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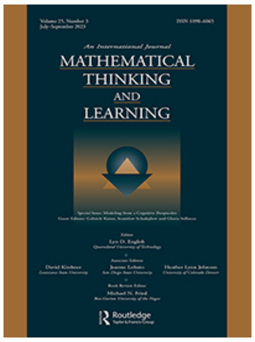

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Embodied learning at a distance: from sensory-motor experience to constructing and understanding a sine graph

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ABSTRACT

Educational technologies develop quickly. Which functions of face-to-face education can be substituted by technology for distance learning? One of the risks of online education is the lack of embodied interactions. We investigate what embodied interactive technologies might offer for teaching trigonometry when learning at a distance. In a multiple case study, we analyze the potential of embodied action-based design for fostering conceptual understanding of a sine graph. It appears that independent learning with tablet-based activities leads to acquiring new sensory-motor coordinations. Some students include these new embodied experiences into mathematical discourse and trigonometry problem solving themselves, while others still need some support from a teacher. However, distantly acquired embodied experiences can be easily recalled in a few days after learning and serve well as a substrate for further conceptualization and problem-solving. The results speak for a clear contribution that embodied design might provide for grounding conceptual understanding in distance learning. However, we expect embodied design to be particularly helpful in a blended learning format.

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Embodied design; conceptual understanding; distance learning; mathematics education; perception-action loop; trigonometry

Introduction: a risk of disembodied learning

Digital technologies are coming to play a major part in educational processes. As the recent pandemic situation has shown technological environments might eventually, at some times and places, replace face-to-face interactions with teachers and peers (Bakker & Wagner, 2020). In this paper, we address how technological solutions for online learning might tackle the absence of full-body interaction in a joint space, which normally supports teaching and learning in a classroom. We investigate if contemporary interactive designs can offer an experience that goes beyond studying from books.

One of the main difficulties in studying mathematics from books is that mathematical concepts are expressed through multiple semiotic registers, such as formulas, definitions, and visual inscriptions (Duval, 2006). For example, understanding of trigonometric functions requires flexible operating with trigonometric values in a triangle, unit circle, graphs of trigonometric functions and algebraic expressions (Moore, 2014; Presmeg, 2008). Moreover, neither formal symbols (Steinbring, 2006), nor visual inscriptions (Arcavi, 2003; Presmeg, 1992) per se are meaningful for students. They need to learn cultural ways of perception (Krichevets et al., 2014; Radford, 2010) and cultural ways of acting (Abrahamson & Sánchez-García, 2016) as they come to recognize and embed mathematical meaning into visual notations and other displays.

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In a classroom, new ways of seeing develop in a tight and multimodal collaboration with a teacher or peers. The gestures, voice pitches and rhythmical structures of the multimodal discourse (Nathan & Alibali, 2011; Radford, 2010; Roth, 2008) help students to perceive mathematics in the inscriptions and make the inscriptions meaningful. It is through establishing joint attention with a cultural adult that a student learns to interrelate different semiotic means in mathematics and connect visual inscriptions with verbal expressions (Shvarts, 2018). Overall, embodied aspects of social interaction seem to be crucial for understanding mathematics for many, if not all, students.

In a distance learning situation, multimodal communication is severely cut. The possibilities for establishing an attunement of two bodies in participatory sense-making are limited, especially in the case of communicating through asynchronous exchange of posts and feedback (Maiese, 2013). When students' microphones and cameras are switched off, teachers cannot continuously assess students' state, thus lack opportunities for immediate proximal formative assessment (Erickson, 2007), which is evaluated as critically important. In general, the lack of embodied processes has been considered as the principal disadvantage of online education; as a result, online education might be limited to transitioning formal rules and procedures and might lack the affective involvement and invisible sensual and contextual processes of mastery (Dreyfus, 2008).

However, despite the fundamental character of this critique, lack of embodiment might be framed as a design challenge, rather than a principal constraint of distance learning (Ward, 2018). The question is what kind of new embodiment emerges from the students' bodies expanded by modern technologies (Dall'Alba & Barnacle, 2005). Indeed, systems of distant interaction develop toward incorporating essential aspects of multimodal communication in technologies such as teleconferences and are thus creating a feeling of tele-proximity (Themeli & Bougia, 2016). Moreover, there are attempts to implement full-body presence of the other partner through mixed reality (e.g., Kuechler & Kunz, 2010).

Educational technologies also develop toward incorporating and developing opportunities for embodied experiences in learning mathematics and science. Many of them require full-body interaction, for example, as students enact planets' rotations (Lindgren et al., 2016) or honeybee behavior in a complex biological system (Peppler et al., 2018); alternatively, or high-tech motion and gesture sensor technologies are required (Duijzer et al., 2019; Lindgren et al., 2019; Nemirovsky et al., 2013). Unfortunately, as these technologies are hardly helpful in distance learning situations, as the required environments and devices are not available at home. Tablet-based applications might be more accessible at home and some of them allow for such embodied experiences as sensing numbers and algebraic relations with fingers and digital manipulatives (Baccaglini-Frank & Maracci, 2015; Ladel & Kortenkamp, 2014; Reinschlüssel et al., 2018; Sinclair & Heyd-Metzuyanim, 2014).

The embodied action-based design genre (Abrahamson, 2014) goes beyond manipulatives and directly helps the students to develop cultural forms of acting and perceiving mathematical inscriptions. The designs from this genre make use of continuous feedback on students' movements, thus facilitating particular perception and action, just like, for example, continuous feedback from the ice helps in learning to skate (Turvey, 1977). Design principles of this genre have been implemented as activities on a tablet or a touch pad for learning proportions (Duijzer et al., 2017), parabolas (Shvarts & Abrahamson, 2019), notion of area (Shvarts, 2017) and trigonometry (Alberto et al., 2019; Shvarts et al., 2019). However, so far, these interactive technologies have been applied in laboratory settings, and interaction with a tutor was critical for transitioning from prospectively mathematical sensory-motor coordinations to incorporating embodied experiences into scientific discourse (Flood, 2018; Shvarts & Abrahamson, 2019; Shvarts et al., 2019).

This paper takes the threat of disembodied learning as a design challenge and aims to clarify if embodied action-based designs might be adapted for facilitating an understanding of mathematical inscriptions when learning at a distance from a teacher. We continue a design research on embodied learning of trigonometry (Alberto et al., 2019; Shvarts & Alberto, 2020) and present a task sequence for learning sine graphs as being built from a unit circle adapted to a distance learning situation. We question, (1) *how can embodied action-based design contribute to the mathematical understanding of a sine graph in a distance learning situation* and (2) *what are the limitations of implementing distance* *learning with an embodied action-based design genre*? We start answering these research questions with a theoretical analysis of the literature that leads us to design principles; then, we implemen these design principles in a task sequence and empirically analyze the implementation in a multiple case study.

Theoretical framework and design principles

To investigate how embodied technology can help in learning about a sine graph, let us focus on what we mean by learning a sine graph. Our theoretical approach to learning builds on a cultural-historical approach (Roth & Radford, 2011; Vygotsky, 1978) and a radical embodied enactivist-ecological approach (Abrahamson & Sánchez-García, 2016; Baggs & Chemero, 2018; Maturana & Varela, 1992; Rietveld et al., 2018; Turvey, 1977) to cognition and education, which have been recently acknowledged as congruent (Baggs & Chemero, 2020; Pagnotta, 2018). Further we elaborate on this theoretical approach and explicate how it leads to the design principles.

A body-artifacts functional system for solving a mathematical task

According to these proposals, knowledge – including mathematics – emerges from action and aims to serve action (Maturana & Varela, 1992; Radford, 2013). Unlike other biological organisms, which can store their knowledge and transfer it through generations only in body structures and genetic material, humans use cultural artifacts for this purpose. When particular aspects of cultural practices are stabilized and useful, they become crystalized or reified in cultural artifacts (Radford, 2003; Wenger, 1998), such as definitions, algebraic formulas and visual inscriptions. Mathematical concepts do not exist in a form other than through embodied activities with these artifacts (Roth & Thom, 2009). Such conceptual activity is a micro-genetic moment of re-production and creative alteration of cultural artifacts as forms of knowledge representation (Saxe, 2018).

Each student passes through the educational process to become a creative member of society and to take part in its practices. In this process, students encounter cultural artifacts and develop cultural ways to perceive them (Radford, 2010). The artifacts become incorporated into dynamics of learners' practice, namely in their perception-action loops, such as eating with a spoon or finding a point on a graph (Lockman, 2000; Nemirovsky et al., 2013). Students' bodies and brains together with cultural artifacts – a spoon or a graph – form *body-artifacts functional systems* responsible for instrumented actions (Shvarts et al., 2021). Mathematical actions are repeatedly constituted by a student's body-artifact functional system in solving mathematical tasks. The experience sediments in a student's body structures as brain plasticity enables such transformations (Menary, 2015). This sedimentation of previous actions creates potentiality for future actions. An active constitution of new actions with mathematical artifacts, sedimentation of these actions in *body potentialities*, and further reactivation in future actions lies at the core of the learning process.

Action-based learning

Embodied action-based design genre (Abrahamson, 2014; Abrahamson et al., 2011) provides a framework for including mathematical artifacts in new perception-action loops (Abrahamson & Sánchez-García, 2016). Targeted mathematical relations are exposed to students in the form of a motor problem (Bernstein, 1996). Imagine, the students are about to study the notion of circle. Each student is given two points on a screen and asked to keep one point still with one hand while moving the other point with the other hand. A continuous feedback on the interaction turns the screen either green or red. The students are asked to find and maintain green feedback, thus solving the motor problem. Unbeknownst to the students, the feedback turns green when the distance between the still hand and the moving hand equals to some constant. Based on the visual feedback, students establish new sensory-motor coordinations (in this case, learn to move one hand equidistantly from the other, still hand) and later reflect on their embodied strategies to discover the target mathematical relation and concept (i.e. circle) . Further, the students might be invited to involve the established coordinations in solving mathematical tasks with digital tools, thus fulfilling the principle of embodied

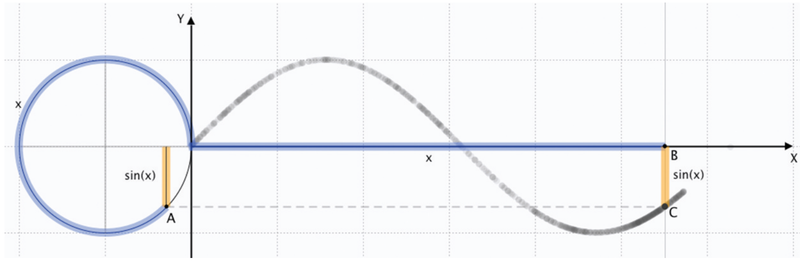

Figure 1. Sine graph is constructed as coordination of two mathematical relations.

instrumentation (Drijvers, 2019). For example, a sine graph crystallizes two sensory-motor coordinations: that of a unit circle's arc with x-coordinate and that of a sine value on a unit circle with y-coordinate on a Cartesian plane (Figure 1). In embodied design, the students would learn to coordinate these relations with their hands along a unit circle and Cartesian plane. Thus, students develop perception-action loops that correspond to cultural forms of perceiving and constructing a sine graph. Later, they might meaningfully use a point running on a unit circle for solving trigonometric equations or building graphs of composite trigonometric functions.

Re-discovering mathematical artifacts

Moreover, to become involved in mathematics as a meaningful practice of structuring and generalizing, students need to participate in an activity that mathematicians do (Freudenthal, 1973). Just as mathematicians are busy with transforming mathematics by creating new definitions, theorems and other cultural artifacts, students need to become involved in re-discovering and establishing mathematical inscriptions for themselves. In this way, students establish theoretical generalization and notice new relations, which are crystalized and reified in the artifacts (Davydov, 1990). From an enactivist-ecological approach, the students actively constitute new artifacts' affordances, thus re-discovering cultural ways of perception and action with the artifacts (Shvarts & Alberto, 2020).

Not surprisingly, these theoretical principles are implemented in the designs of realistic mathematics activities, which are developed to teach mathematics as an activity. Students discover mathematics as an activity in *guided reinventions* (Freudenthal, 1973) and come to the mathematical artifacts themselves, as they build *emergent* mathematical models of the relations iin everyday situations (Gravemeijer, 1999). Ideas of emergent models have been conceptualized for technological innovations as *reversed scaffolding* (Chase & Abrahamson, 2015): students are invited to use a new technological artifact only when they are capable of understanding and maintaining the mathematical relations embedded in the artifact by themselves (for example, active creation of a circle has to precede automatic generation of a circle). Thus, thecnological artifacts reify relations that were previously discovered by the students and support their further thinking process. Applying these ideas to embodied design environments, we '*melt*' a sine graph into sensory-motor coordinations (Shvarts & Alberto, 2020) and do not show a graph per se until the students are capable of building it.

Mathematical discourse and notations in embodied learning

Cultural practices of acting with mathematical artifacts are not limited to spatially coordinated sensory-motor actions, as it would be in case of appropriating artifacts in sport. Students must reflect on their sensory-motor experiences and involve them in mathematical discourse (Roth & Welzel, 2001), thus talk about the sine graph as a depiction of sine values and connect it with $y = \sin(x)$ notation (Presmeg, 2008). New coordinations need to be involved in a system of other semiotic registers (Duval, 2006). Otherwise, new coordinations are at risk of staying at the level of motion regulation and never developing to serve the solving of mathematical tasks (Shvarts et al., 2019).

Designers of many embodiment-inspired technological environments solve the problem of connecting embodied experiences with other semiotic registers in a rather simplistic way. Often, while students actively interact with virtual manipulatives, formal inscriptions – such as numbers or algebraic expressions – are automatically attached and change together with the changes in manipulatives caused by students' actions (Reinschlüssel et al., 2018; Sinclair & Heyd-Metzuyanim, 2014) or given by a nearby adult (Ladel & Kortenkamp, 2014). However, the importance of active translation between different representations for deep understanding has been stressed (Ainsworth, 1999) and research demonstrated limited conceptualization of the connection between representations in case of their automatic linkage (Yerushalmy, 1991).

In teaching practice, embodied experiences are not automatically connected with the discursive means but might serve as further *substrate* (Goodwin, 2017) for communication and thinking (Malinverni et al., 2016). In a classroom, a complex interplay between top-down and bottom-up processes in connecting sensory-motor experiences with formal discourse is present, as students actively participate in establishing connections between spatially articulated actions and disciplinary ways of talking about them (Saxe et al., 2015).

Embodied action-based designers provide an opportunity to actively connect sensory-motor experiences with mathematical discourse and re-discover these mathematical ways of talking in collaboration with a tutor. For this reason, mathematical notations are *postponed* and students have time to develop new coordinations before other mathematical inscriptions might change these new emerging ways of acting (Abrahamson et al., 2011). Moreover, tutors wait for the moments – called *micro-zones of proximal development* – when the students' sensory-motor coordinations have been sufficiently established and then elicit mathematical ideas from the students (Shvarts & Abrahamson, 2019). Tutors carefully build on students' initially vague indexical referencing – such as 'this' and 'that' – toward disciplinary discourse through *multimodal re-voicing* of students' utterances, thus repeating their gestures or verbal expressions in one modality and developing another modality (Flood, 2018; Flood et al., 2016). To prompt students toward re-inventing and re-discovering mathematical ways of talking about their experiences, tutors use students' sensory-motor experiences as *substrate* for explicating mathematical meaning; they co-construct gestures together with the students, invite the students to gesture their embodied ideas, and question these ideas in multimodal ways (Flood et al., 2020).

The complexity of this interaction with a tutor seems to be the main challenge for transforming embodied action-based activities into a distance learning format. Multimodal learning analytics has been applied to distinguish the main phases in the learning process with embodied action-based designs (Ou et al., 2020; Pardos et al., 2018). Implementation of tutor tactics through modeling of gestures has been explored (Abdullah et al., 2017). However, technologies are still far from automatically distinguishing micro-zones of proximal development and providing multimodal support for the students in involving their sensory-motor coordinations into disciplinary discourses.

Embodied action-based design for learning a sine graph at a distance

The learning sequence in this study represents a stage of a larger design research and further develops the previously published designs (Alberto et al., 2019). The entire sequence is built to introduce the students to a sine function on a unit circle and then to teach building a sine graph based on the unit circle. With respect to the theoretical statements presented above, we used four design heuristics in constructing the tasks:

(1) With respect to the principle of action-based learning, we introduced all mathematical relations between unit circle, sine value, and sine graph at first in the form of sensory-motor problems (Abrahamson, 2014; Abrahamson et al., 2011). The students were asked to discover and practice a new sensory-motor coordination between two hands; they manipulated touch points on a tablet and aimed at maintaining continuous green feedback from the screen elements.

(2) With respect to the principle of postponed introduction of mathematical discourse and notations (Abrahamson et al., 2011, Flood et al., 2020), the students were prompted to reflect on their performance in solving a sensory-motor problem and to express their strategies with mathematical discourse.
(3) With respect to the principle of incorporating an artifact into a body-artifacts functional system for solving a mathematical task in the course of embodied instrumentation (Shvarts et al., 2021), the embodied tasks and reflection were followed by mathematical tasks. In these tasks, a student needed to apply acquired coordinations and thier descriptions to solving mathematical tasks.

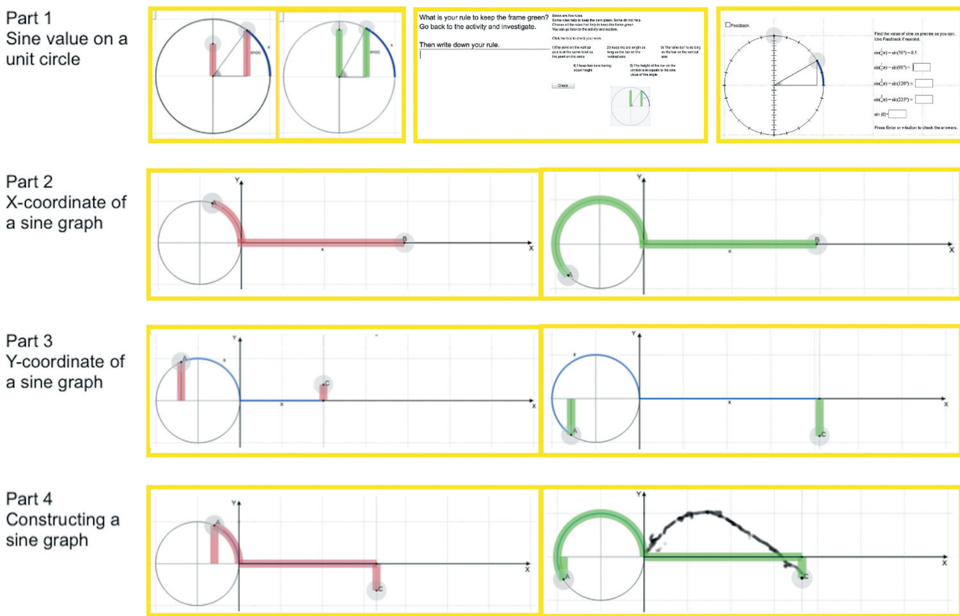

Figure 2. An overview of the task sequence for embodied learning of a sine graph.

Table 1. The structure of the task sequence for embodied learning of a sine graph.

	Type 1 Sensory-motor task	**Type 2** Reflection task	**Type 3** Mathematical task
Part 1 Sine value on a unit circle	Keep green feedback. Students coordinate a point on a unit circle with its horizontal projection on *y*-axis.	Reflect on the height, horizontal alignment and *y*-coordinates as a sine value for the correspondent points on a unit circle and on the *y*-axis.	Find sine values for a point on a unit circle.
Part 2 *X*-coordinate of a sine graph	Keep green feedback. Students coordinate an arc on a unit circle and a distance on the *x*-axis from the origin.	Reflect on the speeds, alignment of two distances and alignment of an arc and *x*-coordinates for the correspondent points on a unit circle and a sine graph.	Position a point on *x*-axes for some given fractions of π. A unit circle with marked parts of 2π can be used.
Part 3 *Y*-coordinate of a sine graph	Keep green feedback. Students coordinate of a sine value on a unit circle and *y*-coordinate on the Cartesian plane.	Reflect on the height, horizontal alignment and *y*-coordinates of the correspondent points on a unit circle and a sine graph.	Put a point on a Cartesian plane; *x*-coordinate is given, a point on a unit circle automatically takes the right position. Students need to find *y*-coordinate and then sine value for the target point.
Part 4 Constructing a sine graph	Keep green feedback. Students join the coordinations from Part 2 and Part 3.	Reflect on how to draw any point on a sine graph.	Put multiple points on a Cartesian plane; *x*- and *y*-coordinates and a point on a unit circle can be freely manipulated. Draw a sine graph.

(4) With respect to the principle of students re-discovering the artifacts and crystallizing their own actions into the artifacts (Shvarts & Alberto, 2020), a sine graph did not appear before the students were able to draw it themselves in the final activities.

There are four major parts in the design sequence; each consists of 8–10 screens to fulfill all design requests. A careful introduction of all four parts is beyond the scope of this paper, so we mention them briefly[1] (Figure 2, Table 1). Each part consisted of three main consequential types of activities: at first (Type 1), some mathematical relations were explored at sensory-motor level using a tablet; then (Type 2), the students were reflecting on their sensory-motor strategies and expressing them in the verbal form, and (Type 3) finally, they were solving mathematical tasks that involved the explored relations with the use of a digital tool. In Type 1 activities (see Figure 2 for each of 4 parts), the students could manipulate two points on the screen: one was movable around the circle and the other one could be moved either vertically (Parts 1 and 3), or horizontally (Part 2) or in both directions (Part 4). The initially red elements on the screen would turn green in case two points were positioned in a coordinated manner. The students did not know in advance when green feedback would appear and they were asked to find and then maintain continuous green feedback, thus discovering the rule that determined the feedback. The digital tools in the activities of Type 3 were similar to the tools in the Type 1 activities; however, the color feedback was not continuous and was given per request. Figure 2 schematically represent all three types for Part 1, and only activities of Type 1 for the other parts. Table 1 briefly describes each type for each part.

Type 2 activities were designed to implement design principle 2. In the previous studies, the involvement of embodied experience into mathematical discourse took place in contingent collaboration with a tutor, who would distinguish the moment when reflection could be productive and guide the discovery process (Flood, 2018; Shvarts & Abrahamson, 2019, Flood et al., 2020). Designing for distance learning, we modeled this process as a combination of a bottom-up and top-down approaches that consisted of 5 steps. (1) A vertical strip in a small rectangle at the top of the screen represented each point on a unit circle: each time a student would coordinate a point on the circle with the point on the Cartesian plane, a green strip appeared on the rectangle. The students were tasked to fill the entire rectangle, thus securing sufficient practice of embodied coordination; at the same time, it was their responsibility to fulfill the instruction before switching to the next task. (2) A student assessed a given (wrong) rule as a possible description of their sensory-motor strategy. (3) A student reflected on their activity and provided their own description in a free written form. (4) A student chose from the given rules which rule fitted their own rule the best. (5) A student chose the correct rules that would describe successful strategies in a multiple choice task. In steps (4) and (5), there were three correct descriptions that progressed toward more and more advanced mathematical discourse. Figure 3 presents these Type 2 activities for Part 1.

Methodology

Data collection

In the paper, we report on a multiple-case study (Miles & Huberman, 1994) within a broader design research (Bakker, 2018) on teaching trigonometry with embodied designs. This study was conducted during the Covid-19 outbreak, when schools in the Netherlands were closed. We provided the students with embodied resources for understanding that were lacking due to the missing interaction with a teacher; as such, this was a genuine investigation of distance learning in an ecological situation. All interactions with the participants were at a distance through the Internet. The students were invited to participate in the study by their school teachers, who were informed by an advertisement at http://www.wiskundebrief.nl/ (a news website for mathematics teachers in the Netherlands) and through the Freudenthal Institute's channels of information. The design study unfolded in iterative cycles of improving designs based on the students' performance and feedback. Here, we report the data from 8 cases that we collected during the fourth empirical tryout in a distance learning format.

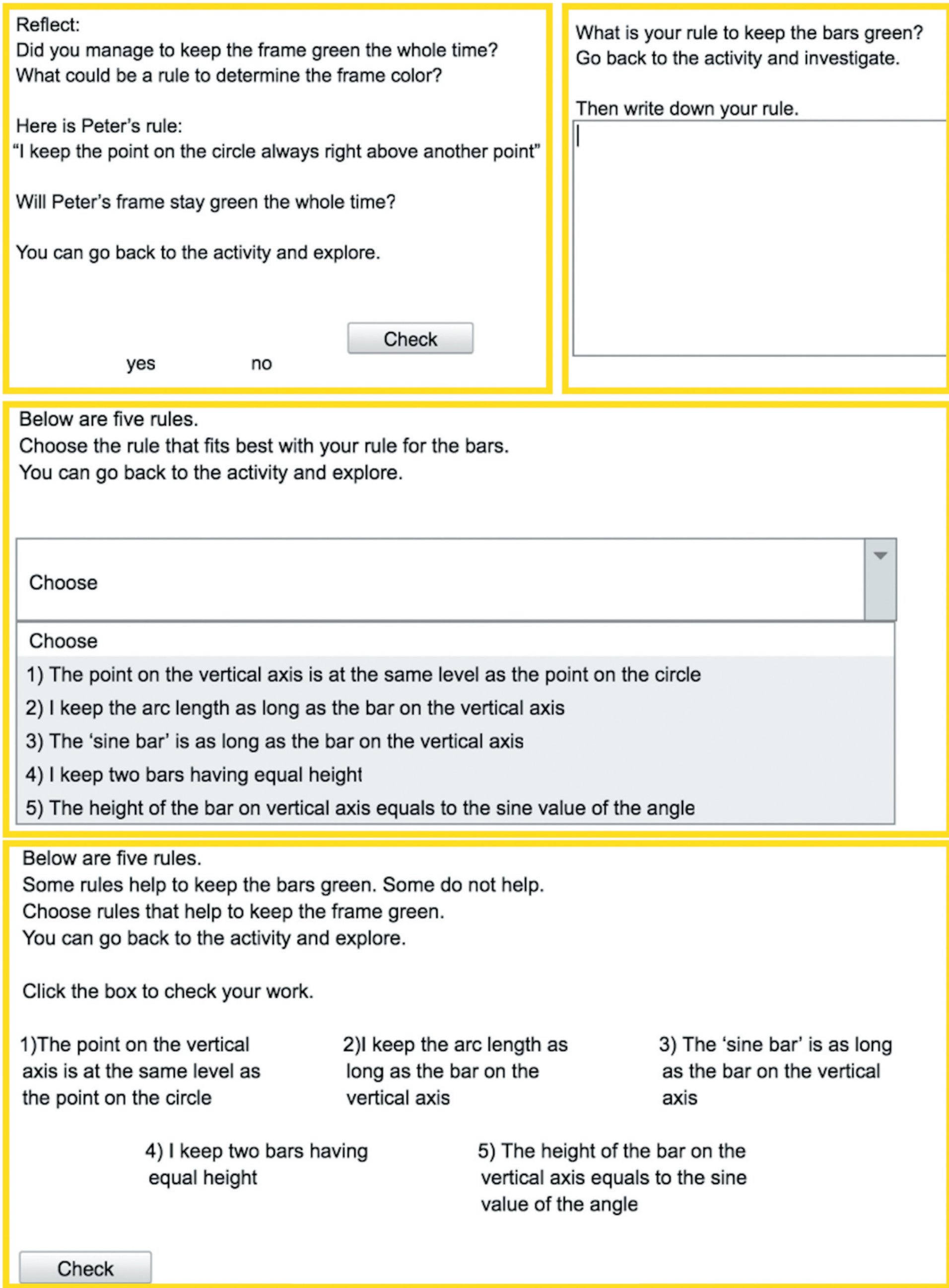

Figure 3. Combined approach to eliciting and introducing mathematical discourse.

The data collection consisted of two major parts: (1) independent distance learning of trigonometry on a tablet with the interactive embodied designs described above, (2) a clinical interview with a participant by the second author of this paper. All the students passed online learning and interview phases.

The task sequence for independent learning started from a short rehearsal of the main preliminary knowledge, i.e., students answered a few questions on a sine function in a right triangle and representation of a unit circle length in parts of π. Then, the students passed pretest tasks where they were tasked to draw a graph of function $y = \sin(x)$ and to find the sine values of three points on a unit circle. Further, they accomplished all four parts of the embodied design sequence followed by a posttest. The posttest included near transfer tasks similar to the tasks in the pretest and tasks that they had encountered in the learning sequences, as well as a far transfer task where the students were required to draw a graph of a function $y= \sin(2x)$. Translated screenshots of the pretest and posttest tasks can be found in Appendix 1. All answers and drawings were stored anonymously in the online mathematical environment, provided by Numworx; we also collected hand logging to have insights on the performance in the embodied tasks.

The semi-structured clinical interviews were conducted in a few days after the distance learning and aimed to investigate the gained knowledge in depth. Following Vygotsky's idea that investigating what a student is capable of achieving together with a more knowledgeable other provides a better understanding of a student's knowledge and abilities (Vygotsky, 1978), we tried "to map the zone of proximal development" (Brown, 1992, p. 157). We particularly questioned whether embodied experience served or could serve as a source for grounding students' conceptual understanding of sine graphs. The interview started from revisiting sensory-motor tasks, by showing a screenshot from embodied tasks and prompting students to recall their performance. Then, the posttest tasks were revisited. The questioning of each task depended on the students' performance in the posttest. If a task was correct, the interviewer would ask a student for an explanation and scaffold the clarification of the details if needed. In case of mistakes in the posttest or vagueness in the explanations, the interviewer would guide a student toward the answer by scaffolding questions, showing screenshots from the sensory-motor tasks and provoking the student to build on their embodied experience by multimodal discursive means. In particular, an interviewer would persistently ask to clarify vague references (Flood et al., 2016), could revoice the students' gestural or verbal expressions (Flood, 2018), check if the interviewer correctly understood the students' explanation by providing multimodal candidate understanding or ask for expressing particular statements with gestures for clarification (Flood et al., 2020). The interviewer aimed to provide as little information to the student as possible and prompted only if the student could not move further after being reminded of sensory-motor tasks. An interview guideline for questioning students' drawings and understanding of a sine graph is provided in Appendix 2. At the end of the interviews, we asked students about their prior learning experience and what they found new in learning trigonometry in this study.

Eight interviews were conducted and recorded in Microsoft Teams software. We used shared whiteboard equipment, so that a student and an interviewer could draw and gesture on a screen together. However, the students were prompted to sit at a distance from a screen as much as possible to create a space for natural gesturing that could preserve more similarity with their embodied experience.

Data analysis

Following the procedure for multiple case studies (Miles & Huberman, 1994), we at first analyzed in depth videos and performance of each case and then made a cross-case analysis. In respect to analyzing the understanding of trigonometry triggered by distance learning implementation of action-based embodied design and the limitations of this implementation, two sub-questions guided our analysis:

1) Were gained embodied experiences helpful in understanding the construction of a sine graph and in solving trigonometry tasks?

2) Did students manage to include their embodied experiences into mathematical discourse and mathematical problem solving already during the distance learning phase?

For a more detailed analysis, the second author coded all students' and interviewer's utterances in drawing a sine graph task in ELAN software with the codes provided in the results section (Table 3). In the cross-case analysis, we used an approach that mixed the case-based and variable-based strategies (Miles & Huberman, 1994, p. 176). After intensive comparison of the cases, we were able to explicate the variables, which helped us to group the students productively and build a general description of all students' experience and possible variations.

Results and discussion

In the analysis, we questioned whether the students managed to arrive at the cultural ways of perception and action with unit circle and sine graph during distance learning, or whether these ways could be easily established during the interview. We were particularly interested in the role of embodied experiences in this new emerging understanding and in the incorporation of these experiences into mathematical discourse and problem-solving.

Table 2 provides an overview of the students' progress through the tasks and of the interview part of drawing a sine graph. For the interview, we depict two variables that appeared to be the most insightful after intensive cross-cases analysis. We postpone the general description of this table for after introducing the detailed analysis.

Table 2. An overview of the student's performance.

	Pre-Test			Task Series												Post-Test			Interview (sine graph)	
	PL	Al	Gr	Embod				Reflect				Rules				Al	Gr	FaTr	SpEm	Scaf
				1	2	3	4	1	2	3	4	1	2	3	4					
Julian																				
Emma																▒				
Timon		▒														▒		x		
Dylan																				
Brent												x								
Maxim		√														√				
Lisa		√														√				
Tara		▒														▒				

Codes:
PL – prior learning experience
Embod – persistence in the sensory-motor tasks
Reflect – cultural adequacy of the personal reflections on sensory-motor strategies
Rules – easiness in the choice of the given rules
Al – success in the algebraic short transfer tasks
Gr – success in a sine graph drawing
FaTr – success in the far transfer task
SpEm – spontaneous inclusion of the embodied experience in the explanations and thinking
Scaf – number of scaffolding questions needed to elicit the full explanation of a sine graph construction
x – missing information due to students missing these tasks; the color is filled based on expectations from other answers and an interview
√ – recalling answer with roots
▒ – use of a calculator

Grey-scale criteria:

	PL	Embod	Reflect	Rules	Algebra	Graph	FaTr	SpEm	Scaf
	None	0-24 % green	'I don't know' or blank	Over 10 attempts or not correct	No correct answers	None	None	0-1	Not achieved
	Knowing a sine graph is wave-shaped	25-50% green	Non-conventional or based on prior learning	6-10 attempts	Less than 50% is correct	General wave	General wave	2-3	7-13
	Taking a trigonometry course	50-90% green	Partially complete	5-3 attempts	50% or more is correct	Almost correct	One parameter correct	4	4-6
	Completed the trigonometry courses	90-100% green	Fully complete	1 or 2 attempts	Fully correct	Correct	Both parameters correct	7	0-2

Perception and construction of a sine graph: Embodied learning potential and limitations

We focus primarily on the analysis of a sine graph construction and perception. Aiming to grasp the potential of embodied experiences as well as its involvement into mathematical problem solving before or during the interview, we developed the coding scheme presented in Table 3:

Table 3. Coding scheme for the interviews.

The interviewer	A student
Asking to explain the answer (Asking)	Answering correctly without explanation (Answer Cor) Answering incorrectly (Answer Incor)
Providing a scaffolding question (Scaffold Quest)	Referring to knowledge as a source of answer (Know) Explaining without referring to the embodied experience (Non-Emb Explain)
Facilitating involvement of the embodied experience by multimodal means (Embod Prompt)	Providing an explanation with the embodied experiences (Embod Explain)
Introducing a screenshot from a sensory-motor task or a posttest (Screenshot)	Thinking with the involvement of the embodied experience (Embod Think) Recalling the embodied experience nonproductively (Embod Non-prod)

Based on coding the interviews and on the students' performance during the distance learning (Table 2), we could distinguish and meaningfully describe three groups of students (see Figure 4).

Group 1: Brent, Dylan, Maxim. The students successfully passed through all the learning activities and could solve all near transfer tasks in the posttest.

Group 2: Lisa, Emma, Timon, Julian. The students successfully passed through all the learning activities and could solve some of the near transfer tasks; their drawings of the sine graph were not fully correct.

Group 3: Tara. The student did not manage to arrive at a fluent sensory-motor coordination in one of the design parts. She solved most of the near transfer tasks based on her pre-knowledge and drew a sine graph as some wave.

From Figure 4, we can see that, for all students, embodied experiences were helpful in understanding the construction of a sine graph. All students, apart from the Group 3 student, spontaneously and productively referred to their embodied experiences for constructing or explaining their drawing: they brought up embodied experiences without the interviewer referring to those experiences in the preceding utterances – see marks "s" in Figure 4 – and these experiences were helpful in explaining the solution. Moreover, for them, 6–10 minutes were sufficient to fully explain or re-draw correctly the sine graph during the interview. All students used their experiences for further thinking in response to an explanation request.

The Group 1 students had included their embodied experiences in mathematical discourse and problem-solving already before the interviews. However, the essential prompts toward embodied experiences from an interviewer were needed to make students fully explicate their understandings answering the questions from the interview guideline (see blue coding in the interviewer's utterances, Figure 4, Group 1 and Appendix 2). For the students in Group 2, more scaffolding questions from the interviewer and more negotiation of meaning were needed (see red coding in the interviewer's utterances and multiple switches between the communicative partners, Figure 4, Group 2). Students spontaneously involved their embodied experience multiple times while answering the interviewer. The interviewer also asked Maxim from Group 1 many scaffolding questions; however, they were addressing his understanding why the radius of a unit circle is one. For the students from Group 2, it was beyond their reach. The student from Group 3 could hardly rely on her insufficient (as we know from Table 2) embodied experience even after substantial support from an interviewer, as reflected in nonproductive embodied descriptions and the duration of time needed to iteratively facilitate graph construction.

Below, we describe one case from each group to showcase how sensory-motor coordination in the embodied activities led (or did not lead) to the cultural – mathematically meaningful – perception and construction of the sine graph.

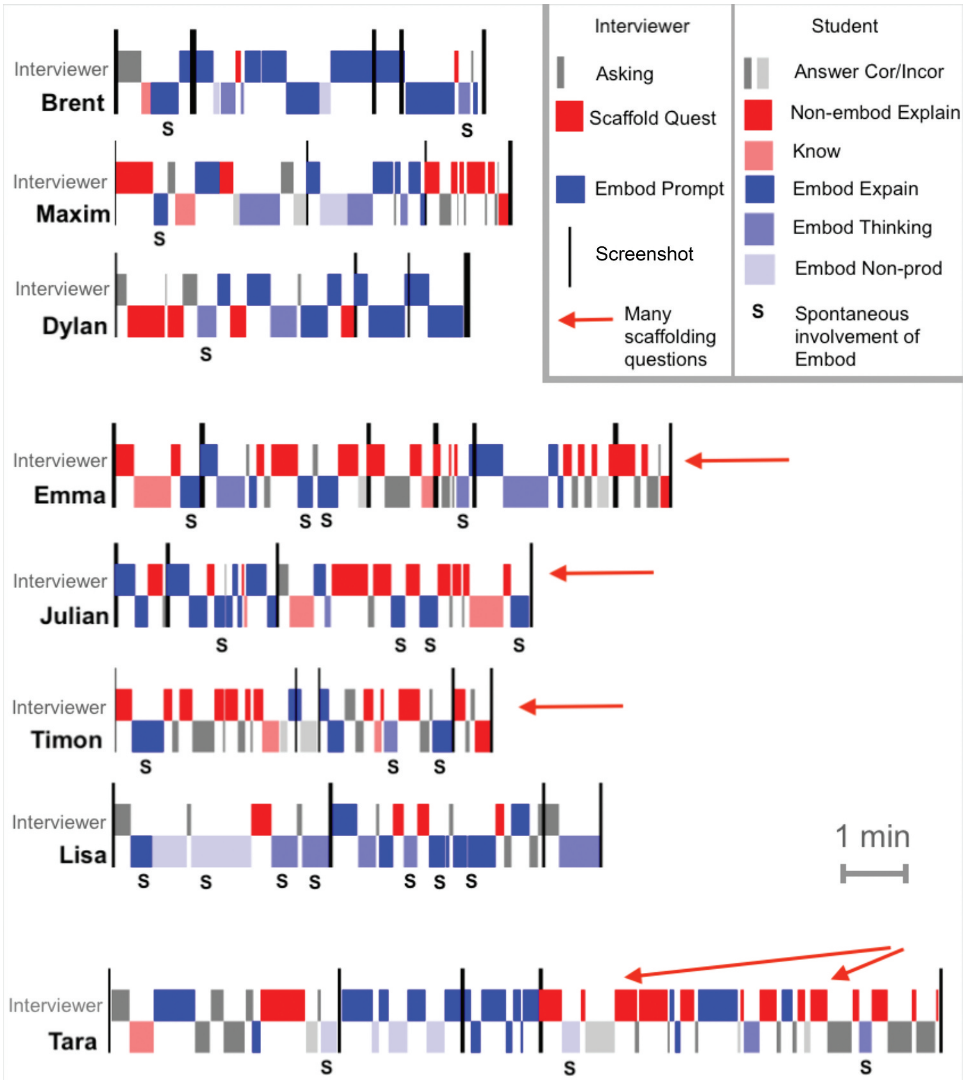

Figure 4. The timelines of the interview parts on the posttest task of drawing a graph of y = sin(x).

Case 1: Brent. Embodied experience grounds the understanding of a sine graph

First, let us bring forth the case of a student who successfully incorporated embodied experiences in scientific discourse. Brent is a 10th grade student (age 16), whose prior knowledge according to the pretest was limited to knowing a sine graph has the form of a wave. In general, Brent completed the task series successfully. He was able to complete sensory-motor tasks; he could easily formulate and choose the rules for 3 parts out of four (he occasionally missed some exercises in part 1). Notable was that the rule that he formulated for the coordination of a unit circle's arc with *x*-coordinate did not match the standard cultural way of talking about this alignment: *'let the horizontal point go 1.5 times faster than the point of the circle.'* However, when he was asked to choose the rules, he easily selected the correct rules. Thus, he was able to change his personal description of the embodied strategy to a cultural one. Brent solved all posttest tasks correctly, with an exception of the far transfer task.

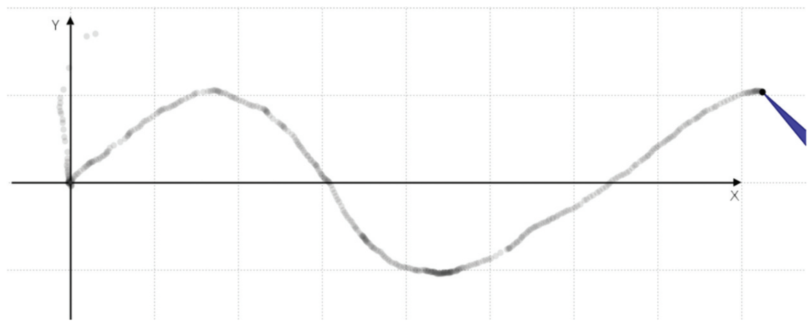

Figure 5. Brent's drawing of a sine graph.

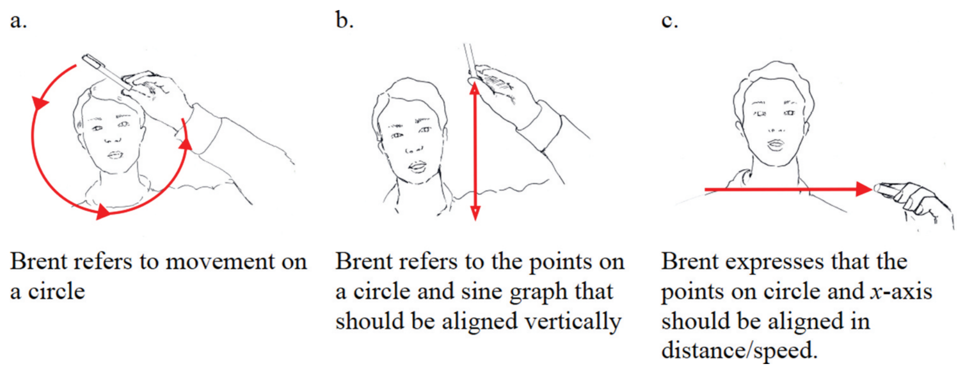

Figure 6. Brent's spontaneous explanation of the sine graph construction.

During the interview,[2] Brent was asked to explain how he drew his sine graph (Figure 5). Brent repeatedly used gestures in his explanation that was clearly grounded in his embodied experiences[3] (Figure 6):

B: *But how I got it, you had that circle* [Figure 6b] *and the whole time you, ehm, went* [Figure 6c] *further.*

And then with that circle [Figure 6a] *up and down* [Figure 6b] *kind of and also that vertical line* [Figure 6b]

And that way I had thought that going to the right [Figure 6c] *and up and down* [Figure 6b], *that idea kind of was behind it.*

His initial expression was not fully comprehensible, so, together with the interviewer, he revisited the screenshots of the sensory-motor tasks. This image from the previous activities appeared to trigger the description of the perception-action loops of his previous performance. He expressed the strategy by drawing the fully clear procedure on the screenshot(Figure 7):

Sensory-motor coordinations, which he acquired in distant embodied learning, have grounded the solution of a mathematical task. His initial description of an embodied strategy *(1,5 times faster)* was transformed into a mathematically correct description of aligning distances (*I also looked at how long this* [arc] *was)*. A sine graph emerged for him as a crystallization of sensory-motor coordinations; its wavy image was meaningful, as it was immediately perceived as an accumulation of relevant coordinations. The sine graph became a part of Brent's body-artifact functional system: it was not merely a wavy line, but an instrument that facilitated Brent's solution of the other posttest tasks that required the use of a sine graph (see Appendix 1 for the tasks).

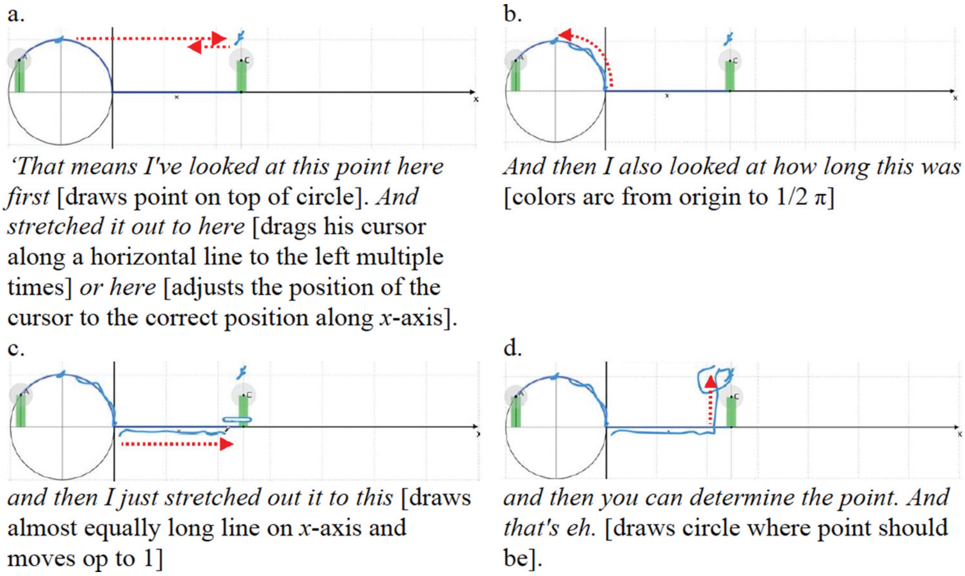

Figure 7. Brent's explanation of the sine graph based on a screenshot from the sensory-motor task.

Case 2: Julian. Embodied experience has potential to ground the understanding of a sine graph

Julian, a 10th grade student, had no prior learning experience of trigonometry beyond the right triangle. He managed to complete the embodied exercises and formulated many correct rules himself. Still, these experiences were not fully incorporated in his mathematical reasoning. He did not connect the wavy shape that he encountered in the tasks with a sine graph and could not draw a graph of $y=\sin(x)$ in the posttest.

However, when engaging in a conversation with the interviewer, Julian could describe his experience, using mathematical discourse and organize the sensory-motor coordinations from the different activities into a functional system of drawing a sine graph. Revisiting the task from Part 2, Julian explained coordination of an arc on the unit circle and x-coordinate:

I: *Do you remember what you did on this task? Could you explain that to me?*

J: *Yeah, I tried to move as fast on the x-axis as on the circle. So that the bar actually stayed green.*

I: *And you say, as fast, what does that mean for the distance they both travel?*

J: *Yeah, just as big, I guess.*

When revisiting Part 3, Julian productively joined his experiences with sensory-motor tasks from Part 1 and Part 3, as he reasoned about the vertical alignment of a unit circle and sine graph (Figure 8).

Further, the interviewer drew the student's attention to the horizontal displacement of the right hand:

I: *Yes, so you held your fingers equally high. And did you do anything more than that?*

J: *Em . . . don't think so. Move along with the circle but . . .*

I: *What do you mean move with the circle?*

J: *Yes, to go around the circle you have to go around.*

I: *So you move that hand round. And did then something happen to that other hand?*

J: *Yeah, it always went a little bit sideways.*

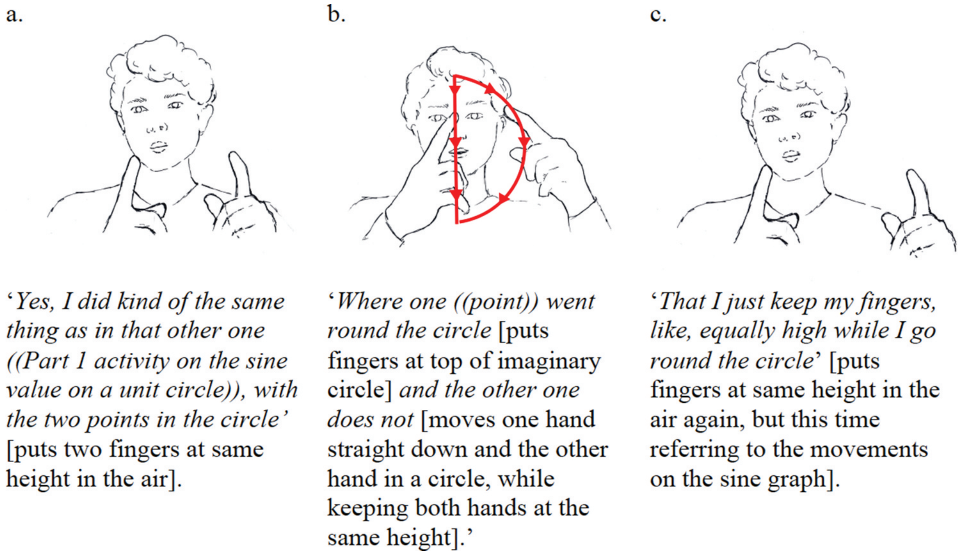

Figure 8. Integrating together sine values on a unit circle and on a sine graph.

I: *And do you know how far to the side?*

J: *Same distance as on the circle.*

Then tutor asked to perform the movement from Part 3 activity and Julian performed a well-coordinated movement (Figure 9).

Further, he drew a correct sine graph (Figure 10) and established direct connection with his embodied experience: '*Yes, what I was just doing with my right hand.*' He explained his graph construction through reasoning about the lengths of a unit circle's arcs and the height on a unit circle.

Although distance learning per se did not lead Julian to the full understanding of a sine graph construction, in a conversation with the tutor the dissociation between embodied experiences and mathematical notation was easily repaired. Embodied experiences worked as *a substrate* (Goodwin, 2017; Flood et al., 2020) for further mathematization. Finally, Julian meaningfully constructed the sine graph and incorporated it into perception-action loops that matched cultural ways of perceiving this visual inscription.

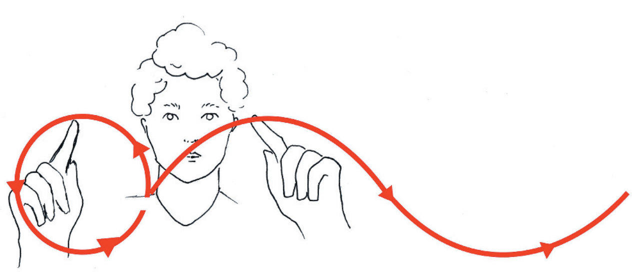

Figure 9. Julian accurately replays the movement from the sensory-motor task of Part 4.

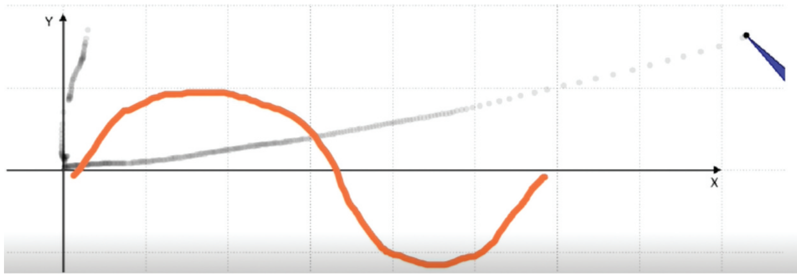

Figure 10. Julian's revised drawing of a sine graph on top of his initial drawing.

Case 3: Tara. Lack of embodied experience leads to recalling a sine graph as an iconic image

Tara is an 11th grade student; she already covered trigonometry at school. Still, at the pretest she drew a general wavy line, which did not match a sine graph. In the sensory-motor tasks, Tara was not persistent (see Table 2). In particular, in Part 2 activities on the relation between an arc's length on the unit circle and x-coordinate, she did not manage to maintain green feedback continuously. Expressing her strategies, Tara only mentioned equal height of the bars that represented y-coordinates, but never equal distances of an arc and x-coordinate. In her posttest, just like in the pretest, algebraic questions were often made correctly, but her drawing of a sine graph did not show improvement (Figure 11).

During the interview, Tara did not explain how she drew a sine graph: '*I don't remember very well. I guess I just knew what a sine graph looked like and I just drew that. I didn't base it on anything.*'

When the interviewer suggested recalling the unit circle that they had discussed in the previous task, Tara could repair her explanation concerning the height of her graph: '*Because the maximum of the unit circle is also 1'.*

The interviewer directed Tara back to the sensory-motor task from Part 2 trying to provoke reasoning about the period of the graph and asked her how she achieved green feedback. However, her gestures do not reveal any coordination between an arc on the unit circle and distance on the x-axis (see Figure 12).

Tara's movement is not productive regarding the mathematical task of drawing a sine graph. Whereas this activity should lead to the conception that the length of the arc on the unit circle should be equal to the distance on the x-axis, Tara's movement has no potential for expressing this mathematical relation. Through multiple scaffolding questions, Tara and the interviewer established a new rule: both hands should move equally fast. However, when asked to draw a new sine graph, Tara does not use this new rule and changes only the magnitude, but not the period, of her graph: "*I don't know, like, how wide it should be. But now ((as she comes to it for the second time)) 1 box up, 1 box down.*"

Tara stuck to the naive way of perceiving and drawing the sine graph as a mere wave. She completed sensory-motor tasks concerning sine value and height, and – although she did not use this experience in drawing the sine graph in a posttest – she could use this experience later in the interview and repair her drawing. However, Tara lacks sensory-motor experience that would coordinate the unit circle's arc and x-coordinate and could provide her with a substrate for mathematical reasoning about the period of the graph.

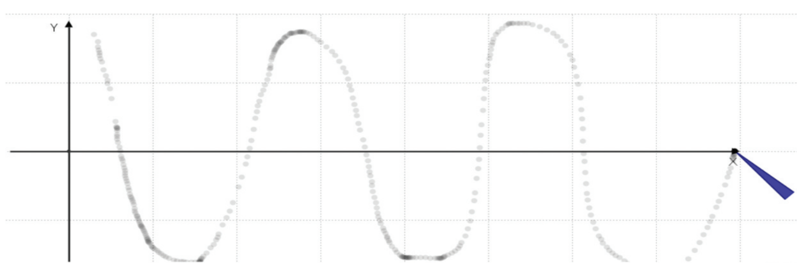

Figure 11. Tara's drawing of a sine graph in the posttest.

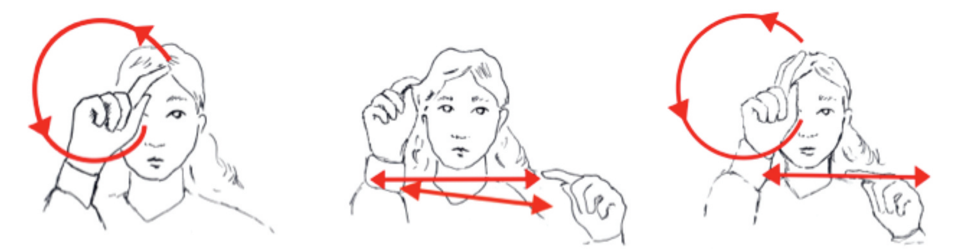

Figure 12. Tara's nonproductive gesturing does not express coordination of an arc and *x*-coordinate.

Embodied learning limitation: Well-established procedures

Tara's case illustrates that prior learning experience can make a student hesitant to incorporate embodied experience in their already established way of reasoning and acting. For Tara, a sine graph was equal to an image of a wave and detached from any experience. She could just recall this image without further reasoning. Throughout the conducted interviews, we have seen multiple other examples of students who relied on already established procedures, which often led them to an answer, but not to an explanation. When finding sine values on the posttest, Timon, for example, used a calculator, '*You can also just fill in your calculator* 'sin$\left(\frac{1}{4}\pi\right)$,' and Lisa knew corresponding sine values by heart, '*Yes. I knew that. I knew the sine value of* $\frac{4}{3}\pi$, *already*.' Knowing an easy procedure for getting an answer provided an alternative way of solving the tasks, without integrating the new embodied experience into the system of their already established mathematical knowledge.

Embodied learning benefit: Solving a far transfer task

For some students, the embodied experience was successfully used in the far transfer task of drawing a graph for $y = \sin(2x)$.

Case 4: Dylan

Dylan only saw a sine graph once before the study. He successfully passed the task series, made a flawless posttest apart from incorrectly depicting the height of the $y = \sin(2x)$ graph, which he easily corrected in the interview (Figure 13).

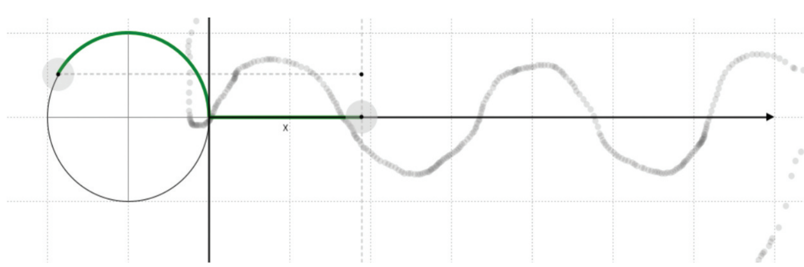

Figure 13. Dylan's graph for y = sin(2x).

Dylan explained his solution:

D: *Because it was, when it is y=sin(2x), then you can, you actually have to, at each x, then I double it and then I look there what it is ((on the unit circle)). So, for example, at x is 1, then instead of searching on the circle for 1, I searched for the distance 2 on the circle. And the height I got with that, I put at ((point)) x ((equals)) 1.*

Dylan's correct explanation is clearly grounded in his embodied experience. He adapted his sensory-motor procedure of locating a point on the sine graph of $y= \sin(x)$ so that it became applicable to $y= \sin(2x)$.

Case 5: Emma

Emma, a 9th grade student without any prior learning experience, correctly drew the graph $y= \sin(2x)$ in the posttest (Figure 14).

I: *So, I was wondering how you came to this answer. What were you thinking?*

E: *Well, I figured because it's twice. So then I assumed it was two waves. And, yes, I just guessed the distances, so I would do it differently now. With the explanation I have now.*

I: *Yeah, okay, and with what you know now could you explain it to me?*

E: *Eh yeah, if I assume with 2x that ((on)) the circle ((the point)) travels twice the distance . . . instead . . . regarding the x-axis.*

I: *And how do you mean?*

In response, she gestures around a circle faster than along a horizontal line (Figure 15).

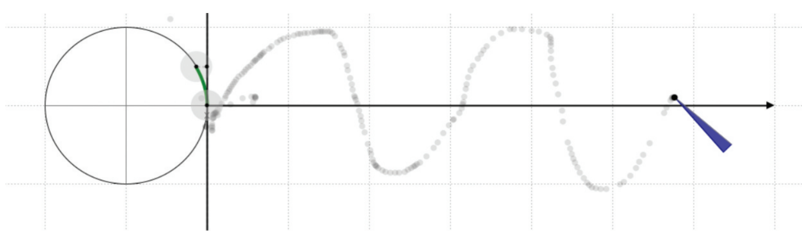

Figure 14. Emma's graph for y = sin(2x).

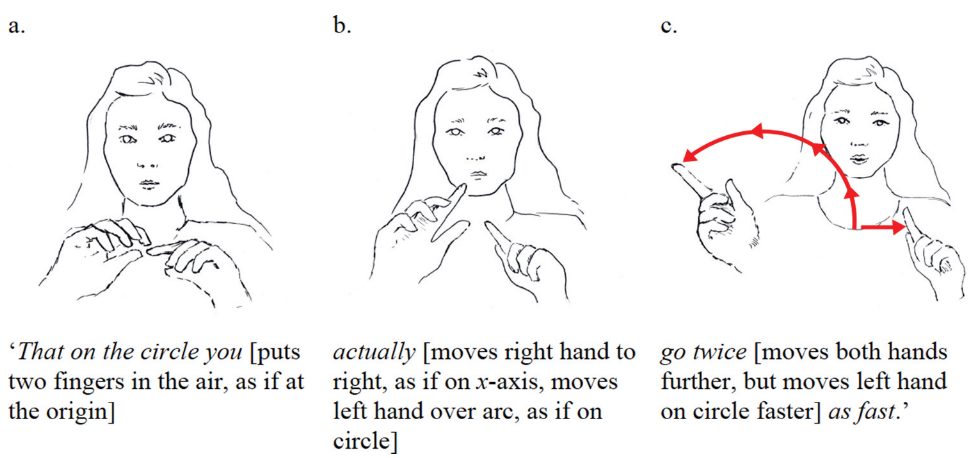

Figure 15. Emma adapts her sensory-motor coordination to the transfer task.

Emma uses the movement she performed earlier for the sine graph for $y= \sin(x)$ to think further about a new graph and adapts this movement to make it congruent with the new formula $y= \sin(2x)$. Emma's embodied experience pushed her to grasping that her hand on the circle will move two times faster compared to her hand on the x-axis, thus generating two waves for the same distance on the x-axis, although she could not yet fully express it verbally.

Although there was no learning material on function composition, a unit circle with a point running along the circle and a point running along the x-axis created sufficient ground for the students to develop their reasoning about the transformation of a sine graph. Dylan could find a mathematical way of expressing the sensory-motor procedure of finding a point on the new graph. Emma's explanation was less mathematically solid; however, her embodied experience was helpful in guessing the culturally anticipated answer. The waves that the students drew were meaningful and incorporated into body-artifact functional systems, as the students could explain their constructions.

Summarizing description

As students passed though the task sequence for embodied trigonometry learning, two students (Dylan, Brent) learned how to construct a sine graph and came to understand sine function on a unit circle without guidance of a teacher in a distance learning format. They could find the sine value of a particular angle, expressed in parts of π on a unit circle or on a sine graph, and also explain their solutions without much prompting from the interviewer. For one student (Maxim), who was studying trigonometry at school at the time of the study, the embodied learning helped to understand interconnection between different visual models in trigonometry. Those students could come up with culturally adequate descriptions of their strategies or change an idiosyncratic description to a cultural one while choosing the rules (Brent). Our combined bottom-up and top-down strategy for gradual incorporating of embodied experiences into mathematical discourse worked well for them. During the interviews, they would rarely involve their embodied experience in explaining the solutions, unless directly prompted by the interviewer.

Four other students (Julian, Emma, Timon, Lisa) were not always able to choose correct mathematical descriptions of their sensory-motor coordinations and incorporate them into mathematical problem-solving on their own. One student (Julian) could not draw a sine graph despite describing most of the sensory-motor coordinations adequately and choosing the rules. However, embodied learning created a substrate which could later be effectively used for further thinking, discussing and reasoning. The students relied on their sensory-motor experiences and brought them into the conversations in a form of spontaneous gestures in the air or on a shared screen. They could repair their lack of connection between embodied experience and mathematical notations in the interviews and later use these experiences in mathematical problem solving and explaining.

One student (Tara) could not overcome an iconic image of a sine graph that she had from earlier education; she also stuck to the procedures that she had previously learned at school for other tasks, such as recalling sine values from memory and using calculator. Her attempts to maintain green feedback in embodied tasks were not persistent. Lisa, another student with a large amount of prior learning experience, showed similar behavior; however, we insisted on spending additional time at the distance learning stage before the interview, which led her to swift inclusion of embodied experiences into mathematical reasoning and problem-solving during the interview. We suggest that well-established procedures might hinder students' readiness for incorporation of new embodied experiences into their system of mathematical knowledge.

Conclusions

Online learning has potential for providing education for those who do not have access to ordinary classrooms or temporarily cannot visit schools. However, the sufficiency of digital interactions for deep understanding of mathematics is questioned from the embodied perspectives to learning:

conceptual understanding is grounded in embodied experience, which might be cut at a distance. In this paper, we took this limitation as a design challenge (Ward, 2018) and investigated (1) how embodied action-based design can contribute to the mathematical understanding of a sine graph in a distance learning situation and (2) what the limitations of implementing an embodied action-based design genre in distance learning format are.

Theoretical analysis led us to four design principles for the embodied design (Abrahamson, 2014) that would facilitate conceptual understanding: (1) presenting mathematical relations in the form of a motor problem; (2) prompting reflection on the sensory-motor strategies and introducing their description in mathematical discourse; (3) incorporating new sensory-motor coordination in mathematical problem solving; (4) 'melting' mathematical artifacts and postponing their introduction to the moment when students are capable of re-inventing them. As we adapted these principles for distance learning, we saw principle (2) as the most difficult for implementing without collaboration with a tutor. We used a combined bottom-up and top-down approach that modeled gradual transition from personal reflections on sensory-motor strategies to mathematical discourse in multimodal collaboration with a tutor (Flood, 2018, Flood et al., 2020): students progressed from freely describing their strategies to choosing a mathematical rule that resembled their description and then to a multiple choice from the given set of rules.

We implemented these design principles in a series of embodied action-based designs for distance learning of a sine graph. An empirical tryout revealed three main sets of evidence that the gained embodied experiences grounded students' mathematical understanding. Firstly, most of the students avoided the prototypical images, such as a general wave for a sine graph (Presmeg, 1992) and arrived at understanding of a sine graph as related to the relevant operations on a unit circle, which they had performed in embodied tasks. The students explained how a sine graph is constructed, using gestures and verbal references to the embodied experiences from their distance learning (see section 5.1). Secondly, some students could successfully reason in solving a far transfer task on function composition relying on their embodied experiences (Section Figure 13). Thus, embodied learning seems to provide particular benefit for deep understanding and creative reasoning, as function composition is traditionally considered as a hard topic to comprehend (Meel, 2003). Thirdly, in the cases when the students could not connect their embodied experiences with solving mathematical tasks at a distance, they spontaneously brought these experiences into the conversations with the interviewer (Section Figure 4, Figure 4). Thus, the embodied coordinations might ground joint attention with an interviewer and become a part of sine graph understanding even in a few days after sensory-motor practice.

At the same time, there are two main limitations of implementing embodied action-based designs at a distance. First of all, a combined bottom-up and top-down approach to connecting sensory-motor experience with mathematical discourse was only partially effective (see Table 2, columns on Reflection and Rules). Some students needed to revisit embodied experiences together with an interviewer in order to include them in mathematical problem solving. Yet, bridging these experiences with mathematical tasks and discourse was still possible after a few days. Further study is needed to answer if this limitation is fundamental or can be resolved by design improvements. At the next stage of the design research, we consider decreasing the complexity of multiple-choice tasks on Rules, which proved to be very difficult (see Table 2), thus making the transition from sensory-motor experience to mathematical discourse smoother.

Another limitation lies in a resistance that students with large prior learning experience exibit to such non-traditional embodied tasks (see section 5.4 and the case of Tara). Due to lack of persistence and too early departure from sensory-motor tasks (see Table 2, columns *Embod* for sensory-motor tasks, Tara), the target sensory-motor coordinations might not emerge, thus leading to insufficient embodied experience for further conceptualization. This result highlights a need for multimodal analytics approaches that would distinguish appropriate moments for progressing from sensory-motor tasks to reflection (Abdullah et al., 2017). Additionally, previously established procedures make students hesitant to include new embodied experiences into their system of mathematical knowledge. Introducing embodied technologies early in the learning particular topic might be helpful in gaining the understanding and avoiding the rigid use of calculators and memorized answers.

At the theoretical level, we see conceptual understanding as building a flexible and widely integrated system of coordinated *perception-action loops* with a variety of artifacts, such as visual inscriptions, mathematical discourse, and algebraic notations. In distance learning, sensory-motor tasks within embodied action-based design genre helped in establishing new sensory-motor coordinations between a unit circle and points on a Cartesian plane. However, a special effort is needed for including these embodied coordinations in perception and action with other mathematical inscriptions. In some cases, sensory-motor coordinations was included in mathematical discourse without any interaction with a teacher: the students were sufficiently guided by the online system. However, in other cases, collaboration with an interviewer was critical for coordinating embodied experiences and mathematical inscriptions into an integrated *body-artifacts functional system*. Yet, this interaction did not require much time, as it was based on students' previous sensory-motor experiences that provided a *substrate* for collaboration and grounded joint attention with an interviewer. Importantly, embodied experiences preserved in students' bodies for a few days and facilitated future conversations.

Overall, carefully designed embodied learning can be successful at a distance. However, students might need support from a teacher for integrating their embodied experiences into a broader system of mathematical knowledge. Yet, the time of communication with a teacher might be essentially shortened if students gain embodied experience earlier in independent learning. Thus, embodied action-based design is promising for blended learning formats, when embodied experiences can later be included into mathematical discourse in an online or face-to-face class.

Notes

1. The tasks can be found at https://embodieddesign.sites.uu.nl/distant-learning/ (see tasks for multi-touch technology). The implementation is realized in the Numworks (www.numworx.nl/en/) learning environment.
2. All episodes are transcribed as follows:
   text Citation
   (. . .) Text leaved out
   ((text)) author's comment
   [text] description of bodily action
3. All graphic sketches are based on the mirrored screenshots from the videos. The sketches made by Gitte van Helden.

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Ethics approval

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Appendix 1. Pretest and posttest tasks

1. Pretest

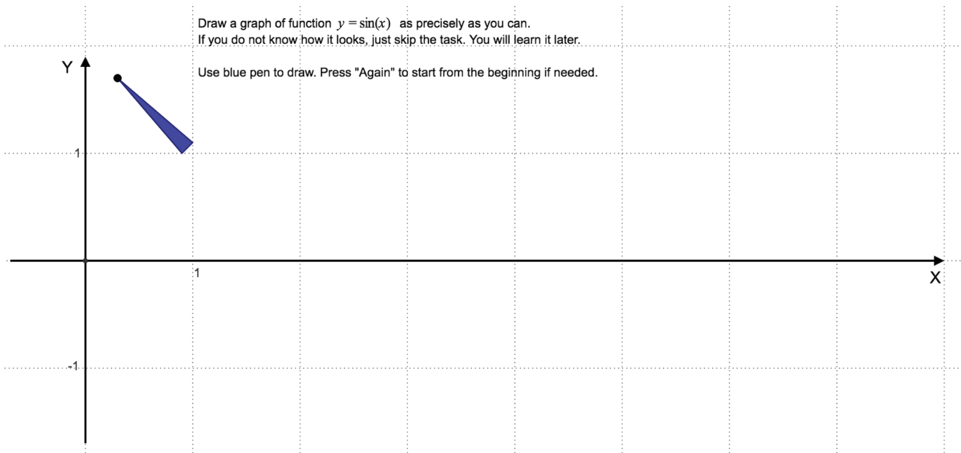

Figure 16. Task 1.

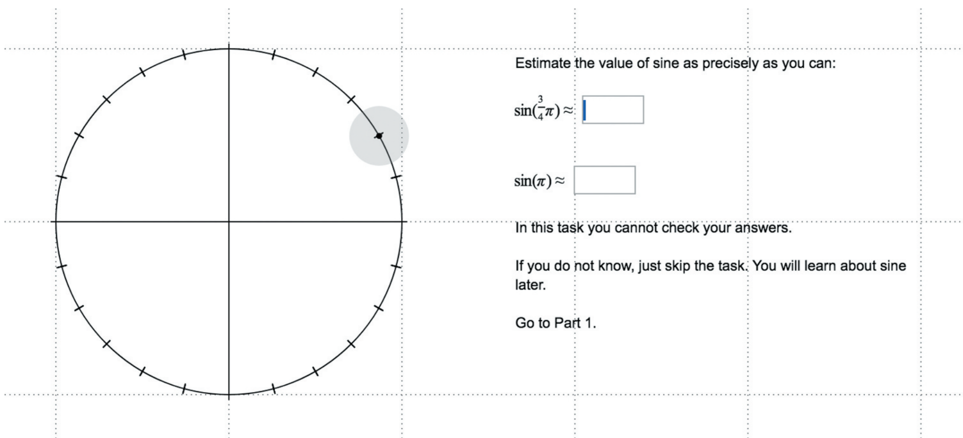

Figure 17. Task 2.

2. Posttest

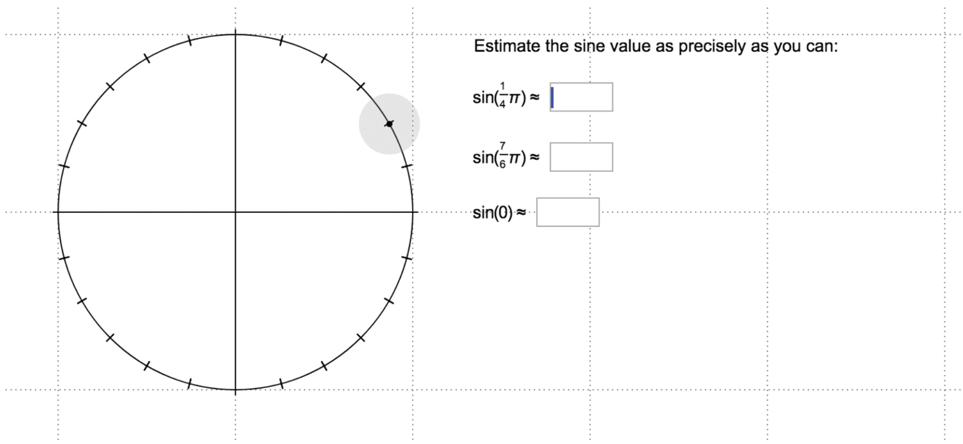

Figure 18. Task 1.

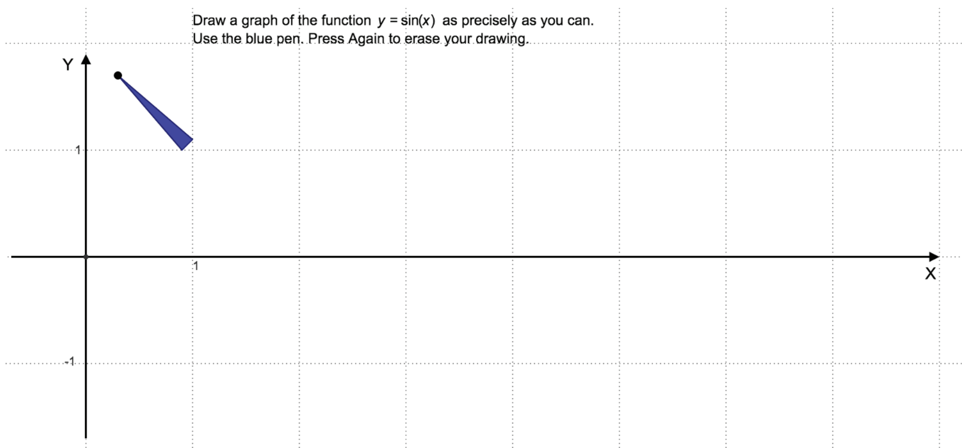

Figure 19. Task 2.

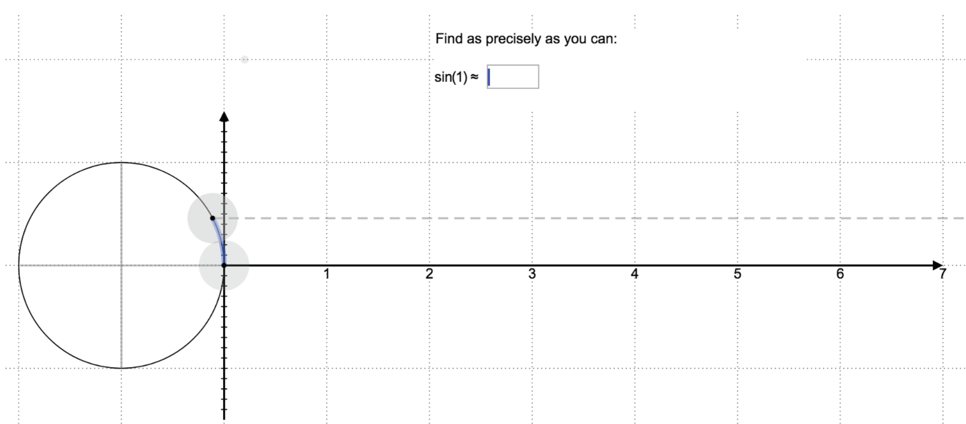

Figure 20. Task 3.

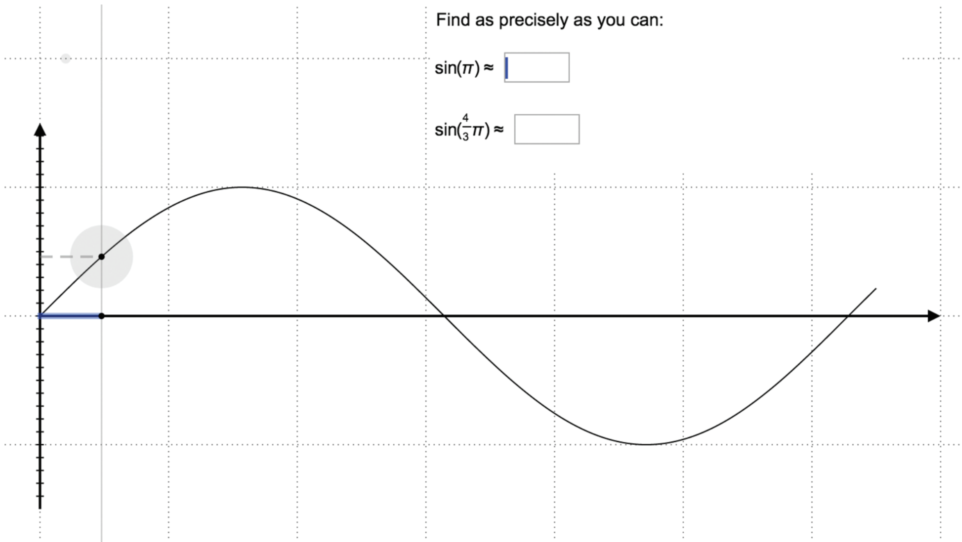

Figure 21. Task 4.

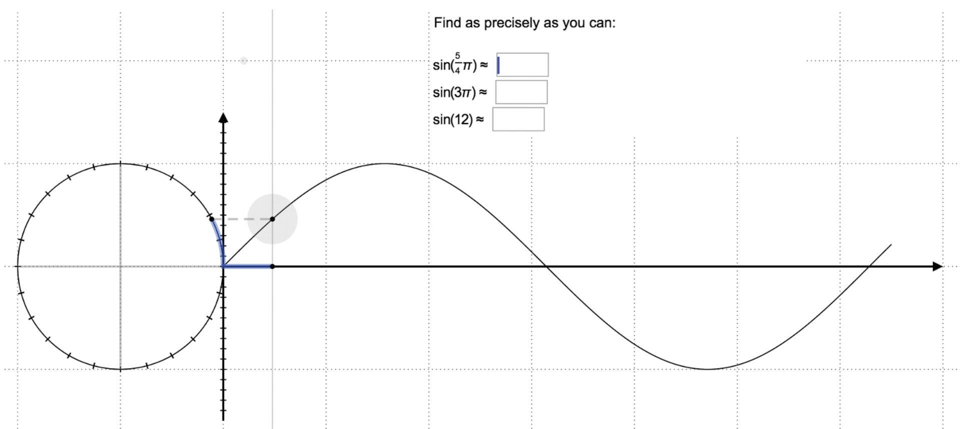

Figure 22. Task 5.

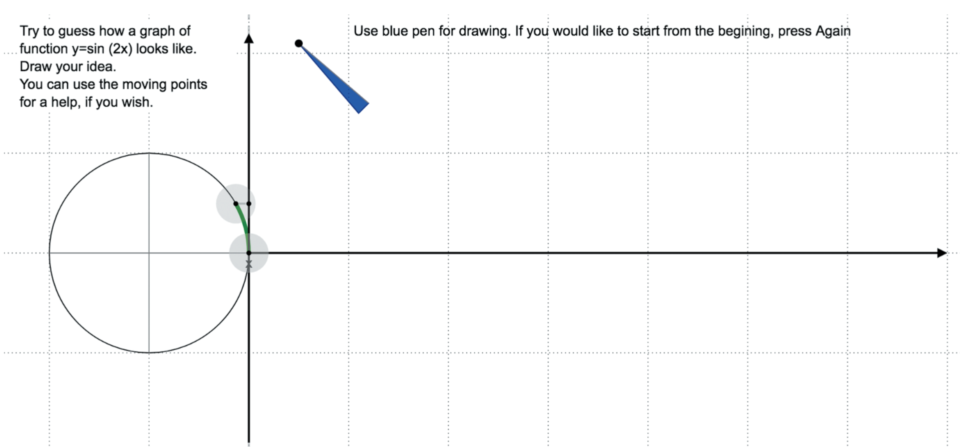

Figure 23. Task 6.

Appendix 2. Interview guidline for the task of drawing a sine graph

Questioning a drawn sine graph:

1) «Could you please explain to me how you draw this graph?»
2) «In particular, where does the graph cross the X-axis? And why?»
   In case there is no answer, return to the screenshot from the sensory-motor activity, Part 2.
   Approximate answer: the length on the X-axis matches the arc on the unit circle, which reaches π.
3) «How high do you go along the Y-axis when you draw the graph? And why?»
   In case there is no answer, return to the screenshot from the sensory-motor activity, Part 3.
   Approximate answer: sine graph goes as high on the coordinate plane as on the unit circle.
4) «What does this height along the Y-axis mean?»
   In case there is no answer: return to the screenshot from the sensory-motor activity, Part 1.
   Approximate answer: The height is sine, you put an arc along the X-axis, and find the sine value along the Y-axis.

Scaffolding questions if no graph was drawn or if the graph is fully incorrect:

«Let us think of sine for some arcs on a unit circle: Imagine you went one quarter of a circle, π/2 along the unit circle. How far you should be on the X-axis? And how high?»

In case of difficulties: go back to the sensory-motor activities.

The same questioning can be repeated for the arcs π and π/4 (pay attention that the sine value of π/4 is higher than 0.5).