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ON THE REPRESENTATION OF THE NATURAL NUMBERS BY POWERS OF THE GOLDEN MEAN

MICHEL DEKKING AND AD VAN LOON

Abstract. In a base phi representation, a natural number is written as a sum of powers of the golden mean φ. There are many ways to do this. Well known is the standard representation, introduced by George Bergman in 1957, where a unique representation is obtained by requiring that no consecutive powers, φ^n and φ^{n+1}, occur in the representation. In this paper, we introduce a new representation by allowing that the powers φ^0 and φ^1 may occur at the same time, but no other consecutive powers. We then argue that this representation is much closer to the classical representation of the natural numbers by powers of an integer than Bergman's standard representation.

1. Introduction

A natural number N is written in base phi if N has the form

$$N = \sum_{i=-\infty}^{\infty} a_i \varphi^i,$$

where the a_i are arbitrary nonnegative numbers, and where $\varphi := (1+\sqrt{5})/2$ is the golden mean.

There are infinitely many ways to write a number N as a sum of powers of φ. In 1957, George Bergman ([2]) proposed restrictions on the digits a_i, which entail that the representation becomes unique. This is generally accepted as *the* representation of the natural numbers in base phi. A natural number N is written in the Bergman representation if N has the form

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1, and where $d_{i+1}d_i = 11$ is not allowed. Similarly for base 10 numbers, we write these representations as

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

Here L is the largest positive, and R is the smallest negative power of φ that occurs.

The goal of the present paper is to introduce a new representation, which we tendentiously call the canonical representation, which has properties that are much closer to the classical representation of the natural numbers by powers of an integer than the Bergman representation. The *canonical representation* of a natural number N by powers of φ has the form

$$N = \sum_{i=-\infty}^{\infty} c_i \varphi^i,$$

with digits $c_i = 0$ or 1, and where $c_{i+1}c_i = 11$ is not allowed, *except* that $c_1 c_0 = 11$, as soon as this is possible. We write these representations as

$$\gamma(N) = c_L c_{L-1} \dots c_1 c_0 \cdot c_{-1} c_{-2} \dots c_{R+1} c_R.$$

Note that to obtain the canonical representation of N, one first determines if there exists a representation of N with $c_1c_0 = 11$, and no other $c_{i+1}c_i = 11$, and if this is not the case, then $\gamma(N) = \beta(N)$.

The following table compares the two representations. Most of the time, $\gamma(N) = \beta(N)$. The sequence $N = 3, 7, 10, \ldots$ for which $\gamma(N) \neq \beta(N)$ is characterized in Proposition 3.3.

N	$\beta(N)$	$\gamma(N)$
1	1·0	1·0
2	10·01	10·01
3	100·01	11·01
4	101·01	101·01
5	1000·1001	1000·1001
6	1010·0001	1010·0001

N	$\beta(N)$	$\gamma(N)$
7	10000·0001	1011·0001
8	10001·0001	10001·0001
9	10010·0101	10010·0101
10	10100·0101	10011·0101
11	10101·0101	10101·0101
12	100000·101001	100000·101001

We now come to the heart of the matter. Why does the representation $\gamma(\cdot)$ deserve[1] to be called canonical? The evidence for this is two-fold. Representations of the natural numbers in number systems can have two important characteristics. These two characteristics might be indicated as 'horizontal', and 'vertical'. Here 'horizontal' refers to the length of the representations discussed in Section 4, and 'vertical' refers to Section 6. On the one hand, these characteristics are shared by the canonical base phi representation, *and* by the classical base b representation—where, one has to take into account that in base b there are only digits with nonnegative indices. On the other hand, neither one of these characteristics is shared by the Bergman representation.

2. Addition of Base Phi Representations

When N and N' are two natural numbers, with base phi representations $a_L \ldots a_R$ and $a'_{L'} \ldots a'_{R'}$, where we allow the digits a_i, a'_i to be arbitrary nonnegative numbers, then we obtain a base phi representation of $N + N'$, with digits $a_i + a'_i$ for $\max(L, L') \leq i \leq \min(R, R')$, supplementing missing digits by 0's.

For instance, if we add the Bergman representations $10 \cdot 01$ and $100 \cdot 01$ of the numbers 2 and 3, we see this as $010 \cdot 01 + 100 \cdot 01 = 110 \cdot 02$, which is a base phi representation of 5.

In this paper, we consider only $\beta(N) + \beta(N')$ and $\gamma(N) + \gamma(N')$. Note that in general, $\beta(N) + \beta(N') \neq \beta(N + N')$, and similarly for $\gamma(\cdot)$. Because they represent, nevertheless, the same number, we will write $\beta(N) + \beta(N') \doteq \beta(N + N')$, and similarly for $\gamma(\cdot)$.

When we add two numbers in Bergman or canonical representation, then, in general, there is a carry to the left and (two places) to the right. For example,

$$\gamma(5) = \gamma(4+1) \doteq \gamma(4) + \gamma(1) = 101 \cdot 01 + 1\cdot \doteq 102 \cdot 01 \doteq 110 \cdot 02 = 1000 \cdot 1001.$$

Here we used twice that $2\varphi^n = \varphi^{n+1} + \varphi^{n-2}$ for all integers n, a direct consequence of $\gamma(2) = 10 \cdot 01$. Note that there is not only a *double carry*, but that we also have to get rid of the 11's (except if $c_1c_0 = 11$), by replacing them with 100's. This is allowed because of the equation $\varphi^{n+2} = \varphi^{n+1} + \varphi^n$. We call this operation a *golden mean flip*.

[1]The word 'canonical' for our expansion seems to conflict with a free interpretation of Occam's razor: a principle formulated by the 14th century Franciscan friar William of Ockham. Called Ockam's razor (often spelled Occam's razor), it advises you to seek the more economical solution. Occam's Razor is the principle that, "non sunt multiplicanda entia praeter necessitatem" [i.e., "don't multiply the agents in a theory beyond what's necessary."]"

3. Existence and Uniqueness

The key to the existence and uniqueness of the canonical representation is the following lemma.

Lemma 3.1. *A natural number N has a canonical representation $\gamma(N)$ with $c_1c_0 = 11$ if and only if N has a Bergman representation $\beta(N)$ with $d_1d_0 \cdot d_{-1} = 00 \cdot 0$.*

Proof. The proof is based on the analysis of the Bergman representation from the paper [4].

Let $\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R$. In [4], the natural numbers N are coded by four letters $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ according to a coding function T as follows:

We let

$$T(N) = \mathrm{A} \text{ iff } d_1d_0(N) = 10, \quad T(N) = \mathrm{B} \text{ iff } d_1d_0 \cdot d_{-1}(N) = 00 \cdot 0,$$
$$T(N) = \mathrm{C} \text{ iff } d_0(N) = 1, \qquad T(N) = \mathrm{D} \text{ iff } d_1d_0 \cdot d_{-1}(N) = 00 \cdot 1.$$

This leads to the following scheme.

N	$\beta(N)$	$T(N)$
1	1	C
2	$10 \cdot 01$	A
3	$100 \cdot 01$	B
4	$101 \cdot 01$	C
5	$1000 \cdot 1001$	D
6	$1010 \cdot 0001$	A
7	$10000 \cdot 0001$	B
8	$10001 \cdot 0001$	C

N	$\beta(N)$	$T(N)$
9	$10010 \cdot 0101$	A
10	$10100 \cdot 0101$	B
11	$10101 \cdot 0101$	C
12	$100000 \cdot 101001$	D
13	$100010 \cdot 001001$	A
14	$100100 \cdot 001001$	B
15	$100101 \cdot 001001$	C
16	$101000 \cdot 100001$	D

N	$\beta(N)$	$T(N)$
17	$101010 \cdot 000001$	A
18	$1000000 \cdot 000001$	B
19	$1000001 \cdot 000001$	C
20	$1000010 \cdot 010001$	A
21	$1000100 \cdot 010001$	B
22	$1000101 \cdot 010001$	C
23	$1001000 \cdot 100101$	D
24	$1001010 \cdot 000101$	A

Note that in this table, A is always followed by B, and that B is always preceded by A. That this is true for all natural numbers N follows directly from Theorem 5.2 in [4]. With these ingredients, we can now give a proof of the lemma. The first step is to prove the following claim. Here we use that according to Remark 5.4 in [4], Bergman representations with $d_1d_0 \cdot d_{-1}(N) = 10 \cdot 1$ cannot occur.

CLAIM: A natural number N has a canonical representation $\gamma(N)$ with $c_1c_0 = 11$ if and only if $N-1$ has a Bergman representation $\beta(N-1)$ with $d_2d_1d_0 \cdot d_{-1} = 010 \cdot 0$, in other words: $N-1$ has type A.

[Proof of Claim $\Leftarrow$]. Suppose $N-1$ has a Bergman representation $\beta(N-1)$ with $d_2d_1d_0 \cdot d_{-1} = 010 \cdot 0$. When we add 1, we find that $\beta(N) \doteq 1 \dots 011 \cdot 0 \dots 1$, where $\doteq$ means that we obtain a representation of N, but not the Bergman representation. But then, clearly we have obtained a representation of N with $c_1c_0 = 11$, but with no other occurrences of 11.

[Proof of Claim $\Rightarrow$]. Suppose $\gamma(N) = 1 \dots 011 \cdot 0 \dots 1$. When we perform a golden mean flip, we obtain $\gamma(N) \doteq 1 \dots 100 \cdot 0 \dots 1$. Possibly, we have to perform more golden mean flips to obtain a representation of N with no 11. In any case, the result will be of the form $1 \dots 00 \cdot 0 \dots 1$. By the unicity of the Bergman representation, we have found that $\beta(N) = 1 \dots 00 \cdot 0 \dots 1$. So N is of type B. But then, given above, $N-1$ must be of type A.

The lemma now simply follows because A is always followed by B in the T-coding of the natural numbers. $\square$

Proposition 3.2. *The canonical representation of a natural number is unique.*

Proof. Suppose N has canonical representations with $c_1c_0 \neq 11$. By Lemma 3.1, these representations correspond 1-to-1 to Bergman representations of N, so uniqueness follows from the uniqueness of the Bergman representation.

Suppose N has canonical representations with $c_1c_0 = 11$. Changing $c_0 = 1$ to $c_0 = 0$, these representations correspond 1-to-1 to Bergman representations of $N-1$. Again, uniqueness follows from the uniqueness of the Bergman representation. □

How many canonical representation are there in which 11 occurs? It follows from the next proposition that this happens for about 28% of the natural numbers.

Proposition 3.3. *The canonical representation is not equal to the Bergman representation, i.e.,* $\gamma(N) \neq \beta(N)$*, if and only if there exists a natural number* n*, such that* $N = \lfloor(\varphi+2)n\rfloor$.

Proof. Because by Lemma 3.1, $\gamma(N) \neq \beta(N)$ if and only if N is of type B, Theorem 5.1 in [4] gives the result. □

4. The Length of Representations

In this section, we compare the lengths $L + |R| + 1$ of the canonical representations $\gamma(N) = c_L \dots c_0 \cdot c_{-1} \dots c_R$ and the Bergman representations $\beta(N) = d_L \dots d_0 \cdot d_{-1} \dots d_R$.

Note that in the classical base b representation, the natural numbers are partitioned into intervals $B_n := [b^{n-1}, b^n - 1]$, where the representation of a number N has n digits if and only if $N \in B_n$. For base phi representations, the role of b^n is taken over by the Lucas numbers L_n, where $L_0, L_1, L_2, \dots = 2, 1, 3, \dots$ are defined by $L_0 := 2$, $L_1 := 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

It is therefore important to know the representations of the Lucas numbers. The formulas (4.3) for $\gamma(L_{2n}+1)$ and (4.4) for $\gamma(L_{2n+1}+1)$ will be useful in Section 6.

Lemma 4.1. *For all* $n \geq 1$*, one has*

$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1, \quad \gamma(L_{2n}) = [10]^{n-1}11 \cdot 0^{2n-1}1, \tag{4.1}$$

$$\beta(L_{2n+1}) = \gamma(L_{2n+1}) = 1[01]^n \cdot [01]^n, \tag{4.2}$$

$$\beta(L_{2n}+1) = \gamma(L_{2n}+1) = 10^{2n-1}1 \cdot 0^{2n-1}1, \tag{4.3}$$

$$\beta(L_{2n+1}+1) = \gamma(L_{2n+1}+1) = 10^{2n+1} \cdot [10]^n01. \tag{4.4}$$

Proof. The expressions for $\beta(L_{2n})$ and $\beta(L_{2n+1})$ are well-known (see, e.g., [4]), and easy to prove: they follow directly from $L_{2n} = \varphi^{2n} + \varphi^{-2n}$, and the recursion $L_{2n+1} = L_{2n} + L_{2n-1}$.

When we perform n golden mean flips on $[10]^{n-1}11\cdot0^{2n-1}1$, we obtain $10^{2n}\cdot0^{2n-1}1 = \beta(L_{2n})$. This implies the expression for $\gamma(L_{2n})$.

The equality $\gamma(L_{2n+1}) = \beta(L_{2n+1})$ follows by an application of Lemma 3.1, because $\beta(L_{2n+1})$ is of type C.

The expression for $\beta(L_{2n}+1)$ and $\gamma(L_{2n}+1)$ follows immediately from Lemma 3.1 by adding 1 to the Bergman expansion of L_{2n} in equation (4.1), which yields a valid Bergman expansion for $L_{2n}+1$, which is of type C.

We leave the proof of equation (4.4) to the reader, see also Lemma 3.3 (2) in [10]. □

What are the intervals of constant expansion length for the Bergman representation?
As in [4], we define the so called *Lucas intervals* $\Lambda_{2n} := [L_{2n}, L_{2n+1}]$ and $\Lambda_{2n+1} := [L_{2n+1} + 1, L_{2n+2} - 1]$.
The next result is Theorem 2.1 in [10], derived from Theorem 1 in [8].

Proposition 4.2. *The intervals of constant expansion length for the Bergman expansion are the Lucas intervals Λ_n, $n \geq 1$. More precisely: if $\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R$, then the left most index $L = L(N)$ and the right most index $R = R(N)$ satisfy*

$$L(N) = 2n = -R(N) \text{ iff } N \in \Lambda_{2n}, \quad L(N) = 2n+1,\ -R(N) = 2n+2 \text{ iff } N \in \Lambda_{2n+1}.$$

What is 'wrong' with the Bergman Lucas intervals when we compare them with the intervals B_n of constant expansion length for base b? Answer: The odd index intervals are too small compared with the even index intervals: $|\Lambda_{2n}| = L_{2n-1} + 1$, and $|\Lambda_{2n+1}| = L_{2n} - 1$.

Our next task is to determine the intervals of constant expansion length for the canonical representation.
We define the *canonical Lucas intervals*

$$\Gamma_0 := \{1\}, \quad \Gamma_n := [L_n + 1, L_{n+1}] \quad \text{for } n \geq 1.$$

So $\Gamma_1 = [2, 3]$, $\Gamma_2 = \{4\}$, $\Gamma_3 = [5, 7]$, $\Gamma_4 = [8, 11]$, etc.
Note that $|\Gamma_n| = L_{n+1} - L_n$ for all $n \geq 1$, an expression that is similar to $|B_n| = b^n - b^{n-1}$ for the classical base b expansion.

Proposition 4.3. *The intervals of constant expansion length for the canonical expansion are the canonical Lucas intervals Γ_n, $n \geq 1$. More precisely: if $\gamma(N) = c_L c_{L-1} \dots c_1 c_0 \cdot c_{-1} c_{-2} \dots c_{R+1} c_R$, then the left most index $L = L(N)$ and the right most index $R = R(N)$ satisfy*

$$L(N) = 2n = -R(N) \text{ iff } N \in \Gamma_{2n}, \quad L(N) = 2n+1,\ -R(N) = 2n+2 \text{ iff } N \in \Gamma_{2n+1}.$$

Proof. Directly from Lemma 4.1, we see that $|\gamma(L_{2n})| = |\beta(L_{2n})| - 1$. Therefore, we have to move the first number L_{2n} from $\Lambda_{2n} = [L_{2n}, L_{2n+1}]$ to $\Lambda_{2n-1} = [L_{2n-1} + 1, L_{2n} - 1]$ as a first step to obtain the intervals of constant length expansion for the canonical expansion. This leads exactly to the intervals Γ_n. It remains to see that this first step is the only change we have to make, i.e., that $|\gamma(N)| = |\beta(N)|$ for all $N \neq L_{2n}$. To prove this, note that we can transform the canonical representation to the Bergman representation by a number of golden mean flips, starting with replacing $011 \cdot 0$ in $\gamma(N)$ by $100 \cdot 0$. A second golden mean flip will follow if and only if $1011 \cdot 0$ occurs in $\gamma(N)$, and then $0000 \cdot 0$ occurs in $\beta(N)$. Now either this process stops before reaching the left end of $\gamma(N)$ and then $|\gamma(N)| = |\beta(N)|$, or it continues to the left end, and then $\beta(N) = 10 \dots 0 \cdot d_{-1} \dots d_R$. But, by Lemma 4.1, this information suffices to conclude that $N = L_{2n}$ for some natural number n. This follows from the observation in [6] that, in general, the β^+–part of an expansion $\beta(N) = \beta^+(N) \cdot \beta^-(N)$ determines N, because the β^--part codes a real number smaller than 1. □

5. The Recursive Structure Theorem

To obtain recursive relations for the Bergman representation is relatively simple for the intervals Λ_{2n}, but the intervals $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ have to be divided into three subintervals. These three intervals are

$$I_n := \quad [L_{2n+1} + 1,\ L_{2n+1} + L_{2n-2} - 1], \tag{5.1}$$

$$J_n := \quad [L_{2n+1} + L_{2n-2},\ L_{2n+1} + L_{2n-1}], \tag{5.2}$$

$$K_n := \quad [L_{2n+1} + L_{2n-1} + 1,\ L_{2n+2} - 1]. \tag{5.3}$$

It will be convenient to use the free group versions of words of 0's and 1's. This means that we will write, for example, $(01)^{-1}0001 = 1^{-1}001$. We can then formulate the following result from [5].

Theorem 5.1. [Recursive structure theorem for the Bergman representation]
I *For all* $n \geq 1$ *and* $k = 1, \ldots, L_{2n-1}$, *one has* $\beta(L_{2n}+k) = \beta(L_{2n})+\beta(k) = 10\ldots0\,\beta(k)\,0\ldots01$.
II *For all* $n \geq 2$ *and* $k = 1, \ldots, L_{2n-2} - 1$,

$$I_n : \quad \beta(L_{2n+1} + k) = 1000(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}1001,$$

$$K_n : \quad \beta(L_{2n+1} + L_{2n-1} + k) = 1010(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}0001.$$

Moreover, for all $n \geq 2$ *and* $k = 0, \ldots, L_{2n-3}$,

$$J_n : \quad \beta(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1}\beta(L_{2n-2} + k)(01)^{-1}001001.$$

Because the canonical Lucas intervals are only—literally—marginally different from the Bergman Lucas intervals, we can transform Theorem 5.1 into a similar result for the canonical representation.

This time, the interval $\Gamma_{2n+1} = [L_{2n+1} + 1, L_{2n+2}]$ has to be divided into the three subintervals

$$I_n := [L_{2n+1} + 1,\ L_{2n+1} + L_{2n-2}], \tag{5.4}$$

$$J_n := [L_{2n+1} + L_{2n-2} + 1,\ L_{2n+1} + L_{2n-1}], \tag{5.5}$$

$$K_n := [L_{2n+1} + L_{2n-1} + 1,\ L_{2n+2}]. \tag{5.6}$$

The result becomes the following.

Theorem 5.2. [Recursive structure theorem for the canonical representation]
I *For all* $n \geq 1$ *and* $k = 1, \ldots, L_{2n-1}$, *one has* $\gamma(L_{2n} + k) = 10^{2n} \cdot 0^{2n-1}1 + \gamma(k) = 10\ldots0\,\gamma(k)\,0\ldots01$.
II *For all* $n \geq 2$ *and* $k = 1, \ldots, L_{2n-2}$,

$$I_n : \quad \gamma(L_{2n+1} + k) = 1000(10)^{-1}\gamma(L_{2n-1} + k)(01)^{-1}1001,$$

$$K_n : \quad \gamma(L_{2n+1} + L_{2n-1} + k) = 1010(10)^{-1}\gamma(L_{2n-1} + k)(01)^{-1}0001.$$

Moreover, for all $n \geq 2$ *and* $k = 1, \ldots, L_{2n-3}$,

$$J_n : \quad \gamma(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1}\gamma(L_{2n-2} + k)(01)^{-1}001001.$$

Proof. These statements follow directly from Theorem 5.1, except that we have to do an extra check for the exceptional numbers $N = L_{2n}$, and for the endpoints of the intervals I_n and K_n in (5.4) and (5.6).

The numbers $N = L_{2n+2}$ *are* the endpoints of the intervals K_n, and the recursion formula above remains valid for $k = L_{2n-2}$, as we can see by an application of Lemma 4.1:

$$\gamma(L_{2n+2}) = [10]^n 11 \cdot 0^{2n+1}1 = 10[10]^{n-1}11 \cdot 0^{2n-1}001 = 1010(10)^{-1}\gamma(L_{2n})(01)^{-1}0001.$$

The endpoint of I_n is equal to $L_{2n+1}+L_{2n-2} = L_{2n}+L_{2n-1}+L_{2n-2} = 2L_{2n}$. For the recursion formula above to remain valid for $k = L_{2n-2}$ in the I_n case, we therefore have to prove that

$$\begin{aligned}
\gamma(2L_{2n}) &= 1000(10)^{-1}\gamma(L_{2n-1} + L_{2n-2})(01)^{-1}1001 \\
&= 1000(10)^{-1}\gamma(L_{2n})(01)^{-1}1001 \\
&= 1000(10)^{-1}[10]^{n-1}11 \cdot 0^{2n-1}1(01)^{-1}1001 \\
&= 1000[10]^{n-2}11 \cdot 0^{2n-2}1001.
\end{aligned}$$

We leave the proof of this canonical expansion of $2L_{2n}$ as a (nontrivial) exercise to the reader (Hint: pass to the Bergman expansion, and exploit the unicity of the canonical expansions. See also page 3 of [5]) □

Remark 5.1. *It is important to observe the close relationship between the intervals I_n, J_n, K_n in* (5.4), (5.5), (5.6) *and the canonical Lucas intervals:*

$$I_n = \Gamma_{2n-1} + L_{2n}, \quad J_n = \Gamma_{2n-2} + L_{2n+1}, \quad K_n = \Gamma_{2n-1} + L_{2n+1}.$$

Here we use the notation $A + x = \{a + x : a \in A\}$ for a set of real numbers A and a real number x.

We end this section with a typical application of the Recursive Structure Theorem, a lemma that will be useful in the next section.

Lemma 5.3. a) *For all $n \geq 2$, one has $c_{-2n+3}(N) = 0$ for all N from Γ_{2n}.*

b) *For all $n \geq 2$, one has $c_{-2n+1}(N) = 1$ for the first L_{2n-1} numbers N from Γ_{2n+1}, and $c_{-2n+1}(N) = 0$ for the last L_{2n-2} numbers N from Γ_{2n+1}.*

Proof of a). By the Recursive Structure Theorem, Part I, the expansions in $\Gamma_{2n} = [L_{2n} + 1, L_{2n+1}]$ look like the expansions in the interval $[1, L_{2n-1}]$. This interval is a union of $\Gamma_0, \Gamma_1, \dots, \Gamma_{2n-1}, \Gamma_{2n-2}$. Except for the last two, the numbers N from these intervals have canonical expansions that have a right endpoint $R(N)$ with $-R(N) \leq 2n-4$. So automatically, we have $c_{-2n+3}(N) = 0$ for the N from these intervals. From Proposition 4.3, we see that the last two intervals Γ_{2n-1} and Γ_{2n-2} contain only numbers N with $-R(N) = 2n-2$. But then, the digit $c_{-2n+3}(N)$ directly to the left of $c_{-2n+2}(N) = 1$ must be equal to 0.

Proof of b). From Proposition 4.3, we have $-R(N) = 2n+2$ for all $N \in \Gamma_{2n+1} = I_n \cup J_n \cup K_n$.

Using Remark 5.1, we see that the interval $I_n \cup J_n$ has length $|\Gamma_{2n-1}| + |\Gamma_{2n-2}| = L_{2n-2} + L_{2n-3} = L_{2n-1}$. We see directly from the Recursive Structure Theorem, Part II, that the expansions of the numbers N in $I_n \cup J_n$ have $c_{R+3}(N)c_{R+2}(N)c_{R+1}(N)c_R(N) = 1001$. Here $R(N) + 3 = -2n - 2 + 3 = -2n + 1$. On the other hand, we see that the expansions in the interval K_n all have in $c_{R+3}(N)c_{R+2}(N)c_{R+1}(N)c_R(N) = 0001$. These two observations imply part b). □

6. Vertical Runs

When we make a table of the classical base b expansions of the natural numbers, one observes a regular structure of the runs of the digits in the columns of the table. As an example, consider the case $b = 2$ of the binary expansion.

N	expansion	N	expansion	N	expansion
0	0	8	1000	16	10000
1	1	9	1001	17	10001
2	10	10	1010	18	10010
3	11	11	1011	19	10011
4	100	12	1100	20	10100
5	101	13	1101	21	10101
6	110	14	1110	22	10110
7	111	15	1111	23	10111

In digit position i, for $i \geq 0$, only runs of 2^i 1's occur—separated by runs of 2^i 0's.

For the Bergman expansion, there is no such regularity: vertical runs of 1's of length 1,2,3,4,5,6, and 7 do occur. This is completely different for the canonical expansion: see Theorem 6.2. Part of the proof of this theorem is provided by the following lemma, which compares digits of the numbers at the end and the beginning of canonical Lucas intervals.

Lemma 6.1. *Let N have canonical expansion $\gamma(N) = c_L(N)\dots c_R(N)$, where we add 0's when comparing two expansions, for example $c_{2n+1}(L_{2n}) = 0$. Then for all $n \geq 1$:*

[OE] $c_i(L_{2n}) = 1$ *and* $c_i(L_{2n} + 1) = 1$ happens if and only if $i = 0$ or $i = -2n$,

[EO] $c_i(L_{2n+1}) = 1$ *and* $c_i(L_{2n+1} + 1) = 1$ does not happen for any $i = -2n-2, \dots, 2n+2$.

Proof. This follows directly from Lemma 4.1. See, e.g., equation (4.1) and equation (4.3) for [OE]. □

In Lemma 6.1, [OE] refers to the indices of the two successive Lucas intervals Γ_{2n-1}, Γ_{2n}, [EO] to the indices of the two successive Lucas intervals Γ_{2n}, Γ_{2n+1}.

Theorem 6.2. *In the canonical base phi expansion of the natural numbers, only vertical runs of 1's with length a Lucas number occur, and all Lucas numbers occur as a run length. More precisely: in digit position i, only runs of length L_{i-1} occur when $i \geq 1$, and only runs of length L_{-i} occur when $i \leq 0$.*

Proof. The proof will be divided into five parts: $i = 0$, $i = -1$, $i > 0$, $i < -1$ and i even, and $i < -1$ and i odd. In the last three cases, we partition the natural numbers in canonical Lucas intervals, and use the Recursive Structure Theorem 5.2. For $i < -1$, the situation is more complicated than for $i > 1$, which forces us to consider i even and i odd separately.

Part 1: $\mathbf{i = 0}$. Then $L_{-i} = L_0 = 2$. We remark here that we did not mention in the statement of the theorem that for $i = 0$ the first run deviates from the pattern: it has length 1, and this does not change if we would add $N = 0$ to the table.

According to Theorem 5.1 in [4], one has $d_0(N) = 1$ if and only if $N = \lfloor n\varphi \rfloor + 2n + 1$ for some natural number n. According to Lemma 3.1, $\gamma(N) \neq \beta(N)$ if and only if there exists a natural number n such that $N = \lfloor n\varphi \rfloor + 2n$. If we combine these two statements, we see that all the runs of 1's (except the first one) have length $2 = L_0$ in digit position $i = 0$.

Part 2: $\mathbf{i = -1}$. From Remark 5.1 from [4], we have that for the Bergman representation, $d_{-1}(N) = 1$ if and only if $N = 3\lfloor n\varphi \rfloor + n + 1$ for some natural number n. Because by Proposition 3.3, $\gamma(N) \neq \beta(N)$ if and only if N is of type B, which has $d_1 d_0 \cdot d_{-1}(N) = 00\cdot 0$, we can deduce that also for the canonical representation, $c_{-1}(N) = 1$ if and only if $N = 3\lfloor n\varphi \rfloor + n + 1$ for some natural number n. This obviously implies that the runs of 1's at digit position $i = -1$ have length $L_1 = 1$.

The proofs of Parts 3, 4, and 5 are based on the Recursive Structure Theorem. When we perform the induction, we have to prove that runs of 1's do not extend beyond the intervals that are produced by the induction. In the following, we will show that this holds for all digit positions with the exception of $i = 0$ and certain positions at the left end and right end of the expansion. Note that we can ignore the case $i = 0$, because it has already been addressed in Part 1 of the proof. For canonical Lucas intervals Γ_{2n+2}, this "isolated run property" is considered in [*], for canonical Lucas intervals Γ_{2n+1} in [**].

[*] For the Recursive Structure Theorem, Part I, we use that the runs of 1's in column i in the interval Γ_{2n+2} are a copy of the runs of 1's in the interval $[1, L_{2n+1}]$, *except* for the left most column, which corresponds to digit position $L = 2n + 2$, and the right most column, corresponding to digit position $R = -2n - 2$.

We still have to check that no new runs are created at the beginning or the end of the interval Γ_{2n+2}. This is obvious for the beginning, because $c_i(1) = 0$ for $i \neq 0$, and for the end, it follows from Lemma 6.1 [EO].

[**] For the Recursive Structure Theorem, Part II, we use that the runs of 1's in column i in the interval Γ_{2n+1} are a copy of the column of digit i lying in the intervals I_n, J_n, and K_n, except for the three leftmost columns, and the five right most columns. These exceptions correspond to digit positions $L = 2n+1$, $L-1 = 2n$, and $L-3 = 2n-1$, at the left, and at the right to positions with indices $R+3 = -2n+1$, $R+2 = -2n$, $R+1 = -2n-1$, $R = -2n-2$ when N is from I_n or K_n, and to positions with indices $R+4$, $R+3$, $R+2$, $R+1$, R when N is from J_n. Here you use computations like $1000(10)^{-1} = 1001^{-1}$.

This time, we still have to check that no new runs are created at the beginning or the end of the intervals I_n, J_n, and K_n.

For the beginning and the end of $\Gamma_{2n+1} = I_n \cup J_n \cup K_n$, this follows again from Lemma 6.1 [EO], respectively Lemma 6.1 [OE], except for $i = -2n-2$.

By Remark 5.1, the digits in column i of I_n, J_n, and K_n are equal to the corresponding digits in column i of the intervals Γ_{2n-1}, Γ_{2n-2}, and Γ_{2n-1}.

There will be no new run created on the boundary between I_n and J_n. We have that Γ_{2n-1} (the shift of I_n) ends with $N = L_{2n}$, and $c_i(L_{2n}) = 0$ by equation (4.1), when c_i is not the digit of one of the last four columns.

Finally, there will be no new run created on the boundary between J_n and K_n, which are shifts of the two successive intervals Γ_{2n-2} and Γ_{2n-1}, by Lemma 6.1 [EO].

Part 3: $\mathbf{i \geq 1}$.

We first illustrate how this works for the case $i = 1$. The column of digit position 1 starts with a run of length $L_{i-1} = L_0 = 2$ in $\Gamma_1 = [2, 3]$. Then a 0 follows in $\Gamma_2 = \{4\}$, followed by another run of length 2 in $\Gamma_3 = [5, 6, 7]$. Suppose one has proved that only runs of length 2 occur in the Lucas intervals $\Gamma_1, \ldots, \Gamma_m$ for some natural number m. We then proceed by induction, distinguishing the cases $m = 2n$ and $m = 2n+1$.

We start with the case $m = 2n+1$. Then, the next interval is Γ_{2n+2}. By the Recursive Structure Theorem Part I, the column of digit position 1 lying in this interval is a copy of the column of digit 1 lying in the interval $[1, L_{2n+1}]$. Therefore, by the induction hypothesis and [**], there will be only runs of length 2 in this part of the column.

For the case $m = 2n$, the next interval is Γ_{2n+1}. By the Recursive Structure Theorem Part II, the column of digit position 1 lying in this interval is a copy of the column of digit 1 lying in the intervals I_n, J_n, and K_n. Therefore, by the induction hypothesis and [*], there will be only runs of length 2 in this part of the column. This ends the proof of the case $i = 1$.

We next consider the case i for arbitrary $i \geq 2$. The proof is similar to the proof of the case $i = 1$. The main complication is the change in the digits occurring at the left most part of the expansion in the Recursive Structure Theorem. This is solved by giving the induction more attention at the start.

The first run of 1's in digit column i starts at the number $N = L_i + 1$ in Γ_i, because all $c_i(N) = 1$ for $N \in \Gamma_i$ (see $i = 2n$ and $i = 2n+1$ in Proposition 4.3). This run has length L_{i-1}, because $|\Gamma_i| = L_{i-1}$ and by Lemma 6.1.

Next, for all N in Γ_{i+1}, one has $c_i(N) = 0$, simply because $c_{i+1}(N) = 1$. So, no runs of 1's occur in Γ_{i+1}.

We then pass to Γ_{i+2}.

Suppose i is <u>even</u>. Then, by the Recursive Structure Theorem Part I, the column of digit position i lying in the interval Γ_{i+2} is a copy of the column of digit i lying in the interval $[1, L_{i+1}]$. But we know already that this part only contains runs of 1's of length L_{i-1} (actually there is a single run, lying in Γ_i). Then also, Γ_{i+2} will only have a run of 1's of length L_{i-1}. Moreover, this *is* the length of that run by Lemma 6.1 [EO].

Suppose i is odd. Then, by the Recursive Structure Theorem Part II, the column of digit position i lying in the interval Γ_{i+2} can be obtained from the column of digit i lying in the intervals Γ_i, Γ_{i-1}, and Γ_i. In the first case, all the L_{i-1} 1's turn into 0's, in the second case, we obtain only 0's, simply because the right most 1 of the expansions in Γ_{i-1} is in column $i-1$, and in the third case, all the L_{i-1} 1's turn into 1's. Moreover, in this last case, this *is* a run of length L_{i-1} by an application of Lemma 6.1 [OE]. Conclusion: also Γ_{i+2} will only have a run of 1's of length L_{i-1}.

Suppose one has proved that only runs of length L_{i-1} occur in the Lucas intervals Γ_i, Γ_{i+1}, $\dots$, Γ_m for some natural number $m \geq i+2$. We then proceed by induction, distinguishing again the cases $m = 2n$ and $m = 2n+1$.

We start with the case $m = 2n+1$. Then, the next interval is Γ_{2n+2}. By the Recursive Structure Theorem Part I, the column of digit position i lying in this interval is a copy of the column of digit i lying in the interval $[1, L_{2n+1}] = [1, L_m]$, except for the leftmost column. This column has digit position $2n+2$. But $i+2 \leq m = 2n+1$, therefore, by the induction hypothesis and [*], there will be only runs of length L_{i-1} in this part of column i.

For the case $m = 2n$, the next interval is Γ_{2n+1}. By the Recursive Structure Theorem Part II, the column of digit position $i > 0$ lying in this interval is a copy of the column of digit i lying in the intervals I_n, J_n, and K_n, except for the three leftmost columns. These three have indices $L = 2n+1$, $L-1 = 2n$, and $L-3 = 2n-1$. But $2n = m \geq i+2$, i.e., $i \leq 2n-2$. Therefore, by the induction hypothesis and [**], there will be only runs of length L_{i-1} in this part of the column.

Part 4: $\mathbf{i < -1, i}$ even.

Suppose $-i = 2j$ is even. From Proposition 4.3, we obtain that the first run of numbers N with digit c_{-2j} equal to 1 starts with all numbers N in the interval Γ_{2j-1}, and then *continues* in the interval Γ_{2j}. The run will not continue in the next interval Γ_{2j+1} by Lemma 6.1. Conclusion: the first run of 1's in column $-2j$ has length

$$|\Gamma_{2j-1}| + |\Gamma_{2j}| = L_{2j-2} + L_{2j-1} = L_{2j} = L_{-i}.$$

Next, we consider the interval Γ_{2j+1}. By the Recursive Structure Theorem Part II, because for I_n and K_n, the last three digits in the replacement equation are 001, there will be 0's in the corresponding parts of column i. The same is true for the part corresponding to the interval J_n. So, no runs of 1's occur in Γ_{2j+1}. We then pass to Γ_{2j+2}.

Here, by the Recursive Structure Theorem Part I, the column of digit position i lying in the interval Γ_{2j+2} is a copy of the column of digit i lying in the interval $[1, L_{2j+1}]$. But we know already that this part only contains runs of 1's of length L_{-i} (actually there is a single run, lying in the union of Γ_{2j-1} and Γ_{2j}). Then also, Γ_{2j+2} will only have a run of 1's of length L_{-i}. Moreover, this *is* the length of that run by Lemma 6.1 [EO].

Suppose one has proved that only runs of length L_{-i} occur in the Lucas intervals Γ_{-i+2}, $\dots$, Γ_m for some natural number $m \geq 2j+2$. We then proceed by induction, distinguishing again the cases $m = 2n$ and $m = 2n+1$.

We start with the case $m = 2n+1$. Then, the next interval is Γ_{2n+2}. By the Recursive Structure Theorem Part I, the column of digit position i lying in this interval is a copy of the column of digit i lying in the interval $[1, L_{2n+1}]$, except for the rightmost column. This column has digit position $-i = 2n+2$. But, $-i = 2j \leq m-2 = 2n-1$. Therefore, by the induction hypothesis and [**], there will be only runs of length L_{-i} in this part of the column.

For the case $m = 2n$, the next interval is Γ_{2n+1}. By the Recursive Structure Theorem Part II, the column of digit position i lying in this interval is a copy of the column of digit i lying in the intervals I_n, J_n, and K_n, except for the four rightmost columns for I_n and K_n, and the five rightmost columns for J_n. But $-i \leq R-5 = 2n-3$, because $2n = m \geq 2j+2 = -i+2$. Therefore, by the induction hypothesis, and [*], there will be only runs of length L_{-i} in this part of the column. Here we need $i \neq -(2n+2)$. This is satisfied because $-i < 2n$.

Part 5: $\mathbf{i < -1}$, i odd.
Suppose $-i = 2j+1$ is odd. Where does the first run of 1's at digit position $i = -2j-1$ occur? This is more complicated than in all previous cases where this happened at position L (for $i > 0$) or position R (for $i < 0$, i even).

CLAIM: The first run of 1's occurs in column $R+3$ in Γ_{2j+3}, where all numbers have $R = -2j-4$, as given in Proposition 4.3.

Indeed, note that Γ_{2j+1} and Γ_{2j+2} would be the first two candidates for the occurrence of 1's at position $-(2j+1)$, but that both intervals have numbers N with $R(N) = -(2j+2)$, and so there will be 0's at position $-(2j+1)$, because 11 does not occur. The next candidate is the interval Γ_{2j+3}. Here we use Lemma 5.3, Part b), with $n = j+1$. This lemma gives that $c_{-2j-1}(N) = 1$ for the first $L_{2j+1} = L_{-i}$ numbers N from Γ_{2j+3}, and $c_{-2j-1}(N) = 0$ for the remaining numbers N. This proves the claim above.

Next, consider the interval Γ_{2j+4}. By the Recursive Structure Theorem Part I, the column of digit position i lying in this interval is a copy of the column of digit i lying in the interval $[1, L_{2j+3}] = \Gamma_0 \cup \cdots \cup \Gamma_{2j+2}$ (except for the rightmost column). But the first run of 1's occurs in column $R+3$ in Γ_{2j+3}, so there are no 1's at all in column $R+3$ in Γ_{2j+4}.

Next, suppose one has proved that only runs of length L_{-i} occur in the canonical Lucas intervals $\Gamma_{-i+2}, \Gamma_{-i+1}, \ldots, \Gamma_m$ for some natural number $m \geq 2j+4$. We then proceed by induction, distinguishing again the cases $m = 2n$ and $m = 2n+1$.

We start with the case $m = 2n+1$. Then, the next interval is Γ_{2n+2}. By the Recursive Structure Theorem Part I, the column of digit position i lying in this interval is a copy of the column of digit i lying in the interval $[1, L_{2n+1}] = [1, L_m]$, except for the rightmost column with index $R = -2n-2$. So, we need that $-i < -R = 2n+2$, which holds if and only if $2j+1 < m$. This is certainly satisfied.

Therefore, by the induction hypothesis and [*], there will be only runs of length L_{-i} in this part of the column of digit i.

For the case $m = 2n$, the next interval is Γ_{2n+1}, with $R = -2n-2$. By the Recursive Structure Theorem Part II, the column of digit position i lying in this interval is a copy of the column of digit position i lying in the intervals I_n, J_n, and K_n, except for the columns with indices $R+3$, $R+2$, $R+1$, R, when N is from I_n or K_n, and except for the columns with indices $R+4$, $R+3$, $R+2$, $R+1$, R, when N is from J_n. So we need that $-i < -R-4 = 2n+2-4 = m-2$, which holds if and only if $2j+1 < m-2$, which is satisfied because $m \geq 2j+4$.

Therefore, by the induction hypothesis and [**], there will be only runs of length L_{-i} in this part of the column. □

7. Final Remarks

7.1. **Two-dimensional Characteristic.** There is a third characteristic of representations of the natural numbers, which might be labelled as 'two-dimensional'. For the canonical expansion, this amounts to the observation that the lengths of the runs of 1's spread over the table of expansions in chains of consecutive Lucas numbers, cf. Figure 1 for the left-most chain in the positive part, and the third chain in the negative part of the table of expansions. These chains are finite, except the first one, which starts at $N = 2$, and consists of the runs of the c_L digits.

The pattern consists of two kinds of chains:

(1) At the left side, i.e., $i > 0$, the lengths of the links in the chains follow $(L_n)_{n\geq 0} = 2, 1, 3, 4, 7, 11, \ldots$.
(2) At the right side, i.e., $i \leq 0$, the lengths of the links in the chains follow $(L_n)_{n\leq 0} = 2, -1, 3, -4, 7, -11, \ldots$.

Here the sign of the length indicates the direction in which the link goes.

(L_n)	c_5	c_4	c_3	c_2	c_1	c_0	c_{-1}	c_{-2}	c_{-3}	c_{-4}	c_{-5}	(L_n)
2▼					1					1		7▼
					1					1		
1▼				1						1		
3▼			1							1		
			1					1		1		3▼
			1			1		1		1		2▼
4▼		1				1		1		1		
		1					1		1		1	−1▲
		1							1		1	
		1							1		1	
7▼	1								1		1	−4▲
	1										1	
	1										1	
	1										1	
	1										1	
	1										1	
	1										1	
											1	−11▲

FIGURE 1. Two typical chains. Leftmost column: $L_n, n \geq 0$, rightmost column: $L_n, n \leq 0$.

7.2. **Positions.** We conjecture that the positions at which the 1's occur in the column of digit i are given by unions of generalized Beatty sequences. Generalized Beatty sequences, defined in [1], are sequences $V(p, q, r) = (V_n)$ of the form $V_n = p\lfloor n\alpha\rfloor + q\,n + r$, $n \geq 1$, where α is a real number, and p, q, and r are integers. Note that this has been proved in our paper for $i = 0$ and $i = -1$.

7.3. **Generalizations.** We believe that for irrational numbers other than the golden mean, our approach makes sense.

A natural class of numbers to consider are the quadratic Pisot units that are the largest root β of a polynomial of the form $X^2 - aX - 1$, $a > 1$. It is known that every natural number has a finite β-expansion. The sum of two expansions can have a right carry of length 2 at most, see [3] and [9]. It is possible to define the analogue of Lucas numbers by $L_n = \beta^n + \bar{\beta}^n$, where $\bar{\beta}$ is the algebraic conjugate of β.

These numbers are also known as the metallic means, and are contained in the class of numbers given in Theorem 2 of [7]. We shortly discuss the case $a = 2$, which gives the silver mean $\sigma := 1 + \sqrt{2}$.

The standard representation of the natural numbers in base σ is given by

$$N = \sum_{i=-\infty}^{\infty} d_i \sigma^i,$$

with digits $d_i = 0, 1$ or 2, and where $d_{i+1}d_i = 21$ or 22 is not allowed.

The role of the Lucas numbers (L_n) is now taken over by the Pell-Lucas numbers $(L_n^{\mathrm{P}}) = 2, 2, 6, 14, 34, \ldots$, defined by

$$L_0^{\mathrm{P}} = 2, \quad L_1^{\mathrm{P}} = 2, \quad L_{n+2}^{\mathrm{P}} = 2L_{n+1}^{\mathrm{P}} + L_n^{\mathrm{P}} \quad \text{for } n = 0, 1, 2, \ldots$$

We write $\beta^{\mathrm{P}}(N)$ for the standard expansion of N in base σ, and $\gamma^{\mathrm{P}}(N)$ for the canonical expansion of N in base σ. This time, canonical means that the digits are $c_i = 0, 1$ or 2, and that $c_{i+1}c_i = 21$ or 22 is not allowed, *except* that $c_1c_0 = 21$, as soon as this is possible.

The following table displays these two representations.

N	$\beta^{\mathrm{P}}(N)$	$\gamma^{\mathrm{P}}(N)$
1	1·0	1·0
2	2·0	2·0
3	10·11	10·11
4	11·11	11·11
5	20·01	20·01
6	100·01	21·01

N	$\beta^{\mathrm{P}}(N)$	$\gamma^{\mathrm{P}}(N)$
7	101·01	101·01
8	102·01	102·01
9	110·12	110·12
10	111·12	111·12
11	120·02	120·02
12	200·02	121·02

N	$\beta^{\mathrm{P}}(N)$	$\gamma^{\mathrm{P}}(N)$
13	201·02	201·02
14	202·02	202·02
15	1000·2011	1000·2011
16	1001·2011	1001·2011
17	1010·1011	1010·1011
18	1011·1011	1011·1011

We conjecture that the sequence of natural numbers $6, 12, 20, 26, 34, \ldots$ for which $\beta^{\mathrm{P}}(N) \neq \gamma^{\mathrm{P}}(N)$ is equal to the generalized Beatty sequence (with $\alpha = \sigma$) $V(2, 2, 0) = (2\lfloor n(\sigma + 1)\rfloor)$.

What are the intervals of constant expansion length for the two representations by powers of σ?
In the same way as in Section 4, we define the *Pell-Lucas intervals* and the *canonical Pell-Lucas intervals*:

$$\Lambda_0^{\mathrm{P}} := \{1, 2\}, \qquad \Lambda_{2n}^{\mathrm{P}} := [L_{2n}^{\mathrm{P}}, L_{2n+1}^{\mathrm{P}}] \text{ for } n \geq 1, \quad \Lambda_{2n+1}^{\mathrm{P}} := [L_{2n+1}^{\mathrm{P}} + 1, L_{2n+2}^{\mathrm{P}} - 1] \text{ for } n \geq 0$$

$$\Gamma_0^{\mathrm{P}} := \{1, 2\}, \qquad \Gamma_n^{\mathrm{P}} := [L_n^{\mathrm{P}} + 1, L_{n+1}^{\mathrm{P}}] \text{ for } n \geq 1$$

We conjecture that the Λ_n^{P} are the intervals of constant expansion length of the standard silver mean representation, and that the Γ_n^{P} are the intervals of constant expansion length of the canonical silver mean representation.

We next consider vertical runs. Here there are runs of 1's and runs of 2's. We conjecture that in the column of digit $i > 0$, there are only runs of 1's of length L_i^{P}, followed directly by runs of 2's of length L_{i-1}^{P}.

We also conjecture that for $i < 0$, there are either runs of 1's of length L_{-i}^{P} or runs of 1's of length L_{-i}^{P} directly followed by runs of 2's of length L_{-i}^{P}. In the odd columns, the order of the runs of 1's and runs of 2's is reversed. So, for example, for $i = -1$ between 0's, there are only blocks 11 and 2211. Note that these reversals are in line with the changes of direction of the base phi expansions observed at the end of Section 7.1.

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