(n), \ldots, \mu_{t-1}(n)\} = \emptyset$.

Then there exists an arc function ϕ such that

(A) $N \cdot \phi(\beta) \subseteq N \cdot \overline{\phi(\beta)} \subseteq \psi(\beta)$ for each $\beta \in \operatorname{supp}\psi$;
(B) $\sum_{\beta \in \operatorname{supp} g_l(n)} g_l(n)(\beta)\phi(\beta) \subseteq W_l$ for each $l < t$;
(C) $\sum_{\beta \in \operatorname{supp} f_{i,j,k}(n)} f_{i,j,k}(n)(\beta) \cdot \phi(\beta) \subseteq V_{i,j,k}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$;
(D) $\delta(\phi(\beta)) = \rho$ for each $\beta \in \operatorname{supp}\phi$ and

(E) $\operatorname{supp}\phi$ *can be chosen to be any finite set containing*

$$\operatorname{supp}\psi \cup \bigcup_{0\le i<s, 0\le j<r_i, 0\le k<r_{i,j}} \operatorname{supp} f_{i,j,k}(n) \cup \bigcup_{0\le l<t} \operatorname{supp} g_l(n). \quad \square$$

Now we are ready to prove Lemma 5.2.

Proof of Lemma 5.2. Write $\mathcal{F} = \{u_0, \dots u_{q-1}\}$ with no repetition. Let $\mathcal{S}$ be an integer stack on $A' \in p$ and let N be a positive integer such that $\left(\frac{1}{N}\mathcal{S}, A', p\right)$ is associated to $\mathcal{F}$.

As in Definition 5.1 the components of $\mathcal{S}$ will be denoted s, t, M, $(r_i : i < s)$, $(r_{i,j} : i < s, j < r_i)$, $(f_{i,j,k} : i < s, j < r_i, k < r_{i,j})$, $(g_l : l < t)$, $(\xi_i : i < s)$, $(\mu^p : i < t)$ and $(\theta_{i,j,k} : 0 \le i < s, 0 \le j < r_i, k < r_{i,j})$.

We write $\{f_{i,j,k} : i < s_p, j < r_i, k < r_{i,j}\} \cup \{g_l : l < t\}$ as $\{v_0, \dots, v_{q-1}\}$.

Let $\mathcal{M}$ be the $q \times q$ matrix of integer numbers such that $Nu_i(n) = \sum_{j<q} \mathcal{M}_{i,j} v_j(n)$ for all $n \in A$ and $i < q$.

By (2) in Lemma 5.4, each v_j is an integer combination of the u_i's, therefore the inverse matrix of $\frac{1}{N}\mathcal{M}$, which we denote by $\mathcal{N}$, has integer entries.

Let $\epsilon' = \epsilon \cdot (\sum_{i,j<l} |\mathcal{M}_{i,j}|)^{-1}$ and $\gamma < \epsilon'/(3MNs)$. Let L be larger than or equal to the maximum of the set $\{L(\theta_{i,j,0}, \dots, \theta_{i,j,r_{i,j}-1}, \epsilon'/24) : i < s, j < r_i\}$.

For each $n \in A'$, let $\delta_n < \frac{1}{2}$ be such that:

$$\epsilon' > 3N \cdot \max\big(\{\|g_l(n)\| : 0 \le l < t\} \cup \bigcup \{\|f_{i,j,k}(n)\| : 0 \le i < s, 0 \le j < r_i, 0 \le k < r_{i,j}\}\big) \cdot \frac{\delta_n}{N}$$

We note that both N's above cancel but we write this way as we will use δ_n/N in the place of ρ in item c) of Lemma 5.6.

Let $r = \max\{r_{i,j} : 0 \le i < s, 0 \le j < r_i\}$. Let A be the set of n's in A' such that:

- $|f_{i,r_i-1,0}(n)(\xi_i(n))|\gamma > 3L$ for each $0 \le i < s$,
- $|f_{i,j-1,0}(n)(\xi_i(n))| \cdot \dfrac{\epsilon'}{6\sqrt{r_{i,j}}|f_{i,j,0}(n)|} > 3L$ for each $0 \le i < s$ and $0 < j < r_i$,
- $\left|\theta_{i,j,k} - \dfrac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j,0}(n)(\xi_i(n))}\right| < \dfrac{\epsilon'}{24\sqrt{r}L}$ for each $i < s, j < r_i$ and $k < r_{i,j}$, and
- $E \cap \{\mu_0(n), \dots, \mu_{t-1}(n)\} = \emptyset$.

Notice that A is cofinite in A', therefore $A \in p$.

We claim this A and this sequence $(\delta_n : n \in A)$ work.

Fix $n \in A$.

Let $(U_f : f \in \mathcal{F})$ be a family of arcs of length ϵ and let ϱ be an arc function of length at least ϵ with $\operatorname{supp}\varrho \subseteq E$. We rewrite the family of arcs as $(U_i : i < q)$, where $U_i = U_{f_i}$ for each $i < q$. For each $i < q$ let y_i be a real such that $y_i + \mathbb{Z}$ is the center of U_i. Let $z_j = \sum_{i<q} \mathcal{N}_{j,i} \frac{y_i}{N}$ and, for each j let R_j be the arc of center z_j and length ϵ'. Since $\mathcal{N}$ is a matrix of integers, $z_j + \mathbb{Z} = \sum_{i<q} \mathcal{N}_{j,i}(\frac{y_i}{N} + \mathbb{Z})$. Then the arc $\sum_{j<q} \mathcal{M}_{i,j} R_j$ is a subset of U_i for each $i < q$.

Now we aim to apply Lemma 5.6. Set $\psi = \varrho$, $\rho = \delta_n/N$ and ϵ' in the place of ϵ. For $i < s$, $j < r_i$, $k < r_{i,j}$ we put $V_{i,j,k} = R_x$ if $f_{i,j,k} = v_x$ for some $x < q$, and for $j < t$ we put $W_j = R_x$ if $g_j = v_x$ for some $x < q$.

Then there exists an arc function $\tilde{\psi_n}$ such that

(A) $N\tilde{\psi_n} \subseteq N\overline{\tilde{\psi_n}} \subseteq \varrho(\beta)$ for each $\beta \in \operatorname{supp}\psi$;
(B) $\sum_{\beta \in \operatorname{supp} g_l(n)} g_l(n)(\beta)\tilde{\psi_n}(\beta) \subseteq W_l$ for each $l < t$;
(C) $\sum_{\beta \in \operatorname{supp} f_{i,j,k}(n)} f_{i,j,k}(n)(\beta) \cdot \tilde{\psi_n}(\beta) \subseteq V_{i,j,k}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$;

(D) $\delta(\tilde{\psi_n}(\beta)) = \delta_n/N$ for each $\beta \in \operatorname{supp}\tilde{\psi_n}$ and
(E) $\operatorname{supp}\tilde{\psi_n}$ is equal to

$$\bigcup_{0\le i<s, 0\le j<r_i, 0\le k<r_{i,j}} \operatorname{supp} f_{i,j,k}(n) \cup \bigcup_{0\le l<t} \operatorname{supp} g_l(n) \cup E = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f(n) \cup E.$$

Let $\psi_n = N\tilde{\psi_n}$. By (A), $\psi_n \le \varrho$. By (E) and (D), $\operatorname{supp}\psi_n = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f(n) \cup E$ and for each $\beta \in \operatorname{supp}\psi_n$, we have $\delta(\psi_n(\beta)) = \delta_n$. Let $S = \operatorname{supp}\psi_n$. Now notice that given $u_i \in \mathcal{F}$ we have:

$$\begin{aligned}\sum_{\mu\in\operatorname{supp} u_i} u_i(n)(\mu)\psi_n(\mu) &= \sum_{\mu\in S} u_i(n)(\mu)N\tilde{\psi_n}(\mu)\\ &= \sum_{\mu\in S}\left(\sum_{j<q}\mathcal{M}_{i,j}v_j(n)(\mu)\right)\tilde{\psi_n}(\mu)\\ &= \sum_{j<q}\mathcal{M}_{i,j}\left(\sum_{\mu\in S} v_j(n)(\mu)\tilde{\psi_n}(\mu)\right)\end{aligned}$$

Then by (B), (C) and the definitions of the W_l's and $V_{i,j,k}$'s:

$$\sum_{\mu\in\operatorname{supp} u_i} u_i(n)(\mu)\psi_n(\mu) = \sum_{\mu\in S} u_i(n)(\mu)N\tilde{\psi_n}(\mu) \subseteq \sum_{j<q}\mathcal{M}_{i,j}R_j \subseteq U_i.$$

As intended. □

6. Final comments

The method to construct countably compact free Abelian groups came from the technique to construct countably compact groups without non-trivial convergent sequences. It is not known if there is an easier method to produce countably compact group topologies on free Abelian groups if we do not care if the resulting topology has convergent sequences.

In fact, even to produce a countably compact group topology with convergent sequences in non-torsion groups it is used a modification of the technique to construct countably compact groups without non-trivial convergent sequences, see [1] and [2].

The first examples of countably compact groups without non-trivial convergent sequences were obtained by Hajnal and Juhász [10] under CH. E. van Douwen [6] obtained an example from MA and asked for a ZFC example. Other examples were obtained using $\mathrm{MA}_{\mathrm{countable}}$ [14], a selective ultrafilter [9] and in the Random real model [18]. Only recently, Hrušak, van Mill, Shelah and Ramos obtained an example in ZFC ([12]).

This motivates the following questions in ZFC:

Question 6.1. Are there large countably compact groups without non-trivial convergent sequences in ZFC?

The example of Hrušak et al., which has size continuum, uses an almost disjoint family of cardinality $\mathfrak{c}$ to define $\mathfrak{c}$ ultrafilters in ZFC that will lead to the construction. Tomita and Trianon-Fraga modified this example using $2^{\mathfrak{c}}$ incomparable weak P-points to obtain an example of cardinality $2^{\mathfrak{c}}$ [27]. Here there is a new limitation, since there are only $2^{\mathfrak{c}}$ ultrafilters on ω. Thus, in the previous question "large" now means cardinality strictly greater than $2^{\mathfrak{c}}$.

Question 6.2. Is there a countably compact free Abelian group in ZFC? A countably compact free Abelian group without non-trivial convergent sequences in ZFC?

It is still open if there exists a torsion-free group in ZFC that admits a countably compact group topology without non-trivial convergent sequences. If such example exists then there is a countably compact group topology without non-trivial convergent sequences in the free Abelian group of cardinality $\mathfrak{c}$ (see [24] or [26]).

Question 6.3. Is there a both-sided cancellative semigroup that is not a group that admits a countably compact semigroup topology (a Wallace semigroup) in ZFC?

The known examples were obtained in [17] under CH, in [20] under $\mathrm{MA}_{\text{countable}}$, in [16] from $\mathfrak{c}$ incomparable selective ultrafilters and in [2] from one selective ultafilter. The last two use the known fact that a free Abelian group without non-trivial convergent sequences contains a Wallace semigroup, which was used in [17]. The example in [20] is a modification of [11].

7. Appendix: a full proof for Lemma 5.4

In this section we prove Lemma 5.4. We have decided to write it in this appendix instead of in the middle of the text since altought its proof is a small modification of Lemma 7.1. of [25], the proof is rather long (as the original is long).

Lemma 5.4 states the existence of a stack associated to a sequence of functions with the following property with respect to a selective ultrafilter:

Definition 7.1. Let $\mathcal{U}$ be an ultrafilter. We say that a finite sequence $(f_i : i < p)$ of elements of $(\mathbb{Z}^{(\mathfrak{c})})^\omega$ is *admissible* with respect to $\mathcal{U}$ iff for each $c_0, \dots c_{p-1} \in \mathbb{Z}$ not all 0, there exists $D \in \mathcal{U}$ such that $(\sum_{i<p} c_i f_i(n) : n \in D)$ is injective.

This definition is not present in [25]. We introduced it as a shorthand for property b) of Lemma 4.1. of that paper.

To prove the existence of a stack associated to an admissible sequence $(f_q : q < p)$, we must construct it. In order to do that, we "change" the f_q's into $f_{i,j,k}$'s that have the properties of the stack without modifying the $\mathbb{Q}$-vector space they generate, and leaves some modified functions g_q^0 that correspond to the f_q's that could not be modified to become an $f_{i,j,k}$. This role is played by Lemma 7.2 (Lemma 5.4. of [25]). The "output" of this lemma has a very heavy notation that is a bit different than the one from the definition of stack (that is already heavy), thus we need a lemma to refine it - Lemma 7.3, which is a very small improvement of Lemma 5.5. of [25].

Then we need a second lemma that transforms these functions in the second half of the stack (the sequences g_l). This is done by Lemma 7.4 (Lemma 6.1. of [25]). Again, this needs to be refined, and Lemma 7.5, which is a very small improvement of Lemma 6.1. of [25], does this job.

Finally, we apply Lemmas 7.3 and 7.5 conveniently to prove Lemma 5.4.

We emphasize that no new mathematical content is present in this section as the original construction for the stack is not modified at all, we just keep track of some features presented in that very same construction through the Lemmas to observe that without modifying it, (2) of Lemma 5.4 also holds. That's why we are presenting these arguments in an appendix and not in a regular section.

Recall that if $a \in \mathbb{Z}^{(\mathfrak{c})}$, $|a| = \max\{a(\xi) : \xi \in \mathbb{Z}\}$.

We start by assuming, the following Lemma 5.4. of [25] as stated below.

Lemma 7.2 *(Lemma 5.4. of [25]). Let $\mathcal{U}$ be a selective ultrafilter, $p \in \omega$ and $(f_0, \ldots, f_{p-1})$ be an admissible sequence with respect to $\mathcal{U}$.*

Suppose that there exists $i < p$ and $D^* \in \mathcal{U}$ such that $\{|f_i(n)| : n \in D^*\}$ is strictly increasing.

There exists:

- *a positive natural number s,*
- *a finite sequence of positive natural numbers $(r_i)_{i<s}$,*
- *the lexicographical order of $\bigcup_{i<s}\{i\} \times r_i$, denoted by $\prec$,*
- *$B \in \mathcal{U}$,*
- *a sequence of ordinals smaller than $\mathfrak{c}$, $(\xi_i)_{i<s}$,*
- *a family of nonempty subsets of p, $(J_{i,j} : i < s \text{ and } j < r_i)$,*
- *a family of sequences into $\mathbb{Z}^{(\mathfrak{c})}$, $(f_q^{i,j} : i < s, j < r_i \text{ and } q \in p \setminus \bigcup\{J_{i^*,j^*} : (i^*, j^*) \prec (i,j)\})$,*
- *a family of sequences into $\mathbb{Z}^{(\mathfrak{c})}$, $(g_q^0 : q \in p \setminus \bigcup\{J_{i,j} : i < s \text{ and } j < r_i\})$*
- *a family of real numbers, $(\theta_q^{i,j} : i < s, j < r_i \text{ and } q \in J_{i,j})$,*
- *a family of elements of $\mathbb{Z}^{(\mathfrak{c})}$, $(\sigma_q^{i,j} : i < s, j < r_i, q \in p \setminus \bigcup\{J_{i^*,j^*} : (i^*, j^*) \preceq (i,j)\})$*
- *a family of positive integers $(N_q^{i,j} : i < s, j < r_i \text{ and } q < p)$,*

such that for every $i < s$ and $j < r_i$,

i) $(J_{i,j} : i < s \text{ and } j < r_i)$ are pairwise disjoint subsets of p,
ii) $f_q^{0,0} = f_q$ for each $q < p$,
iii) if (i^+, j^+) is the successor of (i,j) then $f_q^{i^+,j^+} = N_q^{i,j}.f_q^{i,j} - \sigma_q^{i,j}$ for each $q \in p \setminus \bigcup\{J_{i',j'} : (i',j') \preceq (i,j)\}$;
iv) $\sigma_q^{i,j}$ is an integer combination of $(f_{q^*}^{i,j} : q^* \in J_{i,j})$ for each $q \in p \setminus \bigcup\{J_{i',j'} : (i',j') \preceq (i,j)\}$,
v) if $(0,0) \prec (i,j)$ then $(\prod_{(i^\#,j^\#)\prec(i,j)} N_q^{i^\#,j^\#}).f_q - f_q^{i,j}$ is an integer combination of $(f_{q^*}^{i^*,j^*} : (i^*,j^*) \prec (i,j)$ and $q^* \in J_{i^*.j^*})$ for each $q \in p \setminus \bigcup\{J_{i',j'} : (i',j') \preceq (i,j)\}$,
vi) For every $q, q^* \in J_{i,j}$, if $q \neq q^*$ then either for every $n \in B$, $|f_q^{i,j}(n)(\xi_i(n))| > |f_{q^*}^{i,j}(n)(\xi_i(n))|$ or for every $n \in B$, $|f_q^{i,j}(n)(\xi_i(n))| < |f_{q^*}^{i,j}(n)(\xi_i(n))|$,
vii) $(\theta_q^{i,j} : q \in J_{i,j})$ is a linearly independent subset of $\mathbb{R}$ as a $\mathbb{Q}$-vector space,
viii) $\frac{f_q^{i,j}(n)(\xi_i(n))}{f_{q^*}^{i,j}(n)(\xi_i(n))} \xrightarrow{n\in B} \theta_q^{i,j}$ monotonically, where $q^* \in J_{i,j}$ is such that $|f_{q^*}^{i,j}(n)(\xi_i(n))| \geq |f_q^{i,j}(n)(\xi_i(n))|$ for each $n \in B$ and $q \in J_{i,j}$,
ix) $|f_q^{i,j}(\xi_i(n))| \xrightarrow{n\in B} +\infty$ for each $i < s$, $j < r_i$ and $q \in J_{i,j}$;
x) for each integer j^* such that $j < j^* < r_i$, each $q \in J_{i,j}$ and each $q^* \in p \setminus \bigcup\{J_{i',j'} : (i',j') \preceq (i,j)\}$, $\frac{(N_{q^*}^{i,j}.f_{q^*}^{i,j} - \sigma_{q^*}^{i,j})(n)(\xi_i(n))}{f_q^{i,j}(n)(\xi_i(n))} \xrightarrow{n\in B} 0$ monotonically
xi) $\{f_q^{i,j}(\xi_{i^*}(n)) : n \in A_{i,j}\}$ is bounded in $\mathbb{Z}$ for each $0 \leq i^* < i$ and $q \in p \setminus \bigcup\{J_{i',j'} : i' \leq i^* \text{ and } j' < r_{i'}\}$;
xii) $g_q^0 = N_q^{s-1,r_{s-1}-1}.f_q^{s-1,r_{s-1}} - \sigma_q^{s-1,r_{s-1}-1}$ for each $q \in p \setminus \bigcup\{J_{i,j} : i < s \text{ and } j < r_i\}$;
xiii) $\{|g_q^0(n)| : n \in B\}$ is bounded in $\mathbb{Z}$ for each $q \in p \setminus \bigcup\{J_{i,j} : i < s \text{ and } j < r_i\}$ and
xiv) $(\prod_{i<s,j<r_i} N_q^{i,j}.f_q) - g_q^0$ is an integer combination of $(f_{q^*}^{i^*,j^*} : i^* < s, j^* < r_{i^*}$ and $q^* \in J_{i^*.j^*})$ for each $q \in p \setminus \bigcup\{J_{i',j'} : i' < s \text{ and } j' < r_{i'}\}$.

Even though the statement above is not exactly the same as in [25], we just fixed some typos and imprecisions that appeared there. We explain the changes below to avoid creating confusion. We explain the changes below:

(1) We reordered the items in the statement in a order where they make more sense, and omitted the $A_{i,j}$'s, replacing them all by B, since they are not important when applying the result (only in the proof). Of course, all the statements in which $A_{i,j}$ appeared there are still true when replaced by B since $B \subseteq A_{i,j}$.
(2) Not all $N_p^{i,j}$'s are necessary since not all of them appear on i)-xiv). In the statement that appears in [25] the domain of this family does not appear, which may cause some confusion, but, of course, all the relevant $N_p^{i,j}$'s appear in the proof. The domain we presented here is larger than necessary. Formally, one may let $N_p^{i,j}$ be their favorite number for the triples (i, j, p) that are irrelevant since they do not play any role.
(3) Condition xii) was wrong in [25] (it is listed as condition xiv) there). However, that was just a typo and the proof of this lemma that is presented there proves our current condition xii). The incorrect statement there did not create problems in the previous paper: it was only needed to prove Lemma 5.5. of that paper, and we will prove a version of it below, thus assuring the reader that the incorrect statement was not necessary for the other statements of that paper to hold.
(4) The fact that s, the r_i's and the $J_{i,j}$'s are nonempty/nonzero were clearly present in the proof there but they were only implicit in the statement. We decided to make this explicit in this paper to avoid making confusion.

Thus we will not reproduce its proof here as the proof would be a full copy from the one in [25].

We will use the preceding lemma to prove the following, which is a modified version of Lemma 5.5. of [25]. The proof is the same as in the previous one. We just add condition (E)-(G) to the statement and argue that it also holds by observing the properties given by the previous lemma. No new ideas are used in this modification.

Lemma 7.3 *(A version of Lemma 5.5. of [25]). Let $\mathcal{U}$ be a selective ultrafilter, $p \in \omega$ and $(f_0, \dots, f_{p-1})$ be an admissible sequence with respect to $\mathcal{U}$*

Suppose that there exists $i < p$ and $D^* \in \mathcal{U}$ such that $(|f_i(n)| : n \in D^*)$ is strictly increasing.

Then there exists:

- *Positive natural number s and N',*
- *A finite sequence of positive natural numbers $(r_i)_{i<s}$,*
- *A finite family of positivenatural numbers $(r_{i,j} : i < s, j < r_i)$,*
- *$B \in \mathcal{U}$,*
- *a sequence of ordinals smaller than $\mathfrak{c}$, $(\xi_i)_{i<s}$,*
- *an infinite subset of ω, B,*
- *a subset of p, J,*
- *a family of sequences into $\mathbb{Z}^{(\mathfrak{c})}$, $(f_{i,j,k} : i < s, j < r_i$ and $k < r_{i,j})$,*
- *positive integers M' and N',*
- *a family of sequences into $\mathbb{Z}^{(\mathfrak{c})}$, $(g_q^0 : q \in p \setminus J\})$*
- *a family of real numbers, $(\theta_{i,j,k} : i < s, j < r_i$ and $k < r_{i,j})$*

such that

(A) conditions (5)-(10) are satisfied in the definition of the stack for M' instead of M and B instead of A,
(B) $N'.f_q$ is an integer combination of $\{f_{i,j,k} : i < s, j < r_i$ and $k < r_{i,j}\}$ for each $q \in J$,
(C) $\{|g_q^0(n)| : n \in B\}$ is bounded in $\mathbb{Z}$ for each $q \in p \setminus J$,
(D) For each $q \in p \setminus J$ there exists a positive divisor K of N' so that $K.f_q - g_q^0$ is an integer combination of $(f_{i,j,k} : i < s, j < r_i$ and $k < r_{i,j})$
(E) For every $i < s$, $j < r_i$ and $k < r_{i,j}$, $f_{i,j,k}$ is an integer combination of $(f_q : q \in J)$.

(F) For every $q \in p \setminus J$ then g_q^0 is an integer combination of $\{f_i : i \in J\} \cup \{f_q\}$ whose coordinate corresponding to f_q is nonzero.

(G) The family $(g_q^0 : q \in m \setminus J)$ is admissible with respect to $\mathcal{U}$.

Proof. Apply Lemma 7.2 to obtain the objects satisfying properties $i)$-$xiv)$. Set $r_{i,j} = |J_{i,j}|$ for each $i < s$ and $j < r_i$.

By property $iv)$, we can enumerate $\{f_q^{i,j} : q \in J_{i,j}\}$ as $\{f_{i,j,k} : k < r_{i,j}\}$ for each $i < s$ and $j < r_i$ so that $(*)$ if $k < k^*$ then $|f_{i,j,k}(n)(\xi_i(n))| > |f_{i,j,k^*}(n)(\xi_i(n))|$ for each $n \in B$.

Let $J = \bigcup\{J_{i,j} : i < s, j < r_i\}$ and $N' = \prod_{i<s,j<r_i,q<p} N_q^{i,j}$.

(A):

- Condition (5) of the definition of stack follows from vii).
- Condition (6) of the definition of stack follows from viii).
- Condition (7) of the definition of stack follows from iv).
- Condition (8) of the definition of stack is satisfied by (*).
- Condition (9) of the definition of stack follows from iii) and x) if $j = j^* + 1$. For the general case, proceed by finite induction and use the fact that if $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ and $(c_n)_{n\in\omega}$ are sequences of positive real numbers such that $\frac{a_n}{b_n}$ and $\frac{b_n}{c_n}$ converge monotonically to 0, then so does $\frac{a_n}{c_n} = \frac{a_n}{b_n}\frac{b_n}{c_n}$.
- Condition (10) of the definition of stack is satisfied as follows: by $xi)$ it follows that $\{f_{i,j,k}(\xi_{i^*}(n)) : i < s, j < r_i, k < r_{i,j}$ and $n \in B\}$ is bounded. Let $M_{i,j,k}$ be a positive integer such that the set above is contained in $[-M_{i,j,k}, M_{i,j,k}]$ and let $M' = \max\{M_{i,j,k} : i < s, j < r_i, k < r_{i,j}\}$. Then property 10) is satisfied.

(B):

First, if $q \in J_{0,0}$ then $f_q = f_q^{0,0} = f_{0,0,k}$ for some $k \in r_{0,0}$. So we only need to work with $q \notin J_{0,0}$, that is, $(0,0) \prec (i,j)$.

By $xii)$, we have that $\left(\prod_{(i^\#,j^\#)\prec(i,j)} N_q^{i^\#,j^\#}\right).f_q - f_q^{i,j}$ is an integer combination of $(f_{q^*}^{i^*,j^*} : (i^*,j^*) \prec (i,j)$ and $q^* \in J_{i^*,j^*})$ for each $q \in p \setminus \bigcup\{J_{i',j'} : (i',j') \preceq (i,j)\}$.

Therefore, the function $(\prod_{(i^\#,j^\#)\prec(i,j)} N_q^{i^\#,j^\#}).f_q$ is an integer combination of $\{f_{q^*}^{i^*,j^*} : (i^*,j^*) \preceq (i,j)$ and $q^* \in J_{i^*,j^*}\}$. Rewriting this, it follows that the function $(\prod_{(i^\#,j^\#)\prec(i,j)} N_{i^\#,j^\#,k}).f_q$ is an integer combination of $\{f_{i^*,j^*,k^*} : (i^*,j^*) \preceq (i,j)$ and $k^* \in r_{i^*,j^*}\}$.

It follows that $N'.f_q$ is an integer combination of $\{f_{i^*,j^*,k^*} : i < s, j < r_i$ and $k < r_{i,j}\}$ since $\prod_{(i^\#,j^\#)\prec(i,j)} N_{i^\#,j^\#,k}$ divides N'.

Conditions (C) and (D) follows from $xiii)$ and $xvi)$ respectively.

(E) follows by induction on pairs (i,j) using the relation $\prec$ by proving this for every $k < r_{i,j}$ at the same time at each inductive step by using ii), iii) and iv).

(F) if $s = 1$ and $r_0 = 1$, then by xii) $g_q^0 = N_q^{0,0} f_q^{0,0} = N_q^{0,0} f_q$. Otherwise, the thesis follows from xii), iv), v) and (E).

(G):

Assume by contradiction that this family is not admissible. Then by the selectiveness of $\mathcal{U}$ there exists $Z \in \mathcal{U}$ with $Z \subseteq B$ and $(c_q : q \in m \setminus J)$ not all 0 such that $\sum_{q\in m\setminus J} c_q g_q^0$ is constant when restricted to Z. By (F), this sum can be rewritten as $\sum_{q\in m\setminus J} K_q c_q f_q + \delta$, where δ is an integer combination of $(f_q : q \in J)$ and the K_q's are nonzero integers. Thus, by the admissibility of the original family and the positivity of the N's, for every $q \in m \setminus J$ we have $c_q = 0$, a contradiction. $\square$

Now we start to treat the construction of the second part of the stack. We will need Lemma 6.1. of [25].

Lemma 7.4 *(Lemma 6.1. of [25]). Let $\mathcal{U}$ be a selective ultrafilter, $p \in \omega$ and $(f_0, \dots, f_{m-1})$ be an admissible sequence with respect to $\mathcal{U}$. Let $D \in \mathcal{U}$.*

If $\{|f_j(n)| : n \in D$ and $j < m\}$ is a bounded subset of $\mathbb{N}$ then there exists:

- *A family $(N^i_j : 0 \le i \le j < m)$ of natural numbers,*
- *A family $(g^i_j : 0 \le i \le j < m)$ of sequences into $\mathbb{Z}^{(\mathfrak{c})}$,*
- *$C \in \mathcal{U}$,*
- *$\mu_i : C \longrightarrow \mathfrak{c}$,*
- *A family $(\sigma^i_j : 0 \le i \le j < m)$ of elements of $\mathbb{Z}^{(\mathfrak{c})}$,*

satisfying that, for every $i, j < m$ with $i \le j$:

a) $\mu_i(n) \in \operatorname{supp} g^i_i(n)$ for each and $n \in C$,
b) $g^i_j(n)(\mu_i(n)) = N^i_j$ for each $n \in C$,
c) If $i < j$, $g^{i+1}_j = N^i_i.g^i_j - N^i_j.g^i_i$,
d) If $i < j$, $\mu_i(n) \notin \operatorname{supp}(g^{i+1}_j)(n)$,
e) $(\mu_i(n) : n \in C$ and $i < m)$ is pairwise distinct family (injective family),
f) The finite sequence $(g^0_0, \dots, g^{i-1}_{i-1}, g^i_i, g^i_{i+1}, g^i_{i+2}, \dots, g^i_{m-1})$ is admissible with respect to $\mathcal{U}$,
g) $g^0_j = f_j$ and if $i > 0$ then $\prod_{i^* < i} N^{i^*}_{i^*}.f_j - g^i_j$ is an integer combination of $(g^0_0, \dots g^{i-1}_{i-1})$.

In the statement above we have addressed some imprecisions and typos found in the original statement, but it is the original Lemma 6.1. of [25].

Now we modify Lemma 6.2. of [25] to keep track of the important content of Lemma 6.1. of [25] needed to prove (2) of Lemma 5.4.

Lemma 7.5 *(A version of Lemma 6.2. of [25]). Let $\mathcal{U}$ be a selective ultrafilter, $p \in \omega$ and $(f_0, \dots, f_{m-1})$ be an admissible sequence with respect to $\mathcal{U}$. Let $D \in \mathcal{U}$.*

If $\{|f_j(n)| : n \in D$ and $j < m\}$ is bounded, then there exist:

- *positive natural numbers N'' and M'',*
- *A family $(g^i : i < m)$ of sequences into $\mathbb{Z}^{(\mathfrak{c})}$,*
- *$C \in \mathcal{U}$, and*
- *$\mu_i : C \longrightarrow \mathfrak{c}$,*

such that:

A) Conditions $(1) - (3)$ in the definition of the stack are satisfied,
B) For every $i < m$, g_i is a integer combination of $(f_i : i < m)$.
C) $|g_i(n)| \le M''$ for each $i < m$ and $n \in C$,
D) $N''.f_j$ is an integer combination of $\{g_0, \dots g_{m-1}\}$ for each $0 \le i < j < m$, and

Proof. Apply Lemma 7.4. Using the same notation, let $g_i = g^i_i$ for each $i < m$ and $N'' = \prod_{i<m} N^i_i$.

A) Conditions (1) and (3) in the definition of the stack follows from Properties 7.4 *a*) and 7.4 *e*). We verify (2) inductively by showing that:

$$\forall i < m\, (i' < i \implies \forall j < m\, (i \le j \implies \forall n \in C\, g^j_i(\mu_{i'}(m) = 0)).$$

Thus assume $i < m$ with $i' < i$.

Case 1 (base cases) $i'+1=i$. In this case, given j such that $i \leq j < m$ and given $n \in C$, it follows from c) that $g^i_j(\mu_{i'}(n)) = g^{i'+1}_j(\mu_{i'}(n)) = N^{i'}_{i'} g^{i'}_j(\mu_{i'}(n)) - N^{i'}_j g^{i'}_{i'}(\mu_{i'}(n))$ which is, by b), $N^{i'}_{i'} N^{i'}_j - N^{i'}_j N^{i'}_{i'} = 0$.

Case 2 (induction step) $i'+1 < i$. In this case, given j such that $i \leq j < m$ and given $n \in C$, it follows from c) that $g^i_j(\mu_{i'}(n)) = g^{(i-1)+1}_j(\mu_{i'}(n)) = N^{i-1}_{i-1} g^{i-1}_j(\mu_{i'}(n)) - N^{i-1}_j g^{i-1}_{i-1}(\mu_{i'}(n))$ which is, by induction hypothesis, $N^{i-1}_{i-1} \cdot 0 - N^{i-1}_j \cdot 0 = 0$.

B) By induction, we show that:

$$\forall i < m \, \forall j < m \, (i \leq j \implies g^i_j \text{ is a linear combination of } (f_0, \dots, f_p)) = 0.$$

This follows easily from the first part of g) and c).

C) The function g_i is a combination of $(f_0, \dots, f_{m-1})$ by B), therefore, since $C \subseteq D$, $\{g_i(n) : n \in C\}$ is bounded for each $i < m$. Let M'' be a positive integer such that $|g_i(n)| \leq M''$ for each $i < m$ and $n \in C$. Then condition $B)$ is satisfied.

D) By the first part of g), $f_0 = g_0$. Given $i > 0$, by the second part of g) applied to $j = i$ it follows that $\prod_{i^*<i} N^{i^*}_{i^*} f_i$ is an integer combination of $(g_0, \dots, g_i)$. Thus so is $N' f_i$ as $\prod_{i^*<i} N^{i^*}_{i^*}$ divides N'. □

Now we are ready to prove the modified version of Lemma 7.1. of [25] that is used in this paper. The proof is about the same as for the original one.

Lemma 7.6 *(A version of Lemma 7.1. of [25]). Let $h_0, \dots, h_{m-1}$ be sequences in $\mathbb{Z}^{(\mathfrak{c})}$ and $\mathcal{U} \in \omega^*$ be a selective ultrafilter so that $\{[h_0]_\mathcal{U}, \dots, [h_{m-1}]_\mathcal{U}\} \cup \{[\chi_{\vec{\alpha}}]_\mathcal{U} : \alpha < \mathfrak{c}\}$ is linearly independent in the vector space $\mathbb{Q}^{(\mathfrak{c})}/\mathcal{U}$. Then there exists $A \in \mathcal{U}$, $N \in \omega \setminus \{0\}$ and a stack $\frac{1}{N}\mathcal{S}$ on A such that:*

(1) If $\mathbb{Z}^{(\mathfrak{c})}$ is given a group topology and the elements of the stack $\frac{1}{N}\mathcal{S}$ have a $\mathcal{U}$-limit in $\mathbb{Z}^{(\mathfrak{c})}$ then h_i has a $\mathcal{U}$-limit in $\mathbb{Z}^{(\mathfrak{c})}$ for each $0 \leq i < m$.

(2) For each $i < m$, $h_i|_A$ is an integer combination of the elements of the sequences of the stack $\frac{1}{N}\mathcal{S}$ restricted to A. On the other hand, each sequence of the integer stack $\mathcal{S}$ restricted to A is an integer combination of $\{h_0, \dots, h_{m-1}\}$ restricted to A.

We will say in this case that the finite sequence $\{h_0, \dots, h_{m-1}\}$ is associated to $(\frac{1}{N}\mathcal{S}, A, \mathcal{U})$.

Proof. First we show that for each $c_0, \dots c_{m-1} \in \mathbb{Z}$ not all 0, there exists $D \in \mathcal{U}$ such that $(\sum_{i<p} c_i h_i(n) : n \in D)$ is injective. By the selectivity of $\mathcal{U}$, we know that there exists a D which makes $(\sum_{i<p} c_i h_i(n) : n \in D)$ either constant or injective. However it cannot be constant since $\{[h_0]_\mathcal{U}, \dots, [h_{m-1}]_\mathcal{U}\} \cup \{[\chi_{\check{\alpha}}]_\mathcal{U} : \alpha < \mathfrak{c}\}$ is linearly independent.

Case 1. There exists $i < p$ and $D^* \in \mathcal{U}$ such that $\{|h_i(n)| : n \in D^*\}$ is strictly increasing. In this case, apply Lemma 7.3 on $\{h_0, \dots, h_{m-1}\}$ and D to obtain $s, r_0, \dots, r_{s-1}$, $\{r_{i,j} : i < s \text{ and } j < r_i\}$, $\xi_0, \dots, \xi_{s-1}$, B, J, $(f_{i,j,k} : i < s, j < r_i \text{ and } k < r_{i,j})$, $(\theta_{i,j,k} : i < s, j < r_i \text{ and } k < r_{i,j})$, $(g^0_q : q \in p \setminus \bigcup\{J : i < s \text{ and } j < r_i\})$, M' and N' satisfying properties (A)-(G) of Lemma 7.3.

Subcase 1.a. If $p \setminus J = \emptyset$, let $M = M'$ and $N = N'$. Then $\{f_{i,j,k} : i < s, j < r_i \text{ and } k < r_{i,j}\}$, M, $(\xi_i : i < s)$, $(\theta_{i,j,k} : i < s, j < r_i \text{ and } k < r_{i,j})$ form an integer stack $\mathcal{S}$ on $A = B$ with $t = 0$ and $(g_i : i < t) = \emptyset$.

Subcase 1.b. Not subcase 1.a. Enumerate $p \setminus J$ as $q_0, \dots, q_{t-1}$. Apply Lemma 7.5 on the family $(g^0_{q_0}, \dots, g^0_{q_{t-1}})$ and B to obtain a set $C \in \mathcal{U}$, N'', M'', $(g_i : i < t)$ and $\mu_i : C \longrightarrow \mathfrak{c}$ for each $i < t$ satisfying properties (A)-(C) of Lemma 7.5.

Let $M = \max\{M', M''\}$ and $N = N'.N''$.

Then $(f_{i,j,k} : i < s, j < r_i \text{ and } k < r_{i,j})$, $(g_i : i < t)$, M, $(\xi_i : i < s)$, $(\theta_{i,j,k} : i < s, j < r_i \text{ and } k < r_{i,j})$ and $(\mu_l : l < t)$ form an integer stack $\mathcal{S}$ on $A = C$.

Case 2. Not Case 1. Then $s = 0$ and $(f_{i,j,k} : i < s, j < r_i, k < r_{i,j}) = \emptyset$. We follow use the enumeration of p and proceed as in subcase 1.b.

Then these with $(g_i : i < t)$, M'', $(\xi_i : i < s)$, $(\mu_l : l < t)$ form an integer stack $\mathcal{S}$ on $A = C$.

In any case, for every $i < m$, $h_i|C$ is an integer combination of the stack $\frac{1}{N}.\mathcal{S}$ on A (due to Lemma 7.3 (B), (D) and Lemma 7.5 D)). Therefore, if this stack has $\mathcal{U}$-limits in $\mathbb{Z}^{(\mathfrak{c})}$ then $\{f_0, \dots, f_{p-1}\}$ also has a $\mathcal{U}$-limit. Moreover, every element of the stack is an integer combination of the original sequence due to Lemma 7.3 (E), (F) and Lemma 7.5 B). This concludes the proof. $\square$

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