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Carlet, Claude; Picek, Stjepan

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ON THE EXPONENTS OF APN POWER FUNCTIONS AND SIDON SETS, SUM-FREE SETS, AND DICKSON POLYNOMIALS

Claude Carlet*

Department of informatics, University of Bergen, Norway

Stjepan Picek

Delft University of Technology, The Netherlands

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Abstract. We derive necessary conditions related to the notions, in additive combinatorics, of Sidon sets and sum-free sets, on those exponents $d \in \mathbb{Z}/(2^n-1)\mathbb{Z}$, which are such that $F(x) = x^d$ is an APN function over $\mathbb{F}_{2^n}$ (which is an important cryptographic property). We study to what extent these new conditions may speed up the search for new APN exponents d. We provide results up to $n = 48$, denoting the number of possible APN exponents after each necessary condition for a function to be APN.

We also show a new connection between APN exponents and Dickson polynomials: $F(x) = x^d$ is APN if and only if the reciprocal polynomial of the Dickson polynomial of index d is an injective function from $\{y \in \mathbb{F}_{2^n}^*; tr_n(y) = 0\}$ to $\mathbb{F}_{2^n} \setminus \{1\}$. This also leads to a new and simple connection between Reversed Dickson polynomials and reciprocals of Dickson polynomials in characteristic 2 (which generalizes to every characteristic thanks to a small modification): the squared Reversed Dickson polynomial of some index and the reciprocal of the Dickson polynomial of the same index are equal.

1. **Introduction.** There is a significant number of works investigating APN *Almost Perfect Nonlinear* functions (see, e.g., [3]) and, in particular, APN power functions, i.e., functions of the form $F(x) = x^d$ where $d \in \mathbb{Z}/(2^n-1)\mathbb{Z}$. While we know a number of exponent values d that result in APN power functions (see Table 1), there is no substantial progress (e.g., new exponent values) for a number of years. One of the core reasons for this lack of new results is the computational complexity required to test large values d in $\mathbb{F}_{2^n}$. While new APN functions would not have immediate use in cryptography, finding new APN exponents or confirming there are no new APN exponents for a certain value n would significantly impact the APN research. Informally, we can divide the research on the new APN power functions into two directions.

1. **Reducing the number of possible APN exponents**. For a value n, there are d values that are possible exponents, where $d \in \mathbb{Z}/(2^n-1)\mathbb{Z}$. Thus, the goal is to efficiently recognize such values d that will not result in an APN function.

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TABLE 1. Known APN exponents on $\mathbb{F}_{2^n}$ up to equivalence and inversion.

Functions	Exponents d	Conditions
Gold	2^i+1	$\gcd(i,n)=1$
Kasami	$2^{2i}-2^i+1$	$\gcd(i,n)=1$
Welch	2^t+3	$n=2t+1$
Niho	$2^t+2^{\frac{t}{2}}-1$, t even $2^t+2^{\frac{3t+1}{2}}-1$, t odd	$n=2t+1$
Inverse	$2^{2t}-1$	$n=2t+1$
Dobbertin	$2^{4t}+2^{3t}+2^{2t}+2^t-1$	$n=5t$

2. **Speed-up the evaluation if a power function is APN**. This direction concentrates on checking if the differential uniformity for a power function equals 2.

In this paper, we concentrate on the first direction. More precisely, we study the so-called *APN exponents* in fields $\mathbb{F}_{2^n}$, that is, those values $d \in \mathbb{Z}/(2^n-1)\mathbb{Z}$ such that the corresponding *power function* $F(x)=x^d$ over $\mathbb{F}_{2^n}$ is (APN). A function from $\mathbb{F}_{2^n}$ to itself is called APN [11, 2, 10] if, for every nonzero $a \in \mathbb{F}_{2^n}$ and every $b \in \mathbb{F}_{2^n}$, the equation $F(x)+F(x+a)=b$ has at most two solutions. Equivalently, the system of equations $\begin{cases} x+y+z+t=0 \\ F(x)+F(y)+F(z)+F(t)=0 \end{cases}$ has for only solutions quadruples (x,y,z,t) whose elements are not all distinct (i.e., are pairwise equal). Recall that changing d into one of its conjugates 2^jd corresponds to changing $F(x)$ into a linearly equivalent APN function, which preserves APNness. The APN exponents then constitute a union of cyclotomic classes of 2 mod 2^n-1. The known APN exponents (Gold, Kasami, Welch, Niho, Inverse, and Dobbertin) are all those exponents which are the conjugates of those given in Table 1 below, or of their inverses when they are invertible in $\mathbb{Z}/(2^n-1)\mathbb{Z}$. Note that i (in the definitions of Gold and Kasami exponents) can always be taken lower than $n/2$ (thanks to conjugacy).

It has been proved by Dobbertin (as described in the survey chapter [3], to which we refer for more information on APN functions) that an exponent can be APN only if $gcd(d,2^n-1)$ equals 1 if n is odd and 3 if n is even. We shall show in Section 2 that for all exponents given in Table 1, we have $gcd(d-1,2^n-1)=1$. This corresponds to the fact that the related functions F have 0 and 1 as only fixed points, since $x \in \mathbb{F}_{2^n}$ is a nonzero fixed point of function $F(x)=x^d$ if and only if $x^{d-1}=1$.

It happens for some cyclotomic classes that the property $gcd(d-1,2^n-1)=1$ be true for any element in the cyclotomic class, or equivalently that $gcd(d-2^j,2^n-1)=1$ for every $j=0,\ldots,n-1$. We list in Table 2, for the (known) APN exponents of Table 1 up to $n=32$, when $gcd(d-2^j,2^n-1)=1$ is true for every $j=0,\ldots,n-1$. The proportion of such exponents is large. Since such property is unlikely for random exponents satisfying Dobbertin's observation recalled above, we can hope that some other property can be found, which would explain such large proportion, and could maybe ease the search for APN exponents outside the main classes. This other property cannot be that $gcd(d-1,2^n-1)=1$ for all APN exponents d, which

TABLE 2. $gcd(d-2^j, 2^n-1)=1$ for every $j=0,\ldots,n-1$.

Class name	Value
	$(n\|i)$; $i \leq n/2$
Gold	$(3\|1),(5\|1,2),(6\|1),(7\|1,2,3),(9\|1,2,4),(11\|2,4,5)$ $(13\|1,2,3,4,5,6),(14\|1,3,5),(15\|1,2,4,7),(17\|1,2,3,4,5,6,7,8),$ $(19\|1,2,3,4,5,6,7,8,9),(21\|1,2,4,5,8,10),(22\|5,7,9),$ $(23\|2,5,7,8,9,10),(25\|1,2,3,4,6,7,8,9,11,12),(26\|1,3,5,7,9,11)$ $(27\|1,2,4,5,7,8,10,11,13),(29\|1,2,3,4,5,6,7,8,9,10,11,12,14)$ $(31\|1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$
Kasami	$(3\|1),(5\|1,2),(6\|1),(7\|1,2,3),(9\|1,2,4),(11\|3,4),(13\|1,2,3,4,5,6)$ $(14\|1,3),(15\|1,2,4,7),(17\|1,2,3,4,5,6,7,8),(19\|1,2,3,4,5,6,7,8,9),$ $(21\|1,4,5,8,10),(22\|3,7),(23\|2,3,6,8,9,11),(25\|1,2,3,4,6,7,8,9,11,12),$ $(26\|1,3,5,7,9,11),(27\|1,2,4,5,7,8,10,11,13),$ $(29\|1,2,3,5,6,7,8,9,10,11,12,13,14),$ $(31\|1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$
	n
Welch	$3,5,7,9,13,15,17,19,23,25,27,31$
Niho	$3,5,7,9,11,13,15,17,19,21,23,25,27,29,31$
Dobbertin	$5,15,25$
Inverse	$3,5,7,9,11,13,15,17,19,21,23,25,27,29,31$

would imply $gcd(d-2^j,2^n-1)=1$ for all j, since we see in Table 2 that some cyclotomic classes do not satisfy this.

In this paper, we find a new property relating APN exponents to Sidon sets and sum-free sets (two well-known notions in additive combinatorics [1, 6, 12]; see the definitions in Section 3): for every APN exponent d and every integer j, the multiplicative subgroup of $\mathbb{F}_{2^n}$ of order $gcd(d-2^j,2^n-1)$ is a Sidon set and a sum-free set. Note that the relationship between APN functions and Sidon sets is not new: by definition, an (n,n)-function is APN if and only if its graph is a Sidon set (see Section 3). The relationship we establish in this paper is different and gives more insight into APN exponents.

We study the consequences of searching for new APN exponents, which is a sensitive open question on which the research is being stuck for almost 20 years. We do not find new APN exponents, but we show that d is an APN exponent if and only if the function equal to the reciprocal of the Dickson polynomial $D_d(X,1)$ is injective from $\{y\in\mathbb{F}_{2^n}^*; tr_n(y)=0\}$ to $\mathbb{F}_{2^n}\backslash\{1\}$, where $tr_n(x)=x+x^2+\cdots+x^{2^{n-1}}$ is the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2$. Finally, we show a very simple new relationship (which generalizes to every characteristic after a small modification) between Reversed Dickson polynomials and the reciprocals of Dickson polynomials: for every positive integer d, the Reversed Dickson polynomial $D_{2d}(1,X)$ of index $2d$ and the reciprocal of the Dickson polynomial $D_d(X,1)$ of index d are equal.

2. **On the Exponents of Table 1.** The value $gcd(d-1,2^n-1)$ for a power function $F(x)=x^d$ is an important parameter. The number of fixed points of F equals $2^{gcd(d-1,2^n-1)}$.

Lemma 2.1. *All the exponents d in Table 1 satisfy $gcd(d-1,2^n-1)=1$.*

Proof. In the case of Gold functions $F(x) = x^{2^i+1}$, where $gcd(i,n) = 1$, we have $gcd(d-1, 2^n-1) = gcd(2^i, 2^n-1) = 1$.
In the case of Kasami functions $F(x) = x^{2^{2i}-2^i+1}$, where $gcd(i,n) = 1$, we have $gcd(d-1, 2^n-1) = gcd(2^i-1, 2^n-1) = 2^{gcd(i,n)} - 1 = 1$.
In the case of Welch function $F(x) = x^{2^t+3}$, we have according to the Gauss theorem (which states that if a divides bc and is co-prime with b then it divides c): $gcd(d-1, 2^n-1) = gcd(2^{t-1}+1, 2^{2t+1}-1) = \frac{gcd(2^{2t-2}-1, 2^{2t+1}-1)}{gcd(2^{t-1}-1, 2^{2t+1}-1)} = \frac{2^{gcd(2t-2,2t+1)}-1}{2^{gcd(t-1,2t+1)}-1} = \frac{2^{gcd(t-1,2t+1)}-1}{2^{gcd(t-1,2t+1)}-1} = 1$.
In the case of Niho functions:
- $F(x) = x^{2^t+2^{\frac{t}{2}}-1}$, t even, we have, applying the Euclidean algorithm: $gcd(d-1, 2^n-1) = gcd(2^t+2^{\frac{t}{2}}-2, 2^{2t+1}-1) = gcd(2^t+2^{\frac{t}{2}}-2, -5\cdot 2^{t/2+1}+11) = gcd(2^2\cdot 5^2\cdot(2^t+2^{\frac{t}{2}}-2), 5\cdot 2^{t/2+1}-11) = gcd(5\cdot 2^{t/2+1}-11, 31) = 1$, since 31 divides $2^{2t+1}-1$ if and only if $2t+1 \equiv 0 \pmod 5$ and the only possibility for that is $t \equiv 2 \pmod 5$, $\frac{t}{2} \equiv 1 \pmod 5$ and $2^t + 2^{t/2} - 2 \equiv 4 \not\equiv 0 \pmod{31}$;
- $F(x) = x^{2^t+2^{\frac{3t+1}{2}}-1}$, t odd, we have $gcd(d-1, 2^n-1) = gcd(2^{\frac{3t+1}{2}}+2^t-2, 2^{2t+1}-1) = gcd(2^{\frac{3t+1}{2}}+2^t-2, 2^t+2^{\frac{t+3}{2}}-3) = gcd(2^t+2^{\frac{t+3}{2}}-3, 9\cdot 2^{\frac{t+1}{2}}-11) = gcd(2\cdot 9^2\cdot(2^t+2^{\frac{t+3}{2}}-3), 9\cdot 2^{\frac{t+1}{2}}-11) = gcd(9\cdot 2^{\frac{t+1}{2}}-11, 31) = 1$, since, again, 31 divides $2^{2t+1}-1$ if and only if $2t+1 \equiv 0 \pmod 5$ and the only possibility for that is $t \equiv 2 \pmod 5$, $\frac{3t+1}{2} \equiv 1 \pmod 5$ and $2^t + 2^{\frac{3t+1}{2}} - 2 \equiv 4 \not\equiv 0 \pmod{31}$.
In the case of the APN Inverse function $F(x) = x^{2^{2t}-1}$, we have, by the Euclidean algorithm: $gcd(d-1, 2^n-1) = gcd(2^{2t-1}-1, 2^{2t+1}-1) = 2^{gcd(2t-1,2t+1)} - 1 = 1$.
In the case of Dobbertin APN function $F(x) = x^d$, where $d = 2^{4t}+2^{3t}+2^{2t}+2^t-1$ and $n = 5t$, we could calculate $gcd(d-1, 2^n-1)$ by applying again the Euclidean algorithm but more simply we have $gcd(d-1, 2^n-1) = gcd(d-1, (2^t-1)(d+2))$, and since $d \equiv 3 \pmod{(2^t-1)}$, and $d-1$ is then co-prime with 2^t-1, we obtain then $gcd(d-1, 2^n-1) = gcd(d-1, d+2) = gcd(d-1, 3)$, which equals 1 if n is odd (because we know that 3 does not divide 2^n-1 in this case) and which equals $gcd(2,3) = 1$ if n is even (since, t being then even, we have $2^{4t}, 2^{3t}, 2^{2t}, 2^t$ all congruent with 1 mod 3 and then $d-1 \equiv 2 \pmod 3$).
Then $gcd(d-1, 2^n-1) = 1$ in all cases. □

Hence, all the corresponding APN functions have 0 and 1 as only fixed points.

Remark 1. If d is invertible mod 2^n-1 and d' is its inverse, then $gcd(d-1, 2^n-1)$ equals 1 if and only if $gcd(d'-1, 2^n-1)$ equals 1, since a permutation has the same number of fixed points as its compositional inverse.

3. **Sidon Sets and Sum-free Sets.** We saw in Section 2 that the known APN exponents might have a property not covered by the Dobbertin observation (recalled in the introduction). We also saw in introduction that such property (to be found) cannot be that $gcd(d-1, 2^n-1) = 1$, since this would imply $gcd(d-2^j, 2^n-1) = 1$ for every $j \in \mathbb{Z}/n\mathbb{Z}$, which is already not true (for some n) for the simplest known APN exponent 3. In this section, we show that every APN exponent (known or unknown) satisfies a property that deals with the numbers $gcd(d-2^j, 2^n-1)$, $j \in \mathbb{Z}/n\mathbb{Z}$, in a more subtle way. We first need to recall two definitions from additive combinatorics.
Definition 3.1. [1] A subset of an additive group $(G,+)$ is called a *Sidon set* if it does not contain elements x, y, z, t, at least three of which are distinct, and such that $x + y = z + t$.

This notion is due to S. Sidon [1]. It is preserved by (additive) equivalence, that is, if S is a Sidon set in $(G,+)$ and A is a permutation of G such that $A(x+y)=A(x)+A(y)$, then $A(S)$ is a Sidon set. The notion is also preserved by translation. Of course, any set included in a Sidon set is a Sidon set.
This definition is also relevant in characteristic 2. In such characteristic, we have more simply: *A subset of an additive group of characteristic 2 is a Sidon set if it does not contain four distinct elements x,y,z,t such that $x+y+z+t=0$.* Indeed, if two elements are equal, then there cannot be three distinct elements among x,y,z,t such that $x+y+z+t=0$.

Remark 2. By definition, an (n,n)-function F is APN if and only if its graph $\mathcal{G}_F=\{(x,F(x));x\in\mathbb{F}_{2^n}\}$ is a Sidon set in $(\mathbb{F}_{2^n}^2,+)$. Hence, APN functions correspond to a subclass of Sidon sets in $(\mathbb{F}_{2^n}^2,+)$: those S such that, for every $x\in\mathbb{F}_{2^n}$, there exists a unique $y\in\mathbb{F}_{2^n}$ such that $(x,y)\in S$.

Remark 3. A subset S of an additive group $(G,+)$ is a Sidon set if and only if, denoting by P_S the set of pairs in S, the mapping $\{x,y\}\in P_S\mapsto x+y$ is one-to-one. The size $|S|$ is then (see e.g., [1]) such that $\binom{|S|}{2}=\frac{|S|\,(|S|-1)}{2}\leq|G|-1$, since otherwise the number of pairs $\{x,y\}$ included in S would be strictly larger than the number of nonzero elements of G; at least two different pairs $\{x,y\}$ and $\{x',y'\}$ would then have the same sum and these two pairs would in fact be disjoint (if, for instance $x=x'$, then $y\neq y'$ and $x+y\neq x'+y'$, a contradiction).
Definition 3.2. [6, 12] A subset S of an additive group $(G,+)$ is called a *sum-free set* if it does not contain elements x,y,z such that $x+y=z$ (i.e., if $S\cap(S+S)=\emptyset$).

This notion is due to P. Erdös.

Remark 4. A subset S of an additive group $(G,+)$ is sum-free if and only if, denoting again by P_S the set of pairs in S, the mapping $\{x,y\}\in P_S\mapsto x+y$ is valued outside S. The size $|S|$ is then (see, e.g., [6, 12]) smaller than or equal to $\frac{|G|}{2}$ because the size of $S+S$ is at least the size of S (since G is a group), and if $|S|>\frac{|G|}{2}$ then the two sets $S+S$ and S have sizes whose sum is strictly larger than the order of the group, and they necessarily have a non-empty intersection. A basic example of a sum-free set in $\mathbb{F}_{2^n}$, which achieves this bound $|S|\leq\frac{|G|}{2}$ with equality, is any affine hyperplane (i.e., the complement of any linear hyperplane).

Remark 5. The size $|S|$ of a sum-free Sidon set satisfies $\frac{|S|\,(|S|+1)}{2}\leq|G|-1$, since otherwise, the number of pairs $\{x,y\}\in P_S$ would be strictly larger than the number of nonzero elements of $G\setminus S$. Note that, in characteristic 2, if S is a Sidon-sum-free set, then $S\cup\{0\}$ is a Sidon set, which gives again the same bound by using Remark 3.

4. APN Exponents, Sidon Sets, and Sum-free Sets.

We now give the new property valid for all APN exponents related to Sidon sets and sum-free sets.

Theorem 4.1. *For every positive integers n and d and for every $j\in\mathbb{Z}/n\mathbb{Z}$, let $e_j=gcd(d-2^j,2^n-1)\in\mathbb{Z}/(2^n-1)\mathbb{Z}$, and let G_{e_j} be the multiplicative subgroup $\{x\in\mathbb{F}_{2^n}^*;x^{d-2^j}=1\}=\{x\in\mathbb{F}_{2^n}^*;x^{e_j}=1\}$ of order e_j. If function $F(x)=x^d$ is APN over $\mathbb{F}_{2^n}$, then, for every $j\in\mathbb{Z}/n\mathbb{Z}$, G_{e_j} is a Sidon set in the additive group $(\mathbb{F}_{2^n},+)$ and is also a sum-free set in this same group. Moreover, for every $k\neq j$, if $x\in G_{e_k}$, $y\in G_{e_j}$, $x\neq y$ and $x\neq y^{-1}$, then we have $(x+1)^{d-2^k}\neq(y+1)^{d-2^j}$.*

[1] His last name is often also spelled as Szidon.

Proof. Using the same idea as the one used by Dobbertin for showing the observation recalled in the introduction, for every $x \in G_{e_j} \setminus \{1\}$, we introduce the unique $s \in \mathbb{F}_{2^n}^* \setminus \{1\}$ such that $x = \frac{s}{s+1}$, that is, $s = \frac{x}{x+1}$. Then $x^{d-2^j} = 1$ implies $s^{d-2^j} + (s+1)^{d-2^j} = 0$, which implies after multiplication by $s^{2^j} + 1 = (s+1)^{2^j}$ that $s^d + (s+1)^d = s^{d-2^j} = (s+1)^{d-2^j} = \frac{1}{(x+1)^{d-2^j}}$. Note that if $s = \frac{x}{x+1}$ and $s' = \frac{x'}{x'+1}$, with $x \neq 1$ and $x' \neq 1$, then we have $s = s'$ if and only if $x = x'$ (since function $\frac{x}{x+1}$ is bijective, being involutive) and we have $s = s' + 1$ if and only if $x' = x^{-1}$, since $\frac{x}{x+1} + 1 = \frac{x^{-1}}{x^{-1}+1}$.
Suppose that G_{e_j} is not a Sidon set, then let x, y, z, t be distinct elements of G_{e_j} such that $x+y = z+t$. Making the changes of variables $x \to xt, y \to yt, z \to zt$ and dividing the equality by t, we obtain distinct elements x, y, z of $G_{e_j} \setminus \{1\}$ such that $x+y+z = 1$. Making now the change of variable $y \to zy$, we obtain elements x, y, z in $G_{e_j} \setminus \{1\}$ such that $x + 1 = z(y+1)$, $x \neq y$ and $x \neq y^{-1}$ (indeed, the condition $y = 1$ in the new setting corresponds to the condition $y = z$ in the former setting, the condition $x = y$ in the new setting is equivalent (thanks to $x + 1 = z(y+1)$) to $z = 1$ in both settings, and the condition $x = y^{-1}$ in the new setting, that is (thanks to $x + 1 = z(y+1)$ again), $zy = 1$, is equivalent to $y = 1$ in the former setting). We have then $\frac{1}{(x+1)^{d-2^j}} = \frac{1}{(y+1)^{d-2^j}}$ and since $x \neq y$ and $x \neq y^{-1}$, we have $\frac{x}{x+1} \neq \frac{y}{y+1}$ and $\frac{x}{x+1} \neq \frac{y}{y+1} + 1$ and this gives 4 distinct solutions to the equation $s^d + (s+1)^d = \frac{1}{(x+1)^{d-2^j}}$, a contradiction with the APNness of F.
Suppose that G_{e_j} is not sum-free, that is, $G_{e_j} \cap (G_{e_j} + G_{e_j}) \neq \emptyset$, that is without loss of generality since G_{e_j} is a multiplicative group, $G_{e_j} \cap (G_{e_j} + 1) \neq \emptyset$, then let $x \in G_{e_j} \cap (G_{e_j} + 1)$ (which implies $x \neq 0, 1$) and $s = \frac{x}{x+1}$ (with $s \neq 0, 1$ as well), we have then $\frac{1}{(x+1)^{d-2^j}} = 1$ and $s^d + (s+1)^d = 1$ and the equation $z^d + (z+1)^d = 1$ has four solutions $0, 1, s$, and $s+1$ in $\mathbb{F}_{2^n}$, a contradiction.
The last assertion is a direct consequence of the observations made in the first paragraph of the present proof. □

Remark 6. Since for $s = \frac{x}{x+1}$, $x \neq 1$, we have $s^d + (s+1)^d = \frac{x^d+1}{(x+1)^d}$ and since $\frac{x^d+1}{(x+1)^d} = \frac{(x^{-1})^d+1}{(x^{-1}+1)^d}$, the condition "$G_{e_j}$ is sum-free" is in fact a weaker version of the condition "the equation $x^d + 1 = (x+1)^d$ has at most one solution in $\mathbb{F}_{2^n}$, up to the replacement of x by x^{-1}" which is implied by the condition "the equation $x^d + (x+1)^d = 1$ has at most two solutions in $\mathbb{F}_{2^n}$". We shall say more in Subsection 4.1. Note that every element of G_{e_j} satisfies $x^d + 1 = (x+1)^d$ since this equation in G_{e_j} is equivalent to $x^{2^j} + 1 = (x+1)^{2^j}$ which is always true, and this is why G_{e_j} plays an interesting role.

Remark 7. Denoting $e = gcd(d, 2^n - 1)$, we have that G_e itself is a Sidon set since, as recalled above, we have $e = 1$ if n is odd and $e = 3$ if n is even, and $G_1 = \{1\}$, $G_3 = \mathbb{F}_4^*$ are Sidon sets (since they do not contain 4 distinct elements). But G_e is a sum-free set only for n odd, since $\mathbb{F}_4^*$ is not sum-free.

Remark 8. An APN function is APN in any subfield where the function makes sense (i.e., such that $F(x)$ belongs to this subfield when x does). In particular, an APN power function is APN in any subfield. Applying Theorem 4.1 with a divisor r of n in the place of n replaces e_j by $gcd(d - 2^j, 2^r - 1)$ and G_{e_j} by $G_{e_j} \cap \mathbb{F}_{2^r}^*$, so

it gives no additional information since if G_{e_j} is a Sidon-sum-free set in $\mathbb{F}_{2^n}$, then $G_{e_j} \cap \mathbb{F}_{2^r}^*$ is also a Sidon-sum-free set in $\mathbb{F}_{2^r}$.

Remark 9. The condition that G_{e_j} is sum-free for every $j \in \mathbb{Z}/n\mathbb{Z}$ implies that, for every divisor k of n larger than 1, the integer e_j is not divisible by 2^k-1, because otherwise G_{e_j} would contain $\mathbb{F}_{2^k}^*$, and this is contradictory with the condition. For $k>2$, the fact that e_j is not divisible by 2^k-1 is also a consequence of the fact that G_{e_j} is a Sidon set, since it is straightforward that for $k>2$, $\mathbb{F}_{2^k}^*$ is not a Sidon set and any superset is then not one either. In fact, the property of being a Sidon-sum-free set is rather restrictive, and this explains the observations made in the introduction.

Remark 10. We observed that, in characteristic 2, the size $|S|$ of a Sidon-sum-free set S not containing 0 cannot be such that $\binom{|S|+1}{2} = \frac{|S|\,(|S|+1)}{2} > 2^n-1$. We deduce then from the theorem that, if d is an APN exponent, then for every divisor λ of 2^n-1 such that $\binom{\lambda+1}{2} > 2^n-1$ and every $j \in \mathbb{Z}/n\mathbb{Z}$, this number λ does not divide $d-2^j$. Take for instance $n=8$ and $\lambda = \frac{2^8-1}{3} = 85$, we have $\binom{\lambda+1}{2} > 255$ and for every APN exponent d, we have that 85 does not divide $d-1$, $d-2$, $d-4$, $d-8$, $d-16$, $d-32$, $d-64$ nor $d-128$ (all these numbers being taken modulo 255). We can also take $\lambda = \frac{2^8-1}{5} = 51$, we have $\binom{\lambda+1}{2} > 255$ and 51 does not divide $d-1$, $d-2$, $d-4$, $d-8$, $d-16$, $d-32$, $d-64$ nor $d-128$ as well. For this value of n, there are only two possible values for λ, but for some larger values of n, the number of possible λ may be much larger and the condition discriminates then better the candidates d.

4.1. **A general Framework for Deriving Results Similar to Theorem 4.1.** In the proof of Theorem 4.1, we have used that, if $x \in G_{e_j} \setminus \{1\}$ and $s = \frac{x}{x+1}$, then $s^d + (s+1)^d = \frac{1}{(x+1)^{d-2^j}}$. In fact, when relaxing the condition $x \in G_{e_j} \setminus \{1\}$, we still have an interesting identity, which leads to a new characterization of APN exponents:

Proposition 1. *Let n be any positive integer and $F(x)=x^d$ be any power function over $\mathbb{F}_{2^n}$. If $x \neq 1$ and $s = \frac{x}{x+1}$ then $s^d + (s+1)^d = \frac{x^d+1}{(x+1)^d}$, and F is APN if and only if the function $x \mapsto \frac{x^d+1}{(x+1)^d}$ is 2-to-1 from $\mathbb{F}_{2^n} \setminus \mathbb{F}_2$ to $\mathbb{F}_{2^n} \setminus \{1\}$.*

Proof. The first identity is straightforward. Hence, function $x \mapsto \frac{x^d+1}{(x+1)^d}$ is 2-to-1 from $\mathbb{F}_{2^n} \setminus \mathbb{F}_2$ to $\mathbb{F}_{2^n} \setminus \{1\}$ if and only if any equation $s^d+(s+1)^d = b \neq 1$ has at most 2 solutions s in $\mathbb{F}_{2^n}$ (indeed, it has no solution in $\mathbb{F}_2$) and equation $s^d+(s+1)^d = 1$ has only 2 solutions s in $\mathbb{F}_{2^n}$ (which are 0 and 1), that is, F is APN. □

Note that function $x \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2 \mapsto \frac{x^d+1}{(x+1)^d}$ is invariant under the transformation $x \mapsto x^{-1}$. Note also that instead of $s = \frac{x}{x+1}$, we could take $s = \frac{x}{x+1} + 1 = \frac{1}{x+1}$.

Theorem 4.1 can then be revisited as follows: we use the facts that if a function is 2-to-1 over some set, then it is at most 2-to-1 over any subset, and that the expression of $\frac{x^d+1}{(x+1)^d}$ is simplified when $x \in G_{e_j}$, because $x^{d-2^j} = 1$ implies $\frac{x^d+1}{(x+1)^d} = \frac{x^{2^j}+1}{(x+1)^d} = \frac{(x+1)^{2^j}}{(x+1)^d} = \frac{1}{(x+1)^{d-2^j}}$. The nice thing here is that we obtain an expression with the same exponent $d-2^j$ as in the definition of G_{e_j} and this is what leads to the Sidon-sum-free property.

4.2. **On the Relationship Between APN Exponents and Dickson Polynomials.** Recall that, for every positive integer d, functions $x^d+(x+1)^d$ and x^2+x being invariant by the translation $x\mapsto x+1$ and the latter one being 2-to-1, $x^d+(x+1)^d$ equals $\phi_d(x^2+x)$ for some polynomial ϕ_d and $F(x)=x^d$ is APN if and only if function ϕ_d is injective over the hyperplane $H=\{x^2+x; x\in\mathbb{F}_{2^n}\}=\{y\in\mathbb{F}_{2^n}; tr_n(y)=0\}$, where $tr_n(x)=x+x^2+\cdots+x^{2^{n-1}}$ is the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2$. This polynomial ϕ_d is called the *Reversed Dickson polynomial* [8] and equals $D_d(1,X)$ (see, e.g., [8]), where D_d is classically defined by $D_d(X+Y,XY)=X^d+Y^d$.
Similarly, functions $\frac{x^d+1}{(x+1)^d}$ and $x+x^{-1}$ over $\mathbb{F}_{2^n}\setminus\mathbb{F}_2$ being invariant under the transformation $x\mapsto x^{-1}$ and the latter one being 2-to-1, $\frac{x^d+1}{(x+1)^d}$ equals $\psi_d(x+x^{-1})$ for some function ψ_d, which is here characterized by $(\psi_d(y))^2=\frac{D_d(y,1)}{y^d}$, since $\left(\frac{x^d+1}{(x+1)^d}\right)^2=\frac{x^d+x^{-d}}{(x+x^{-1})^d}$. According to Proposition 1, function F is then APN if and only if ψ_d is injective over $\{x+x^{-1}; x\in\mathbb{F}_{2^n}\setminus\mathbb{F}_2\}$, that is, over $\{y\in\mathbb{F}_{2^n}^*; tr_n(y^{-1})=0\}$ and does not take value 1. Note that $\frac{D_d(y^{-1},1)}{(y^{-1})^d}=y^dD_d(y^{-1},1)$ equals the value at y of the reciprocal polynomial of $D_d(X,1)$. Hence:

Proposition 2. *For every positive integers n and d, function $F(x)=x^d$ is APN if and only if the reciprocal polynomial $\widetilde{D_d(X,1)}=X^dD_d(X^{-1},1)$ of the Dickson polynomial $D_d(X,1)$ is injective and does not take value 1 over $H^*=\{y\in\mathbb{F}_{2^n}^*; tr_n(y)=0\}$.*

We have seen that, for $x\in\mathbb{F}_{2^n}\setminus\mathbb{F}_2$, if $s=\frac{x}{x+1}$, that is, $x=\frac{s}{s+1}$ or $s=\frac{1}{x+1}$, that is, $x=\frac{s+1}{s}$, we have $\frac{x^d+1}{(x+1)^d}=s^d+(s+1)^d$. We have then $x+x^{-1}=\frac{s+1}{s}+\frac{s}{s+1}=\frac{1}{s^2+s}$ and therefore $\frac{x^d+1}{(x+1)^d}=\psi_d(x+x^{-1})=\psi_d\left(\frac{1}{s^2+s}\right)=s^d+(s+1)^d=\phi_d(s^2+s)$. Hence, for every $z\in H^*$, $\phi_d(z)=\psi_d(z^{-1})$ and squaring gives $(\phi_d(z))^2=\widetilde{D_d}(z,1)$. In other words, the squared Reversed Dickson polynomial and the reciprocal of Dickson polynomial of a same index take the same value over H and then, given their common degree, are equal to each other (this can also be easily seen as a consequence of the classical recurrence relations satisfied by these two polynomials [8]). We have then:

Proposition 3. *For every positive integer d, the squared Reversed Dickson polynomial of index d (equal to the Reversed Dickson polynomial of index $2d$) and the reciprocal of Dickson polynomial of index d are equal* [2]*. For every $z\neq 0$ such that $tr_1^n(z)=0$, we have then $(\phi_d(z))^2=\widetilde{D_d}(z,1)$, where $\widetilde{D_d}$ is the reciprocal polynomial of the Dickson polynomial D_d of degree d. In particular, we have:*

$$x^d+(x+1)^d=\left(\widetilde{D_d}(x^2+x,1)\right)^{2^{n-1}}.$$

This property allows to deduce the expression of Dickson polynomials with so-called Gold indices: for every integer i, we have $D_{2^i+1}(X,1)=X^{2^i+1}+\sum_{j=1}^{i}X^{2^i+1-2^j}$. Indeed, $x^{2^i+1}+(x+1)^{2^i+1}=x^{2^i}+x+1=1+\sum_{j=0}^{i-1}(x^2+x)^{2^j}$ and therefore $\widetilde{D_{2^i+1}}(X^2+X,1)=1+\sum_{j=1}^{i}(x^2+x)^{2^j}$, $\widetilde{D_{2^i+1}}(X,1)=1+\sum_{j=1}^{i}X^{2^j}$. The values

[2] Xiang-dong Hou [7], informed of this property by the authors, has observed that it can be generalized to any characteristic: $X^dD_d(\frac{1}{X}-2,1)=D_{2d}(1,X)$.

of $D_{2^i+1}(X,1)$ and $D_{2^i-1}(X,1)$ (which are related by $D_{2^i-1}(X,1)+D_{2^i+1}(X,1)=X^{2^i+1}$) are already known from [5], but Proposition 3 also allows to obtain the explicit expressions of other Dickson polynomials; for instance with so-called Kasami indices:

Corollary 1. *For every integer i we have:*

$$D_{4^i-2^i+1}(X,1)=X^{4^i-2^i+1}+X^{4^i+2^i+1}\left(\sum_{j=1}^{i}X^{-2^j}\right)^{2^i+1}.$$

Proof. For every $x\in\mathbb{F}_{2^n}\setminus\mathbb{F}_2$, we have (as already observed and used by Dobbertin):

$$\begin{aligned}
x^{4^i-2^i+1}+(x+1)^{4^i-2^i+1} &= \frac{x^{4^i+1}(x+1)^{2^i}+(x+1)^{4^i+1}x^{2^i}}{(x^2+x)^{2^i}}\\
&= \frac{x^{4^i+1}+x^{4^i+2^i}+x^{2^i+1}+x^{2^i}}{(x^2+x)^{2^i}}\\
&= 1+\frac{(x^{2^i}+x)^{2^i+1}}{(x^2+x)^{2^i}}\\
&= 1+\frac{\left(\sum_{j=0}^{i-1}(x^2+x)^{2^j}\right)^{2^i+1}}{(x^2+x)^{2^i}},
\end{aligned}$$

and therefore, after squaring and denoting $X=x^2+x$, we obtain:

$$\widetilde{D_{4^i-2^i+1}}(X,1)=1+\frac{\left(\sum_{j=1}^{i}X^{2^j}\right)^{2^i+1}}{X^{2^{i+1}}},$$

and then:

$$D_{4^i-2^i+1}(X,1)=X^{4^i-2^i+1}+X^{4^i+2^i+1}\left(\sum_{j=1}^{i}X^{-2^j}\right)^{2^i+1}.$$

This completes the proof. □

Of course we can deduce $D_{4^i+2^i+1}(X,1)$ thanks to the relation $D_{4^i-2^i+1}(X,1)+D_{4^i+2^i+1}(X,1)=D_{2^i}(X,1)D_{4^i+1}(X,1)=X^{2^i}D_{4^i+1}(X,1)$.

The same method applies more generally to $D_{2^j-2^i+1}$ but without the nice factorization above.

Remark 11. The Müller-Cohen-Matthews (MCM) polynomial (see [5]) equals $\sum_{i=0}^{k-1}X^{(2^k+1)2^i-2^k}$ and is a permutation polynomial when $gcd(k,n)=1$ and k is odd. Note that it equals $\frac{\phi(X^{2^k+1})}{X^{2^k}}$, where $\phi(X)=\sum_{i=0}^{k-1}X^{2^i}=1+\left(\widetilde{D_{2^k+1}}(X,1)\right)^{2^{n-1}}$.

5. **Experimental Results.**

5.1. **Sidon and Sum-free Conditions.** Hans Dobbertin and Anne Canteaut have checked by computer investigation that no unclassified APN exponent exists for $n \leq 26$. By unclassified APN exponent, we mean an APN exponent not equal to a Gold, Kasami, Dobbertin, Welch, Niho, or Inverse APN exponent, with n odd in the three latter cases, nor to its inverse mod $2^n - 1$ when it is co-prime with $2^n - 1$ (that is, when n is odd), nor to these exponents multiplied by powers of 2 and reduced modulo $2^n - 1$.

Yves Edel checked the same for $n \leq 34$ and $n = 36, 38, 40, 42$. The main idea for his computer investigation was to:

1. consider all the elements in $\mathbb{Z}/(2^n - 1)\mathbb{Z}$, discard (because of Dobbertin's observation recalled in the introduction) all those which are not co-prime with $2^n - 1$ for n odd and do not have gcd equal to 3 with $2^n - 1$ for n even, and
2. discard (because the restriction to a subfield of an APN power function is an APN power function) all the remaining exponents whose reduction mod $2^r - 1$ is not an APN exponent in $\mathbb{F}_{2^r}$ for some divisor r of n.

Since the checking that no unclassified APN exponent exists had been already done previously for r, the condition "is not an APN exponent in $\mathbb{F}_{2^r}$" could be replaced by "is not a known APN exponent in $\mathbb{F}_{2^r}$". Then, after discarding all known APN exponents in $\mathbb{F}_{2^n}$, the remaining exponents were investigated as possibly new APN exponents; they were gathered in cyclotomic classes, and the APNness of one member of each class was investigated. No unclassified APN exponent could be found. Note that in the rest of the paper, when discussing the subfield condition, we mean the condition as implemented by Yves Edel in his investigation.

In this section, we concentrate on utilizing the same methods as well as our newly developed Sidon and sum-free conditions to derive the number of possible new APN exponents to test and see if the Sidon and sum-free conditions contribute to reducing this number. We use the acronym S for Sidon condition, SF for sum-free condition, and SSF for Sidon-sum-free condition. We shall call "S values" (respectively, SF, SSF values) those divisors e of $2^n - 1$ such that $G_e = \{x \in \mathbb{F}_{2^n}^*; x^e = 1\}$ satisfies S (respectively, SF, SSF).

Next, we propose two techniques to calculate S and SF values and one additional technique to calculate SF values. The first technique for S/SF has high computational complexity but low memory complexity, while the second one for S/SF has low computational complexity but high memory complexity. A trade-off can be considered concerning the available resources. In both techniques, we use a result from [4]: for every divisor e of $2^n - 1$, G_e is a Sidon (respectively, a sum-free) set if and only if, for every $u \in \mathbb{F}_{2^n}^*$ (respectively, for $u = 1$), the polynomial $(X + 1)^e + u$ has at most two zeros in G_e (respectively, has no zero in G_e).

In the first technique, to determine whether a value e is Sidon (respectively, sum-free), we visit all the elements u of $\mathbb{F}_{2^n}^*$ and for each of them we visit all x of G_e (that is, all those powers of a primitive element whose exponents are multiples of $\frac{2^n-1}{e}$) and we:

1. Calculate $(x + 1)^e + u$.
2. Increment a counter for value u when $(x + 1)^e + u = 0$.
3. Keep e as Sidon (S) if, for no value of u, the counter reached more than 2 and as sum-free (SF) if, for $u = 1$, the counter never reached more than 0.

This gives computational complexity equal to $2^n e$. From the memory perspective, at any time, we are required only to keep two counters (one for S and one for SF).

For the second technique, we visit all the elements x of G_e (that is, again, all those powers of a primitive element whose exponents are multiples of $\frac{2^n-1}{e}$) and for each, we:

1. Calculate $(x+1)^e$.
2. Increment a counter in a table for value $(x+1)^e$.
3. Keep e as Sidon (S) if we never reached more than 2 in the table and as sum-free (SF) if, for value 1, we never reached more than 0.

This technique gives the computational complexity of e and the memory complexity of 2^n. Since we require 2 bits to store the value 2 in memory, in total, we need up to 2^{n+1} bits.

Finally, there is a technique (Proposition 5.1 in [4]) to calculate SF that is efficient from both computational and memory perspectives. As such, we consider this technique to be preferred for the SF values, but unfortunately, it cannot be used to obtain the Sidon values.

1. Calculate $gcd(x^e+1,(x+1)^e+1)$.
2. If the remainder is 1, then the value is SF.

This technique is efficient as we are required to calculate in $\mathbb{F}_2$ only, and we need only a single bit to store the result.

We show the results for $n \in [3,31]$ in Tables 3 and 4. Observe that sum-free condition is somewhat more discriminating and enables us to reduce more values e than the Sidon condition.

Calculating the Sidon condition and, to some small extent, the sum-free condition as we proposed is efficient only for relatively small values of n or of e or if a value e is not SSF (since then, we stop the search relatively fast). Indeed, in the cases when a large value e is SSF, and n is large, calculating Sidon (and possibly sum-free) can become too expensive in time and space complexities. Consequently, we arrive at the situation that checking SSF is potentially more expensive than checking if a value d is a new APN exponent. To circumvent that problem, for larger values of n, we do not calculate SSF values but the values we call *Approximate SSF (ASSF)* values. The ASSF values are those values e that are not shown "not SSF" by the results of Carlet and Mesnager given in [4]:

Definition 5.1. The Approximate Sidon-sum-free (ASSF) set is the set consisting of the divisors e of 2^n-1 after discarding the following values:

1. 2^r-1 where $r \geq 2$ divides n.
2. $gcd(2^r+1, 2^n-1)$ where r is odd and n is even.
3. $gcd(2^r+3, 2^n-1)$ where $r \equiv 2 \bmod 3$ and n is a multiple of 3.
4. $gcd(2^r-2^k+1, 2^n-1)$ where n, r and $k-1$ have a common divisor larger than 1.
5. every divisor of 2^n-1 which is a multiple of one of the values described in one of the items above.

Analogous to the definition of ASSF set, we define the *Approximate Sidon* (AS) set and *Approximate sum-free* (ASF) set. More precisely, Approximate Sidon (AS) set is the set consisting of the divisors e of 2^n-1 after discarding the values from Definition 5.1, conditions 1 and 5. Approximate sum-free (ASF) set is the set consisting of the divisors e of 2^n-1 after discarding the values obtained from Definition 5.1, conditions 2, 3, 4, and 5.

TABLE 3. Divisors of $2^n - 1$ which are Sidon-sum-free, part I.

n	Specification	Values
3	S/SF/SSF	1
4	S	1, 3, 5
	SF	1, 5
	SSF	1, 5
5	S/SF/SSF	1
6	S	1, 3, 9
	SF	1
	SSF	1
7	S/SF/SSF	1
8	S	1, 3, 5, 17
	SF	1, 5, 17
	SSF	1, 5, 17
9	S/SF/SSF	1
10	S	1, 3, 11, 33
	SF	1, 11
	SSF	1, 11
11	S	1, 23
	SF	1, 23, 89
	SSF	1, 23
12	S	1, 3, 5, 9, 13, 39, 65
	SF	1, 5, 13, 65
	SSF	1, 5, 13, 65
13	S/SF/SSF	1
14	S	1, 3, 43, 129
	SF	1, 43
	SSF	1, 43
15	S	1, 151
	SF	1, 151
	SSF	1, 151
16	S	1, 3, 5, 17, 257
	SF	1, 5, 17, 257, 1 285
	SSF	1, 5, 17, 257
17	S/SF/SSF	1
18	S	1, 3, 9, 19, 27, 57, 171, 513
	SF	1, 19
	SSF	1, 19

Remark 12. Note that all the SSF values belong to the set of Approximate SSF values, but the ASSF set possibly contains more values.
Still, if we compare the results from Tables 3 and 4 with those obtained from the ASSF calculations, we see there are only a few values of n where SSF and ASSF

TABLE 4. Divisors of $2^n - 1$ which are Sidon-sum-free, part II.

n	Specification	Values
19	S/SF/SSF	1
20	S	1, 3, 5, 11, 25, 33, 41, 55, 123, 205, 275, 1 025
	SF	1, 5, 11, 25, 41, 55, 205, 275, 451, 1 025, 2 255,
	SSF	1, 5, 11, 25, 41, 55, 205, 275, 1 025
21	S	1, 337
	SF	1, 337
	SSF	1, 337
22	S	1, 3, 23, 69, 683, 2 049
	SF	1, 23, 89, 683, 15 709
	SSF	1, 23, 683
23	S	1, 47
	SF	1, 47
	SSF	1, 47
24	S	1, 3, 5, 9, 13, 17, 39, 65, 221, 241, 723, 1 205, 4 097
	SF	1, 5, 13, 17, 65, 221, 241, 1 205, 4 097
	SSF	1, 5, 13, 17, 65, 221, 241, 1 205, 4 097
25	S	1, 601, 1 801
	SF	1, 601, 1 801
	SSF	1, 601, 1801
26	S	1, 3, 2 731, 8 193
	SF	1, 2 731
	SSF	1, 2 731
27	S/SF/SSF	1
28	S	1, 3, 5, 29, 43, 87, 113, 129, 145, 215, 339, 565, 1 247, 3 277, 16 385
	SF	1, 5, 29, 43, 113, 145, 215, 565, 1 247, 3 277, 4 859, 6 235, 16 385, 24 295
	SSF	1, 5, 29, 43, 113, 145, 215, 565, 1 247, 3 277, 16 385
29	S	1, 233, 1 103, 2 089
	SF	1, 233, 1 103, 2 089, 256 999
	SSF	1, 233, 1 103, 2 089
30	S	1, 3, 9, 11, 33, 99, 151, 331, 453, 993, 1 359, 1 661, 2 979, 3 641, 4 983, 10 923, 32 769
	SF	1, 11, 151, 331, 1 661, 3 641
	SSF	1, 11, 151, 331, 1 661, 3 641
31	S/SF/SSF	1

sets are not the same. Naturally, this does not necessarily mean that using ASSF for larger n does not weaken the techniques.

Remark 13. It is possible to improve the computation speed for calculating the SSF set by considering the ASSF set: first, we calculate the ASSF set, and then we check if all those values are indeed SSF values. Trivially, we can exclude values 1 from the check (since we know it is always SSF) and $2^n - 1$ since we know it is never SSF.

Remark 14. When 2^n-1 is a Mersenne prime, there is no need to check SSF since we know value 1 is always SSF, and there is no other strict divisor of 2^n-1.

Remark 15. Based on our experiments and the algorithms' complexities, we recommend the following steps in calculating SSF/ASSF values [3]:

1. Calculate ASSF values.
2. Check those values that are SSF in subfield since they are also SSF in the field (which helps by removing some values but also if further reduction is possible - if all the values are covered in subfield then for sure no further reduction is possible).
3. Calculate SF values with the gcd approach (Proposition 5.1 [4]). Here, one needs to check only those values that are ASSF and not covered by the subfield check.
4. Optional: reduce the number of remaining values by running the first or second algorithm (Sidon condition only).

As our results show small differences between ASSF and SSF, and since the sum-free condition is somewhat more discriminative than the Sidon condition, the first three steps should provide very similar results compared to when added the final step.

5.2. **Calculating the Number of Possibly New APN Exponents.** In this section, we employ all constraints on the possibly new APN exponents d to investigate the computational effort needed to find new APN exponents or discard all possible values d for a certain value of n. We start by recalling all the conditions a value d needs to fulfill to be a possibly new APN exponent. We list the conditions in the order we apply them.

1. Remove any value d such that $gcd(d, 2^n-1) \neq 1$ if n is odd and $gcd(d, 2^n-1) \neq 3$ if n is even.
2. Remove any value d if it is already a known APN exponent.
3. If n is even, keep only one representative of a cyclotomic class with d being an element. Keep the minimal representative of a cyclotomic class. If n is odd, keep only one representative of cyclotomic classes with d and its inverse being the elements. Keep the minimal representative of both cyclotomic classes.
4. Remove any value d such that $gcd(d, 2^r-1)$ is not an APN exponent in $\mathbb{F}_{2^r}$.
5. Remove any value d such that $gcd(d-2^j, 2^n-1)$ is not an SSF value, for some j. If n is too large, replace SSF by ASSF.
6. Remove any value d such that there exists a divisor λ of 2^n-1 such that $\binom{\lambda+1}{2} > 2^n-1$ and there exists $j = 1, \ldots, n-1$ such that λ divides $d-2^j$ (see Remark 10).

Remark 16. Note that if n is a prime, then the subfield condition is useless since there are no subfields to explore.

Remark 17. Since the SSF condition works for all values of n where 2^n-1 is not a Mersenne prime and subfield condition works for all values where n is not prime, we consider the SSF condition to be a more general one since Mersenne primes are rarer than primes.

[3] We assume that n is large enough, e.g., larger than ≈ 30, as, for smaller values, all algorithms are sufficiently efficient.

TABLE 5. Number of possibly new APN exponents, the total number of values to consider for a certain n equals $2^n - 2$, n goes up to 31.

n	$gcd(d, 2^n - 1)$	Not known APN	Cyclotomic rep.	Subfield	SSF
3	6	3	1	1	0
4	4	0	0	0	0
5	30	5	1	1	0
6	12	6	1	0	0
7	126	49	4	4	3
8	64	40	5	5	4
9	432	315	19	6	4
10	300	260	26	21	21
11	1 936	1 683	78	78	66
12	576	540	45	21	21
13	8 190	7 839	302	302	301
14	5 292	5 222	373	226	226
15	27 000	26 685	893	365	365
16	16 384	16 272	1 017	377	370
17	131 070	130 475	3 838	3 838	3 837
18	46 656	46 566	2 587	697	697
19	524 286	523 545	13 778	13 778	13 777
20	240 000	239 840	11 992	1 592	1 512
21	1 778 112	1 777 545	42 326	12 923	12 923
22	1 320 352	1 320 154	60 007	7 834	7 824
23	8 210 080	8 208 999	178 458	178 458	178 434
24	2 211 840	2 211 672	92 153	2 153	2 135
25	32 400 000	32 398 875	647 981	539 979	539 966
26	22 358 700	22 358 414	859 939	36 844	36 844
27	113 467 392	113 466 339	2 101 232	569 069	569 010
28	66 382 848	66 382 540	2 370 805	31 349	31 127
29	533 826 432	533 824 721	9 203 878	9 203 878	9 202 166
30	178 200 000	178 199 760	5 939 992	11 212	11 212
31	2 147 483 646	2 147 481 693	34 636 802	34 636 802	34 636 801

In Table 5, we give results for the number of values d one needs to examine to look for new APN exponents (considering values up to $n = 32$). We note that this list serves only the illustrative purpose of how the SSF constraint reduces the number of values to check. Previous results by Y. Edel [9] show that there are no new APN exponents for those values of n. We can observe as the values of n become larger, and when $2^n - 1$ has many divisors, the SSF condition can discriminate more values.

Next, we list the results for $32 \leq n \leq 48$ in Table 6. Comparing the results with Table 5, we observe SSF (or, to be more precise, AS and SF criteria) discriminate more exponent values. Indeed, for odd values, we see that the SSF condition regularly reduces the number of possible exponents, where for some of the values n, the reduction is significant. For example, for $n = 39$, due to the SSF criterion, we reduce more than 40 000 possible exponent values. Even though the SSF criterion is applied last (and if applied before, e.g., the subfield criterion, it would remove many more exponent values), this represents a significant reduction in the number of possible APN exponents left to test (our experiments show it could reduce the

TABLE 6. Number of possibly new APN exponents, the total number of values to consider for a certain n equals 2^n-2, $32 \leq n \leq 48$.

n	$gcd(d, 2^n-1)$	Not known APN	Cyclotomic rep.	Subfield	SSF
32	1 073 741 824	1 073 741 344	33 554 417	229 361	229 328
33	6 963 536 448	6 963 535 029	105 508 114	6 893 976	6 893 596
34	5 726 448 300	5 726 447 790	168 424 935	764 560	764 560
35	32 524 632 000	32 524 630 145	464 637 581	236620975	236 620 012
36	8 707 129 344	8 707 128 948	241 864 693	58 309	58 279
37	136 822 635 072	136 822 632 297	1 848 954 492	1 848 954 492	1 848 954 380
38	91 625 269 932	91 625 269 286	2 411 191 297	3 407 842	3 407 842
39	465 193 834 560	465 193 832 571	5 964 023 502	127 800 480	127 759 412
40	236 851 200 000	236 851 199 360	5 921 279 984	1 480 304	1 480 210
41	2 198 858 730 832	2 198 858 727 429	26 815 350 336	26 815 350 336	26 815 343 652
42	809 240 108 544	809 240 108 082	19 267 621 621	140 857	140 849
43	8 774 777 333 880	8 774 777 330 139	102 032 294 540	102 032 294 540	102 032 289 465
44	4 417 116 143 616	4 417 116 142 780	100 389 003 245	15 054 317	15 054 285
45	28 548 223 200 000	28 548 223 197 615	317 202 480 005	2 004 543 425	2 004 537 282
46	22 957 042 116 160	22 957 042 115 194	499 066 132 939	65 710 726	65 710 708
47	9 339 802 874 699	9 339 802 872 926	1 449 575 966 170	1 449 575 966 170	1 449 575 962 833
48	36 528 696 852 480	36 528 696 851 760	761 014 517 745	1 096 689	1 096 684

time required to check if exponents are APN for several weeks, depending on the computational power available). Simultaneously, for n even, the SSF criterion does not significantly reduce the possible new APN exponents.

Remark 18. We applied SSF as the last criterion. If applied before (e.g., before the subfield criterion), it would remove significantly more exponent values. Then, the subfield criterion would have only a slight effect.

Remark 19. The exponents removed through SSF are not all the same as exponents removed by any other criterion. Thus, it is impossible to remove any of the criteria and obtain the same results as here.

6. More Properties of APN Exponents.

In this section, we give more results on APN exponents, which are not so nice to state as in Section 4 but may be useful for future works.

6.1. Other Necessary Conditions for an Exponent to be APN.

Proposition 4. *For every positive integers n and d and for every integer j such that $0 \leq j \leq n-1$, let $f_j = gcd(d+2^j, 2^n-1)$. Consider the multiplicative group $G_{f_j} = \{x \in \mathbb{F}_{2^n}^*; x^{d+2^j} = 1\} = \{x \in \mathbb{F}_{2^n}^*; x^{f_j} = 1\}$. If function $F(x) = x^d$ is APN over $\mathbb{F}_{2^n}$, then, for every $j, k \in \mathbb{Z}/n\mathbb{Z}$ and for every elements $x \in G_{f_j} \setminus \{1\}$, $x' \in G_{f_k} \setminus \{1\}$ satisfying $x^{2^j}(x+1)^{d-2^j} = x'^{2^k}(x'+1)^{d-2^k}$, we have $x' = x$ or $x' = x^{-1}$.*

Proof. Writing again $x = \frac{s}{s+1}$, $s = \frac{x}{x+1}$, the identity $x^{d+2^j} = 1$ implies $s^{d+2^j} + (s+1)^{d+2^j} = 0$, that is, $s^{d+2^j} + (s+1)^d(s^{2^j}+1) = 0$, that is, $s^d + (s+1)^d = \frac{(s+1)^d}{s^{2^j}} = \frac{1}{x^{2^j}(x+1)^{d-2^j}}$. Hence, if F is APN, every elements $x \in G_{f_j} \setminus \{1\}$, $x' \in G_{f_k} \setminus \{1\}$ such that $\frac{1}{x^{2^j}(x+1)^{d-2^j}} = \frac{1}{x'^{2^k}(x'+1)^{d-2^k}}$, or equivalently $x^{2^j}(x+1)^{d-2^j} = x'^{2^k}(x'+1)^{d-2^k}$, are such that $x' = x$ or $x' = x^{-1}$. □

Remark 20. The interpretation of Subsection 4.1 is in the present case as follows: if $x^{d+2^j}=1$ then $\frac{x^d+1}{(x+1)^d}=\frac{x^{-2^j}+1}{(x+1)^d}=\frac{x^{2^j}+1}{x^{2^j}(x+1)^d}=\frac{1}{x^{2^j}(x+1)^{d-2^j}}$.

Other similar properties can be derived, but they are more complex (and give then less simple ways of discriminating APN exponents).
For instance, for every integers k,j,d such that $0\leq k<j\leq n-1$, let $e_{k,j}=gcd(d-2^k-2^j,2^n-1)$, and let $G_{e_{k,j}}$ be the multiplicative subgroup $\{x\in\mathbb{F}_{2^n}^*; x^{d-2^k-2^j}=1\}=\{x\in\mathbb{F}_{2^n}^*; x^{e_{k,j}}=1\}$ of order $e_{k,j}$. If function $F(x)=x^d$ is APN over $\mathbb{F}_{2^n}$, then, if $x,y\in G_{e_{k,j}}\backslash\{1\}$, $x\neq y$ and $x\neq y^{-1}$, then we have $\frac{x^d+x^{d-2^k-2^j}+x^{d-2^k+2^j}+x^{d-2^j+2^k}}{(x+1)^d}\neq 1$ and $\frac{x^d+x^{d-2^k-2^j}+x^{d-2^k+2^j}+x^{d-2^j+2^k}}{(x+1)^d}\neq\frac{y^d+y^{d-2^k-2^j}+y^{d-2^k+2^j}+y^{d-2^j+2^k}}{(y+1)^d}$. Indeed, still introducing the unique $s\in\mathbb{F}_{2^n}^*\setminus\{1\}$ such that $x=\frac{s}{s+1}$, we have $s^{d-2^k-2^j}+(s+1)^{d-2^k-2^j}=0$, and multiplying by $(s+1)^{2^k+2^j}$ we obtain $s^d+(s+1)^d=s^{d-2^k-2^j}+s^{d-2^k}+s^{d-2^j}=\frac{x^{d-2^k-2^j}(x+1)^{2^k+2^j}+x^{d-2^k}(x+1)^{2^j}+x^{d-2^j}(x+1)^{2^k}}{(x+1)^d}=\frac{x^d+x^{d-2^k-2^j}+x^{d-2^k+2^j}+x^{d-2^j+2^k}}{(x+1)^d}$.
The rest of the proof is similar to above.
More generally, let k be any integer and let $x^k=1$, $x\neq 1$, $x=\frac{s}{s+1}$, we have $s^k+(s+1)^k=0$ and therefore, by multiplication by $(s+1)^{d-k}$: $s^d+(s+1)^d=\sum_{j=0}^{d-k-1}\binom{d-k}{j}s^{j+k}$, which implies that $x\neq 1$, $y\neq 1$, $x\neq y, x\neq\frac{1}{y}$ and $x^k=y^k=1$ imply $\sum_{j=0}^{d-k-1}\binom{d-k}{j}\frac{x^j}{(x+1)^{j+k}}\neq 1$ and $\sum_{j=0}^{d-k-1}\binom{d-k}{j}\frac{x^j}{(x+1)^{j+k}}\neq\sum_{j=0}^{d-k-1}\binom{d-k}{j}\frac{y^j}{(y+1)^{j+k}}$.

7. **Conclusions.** In this paper, we presented necessary conditions related to Sidon sets and sum-free sets for an element $d\in\mathbb{Z}/(2^n-1)\mathbb{Z}$ to be an APN exponent in $\mathbb{F}_{2^n}$ (we call these conditions the Sidon-sum-free, in brief SSF, conditions). This makes a junction between vectorial Boolean functions for cryptography and additive combinatorics. We also gave a new characterization of such exponents, which can be nicely expressed through Dickson polynomials. We proved that Dickson polynomials in characteristic 2 and Reversed Dickson polynomials of the same index are reciprocal of each other, up to squaring the latter. Since Reversed Dickson polynomials are easier to calculate than Dickson polynomials, this allows simplifying the determination of the expressions of the latter (we gave two examples of such determinations).

The new conditions related to Sidon sets and sum-free sets, in turn, enable us to speed up the search for new APN exponents, i.e., to discriminate even more what could be possible new APN exponents. Although our experimental results show that the improvements can be relatively small, they are nevertheless important from theoretical and practical perspectives. We observe small improvements with our new SSF condition since we apply it after all the other known conditions, and we notice that Edel's subfield condition removes many of the same exponents as the SSF condition. Finally, our results show that the SSF condition should become more discriminative as we increase the value n, especially for those values where n is prime (or even just odd), and 2^n-1 has many divisors. Our experimental results give all results for possible new APN exponents up to $n=48$. To the best of our knowledge, this is the first time such exhaustive analysis has been done.

In future work, we plan to extend our research for new APN exponents for higher n and investigate how to calculate the Sidon values more efficiently. Finally, we plan

to investigate techniques that would enable faster evaluation if a power function is APN.

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E-mail address: claude.carlet@gmail.com
E-mail address: s.picek@tudelft.nl