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An analytical framework for the best–worst method☆

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ABSTRACT

Since the development of the best–worst method (BWM) in 2015, it has become a popular research focus in multi-criteria decision-making. The original optimization problem of the BWM is a nonlinear min–max model that can lead to multiple optimal solutions, while the linear model of the BWM produces a unique solution. The two models need to be solved by optimization software packages. In addition, although the linear model of the BWM can obtain a unique solution, it produces different feasible regions than the nonlinear model of the BWM, and it changes the objective function. This study aims to solve the nonlinear model of the BWM mathematically to obtain the analytical forms of the optimal solutions. First, we transform the original nonlinear model of BWM into an equivalent optimization model driven by the optimally modified comparison vectors. The equivalent BWM provides a solid basis for computing the analytical solutions. Second, for not-fully consistent pairwise comparison systems, we strictly prove that there is only one unique optimal solution with three criteria, and there might be multiple optimal solutions with more than three criteria. We further develop the analytical forms of these unique and multiple optimal solutions and the optimal interval weights. Third, we develop a secondary objective function to select a unique solution for the BWM. The secondary objective function retains all the characteristics of the original nonlinear model of the BWM, and we find the unique solution analytically. Finally, some numerical examples are examined, and a comparative analysis is performed to demonstrate the effectiveness of our analytical solution approach.

1. Introduction

Multi-criteria decision-making (MCDM) is a significant branch of operations research and management science that supports decision-makers (DMs) in resolving problems involving multiple conflicting and incommensurable criteria. An essential part of MCDM is determining the weights of criteria or the priority of alternatives regarding a criterion when the values of alternatives on this criterion are not available [1–3]. The weighting methods usually involve assigning the rating/importance of the criteria directly by the DM (e.g., SMARTS (Simple Multi-Attribute Rating Technique using Swings) [4,5] and DRM (Direct Rating Method) [6]), or making comparisons between pairs of the criteria by the DM (e.g., AHP (Analytical Hierarchy Process) [7], RANCOM (RANking COMparison) [8], and BWM (Best–Worst Method) [9]). The comparison-based weighting methods elicit the criteria weights based on the comparisons made between different pairs of the criteria following the DM's preferences. Among them, the AHP has been the most popular method for MCDM in the past 40 years and has been widely studied in the operations research literature and applied to solve many real-world problems [10–12]. As a pairwise comparison method, the AHP makes a comparison between every pair of criteria or alternatives using a scale of 1 (equally preferred) to 9 (absolutely preferred), and $n(n-1)/2$ pairwise comparisons are collected to construct the preference relations and derive the weights (or priority) of n criteria (or alternatives). However, as the number of alternatives or criteria increases, the number of comparisons increases substantially. Dealing with such redundant comparison information is time-consuming and will lead to inconsistent judgments.

Rezaei [9] argued that the leading cause of inconsistencies in the AHP is the unstructured pairwise comparison procedure. To overcome

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this problem and some others, Rezaei [9] proposed a recently popular and well-founded MCDM method, the best–worst method (BWM). The BWM conducts structured comparisons based on two comparison vectors: the preference for the best criterion (e.g., the most important) over all the other criteria and the preference for all the criteria over the worst criterion (e.g., the least important). The weights of the criteria are obtained by solving a nonlinear min–max model (also known as the nonlinear BWM). Compared to similar MCDM methods, the BWM provides several advantages [13]: (i) compared to AHP that involves $n(n-1)/2$ pairwise comparisons, the BWM is based on $2n-3$ pairwise comparisons. The fewer number of pairwise comparisons not only simplifies the comparison process but also yields more consistent preference information; (ii) compared to other pairwise comparison methods, by identifying the Best and the Worst criteria before conducting pairwise comparisons, the DM already has a clear understanding of the range of evaluation, leading to more reliable pairwise comparisons; and (iii) compared to other similar MCDM methods, including two opposite references (Best and Worst), the BWM can mitigate several cognitive biases including anchoring bias [14] and equalizing bias [15]. Although BWM is more data efficient compared to some similar methods (e.g., AHP), it is less data efficient compared to methods such as SMART and DRM. However, in methods with $n-1$ data points (e.g., SMART, Tradeoff, and DRM), we are unable to check the consistency of the provided data by the DMs [16,17]. Due to the efficiency of the BWM in reducing the number of pairwise comparisons and its satisfactory performance in maintaining consistency between judgments, it has attracted the attention of many scholars, and many studies related to the BWM have been published over the past several years [18–22].

Current studies on the BWM mainly concern theory or application. In terms of application, research has concentrated on the stand-alone BWM [23–25], combinations of the BWM and other decision-making methods [26–28], and their corresponding applications. The BWM has been shown to be effective in various real-world applications, such as urban environment [29], supply chain management [30], agriculture [31], risk assessment [32], and others [33–35]. As this paper does not aim to extend the BWM into application areas, we do not review application studies in detail. Theoretically, most existing studies focus on improving the nonlinear BWM or extending the BWM into uncertain evaluation environments. Rezaei [36] found that the nonlinear BWM might result in multiple optimal solutions. He proposed two models to compute the ranges of the criteria weights and employed interval analysis to rank the criteria. He also proposed a linear BWM model based on the same philosophy as the BWM to find a unique solution. Brunelli and Rezaei [37] developed a multiplicative BWM. The multiplicative BWM leads to a linear optimization problem. Liang et al. [20] dealt with some consistency issues in the BWM. They proposed input-based cardinal and ordinal consistency measurements to check a decision maker's (DM's) consistency level during the preference elicitation process and then established thresholds for the consistency ratios used in the BWM. Liang et al. [38] incorporated the criteria interactions in MCDM using the Choquet integral and developed a nonadditive BWM that considers possible interactions between the criteria. Two main kinds of uncertain techniques combined with the BWM are fuzzy [39–41] and interval values [42–46].

As a nonlinear min–max model, the BWM [9] can yield either a unique optimal solution or multiple optimal solutions, determined by the criteria number and consistency of the pairwise comparison system [18]. Rezaei [36] explained the reason for unique optimality and multi-optimality using linear algebra. He concluded that there is only a unique optimal solution for not-fully consistent problems with three criteria, and there might be multiple optimal solutions for not-fully consistent problems with more than three criteria. Multiple optimal solutions can offer DMs more flexibility in assigning criteria weights, and a unique optimal solution gives a precise and deterministic result, so both have desirable features in guiding DMs to make decisions in specific situations. Rezaei [36] presented two ways to address this issue: one is based on interval analysis, and the other is to use the developed linear BWM. Both methods work well for multi-optimality concerning the nonlinear BWM, and they are also the most commonly-used approaches.

Although many application studies use the BWM, theoretical research related to the BWM is still relatively rare, and some theoretical conclusions have not been proven scrupulously from a mathematical perspective. Tu et al. [47] noted that the BWM is a non-convex optimization model, and it is quite challenging for it to achieve optimal results. A state-of-the-art survey on the BWM [18] also stated that other possible techniques for analyzing the multi-optimality of the nonlinear BWM are interesting and challenging. The following valuable research issues can be studied to improve the simplicity and suitability of the nonlinear BWM in the form of analytical solutions.

- There is a lack of rigorous proofs and analytical approaches concerning unique and multiple optimal solutions for the nonlinear BWM. It is important to analyze the characteristics of the optimal solutions of the nonlinear BWM and derive analytical solutions from a mathematical perspective.
- Interval analysis for the case of multiple optimal solutions constructs a series of programming models to determine the ranges of different criteria weights. It would be more convenient and mathematically sound to give the analytical forms of the lower and upper bounds of the criteria weights.
- Although the linear BWM can derive a unique optimal solution for the criteria weights, its feasible region is different than that of the nonlinear BWM. It is necessary to establish a secondary objective function that keeps the features of the nonlinear BWM and produces unique optimal criteria weights that can be solved analytically.

Therefore, this study presents an equivalent BWM, provides some analytical forms of the optimal solutions of the nonlinear BWM, and constructs a secondary objective function to derive a unique optimal solution for the BWM. In summary, the main contributions of this study are as follows:

- We transform the initial nonlinear BWM model into an equivalent BWM model driven by the optimally modified comparison vectors. The equivalent BWM not only maintains the properties of the solutions in the nonlinear BWM but also has fewer nonlinear constraints and more equality constraints, which can help identify the potentially most inconsistent criteria, determine the adjustment direction and obtain the analytical solution of the model.
- The unique and multiple optimal solutions of the nonlinear BWM with different numbers of criteria and their analytical forms are proven. The analytical forms of the optimal solutions include the optimal objective function value, feasible optimal modified comparison vectors, and optimal intervals of the criteria weights. These results yield more convenient and effective solution methods for the nonlinear programming without using optimization software and offer a good reference value for modifying inconsistent comparison vectors.
- To derive a unique optimal solution for the BWM from multiple optimal solutions, we construct a secondary objective function that minimizes the maximum modified deviation of each criterion under the condition that the feasible region is the same as that of the nonlinear BWM. We also prove that its optimal solution is unique and can be solved analytically. The secondary objective function retains all the features of the nonlinear BWM and reduces the adjustment amplitude of the original comparison vectors, leading to a satisfactory unique optimal solution.

The remainder of the paper is organized as follows. Section 2 briefly reviews the nonlinear BWM, the interval analysis for the nonlinear

BWM, and the linear BWM. Section 3 proposes an analytical framework for the BWM. Section 4 gives some numerical examples and conducts a comparative analysis to demonstrate the effectiveness of our approach. Finally, Section 5 concludes the study.

2. Best–worst method

In this section, we briefly review the nonlinear BWM, the interval analysis for the nonlinear BWM, and the linear model of the BWM. Then, we summarize these three BWM models, analyze their characteristics and indicate the research topics in this paper.

2.1. Nonlinear best–worst method

BWM is a pairwise comparison-based weighting method developed by Rezaei [9], which requires the DM to make pairwise comparisons between the two reference points best and worst and the other criteria to derive the weights using a min–max optimization model. The steps of the BWM are as follows:

Step 1: Determine the decision criteria for the MCDM problem by the DM.

Step 2: Identify the best (e.g. the most important or the most desirable) criterion c_B and the worst (e.g. the least important or the least desirable) criterion c_W by the DM.

Step 3: Determine the preference values $a_{Bj}(j=1,\dots,n)$ of the best criterion c_B over all the other criteria $c_j(j=1,2,\dots,n)$ by a number from $\{1,2,\dots,9\}$ by the DM. Then, obtain the best-to-others (BO) vector as follows:

$$A_B=(a_{B1},a_{B2},\dots,a_{Bn}). \tag{1}$$

Step 4: Express the preference values $a_{jW}(j=1,2...,n)$ between all the criteria $c_j(j=1,2,\dots,n)$ and the worst criterion c_W by the DM. The comparison results are represented as the others-to-worst (OW) vector:

$$A_W=(a_{1W},a_{2W},\dots,a_{nW})^T. \tag{2}$$

Step 5: Calculate the optimal criteria weights $w^*=(w_1^*,w_2^*,\dots,w_n^*)$ by using the following min–max optimization model:

$$\min\max_j\left\{\left|\frac{w_B}{w_j}-a_{Bj}\right|,\left|\frac{w_j}{w_W}-a_{jW}\right|\right\}$$
$$s.t.\begin{cases}\sum_{j=1}^n w_j=1,\\ w_j\in[0,1],\forall j.\end{cases} \tag{3}$$

where w_B, w_j, and w_W are the weights of criterion c_B, criterion c_j, and criterion c_W, respectively.

The optimization model (3) aims to minimize the maximum absolute differences $\left|\frac{w_B}{w_j}-a_{Bj}\right|$ and $\left|\frac{w_j}{w_W}-a_{jW}\right|$ for all criteria j, and it is based on the condition that the optimal weights should satisfy $w_B/w_j=a_{Bj}$ and $w_j/w_W=a_{jW}$ if all the comparisons are consistent. To better solve the above optimization model, model (3) can be transformed into the following programming model:

$$\min\ \xi$$
$$s.t.\begin{cases}\left|\frac{w_B}{w_j}-a_{Bj}\right|\leqslant\xi,\ \forall j,\\ \left|\frac{w_j}{w_W}-a_{jW}\right|\leqslant\xi,\ \forall j,\\ \sum_{j=1}^n w_j=1,\\ w_j\in[0,1],\forall j.\end{cases} \tag{4}$$

For a given sufficiently large ξ, the solution space of model (4) must be non-empty. By solving model (4), one can derive the optimal criteria weights $(w_1^*,w_2^*,\dots,w_n^*)^T$ and the optimal ξ^* value.

For a fully consistent problem, we have a nonhomogeneous linear system with n weight variables and n constraints, so we have a unique optimal solution. It is also obvious based on the relation chains in the consistency definition, $a_{Bj}\times a_{jW}=a_{BW}$, that a problem with two criteria ($n=2$) is always consistent, hence, has a unique solution. For not-fully consistent pairwise comparisons with three criteria, one can always obtain a unique solution, while for not-fully consistent problems with more than three criteria, model (4) might result in multiple optimal solutions [36].

2.2. Interval analysis of the BWM

Regarding multi-optimal solutions to model (4), Rezaei [36] utilized interval analysis to determine the ranges of the weights of different criteria. Specifically, Rezaei [36] proposed the following two programming models to compute the lower and upper bounds of the weight of criterion $c_j(j=1,2,\dots,n)$

$$\min\ w_j$$
$$s.t.\begin{cases}\left|\frac{w_B}{w_j}-a_{Bj}\right|\leqslant\xi^*,\ \forall j,\\ \left|\frac{w_j}{w_W}-a_{jW}\right|\leqslant\xi^*,\ \forall j,\\ \sum_{j=1}^n w_j=1,\\ w_j\in[0,1],\forall j.\end{cases} \tag{5}$$

$$\max\ w_j$$
$$s.t.\begin{cases}\left|\frac{w_B}{w_j}-a_{Bj}\right|\leqslant\xi^*,\ \forall j,\\ \left|\frac{w_j}{w_W}-a_{jW}\right|\leqslant\xi^*,\ \forall j,\\ \sum_{j=1}^n w_j=1,\\ w_j\in[0,1],\forall j.\end{cases} \tag{6}$$

where ξ^* is the optimal objective function value of model (4).

By solving models (5) and (6) for all criteria, one can determine the optimal criteria weights as intervals. Then, one can apply the matrix of the preference degree and the preference matrix to rank the interval weights of the criteria.

2.3. A linear model of the BWM

Although multi-optimality can be desirable in some decision-making problems, in other cases, DMs may prefer to have a unique solution. Instead of minimizing the maximum value in the set $\left\{\left|\frac{w_B}{w_j}-a_{Bj}\right|,\left|\frac{w_j}{w_W}-a_{jW}\right|\right\}$, Rezaei [36] minimized the maximum value in set $\left\{\left|w_B-a_{Bj}w_j\right|,\left|w_j-a_{jW}w_W\right|\right\}$ and formulated the following programming model:

$$\min\max_j\left\{\left|w_B-a_{Bj}w_j\right|,\left|w_j-a_{jW}w_W\right|\right\}$$
$$s.t.\begin{cases}\sum_{j=1}^n w_j=1,\\ w_j\in[0,1],\forall j.\end{cases} \tag{7}$$

The optimization model (7) aims to minimize the maximum absolute differences $\left|w_B-a_{Bj}w_j\right|$ and $\left|w_j-a_{jW}w_W\right|$ for all the criteria j, and it is based on the condition that the optimal weights should satisfy $w_B=a_{Bj}\times w_j$ and $w_j=a_{jW}\times w_W$ if all the comparisons are consistent.

To better solve optimization model (7), one can transform it into the following linear programming (LP) problem:

$$\min \ \xi^L$$
$$s.t. \begin{cases} \left|w_B - a_{Bj} w_j\right| \leqslant \xi^L, \ \forall j, \\ \left|w_j - a_{jW} w_W\right| \leqslant \xi^L, \ \forall j, \\ \sum_{j=1}^{n} w_j = 1, \\ w_j \in [0,1], \forall j. \end{cases} \tag{8}$$

By solving model (8), one can derive the unique optimal criteria weights $(w_1^*, w_2^*, \ldots, w_n^*)^T$ and the optimal ξ^{L*} value. ξ^{L*} is an indicator of the comparison system's consistency. Smaller ξ^{L*} values indicate a higher level of consistency.

3. An analytical framework for the best–worst method

3.1. The equivalent BWM

The BWM derives the weights based on two vectors of pairwise comparisons, the BO vector A_B and OW vector A_W. Given these two vectors, the definition of cardinal consistency for the set of preferences contained in a BWM pairwise comparison system is defined as follows:

Definition 1 ([9,20,36]). A comparison in the BWM is fully consistent when

$$a_{Bj} \times a_{jW} = a_{BW}, \ \forall j = 1, 2, \ldots, n,$$

where a_{Bj}, a_{jW}, and a_{BW} are the preference of criterion c_B over criterion c_j, the preference of criterion c_j over criterion c_W, and the preference of criterion c_B over criterion c_W, respectively.

For a fully consistent comparison system, the ratios of the criteria weights in pairs with the two evaluation vectors A_B and A_W given in Eqs. (1) and (2) must satisfy

$$\frac{w_B}{w_j} = a_{Bj} \text{ and } \frac{w_j}{w_W} = a_{jW}, \ \forall j = 1, 2, \ldots, n. \tag{9}$$

However, not all pairwise comparisons are consistent when collecting data from a DM in a real-world decision-making problem, which is why the BWM uses a min–max strategy to determine the optimal criteria weighting vector $(w_1^*, w_2^*, \ldots, w_n^*)$. Due to the existence of inconsistency, there is at least one criterion that fails to meet Eq. (9), that is,

$$\frac{w_B^*}{w_j^*} \neq a_{Bj} \text{ or } \frac{w_j^*}{w_W^*} \neq a_{jW}, \text{ for a criterion } c_j.$$

In terms of the optimal weight vector $(w_1^*, w_2^*, \ldots, w_n^*)$, one can produce a pair of unique and fully consistent BWM pairwise comparison systems such that

$$\tilde{A}_B = (\tilde{a}_{B1}, \tilde{a}_{B2}, \ldots, \tilde{a}_{Bn}) \text{ and } \tilde{A}_W = (\tilde{a}_{1W}, \tilde{a}_{2W}, \ldots, \tilde{a}_{nW})$$

with

$$\tilde{a}_{Bj} = \frac{w_B^*}{w_j^*} \text{ and } \tilde{a}_{jW} = \frac{w_j^*}{w_W^*}, \ \forall j = 1, 2, \ldots, n.$$

The two generated evaluation vectors $\tilde{A}_B$ and $\tilde{A}_W$ are not entirely identical to the original vector A_B and vector A_W, respectively. However, they constitute a consistent comparison system, and thus they are, in essence, the optimal modified pairwise comparison vectors of the original vectors under the minimax strategy.

Assume that $(w_1^*, w_2^*, \ldots, w_n^*)$ is the optimal weight vector, and $\tilde{A}_B$ and $\tilde{A}_W$ are the two equivalent transformed BO and OW vectors. In line with the idea of the BWM, we aim to find two modified comparison vectors $\tilde{A}_B$ and $\tilde{A}_W$ such that the maximum absolute differences $\left|\tilde{a}_{Bj} - a_{Bj}\right|$ and $\left|\tilde{a}_{jW} - a_{jW}\right|$ for all j are minimized, that is,

$$\min \max_j \left\{ \left|\tilde{a}_{Bj} - a_{Bj}\right|, \left|\tilde{a}_{jW} - a_{jW}\right| \right\}$$

Additionally, considering the cardinal consistency of the modified comparison system and the condition of comparison with a strength that is equal to or greater than 1, we have

$$\tilde{a}_{Bj} \times \tilde{a}_{jW} = \tilde{a}_{BW}, \text{ with } \tilde{a}_{Bj}, \tilde{a}_{jW} \geqslant 1, \ \forall j.$$

Then, we can turn the original nonlinear BWM, which is a criteria weight-based programming model, into an equivalent BWM driven by the optimal modified comparison vectors as follows:

$$\min \max_j \left\{ \left|\tilde{a}_{Bj} - a_{Bj}\right|, \left|\tilde{a}_{jW} - a_{jW}\right| \right\}$$
$$s.t. \begin{cases} \tilde{a}_{Bj} \times \tilde{a}_{jW} = \tilde{a}_{BW}, \ \forall j, \\ \tilde{a}_{Bj}, \tilde{a}_{jW} \geqslant 1, \ \forall j. \end{cases} \tag{10}$$

Model (10) can be transformed into the following model:

$$\min \ \xi$$
$$s.t. \begin{cases} \left|\tilde{a}_{Bj} - a_{Bj}\right| \leqslant \xi, \ \forall j, \\ \left|\tilde{a}_{jW} - a_{jW}\right| \leqslant \xi, \ \forall j, \\ \tilde{a}_{Bj} \times \tilde{a}_{jW} = \tilde{a}_{BW}, \ \forall j, \\ \tilde{a}_{Bj}, \tilde{a}_{jW} \geqslant 1, \ \forall j. \end{cases} \tag{11}$$

By solving model (11), the optimal modified pairwise comparison vectors $\tilde{A}_B$ and $\tilde{A}_W$, and ξ^* are obtained. Homogeneously, the optimal solutions for the equivalent BWM (11) satisfy the following: For not-fully consistent comparisons with three criteria, we always have two unique optimal modified pairwise comparison vectors $\tilde{A}_B$ and $\tilde{A}_W$, while for not-fully consistent problems with more than three criteria, we might have multiple optimal modified pairwise comparison vectors $\tilde{A}_B$ and $\tilde{A}_W$.

Furthermore, based on only the modified BO vector $\tilde{A}_B$ or modified OW vector $\tilde{A}_W$, we can calculate the criteria weights as follows [48].

$$w_j^{BO} = \frac{1}{\tilde{a}_{Bj} \sum_j \frac{1}{\tilde{a}_{Bj}}} \text{ or } w_j^{OW} = \frac{\tilde{a}_{jW}}{\sum_j \tilde{a}_{jW}}. \tag{12}$$

For a fully consistent comparison system, as $\tilde{a}_{Bj} = \frac{\tilde{a}_{BW}}{\tilde{a}_{jW}}$ and $\tilde{a}_{jW} = \frac{\tilde{a}_{BW}}{\tilde{a}_{Bj}}$ hold for all $j = 1, 2, \ldots, n$, using any equation in Eq. (12) can produce the same criteria weight vector. Finally, we can use these weights to rank the criteria or alternatives.

Compared to the original BWM, our newly developed equivalent BWM has the following characteristics:

1. It is an indirect way to determine the criteria weights. Regarding the inconsistent comparison system, the equivalent BWM aims to find two optimal modified comparison vectors to make the comparison system fully consistent in line with the minimax rule of the BWM. This modified system might be easier to understand if the DM is not familiar with the relationship between the criteria weights and comparison vectors.
2. It has fewer nonlinear constraints and more equality constraints, and all these nonlinear constraints are precise equality constraints. Specifically, the BWM has $4n-6$ nonlinear constraints, while the newly developed equivalent BWM has only $n-2$ nonlinear constraints. Additionally, the fractional nonlinear constraints bring difficulty for the analytical solution of the model, while the pairwise multiplicative nonlinear equality constraints offer possible analytical solutions for the BWM.
3. It can guide a DM regarding the modification direction of the optimal adjusted comparison vectors. The equivalent BWM aims to find two modified comparison vectors $\tilde{A}_B$ and $\tilde{A}_W$, where

the maximum absolute adjustments $\left|\tilde{a}_{Bj} - a_{Bj}\right|$ and $\left|\tilde{a}_{jW} - a_{jW}\right|$ for all j are minimized. The desired cardinal consistency constraints for each criterion can help identify the potentially most inconsistent criterion, after which the adjustment direction can be determined and the analytical solution of the model can be obtained.

3.2. Analytical solutions of the equivalent BWM

After proposing the equivalent BWM (10), it is necessary to explore the properties of the optimal solutions and their analytical forms of the model. Rezaei [36] explained the reason for multi-optimality in the BWM in terms of linear algebra and came to the conclusion that there is a unique optimal solution for a not-fully consistent problem with three criteria, and that not-fully consistent pairwise comparison systems with more than three criteria might have multiple optimal solutions. However, theoretically, there is still no analytical form of the optimal solutions for the nonlinear BWM.

Next, we will mathematically prove the above conclusions and give some analytical solutions for the model. Before calculating the analytical solutions, we provide the following theorem to guide us regarding the optimal modified strategy.

Theorem 1. *For a not-fully consistent pairwise comparison system with three criteria c_B, c_J and c_W, J here refers to the criterion which is not the Best or the Worst.*

(1) If $a_{BJ} \times a_{JW} > a_{BW}$, then the optimal modified strategy is

$$(a_{BJ} - \xi_{BJ}) \times (a_{JW} - \xi_{JW}) = a_{BW} + \xi_{BW}, \tag{13}$$

where $0 \leqslant \xi_{BJ} \leqslant a_{BJ} - 1$, $0 \leqslant \xi_{JW} \leqslant a_{JW} - 1$, $\xi_{BW} \geqslant 0$.

(2) If $a_{BJ} \times a_{JW} < a_{BW}$, then the optimal modified strategy is

$$(a_{BJ} + \xi_{BJ}) \times (a_{JW} + \xi_{JW}) = a_{BW} - \xi_{BW}, \tag{14}$$

where $\xi_{BJ} \geqslant 0$, $\xi_{JW} \geqslant 0$, $0 \leqslant \xi_{BW} \leqslant a_{BW} - 1$.

Proof. To prove that the optimal modified strategy for a_{BJ}, a_{JW}, and a_{BW} satisfies Eq. (13) when $a_{BJ} \times a_{JW} > a_{BW}$, we only need to prove that any of the following modified strategies is not optimal, where

(I) $(a_{BJ} + \xi_{BJ}) \times (a_{JW} + \xi_{JW}) = a_{BW} - \xi_{BW}$;
(II) $(a_{BJ} + \xi_{BJ}) \times (a_{JW} + \xi_{JW}) = a_{BW} + \xi_{BW}$;
(III) $(a_{BJ} + \xi_{BJ}) \times (a_{JW} - \xi_{JW}) = a_{BW} + \xi_{BW}$;
(IV) $(a_{BJ} - \xi_{BJ}) \times (a_{JW} - \xi_{JW}) = a_{BW} - \xi_{BW}$;
(V) $(a_{BJ} - \xi_{BJ}) \times (a_{JW} + \xi_{JW}) = a_{BW} + \xi_{BW}$;
(VI) $(a_{BJ} + \xi_{BJ}) \times (a_{JW} - \xi_{JW}) = a_{BW} - \xi_{BW}$;
(VII) $(a_{BJ} - \xi_{BJ}) \times (a_{JW} + \xi_{JW}) = a_{BW} - \xi_{BW}$.

First, strategy (I) is not feasible. Assume that any of the above modified strategies (II)–(VII) is the optimal modified strategy. Then, for strategies (II)–(V), we can find $\xi'_{BJ} = \xi_{BJ} - \Delta_{BJ}$ and $\xi_{BW} = \xi'_{BW} - \Delta_{BW}$ to make strategies (II)–(IV) true as well. For strategies (VI) and (VII), we can find $\xi'_{BJ} = \xi_{BJ} - \Delta_{BJ}$ and $\xi_{JW} = \xi'_{JW} - \Delta_{BW}$ to make strategies (VI) and (VII) true as well. That is, we can find another pair of modification strategies that outperform strategies (VI) and (VII). Thus, none of the above modified strategies (I)–(VII) is the optimal modified strategy, and the optimal modified strategy is Eq. (13) when $a_{BJ} \times a_{JW} > a_{BW}$. The proof of condition (2) is similar to the proof of condition (1), and it is omitted.

Theorem 2. *There is only one unique optimal solution for the BWM under a not-fully consistent comparison system with three criteria, and we have the following:*

(1) The optimal objective function value ξ^* is one root of the following quadric equation:

$$\begin{cases} (a_{BJ} - \xi) \times (a_{JW} - \xi) = a_{BW} + \xi, \ if \ a_{BJ} \times a_{JW} > a_{BW}, \\ (a_{BJ} + \xi) \times (a_{JW} + \xi) = a_{BW} - \xi, \ if \ a_{BJ} \times a_{JW} < a_{BW}, \end{cases} \tag{15}$$

and thus,

$$\xi^* = \begin{cases} \dfrac{a_{BJ} + a_{JW} + 1 - \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2}, \\ \quad if \ a_{BJ} \times a_{JW} > a_{BW}, \\ \dfrac{-(a_{BJ} + a_{JW} + 1) + \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2}, \\ if \ a_{BJ} \times a_{JW} < a_{BW}, \end{cases} \tag{16}$$

which leads to

$$\xi^* = \left| \frac{a_{BJ} + a_{JW} + 1 - \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2} \right|. \tag{17}$$

(2) The unique optimal solution for the modified comparison vectors is

$$\begin{cases} \begin{cases} \tilde{a}_{BJ} = a_{BJ} + \xi^*, \\ \tilde{a}_{JW} = a_{JW} + \xi^*, \ if \ a_{BJ} \times a_{JW} < a_{BW}, \\ \tilde{a}_{BW} = a_{BW} - \xi^*, \end{cases} \\ \begin{cases} \tilde{a}_{BJ} = a_{BJ} - \xi^*, \\ \tilde{a}_{JW} = a_{JW} - \xi^*, \ if \ a_{BJ} \times a_{JW} > a_{BW}, \\ \tilde{a}_{BW} = a_{BW} + \xi^*, \end{cases} \end{cases} \tag{18}$$

and thus the analytical forms of the optimal criteria weights are

$$\begin{cases} \begin{cases} w_J = \dfrac{a_{JW} + \xi^*}{a_{JW} + a_{BW} + 1}, \\ w_B = \dfrac{a_{BW} - \xi^*}{a_{JW} + a_{BW} + 1}, \ if \ a_{BJ} \times a_{JW} < a_{BW}, \\ w_W = \dfrac{1}{a_{JW} + a_{BW} + 1}, \end{cases} \\ \begin{cases} w_J = \dfrac{a_{JW} - \xi^*}{a_{JW} + a_{BW} + 1}, \\ w_B = \dfrac{a_{BW} + \xi^*}{a_{JW} + a_{BW} + 1}, \ if \ a_{BJ} \times a_{JW} > a_{BW}. \\ w_W = \dfrac{1}{a_{JW} + a_{BW} + 1}, \end{cases} \end{cases} \tag{19}$$

Proof. (1) Based on the results of Theorem 1, if $a_{BJ} \times a_{JW} > a_{BW}$, then the optimal modified strategy is

$$(a_{BJ} - \xi_{BJ}) \times (a_{JW} - \xi_{JW}) = a_{BW} + \xi_{BW}.$$

Let $\xi = \max\{\xi_{BJ}, \xi_{JW}, \xi_{BW}\}$; then, we have $\xi_{BJ} \leqslant \xi, \xi_{JW} \leqslant \xi, \xi_{BW} \leqslant \xi$, and thus

$$(a_{BJ} - \xi_{BJ}) \times (a_{JW} - \xi_{JW}) \geqslant (a_{BJ} - \xi) \times (a_{JW} - \xi)$$

and

$$a_{BW} + \xi \geqslant a_{BW} + \xi_{BW}.$$

Combining the above equations, we find that

$$a_{BW} + \xi \geqslant (a_{BJ} - \xi) \times (a_{JW} - \xi).$$

Solving the above inequality, we can obtain

$$\frac{a_{BJ} + a_{JW} + 1 - \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2} \leqslant \xi$$
$$\leqslant \frac{a_{BJ} + a_{JW} + 1 + \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2}.$$

Therefore, the optimal ξ^* is

$$\xi^* = \min\{\xi\} = \frac{a_{BJ} + a_{JW} + 1 - \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2},$$

where equality holds if and only if

$$a_{BW} + \xi = (a_{BJ} - \xi) \times (a_{JW} - \xi).$$

(II) If $a_{BJ} \times a_{JW} < a_{BW}$, then the optimal modified strategy is

$$(a_{BJ} + \xi_{BJ}) \times (a_{JW} + \xi_{JW}) = a_{BW} - \xi_{BW}.$$

Let $\xi = \max\{\xi_{BJ}, \xi_{JW}, \xi_{BW}\}$; then, we have $\xi_{BJ} \leqslant \xi, \xi_{JW} \leqslant \xi, \xi_{BW} \leqslant \xi$, and thus

$$(a_{BJ} + \xi) \times (a_{JW} + \xi) \geqslant (a_{BJ} + \xi_{BJ}) \times (a_{JW} + \xi_{JW})$$

and

$$a_{BW} - \xi_{BW} \geqslant a_{BW} - \xi.$$

Combining the above three equations, we can obtain

$$(a_{BJ} + \xi) \times (a_{JW} + \xi) \geqslant a_{BW} - \xi.$$

Solving the above inequality, we can obtain

$$\xi \geqslant \frac{-(a_{BJ} + a_{JW} + 1) + \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2}$$

or

$$\xi \leqslant \frac{-(a_{BJ} + a_{JW} + 1) - \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2}.$$

Therefore, the optimal ξ^* is

$$\begin{aligned}\xi^* &= \min\{\xi\} \\ &= \frac{-(a_{BJ} + a_{JW} + 1) + \sqrt{(a_{BJ} + a_{JW} + 1)^2 - 4(a_{BJ} \times a_{JW} - a_{BW})}}{2},\end{aligned}$$

where equality holds if and only if

$$a_{BW} - \xi = (a_{BJ} + \xi) \times (a_{JW} + \xi).$$

(2) According to the proof of (1), the optimal ξ^* can be found if and only if one of the following two quadric equations holds:

$$\begin{cases} (a_{BJ} - \xi) \times (a_{JW} - \xi) = a_{BW} + \xi, \ if \ a_{BJ} \times a_{JW} > a_{BW}, \\ (a_{BJ} + \xi) \times (a_{JW} + \xi) = a_{BW} - \xi, \ if \ a_{BJ} \times a_{JW} < a_{BW}. \end{cases}$$

Solving the quadric equation, we can obtain the optimal objective function value ξ^*. Then, the unique optimal solution for the modified comparison vectors is

$$\begin{cases} \begin{cases} \tilde{a}_{BJ} = a_{BJ} + \xi^*, \\ \tilde{a}_{JW} = a_{JW} + \xi^*, \ \ if \ a_{BJ} \times a_{JW} < a_{BW}, \\ \tilde{a}_{BW} = a_{BW} - \xi^*, \end{cases} \\ \begin{cases} \tilde{a}_{BJ} = a_{BJ} - \xi^*, \\ \tilde{a}_{JW} = a_{JW} - \xi^*, \ \ if \ a_{BJ} \times a_{JW} > a_{BW}. \\ \tilde{a}_{BW} = a_{BW} + \xi^*, \end{cases} \end{cases}$$

Based on the above modification rule for the original pairwise comparison vectors with three criteria, the comparison system is fully consistent. Applying Eq. (12), the analytical forms of the optimal criteria weights are

$$\begin{cases} \begin{cases} w_J = \dfrac{a_{JW} + \xi^*}{a_{JW} + a_{BW} + 1}, \\ w_B = \dfrac{a_{BW} - \xi^*}{a_{JW} + a_{BW} + 1}, \ \ if \ a_{BJ} \times a_{JW} < a_{BW}, \\ w_W = \dfrac{1}{a_{JW} + a_{BW} + 1}, \end{cases} \\ \begin{cases} w_J = \dfrac{a_{JW} - \xi^*}{a_{JW} + a_{BW} + 1}, \\ w_B = \dfrac{a_{BW} + \xi^*}{a_{JW} + a_{BW} + 1}, \ \ if \ a_{BJ} \times a_{JW} > a_{BW}. \\ w_W = \dfrac{1}{a_{JW} + a_{BW} + 1}, \end{cases} \end{cases}$$

According to Theorem 2, the optimal objective function value ξ^* of the BWM with three criteria is unique and can be determined analytically using Eq. (16) or (17). Regarding a not-fully consistent comparison system with more than three criteria, we first identify the inconsistent criteria $c_J = \{c_j | a_{Bj} \times a_{jW} \neq a_{BW}\}$ and then divide them into two groups such that

$$c_{J_1} = \{c_j | a_{Bj} \times a_{jW} > a_{BW}\}, c_{J_2} = \{c_j | a_{Bj} \times a_{jW} < a_{BW}\},$$

which we call the upside criteria set and downside criteria set, respectively. We assume that the cardinality of the inconsistent criteria sets is n'.

If $c_{J_1} = c_{J_2} = \emptyset$, then the BWM comparisons are fully consistent. Otherwise, the comparisons are not fully consistent. In this case, we rebuild n' BWM comparison systems with only three criteria (c_B, c_W, and c_j). For the n' BWM comparison systems, we can compute their optimal ξ_j values using Eq. (16).

Considering the characteristics of the pairwise comparison vectors, it is intuitive to identify the potentially most inconsistent criterion in line with the ξ_j values. The larger the ξ_j value is, the more likely criterion c_j is to be the most inconsistent criterion. Specifically, the two criteria $c_{J_1^*}$ and $c_{J_2^*}$ in the upside and downside criteria sets are selected as the potentially most inconsistent criteria, where

$$c_{J_1^*} = \left\{ c_{J_1} | \xi_{J_1^*} = \arg\max_{J_1} \left\{ \xi_{J_1} \right\} \right\},$$

and

$$c_{J_2^*} = \left\{ c_{J_2} | \xi_{J_2^*} = \arg\max_{J_2} \left\{ \xi_{J_2} \right\} \right\}.$$

Next, we explore how the potentially most inconsistent criteria $c_{J_1^*}$ and $c_{J_2^*}$ affect the optimal objective function value ξ^* in the BWM. Naturally, we consider the following three conditions: (1) all the inconsistent criteria are upside criteria; (2) all the inconsistent criteria are downside criteria; and (3) the inconsistent criteria include both upside and downside criteria. This idea gradually moves from two special conditions to the general condition. We provide the following three propositions for the above three conditions.

Proposition 1. *When all the inconsistent criteria are upside criteria, $\xi_{J_1^*}$ is the optimal objective function value of the BWM.*

Proof. To prove this proposition, we only need to prove that $\xi_{J_1^*}$ is the feasible solution of all the other upside comparison systems but that no ξ_{J_1} is a feasible solution of the comparison system with criteria c_B, c_W, and $c_{J_1^*}$. To prove the former result, we only need to prove that there exist two values $\xi'_{J_1}, \xi''_{J_1} \in [-\xi_{J_1^*}, \xi_{J_1^*}]$ that make the following equation true.

$$(a_{BJ_1} - \xi'_{J_1}) \times (a_{J_1 W} - \xi''_{J_1}) = a_{BW} + \xi_{J_1^*}.$$

According to the results in Theorem 2, the optimal objective function values $\xi_{J_1^*}$ and ξ_{J_1} for each comparison system with three criteria satisfy $(a_{BJ_1^*} - \xi_{J_1^*}) \times (a_{J_1^* W} - \xi_{J_1^*}) = a_{BW} + \xi_{J_1^*}$ and $(a_{BJ_1} - \xi_{J_1}) \times (a_{J_1 W} - \xi_{J_1}) = a_{BW} + \xi_{J_1}$. As $\xi_{J_1^*} > \xi_{J_1}$, we have

$$(a_{BJ_1} - \xi_{J_1^*}) \times (a_{J_1 W} - \xi_{J_1^*}) < (a_{BJ_1} - \xi_{J_1}) \times (a_{J_1 W} - \xi_{J_1}) = a_{BW} + \xi_{J_1} < a_{BW} + \xi_{J_1^*},$$

that is,

$$(a_{BJ_1} - \xi_{J_1^*}) \times (a_{J_1 W} - \xi_{J_1^*}) < a_{BW} + \xi_{J_1^*},$$

and

$$(a_{BJ_1} + \xi_{J_1^*}) \times (a_{J_1 W} + \xi_{J_1^*}) > a_{BJ_1} \times a_{J_1 W} + \xi_{J_1^*} > a_{BW} + \xi_{J_1^*}.$$

Combining the above two equations, there must exist two values $\xi'_{J_1}, \xi''_{J_1} \in [-\xi_{J_1^*}, \xi_{J_1^*}]$ such that

$$(a_{BJ_1} - \xi'_{J_1}) \times (a_{J_1 W} - \xi''_{J_1}) = a_{BW} + \xi_{J_1^*}.$$

Thus, $\xi_{J_1^*}$ is the feasible solution of all the other upside comparison systems.

To prove the second result, we shall prove that for any two values $\xi'_{J_1}, \xi''_{J_1} \in [-\xi_{J_1}, \xi_{J_1}]$,

$$(a_{BJ_1^*} - \xi'_{J_1}) \times (a_{J_1^* W} - \xi''_{J_1}) > a_{BW} + \xi_{J_1^*}.$$

Because $\xi_{J_1^*} > \xi_{J_1}$ and $\xi'_{J_1}, \xi''_{J_1} \in [-\xi_{J_1}, \xi_{J_1}]$, we have

$$\begin{aligned}(a_{BJ_1^*} - \xi'_{J_1}) \times (a_{J_1^* W} - \xi''_{J_1}) &\geqslant (a_{BJ_1^*} - \xi_{J_1}) \times (a_{J_1^* W} - \xi_{J_1}) \\ &> (a_{BJ_1^*} - \xi_{J_1^*}) \times (a_{J_1^* W} - \xi_{J_1^*}) = a_{BW} + \xi_{J_1^*},\end{aligned}$$

that is,

$$(a_{BJ_1^*} - \xi'_{J_1}) \times (a_{J_1^* W} - \xi''_{J_1}) > a_{BW} + \xi_{J_1^*}.$$

Thus, there exist no feasible solutions in any interval $[-\xi_{J_1}, \xi_{J_1}]$ that make the comparison system with criteria c_B, c_W, and $c_{J_1^*}$ fully consistent. Therefore, $\xi_{J_1^*}$ is the optimal objective function value of the BWM when all the inconsistent criteria are upside criteria.

Proposition 2. *When all the inconsistent criteria are downside criteria, $\xi_{J_2^*}$ is the optimal objective function value of the BWM.*

Proof. To prove this proposition, we only need to prove that $\xi_{J_2^*}$ is the feasible solution of all the other downside comparison systems but that no ξ_{J_2} value is the feasible solution of the comparison system with criteria c_B, c_W, and $c_{J_2^*}$. The proof is similar to the proof of Proposition 1, and it is omitted.

Proposition 3. *When the inconsistent criteria include both upside and downside criteria, the optimal objective function value ξ^* of the BWM is determined by the following rules:*

(1) If $(a_{BJ_2^*} + \xi_{J_1^*}) \times (a_{J_2^* W} + \xi_{J_1^*}) \geqslant a_{BW} + \xi_{J_1^*}$, then $\xi^* = \xi_{J_1^*}$;

(2) If $(a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^* W} - \xi_{J_2^*}) \leqslant a_{BW} - \xi_{J_2^*}$, then $\xi^* = \xi_{J_2^*}$;

(3) If $(a_{BJ_2^*} + \xi_{J_1^*}) \times (a_{J_2^* W} + \xi_{J_1^*}) < a_{BW} + \xi_{J_1^*}$ or $(a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^* W} - \xi_{J_2^*}) > a_{BW} - \xi_{J_2^*}$, then the optimal objective function value ξ^* of the BWM is the root of the equation

$$(a_{BJ_1^*} - \xi) \times (a_{J_1^* W} - \xi) = (a_{BJ_2^*} + \xi) \times (a_{J_2^* W} + \xi),$$

and thus,

$$\xi^* = \frac{a_{BJ_1^*} \times a_{J_1^* W} - a_{BJ_2^*} \times a_{J_2^* W}}{a_{BJ_1^*} + a_{J_1^* W} + a_{BJ_2^*} + a_{J_2^* W}}.$$

Proof. (1) Based on Propositions 1 and 2, we know that $\xi_{J_1^*}$ and $\xi_{J_2^*}$ are the optimal ξ values of the upside and downside comparison systems, respectively. Then, the following two equations are true:

$$(a_{BJ_1^*} - \xi_{J_1^*}) \times (a_{J_1^* W} - \xi_{J_1^*}) = a_{BW} + \xi_{J_1^*}$$

and

$$(a_{BJ_2^*} + \xi_{J_2^*}) \times (a_{J_2^* W} + \xi_{J_2^*}) = a_{BW} - \xi_{J_2^*}.$$

As

$$\begin{aligned}(a_{BJ_2^*} + \xi_{J_1^*}) \times (a_{J_2^* W} + \xi_{J_1^*}) \geqslant a_{BW} + \xi_{J_1^*} > a_{BW} - \xi_{J_2^*} \\ = (a_{BJ_2^*} + \xi_{J_2^*}) \times (a_{J_2^* W} + \xi_{J_2^*}),\end{aligned}$$

then we can obtain that

$$\xi_{J_1^*} > \xi_{J_2^*},$$

and thus $\xi_{J_1^*}$ is also the feasible solution of all the downside comparison systems.

In the opposite case, we can obtain that

$$(a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^* W} - \xi_{J_2^*}) > (a_{BJ_1^*} - \xi_{J_1^*}) \times (a_{J_1^* W} - \xi_{J_1^*}) = a_{BW} + \xi_{J_1^*},$$

and thus $\xi_{J_2^*}$ is not the feasible ξ value of the pairwise comparison with criterion $c_{J_1^*}$. Then, $\xi_{J_1^*}$ is the optimal objective function value ξ^* of all the comparison systems.

(2) When $(a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^* W} - \xi_{J_2^*}) \leqslant a_{BW} - \xi_{J_2^*}$, as $a_{BW} - \xi_{J_2^*} < a_{BW} + \xi_{J_1^*}$, we have

$$(a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^* W} - \xi_{J_2^*}) < (a_{BJ_1^*} - \xi_{J_1^*}) \times (a_{J_1^* W} - \xi_{J_1^*}).$$

Then, we can obtain that

$$\xi_{J_1^*} < \xi_{J_2^*},$$

and thus $\xi_{J_2^*}$ is also a feasible solution of all the upside comparison systems.

As $(a_{BJ_2^*} + \xi_{J_2^*}) \times (a_{J_2^* W} + \xi_{J_2^*}) = a_{BW} - \xi_{J_2^*}$ and $\xi_{J_1^*} < \xi_{J_2^*}$, we have

$$(a_{BJ_2^*} + \xi_{J_1^*}) \times (a_{J_2^* W} + \xi_{J_1^*}) < (a_{BJ_2^*} + \xi_{J_2^*}) \times (a_{J_2^* W} + \xi_{J_2^*}) = a_{BW} - \xi_{J_2^*},$$

and thus $\xi_{J_1^*}$ is not the feasible ξ value of the pairwise comparison toward criterion $c_{J_2^*}$. Therefore, $\xi_{J_2^*}$ is the optimal objective function value ξ^* of the whole BWM comparison system.

(3) If $(a_{BJ_2^*} + \xi_{J_1^*}) \times (a_{J_2^* W} + \xi_{J_1^*}) < a_{BW} + \xi_{J_1^*}$ or $(a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^* W} - \xi_{J_2^*}) > a_{BW} - \xi_{J_2^*}$, then the optimal modified $\tilde{a}_{BW}$ should satisfy

$$a_{BW} - \xi_{J_2^*} < \tilde{a}_{BW} < a_{BW} + \xi_{J_1^*}.$$

In this case, the optimal $\xi_{J_1^*}$ and $\xi_{J_2^*}$ should satisfy

$$(a_{BJ_1^*} - \xi_{J_1^*}) \times (a_{J_1^* W} - \xi_{J_1^*}) = \tilde{a}_{BW} = (a_{BJ_2^*} + \xi_{J_2^*}) \times (a_{J_2^* W} + \xi_{J_2^*}),$$

and the feasible objective function value ξ of the BWM is $\xi = \max\{\xi_{J_1^*}, \xi_{J_2^*}\}$, which leads to

$$(a_{BJ_1^*} - \xi) \times (a_{J_1^* W} - \xi) \leqslant (a_{BJ_2^*} + \xi) \times (a_{J_2^* W} + \xi).$$

Solving the above inequality, we can obtain that

$$\xi \geqslant \frac{a_{BJ_1^*} \times a_{J_1^* W} - a_{BJ_2^*} \times a_{J_2^* W}}{a_{BJ_1^*} + a_{J_1^* W} + a_{BJ_2^*} + a_{J_2^* W}}.$$

Therefore, the optimal $\xi^* = \min\{\xi\}$ is

$$\xi^* = \min\{\xi\} = \frac{a_{BJ_1^*} \times a_{J_1^* W} - a_{BJ_2^*} \times a_{J_2^* W}}{a_{BJ_1^*} + a_{J_1^* W} + a_{BJ_2^*} + a_{J_2^* W}},$$

where equality holds if and only if

$$(a_{BJ_1^*} - \xi) \times (a_{J_1^* W} - \xi) = (a_{BJ_2^*} + \xi) \times (a_{J_2^* W} + \xi).$$

Synthesizing the results of Propositions 1 to 3, we can derive the analytical form of the unique optimal objective function value ξ^* of the BWM. Furthermore, we can determine the analytical form of the multiple optimal solutions of the BWM. They are expressed in Theorem 3.

Theorem 3. *There might be multiple optimal solutions for the BWM under a not-fully consistent pairwise comparison system with more than three criteria, and we have the following:*

(1) The unique optimal objective function value ξ^* of the BWM is

$$\xi^* = \begin{cases} \xi_{J_1^*}, \ if \ (a_{BJ_2^*} + \xi_{J_1^*}) \times (a_{J_2^*W} + \xi_{J_1^*}) \geqslant a_{BW} + \xi_{J_1^*}, \\ \xi_{J_2^*}, \ if \ (a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^*W} - \xi_{J_2^*}) \leqslant a_{BW} - \xi_{J_2^*}, \\ \dfrac{a_{BJ_1^*} \times a_{J_1^*W} - a_{BJ_2^*} \times a_{J_2^*W}}{a_{BJ_1^*} + a_{J_1^*W} + a_{BJ_2^*} + a_{J_2^*W}}, \ else. \end{cases} \tag{20}$$

(2) The unique optimal modified values in the comparison system are

$$\begin{cases} \begin{cases} \tilde{a}_{BJ_1^*} = a_{BJ_1^*} - \xi_{J_1^*} \\ \tilde{a}_{J_1^*W} = a_{J_1^*W} - \xi_{J_1^*}, \ \ if \ (a_{BJ_2^*} + \xi_{J_1^*}) \times (a_{J_2^*W} + \xi_{J_1^*}) \geq a_{BW} + \xi_{J_1^*}, \\ \tilde{a}_{BW} = a_{BW} + \xi_{J_1^*}, \end{cases} \\ \begin{cases} \tilde{a}_{BJ_2^*} = a_{BJ_2^*} + \xi_{J_2^*} \\ \tilde{a}_{J_2^*W} = a_{J_2^*W} + \xi_{J_2^*}, \ \ if \ (a_{BJ_1^*} - \xi_{J_2^*}) \times (a_{J_1^*W} - \xi_{J_2^*}) \leq a_{BW} - \xi_{J_2^*}, \\ \tilde{a}_{BW} = a_{BW} - \xi_{J_2^*}, \end{cases} \\ \begin{cases} \tilde{a}_{BJ_1^*} = a_{BJ_1^*} - \xi^* \\ \tilde{a}_{J_1^*W} = a_{J_1^*W} - \xi^*, \\ \tilde{a}_{BJ_2^*} = a_{BJ_2^*} + \xi^* \qquad \qquad else. \\ \tilde{a}_{J_2^*W} = a_{J_2^*W} + \xi^*, \\ \tilde{a}_{BW} = (a_{BJ_1^*} - \xi^*) \times (a_{J_1^*W} - \xi^*), \end{cases} \end{cases} \tag{21}$$

(3) The feasible optimal solution for the left modified comparison vectors is

$$\tilde{a}_{jW} \in \left[\max \left\{ \max \left\{ a_{jW} - \xi^*, 1 \right\}, \frac{\tilde{a}_{BW}}{a_{Bj} + \xi^*} \right\}, \min \left\{ a_{jW} + \xi^*, \frac{\tilde{a}_{BW}}{\max \left\{ a_{Bj} - \xi^*, 1 \right\}} \right\} \right] \ with \ \tilde{a}_{Bj} = \frac{\tilde{a}_{BW}}{\tilde{a}_{jW}}, \ \forall \ j \tag{22}$$

or

$$\tilde{a}_{Bj} \in \left[\max \left\{ \max \left\{ a_{Bj} - \xi^*, 1 \right\}, \frac{\tilde{a}_{BW}}{a_{jW} + \xi^*} \right\}, \min \left\{ a_{Bj} + \xi^*, \frac{\tilde{a}_{BW}}{\max \left\{ a_{jW} - \xi^*, 1 \right\}} \right\} \right] \ with \ \tilde{a}_{jW} = \frac{\tilde{a}_{BW}}{\tilde{a}_{Bj}}, \ \forall \ j. \tag{23}$$

Proof. (1) According to the results of Propositions 1 to 3, we can directly summarize the analytical form for the unique optimal objective function value ξ^* of the BWM with Eq. (20).

(2) According to the proof of (1), the optimal ξ^* can be found if and only if one of the following three quadric equations holds

$$\begin{cases} (a_{BJ_1^*} - \xi) \times (a_{J_1^*W} - \xi) = a_{BW} + \xi, \ if \ (a_{BJ_2^*} + \xi) \times (a_{J_2^*W} + \xi) \geqslant a_{BW} + \xi, \\ (a_{BJ_2^*} + \xi) \times (a_{J_2^*W} + \xi) = a_{BW} - \xi, \ if \ (a_{BJ_1^*} - \xi) \times (a_{J_1^*W} - \xi) \leqslant a_{BW} - \xi, \\ (a_{BJ_1^*} - \xi) \times (a_{J_1^*W} - \xi) = (a_{BJ_2^*} + \xi) \times (a_{J_2^*W} + \xi), \ else. \end{cases}$$

Therefore, the unique optimal modified values in the comparison system can be analytically expressed by Eq. (21).

(3) Based on the results of (1) and (2), we know that both the optimal ξ^* and $\tilde{a}_{BW}$ in the BWM are unique and analytical. The optimal modified comparison values $\tilde{a}_{Bj}$ and $\tilde{a}_{jW}$ must satisfy

$$\tilde{a}_{Bj} \in \left[\max \left\{ a_{Bj} - \xi^*, 1 \right\}, a_{Bj} + \xi^* \right],$$
$$\tilde{a}_{jW} \in \left[\max \left\{ a_{jW} - \xi^*, 1 \right\}, a_{jW} + \xi^* \right], \ \forall \ j.$$

Additionally, the optimal modified comparison vectors should be fully consistent so that $\tilde{a}_{Bj} \times \tilde{a}_{jW} = \tilde{a}_{BW}, \ \forall \ j$, so the optimal modified comparison values $\tilde{a}_{Bj}$ and $\tilde{a}_{jW}$ also need to satisfy

$$\tilde{a}_{Bj} = \frac{\tilde{a}_{BW}}{\tilde{a}_{jW}} \in \left[\frac{\tilde{a}_{BW}}{a_{jW} + \xi^*}, \frac{\tilde{a}_{BW}}{\max \left\{ a_{jW} - \xi^*, 1 \right\}} \right],$$
$$\tilde{a}_{jW} = \frac{\tilde{a}_{BW}}{\tilde{a}_{Bj}} \in \left[\frac{\tilde{a}_{BW}}{a_{Bj} + \xi^*}, \frac{\tilde{a}_{BW}}{\max \left\{ a_{Bj} - \xi^*, 1 \right\}} \right], \ \forall j.$$

Then, the feasible regions of the optimal modified comparison values $\tilde{a}_{Bj}$ and $\tilde{a}_{jW}$ should be

$$\tilde{a}_{Bj} \in \left[\frac{\tilde{a}_{BW}}{a_{jW} + \xi^*}, \frac{\tilde{a}_{BW}}{\max \left\{ a_{jW} - \xi^*, 1 \right\}} \right] \cap \left[\max \left\{ a_{Bj} - \xi^*, 1 \right\}, a_{Bj} + \xi^* \right], \ \forall j.$$

$$\tilde{a}_{jW} \in \left[\frac{\tilde{a}_{BW}}{a_{Bj} + \xi^*}, \frac{\tilde{a}_{BW}}{\max \left\{ a_{Bj} - \xi^*, 1 \right\}} \right] \cap \left[\max \left\{ a_{jW} - \xi^*, 1 \right\}, a_{jW} + \xi^* \right], \ \forall \ j.$$

The following inequality must be true:

$$\max \left\{ a_{Bj} - \xi^*, 1 \right\} \times \max \left\{ a_{jW} - \xi^*, 1 \right\} \leqslant \tilde{a}_{BW}$$
$$\Leftrightarrow \begin{cases} \max \left\{ a_{Bj} - \xi^*, 1 \right\} \leqslant \dfrac{\tilde{a}_{BW}}{\max \left\{ a_{jW} - \xi^*, 1 \right\}} \\ \max \left\{ a_{jW} - \xi^*, 1 \right\} \leqslant \dfrac{\tilde{a}_{BW}}{\max \left\{ a_{Bj} - \xi^*, 1 \right\}} \end{cases} \forall j.$$

Thus, the lower and upper bounds of the optimal modified $\tilde{a}_{Bj}$ and $\tilde{a}_{jW}$ values are

$$\inf\{\tilde{a}_{jW}\} = \max \left\{ \max \left\{ a_{jW} - \xi^*, 1 \right\}, \frac{\tilde{a}_{BW}}{a_{Bj} + \xi^*} \right\},$$
$$\sup\{\tilde{a}_{jW}\} = \min \left\{ a_{jW} + \xi^*, \frac{\tilde{a}_{BW}}{\max \left\{ a_{Bj} - \xi^*, 1 \right\}} \right\}, \ \forall j,$$

$$\inf\{\tilde{a}_{Bj}\} = \max \left\{ \max \left\{ a_{Bj} - \xi^*, 1 \right\}, \frac{\tilde{a}_{BW}}{a_{jW} + \xi^*} \right\},$$
$$\sup\{\tilde{a}_{Bj}\} = \min \left\{ a_{Bj} + \xi^*, \frac{\tilde{a}_{BW}}{\max \left\{ a_{jW} - \xi^*, 1 \right\}} \right\}, \ \forall j.$$

Because the optimal $\tilde{a}_{BW}$ value is analytically fixed and the optimal modified $\tilde{a}_{Bj}$ and $\tilde{a}_{jW}$ satisfy $\tilde{a}_{Bj} \times \tilde{a}_{jW} = \tilde{a}_{BW}, \ \forall \ j$, we only need to determine one value between $\tilde{a}_{Bj}$ and $\tilde{a}_{jW}$; then, the feasible optimal solution for the modified comparison vectors is

$$\tilde{a}_{jW} \in \left[\max \left\{ \max \left\{ a_{jW} - \xi^*, 1 \right\}, \frac{\tilde{a}_{BW}}{a_{Bj} + \xi^*} \right\}, \min \left\{ a_{jW} + \xi^*, \frac{\tilde{a}_{BW}}{\max \left\{ a_{Bj} - \xi^*, 1 \right\}} \right\} \right] \text{ with } \tilde{a}_{Bj} = \frac{\tilde{a}_{BW}}{\tilde{a}_{jW}}, \ \forall j$$

or

$$\tilde{a}_{Bj} \in \left[\max \left\{ \max \left\{ a_{Bj} - \xi^*, 1 \right\}, \frac{\tilde{a}_{BW}}{a_{jW} + \xi^*} \right\}, \min \left\{ a_{Bj} + \xi^*, \frac{\tilde{a}_{BW}}{\max \left\{ a_{jW} - \xi^*, 1 \right\}} \right\} \right] \text{ with } \tilde{a}_{jW} = \frac{\tilde{a}_{BW}}{\tilde{a}_{Bj}}, \ \forall j.$$

The original BWM used the optimal objective function value ξ^* as a consistency measurement of a comparison system, where the smaller ξ^* is, the higher the consistency of the comparison system. According to Theorems 2 and 3, the optimal ξ^* value of the BWM is unique and can be determined analytically. Thus, we can derive some interesting properties of this analytical function from a mathematical point of view.

Proposition 4. *The optimal modified function value ξ^* of the BWM satisfies the following:*

(1) ξ^* is a continuous function with respect to the values of $a_{BJ_1^*}$, $a_{J_1^*W}$, and a_{BW}; $a_{BJ_2^*}$, $a_{J_2^*W}$, and a_{BW}; or $a_{BJ_1^*}$, $a_{J_1^*W}$, $a_{BJ_2^*}$ and $a_{J_2^*W}$;

(2) $\xi^* = 0$ if and only if the comparison system is fully consistent;

(3) ξ^* is invariant with respect to a permutation of the indices of the criteria;

(4) For a fully consistent comparison system, moving one of the preferences a_{Bj} or a_{jW} away from its original value in the range $[1, a_{BW}]$ will lead to an increase in the ξ^* value;

(5) If we remove a criterion that is not the best, the worst, or the potentially most inconsistent criterion $c_{J_1^*}$ or $c_{J_2^*}$ from the criteria set, then the ξ^* value of the BWM does not change.

The proofs of the above properties are similar to the proofs in [20], so we omit them here for simplicity. Furthermore, as the optimal modified best to worst value $\tilde{a}_{BW}$ of the BWM has a direct relationship with ξ^*, the optimal analytical form of $\tilde{a}_{BW}$ has the following properties:

Proposition 5. *The optimal modified best to worst value $\tilde{a}_{BW}$ of the BWM satisfies the following:*

(1) $\tilde{a}_{BW}$ is a continuous function with respect to the values of $a_{BJ_1^*}$, $a_{J_1^*W}$, and a_{BW}; $a_{BJ_2^*}$, $a_{J_2^*W}$, and a_{BW}; or $a_{BJ_1^*}$, $a_{J_1^*W}$, $a_{BJ_2^*}$ and $a_{J_2^*W}$;

(2) $\tilde{a}_{BW} = a_{BW}$ if and only if the comparison system is fully consistent;

(3) $\tilde{a}_{BW}$ is invariant with respect to a permutation of the indices of the criteria;

(4) If we remove a criterion that is not the best, the worst, or the potentially most inconsistent criteria from the criteria set, then the $\tilde{a}_{BW}$ value of the BWM does not change.

Theorem 4. *The analytical forms of the lower and upper bounds of the criteria weights are*

$$\begin{cases} \bar{w}_J = \dfrac{\sup\{\tilde{a}_{JW}\}}{\sup\{\tilde{a}_{JW}\} + \sum_{j\neq J}\inf\{\tilde{a}_{jW}\}} \\ \quad = \dfrac{\min\left\{a_{JW}+\xi^*, \frac{\tilde{a}_{BW}}{\max\{a_{BJ}-\xi^*,1\}}\right\}}{\min\left\{a_{JW}+\xi^*, \frac{\tilde{a}_{BW}}{\max\{a_{BJ}-\xi^*,1\}}\right\} + \sum_{j\neq J}\max\left\{\max\{a_{jW}-\xi^*,1\}, \frac{\tilde{a}_{BW}}{a_{Bj}+\xi^*}\right\}}, \\ \underline{w}_J = \dfrac{\inf\{\tilde{a}_{JW}\}}{\inf\{\tilde{a}_{JW}\} + \sum_{j\neq J}\sup\{\tilde{a}_{jW}\}} \\ \quad = \dfrac{\max\left\{\max\{a_{jW}-\xi^*,1\}, \frac{\tilde{a}_{BW}}{a_{BJ}+\xi^*}\right\}}{\max\left\{\max\{a_{jW}-\xi^*,1\}, \frac{\tilde{a}_{BW}}{a_{BJ}+\xi^*}\right\} + \sum_{j\neq J}\min\left\{a_{jW}+\xi^*, \frac{\tilde{a}_{BW}}{\max\{a_{Bj}-\xi^*,1\}}\right\}}, \end{cases} \tag{24}$$

where the unique optimal ξ^* and $\tilde{a}_{BW}$ values are determined by Eqs. (20) and *(21), respectively.*

Proof. According to Theorem 3, the feasible optimal range for the modified $\tilde{a}_{jW}$ is

$$\tilde{a}_{jW} \in \left[\max\left\{\max\{a_{jW}-\xi^*,1\}, \frac{\tilde{a}_{BW}}{a_{Bj}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{jW}+\xi^*, \frac{\tilde{a}_{BW}}{\max\{a_{Bj}-\xi^*,1\}}\right\}\right], \ \forall j.$$

The optimal modified comparison vectors $\tilde{A}_B$ and $\tilde{A}_W$ construct a fully consistent comparison system. Therefore, the criteria weights can be computed based on only modified OW vector $\tilde{A}_W$ as

$$w_J^{OW} = \frac{\tilde{a}_{JW}}{\tilde{a}_{JW} + \sum_{j\neq J}\tilde{a}_{jW}}.$$

w_J^{OW} is a monotonically increasing function of $\tilde{a}_{JW}$ and a monotonically decreasing function of $\tilde{a}_{jW}(j \neq J)$. Because the optimal modified $\tilde{a}_{BW}$ value is unique and analytically determined, the optimal modified $\tilde{a}_{jW}(j \neq B)$ values are mutually independent. The lower bound weight $\underline{w}_J$ of criteria c_J can be derived when $\tilde{a}_{JW}$ reaches its lower bound $\inf\{\tilde{a}_{JW}\}$ while the other $\tilde{a}_{jW}(j \neq J)$ reach their upper bounds $\sup\{\tilde{a}_{jW}\}(j \neq J)$ so that

$$\underline{w}_J = \frac{\inf\{\tilde{a}_{JW}\}}{\inf\{\tilde{a}_{JW}\} + \sum_{j\neq J}\sup\{\tilde{a}_{jW}\}}$$
$$= \frac{\max\left\{\max\{a_{jW}-\xi^*,1\}, \frac{\tilde{a}_{BW}}{a_{BJ}+\xi^*}\right\}}{\max\left\{\max\{a_{jW}-\xi^*,1\}, \frac{\tilde{a}_{BW}}{a_{BJ}+\xi^*}\right\} + \sum_{j\neq J}\min\left\{a_{jW}+\xi^*, \frac{\tilde{a}_{BW}}{\max\{a_{Bj}-\xi^*,1\}}\right\}},$$

and the upper bound $\bar{w}_J$ of the weight of criteria c_J can be derived when the value $\tilde{a}_{JW}$ reaches its upper bound $\sup\{\tilde{a}_{JW}\}$ while the other $\tilde{a}_{jW}(j \neq J)$ values reach their lower bounds $\inf\{\tilde{a}_{jW}\}(j \neq J)$ so that

$$\bar{w}_J = \frac{\sup\{\tilde{a}_{JW}\}}{\sup\{\tilde{a}_{JW}\} + \sum_{j\neq J}\inf\{\tilde{a}_{jW}\}}$$
$$= \frac{\min\left\{a_{JW}+\xi^*, \frac{\tilde{a}_{BW}}{\max\{a_{Bj}-\xi^*,1\}}\right\}}{\min\left\{a_{JW}+\xi^*, \frac{\tilde{a}_{BW}}{\max\{a_{Bj}-\xi^*,1\}}\right\} + \sum_{j\neq J}\max\left\{\max\{a_{jW}-\xi^*,1\}, \frac{\tilde{a}_{BW}}{a_{Bj}+\xi^*}\right\}}.$$

□

Based on Eq. (24), we can analytically calculate the interval weights of the criteria instead of solving the optimization programming models one by one. Similar to the method in [36], we can use the center of the intervals to rank the criteria or alternatives. We can also compare and rank the interval weights based on the preference degree and preference matrix.

3.3. A secondary objective function for determining the unique optimal solution

Interval analysis is effective in helping analyze the multiple optimal solutions in the nonlinear BWM. However, in some cases, DMs prefer to have a unique solution. Next, we present a secondary objective function to enhance multi-optimality, which can preserve the solution characteristics of the nonlinear BWM.

From the previous analytical results, we know that at least three pairs of evaluation values are analytically fixed (criteria $c_{J_1^*}$, $c_{J_2^*}$, c_B, and c_W, which are denoted as J_{Fix}). Because the optimal modified $\tilde{a}_{BW}$ value is unique and analytically determined, the optimal modified pair of values for the optimized criteria are mutually independent. In line with the min–max strategy of the BWM, we aim to determine the optimal modified pairwise values, such that the maximum adjustment deviations $\left|\tilde{a}_{Bj} - a_{Bj}\right|$ and $\left|\tilde{a}_{jW} - a_{jW}\right|$ for each criterion $c_j (j \notin J_{Fix})$ are minimized, where

$$\min\max\left\{\left|\tilde{a}_{Bj} - a_{Bj}\right|, \left|\tilde{a}_{jW} - a_{jW}\right|\right\}, \ \forall j \notin J_{Fix}.$$

In addition, the maximum adjustment range of each evaluation value should satisfy

$$\left|\tilde{a}_{Bj} - a_{Bj}\right| \leqslant \xi^*, \left|\tilde{a}_{jW} - a_{jW}\right| \leqslant \xi^*, \ \forall j \notin J_{Fix}.$$

After the modification, the modified comparison system is fully consistent, where

$$\tilde{a}_{Bj} \times \tilde{a}_{jW} = \tilde{a}_{BW}, \ \forall j \notin J_{Fix}.$$

Therefore, we can establish the following optimization model for each criterion $c_j (j \notin J_{Fix})$

$$\begin{aligned} &\min\max\left\{\left|\tilde{a}_{Bj} - a_{Bj}\right|, \left|\tilde{a}_{jW} - a_{jW}\right|\right\} \\ &s.t. \begin{cases} \left|\tilde{a}_{Bj} - a_{Bj}\right| \leqslant \xi^*, \ \forall j \notin J_{Fix}, \\ \left|\tilde{a}_{jW} - a_{jW}\right| \leqslant \xi^*, \ \forall j \notin J_{Fix}, \\ \tilde{a}_{Bj} \times \tilde{a}_{jW} = \tilde{a}_{BW}, \ \forall j \notin J_{Fix}, \\ \qquad \tilde{a}_{Bj}, \tilde{a}_{jW} \geqslant 1. \end{cases} \end{aligned} \tag{25}$$

Theorem 5. *Model (25) can be transformed into the following two optimization models, where*

(1) If $a_{Bj} \times a_{jW} < \tilde{a}_{BW}$, model (25) is equivalent to the following optimization model

$$\begin{aligned} &\min \eta_j \\ &s.t. \begin{cases} \tilde{a}_{Bj} - a_{Bj} = \eta_{Bj}, \\ \tilde{a}_{jW} - a_{jW} = \eta_{jW}, \\ (a_{Bj} + \eta_{Bj}) \times (a_{jW} + \eta_{jW}) = \tilde{a}_{BW}, \\ 0 \leqslant \eta_{Bj} \leqslant \eta_j, \\ 0 \leqslant \eta_{jW} \leqslant \eta_j. \end{cases} \end{aligned} \tag{26}$$

(2) If $a_{Bj} \times a_{jW} > \tilde{a}_{BW}$, model (25) is equivalent to the following optimization model

$$\min \eta_j \quad s.t. \begin{cases} a_{Bj} - \tilde{a}_{Bj} = \eta_{Bj}, \\ a_{jW} - \tilde{a}_{jW} = \eta_{jW}, \\ (a_{Bj} - \eta_{Bj}) \times (a_{jW} - \eta_{jW}) = \tilde{a}_{BW}, \\ 0 \leqslant \eta_{Bj} \leqslant \eta_j, \\ 0 \leqslant \eta_{jW} \leqslant \eta_j. \end{cases} \tag{27}$$

Proof. To prove that model (26) is equivalent to model (25) when $a_{Bj} \times a_{jW} < \tilde{a}_{BW}$, we only need to prove that to make the modified evaluations consistent, the optimal adjustment strategy for a_{Bj} and a_{jW} is

$$\tilde{a}_{Bj} = a_{Bj} + \eta_{Bj},\ \tilde{a}_{jW} = a_{jW} + \eta_{jW},\ \eta_{Bj}, \eta_{jW} \geqslant 0.$$

First, if $a_{Bj} \times a_{jW} < \tilde{a}_{BW}$, then $(a_{Bj} - \eta_{Bj}) \times (a_{jW} - \eta_{jW}) < \tilde{a}_{BW}$, so the adjustment strategy

$$\tilde{a}_{Bj} = a_{Bj} - \eta_{Bj},\ \tilde{a}_{jW} = a_{jW} - \eta_{jW},\ \eta_{Bj}, \eta_{jW} \geqslant 0$$

is not feasible for the modification. Then, we assume that the optimal adjustment strategy for a_{Bj} and a_{jW} is

$$\tilde{a}_{Bj} = a_{Bj} - \eta_{Bj},\ \tilde{a}_{jW} = a_{jW} + \eta_{jW},\ \eta_{Bj}, \eta_{jW} \geqslant 0$$

or

$$\tilde{a}_{Bj} = a_{Bj} + \eta_{Bj},\ \tilde{a}_{jW} = a_{jW} - \eta_{jW},\ \eta_{Bj}, \eta_{jW} \geqslant 0.$$

Regarding the above two strategies, we can find a pair of feasible η'_{Bj} and η'_{jW}

$$\eta'_{Bj} = \eta_{Bj} - \Delta_{Bj}, \eta'_{jW} = \eta_{jW} - \Delta_{jW}, \Delta_{Bj}, \Delta_{jW} \geqslant 0$$

such that the modified evaluations remain consistent. However, η'_{Bj} and η'_{jW} satisfy

$$\min\max\left\{\eta'_{Bj}, \eta'_{jW}\right\} \leqslant \min\max\left\{\eta_{Bj}, \eta_{jW}\right\},$$

which is contrary to the assumption that the modification rule is the optimal adjustment strategy. Therefore, the optimal adjustment strategy for a_{Bj} and a_{jW} has the form of Eq. (26) when $a_{Bj} \times a_{jW} < \tilde{a}_{BW}$.

In addition, as $(a_{Bj} + \xi^*) \times (a_{jW} + \xi^*) \geqslant \tilde{a}_{BW}$, there must exist $\eta_{Bj}, \eta_{jW} \leqslant \xi^*$ that make the equality $(a_{Bj} + \eta_{Bj}) \times (a_{jW} + \eta_{jW}) = \tilde{a}_{BW}$ true, thus the range of η_{Bj} and η_{jW} is $\eta_{Bj}, \eta_{jW} \geqslant 0$. Consequently, we can obtain that model (25) is equivalent to model (26) if $a_{Bj} \times a_{jW} < \tilde{a}_{BW}$. In the same way, we can prove that model (25) is equivalent to model (27) if $a_{Bj} \times a_{jW} > \tilde{a}_{BW}$, and it is omitted here.

After determining two equivalent programming models (26) and (27) of the optimization model (25), we can analytically solve the two optimization models (26) and (27), and their optimal solutions are shown in Theorem 6.

Theorem 6. *The optimal objective function value η_j^* of model (26) or (27) is one root of the following quadric equation*

$$\begin{cases} (a_{Bj} - \eta_j) \times (a_{jW} - \eta_j) = \tilde{a}_{BW},\ if\ a_{Bj} \times a_{jW} > \tilde{a}_{BW} \\ (a_{Bj} + \eta_j) \times (a_{jW} + \eta_j) = \tilde{a}_{BW},\ if\ a_{Bj} \times a_{jW} < \tilde{a}_{BW} \end{cases} \tag{28}$$

and thus

$$\eta_j^* = \begin{cases} \dfrac{a_{Bj} + a_{jW} - \sqrt{(a_{Bj} + a_{jW})^2 - 4(a_{Bj} \times a_{jW} - \tilde{a}_{BW})}}{2}, \\ \qquad if\ a_{Bj} \times a_{jW} > \tilde{a}_{BW} \\ \dfrac{-(a_{Bj} + a_{jW}) + \sqrt{(a_{Bj} + a_{jW})^2 - 4(a_{Bj} \times a_{jW} - \tilde{a}_{BW})}}{2}, \\ \qquad if\ a_{Bj} \times a_{jW} < \tilde{a}_{BW} \end{cases} \tag{29}$$

Table 1
Comparison vectors for the transportation mode selection.

Criteria	C_1	C_2	C_3
Best criterion: C_3	8	2	1
Worst criterion: C_1	1	5	8

Thus the unique optimal solution for the modified comparison vectors is

$$\begin{cases} \begin{cases} \tilde{a}_{Bj} = a_{Bj} + \eta_j^*, \\ \tilde{a}_{jW} = a_{jW} + \eta_j^*, \end{cases} if\ a_{Bj} \times a_{jW} < a_{BW} \\ \begin{cases} \tilde{a}_{Bj} = a_{Bj} - \eta_j^*, \\ \tilde{a}_{jW} = a_{jW} - \eta_j^*, \end{cases} if\ a_{Bj} \times a_{jW} > a_{BW} \end{cases} \tag{30}$$

Proof. The proof is similar to the proof of Theorem 2, and it is omitted.

4. Example validation

This section provides several numerical examples to validate the feasibility and effectiveness of the proposed analytical framework for the BWM. Specifically, some of the data in the evaluation example are adopted from existing BWM papers. In addition, to verify the correctness of our obtained theoretical results, we make some adjustments to the data based on actual demand. Finally, we make a comparative analysis with existing BWM models and discuss it to illustrate the convenience and benefits of the proposed method.

4.1. Numerical examples

First, considering a not-fully consistent BWM comparison system with three criteria, we use the analytical formulas in Theorem 2 to calculate the unique optimal solution.

Example 1. [9] A company aims to transport its products to a market, and the transportation mode selection problem involves three decision criteria: C_1–load flexibility, C_2–accessibility, and C_3–cost. The company identifies cost (C_3) and load flexibility (C_1) as the best and worst criteria respectively. Table 1 shows the comparison vectors.

With the data in Table 1, as $2 \times 5 > 8$, based on the results Eqs. (16) and (19) of Theorem 2, we have

$$\xi^* = \frac{2 + 5 + 1 - \sqrt{(2 + 5 + 1)^2 - 4(2 \times 5 - 8)}}{2} = 0.2583,$$

$$w_1 = \frac{1}{5 + 8 + 1} = 0.0714, w_2 = \frac{5 - 0.2583}{5 + 8 + 1} = 0.3387,$$
$$w_3 = \frac{8 + 0.2583}{5 + 8 + 1} = 0.5899.$$

Furthermore, we discuss how different a_{32} (a_{Bj}) and a_{21} (a_{jW}) values affect the criteria weights and the ξ^* value. As $a_{BW} = 8$, we select different values for $a_{32}, a_{21} \in \{1, 2, \ldots, 8\}$. Based on the analytical forms (17) and (19) for ξ^* and the optimal criteria weights, we can directly draw their function images. Fig. 1 shows the relationship between a_{32}, a_{21}, and ξ^*. As shown in Fig. 1, when $a_{32} \times a_{21}$ is close to $a_{BW} = a_{31} = 8$, ξ^* is close to zero, which provides high consistency; when $a_{32} \times a_{21}$ is far from $a_{31} = 8$, ξ^* becomes larger, which shows a low consistency. The maximum ξ^* value is 4.4689 when both a_{32} and a_{21} are assigned the maximum value of 8, leading to the most inconsistent situation.

Fig. 2 displays the variation in the weights of the three criteria with different values for a_{32} and a_{21}. As seen from the simulation of the 3D projective view, the weight w_1 of the worst criterion c_1 is only affected by the a_{jW} (a_{21}) value (not by the a_{Bj} (a_{32}) value). Additionally, the weight w_3 of the best criterion c_3 decreases as the a_{jW} (a_{21}) value increases, while the weight w_2 of criterion c_2 increases as the a_{jW} (a_{21})

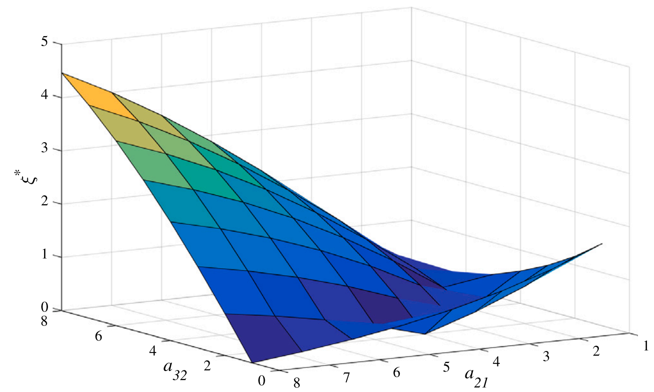

Fig. 1. The variation of ξ^* in a comparison system with three criteria when $a_{BW} = 8$.

Table 2
Comparison vectors for the car purchase.

Criteria	C_1	C_2	C_3	C_4	C_5
Best criterion: C_2	3	1	4	3	8
Worst criterion: C_5	4	8	4	3	1

value increases. The dynamic results of the optimal criteria weights and the ξ^* value are consistent with their mathematical functions, Eqs. (17) and (19).

The above calculation results are the same as the results in [9], which indicates that the results of Theorem 2 are mathematically correct and our method is effective. Instead of solving the programming model one by one with one pair of a_{32} and a_{21} values to obtain the results, we directly substitute them into the analytical functions Eqs. (17) and (19), leading to more general and convenient solutions.

Second, considering that a not-fully consistent BWM comparison system with more than three criteria might have multiple optimal solutions, we use the following three Examples 2–4, which correspond to the three conditions in Propositions 1–3, and then use the analytical formulas in Theorem 3 to compute their multiple optimal solutions.

Example 2. A buyer wants to buy a car, and he considers the following five criteria: C_1–quality, C_2–price, C_3–comfort, C_4–safety, and C_5–style. The buyer identifies price (C_2) and style (C_5) as the best and worst criteria, respectively. The buyer provides his comparison vectors, as shown in Table 2.

First, the inconsistent criteria are C_1, C_3, and C_4, and they are all upside criteria. Based on Eq. (16), we can compute their separate optimal ξ^* values such that

$$\xi_1^* = \frac{3+4+1-\sqrt{(3+4+1)^2-4(3\times 4-8)}}{2} = 0.5359,$$

$$\xi_3^* = \frac{4+4+1-\sqrt{(4+4+1)^2-4(4\times 4-8)}}{2} = 1,$$

$$\xi_4^* = \frac{3+3+1-\sqrt{(3+3+1)^2-4(3\times 3-8)}}{2} = 0.1459.$$

Based on Proposition 1, the unique optimal ξ^* is $\xi^* = \xi_3^* = 1$. Then, the unique optimal modified values $\tilde{a}_{23}$ and $\tilde{a}_{35}$ for criterion C_3 and $\tilde{a}_{25}$ ($\tilde{a}_{BW}$) are

$$\tilde{a}_{23} = a_{23} - \xi^* = 3, \tilde{a}_{35} = a_{35} - \xi^* = 3, \tilde{a}_{25} = a_{25} + \xi^* = 9.$$

Using Eq. (22), the ranges of the optimal feasible solutions for $\tilde{a}_{15}$ and $\tilde{a}_{45}$ are

$$\tilde{a}_{15} \in \left[\max\left\{\max\left\{a_{15}-\xi^*,1\right\}, \frac{\tilde{a}_{25}}{a_{21}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{15}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{21}-\xi^*,1\right\}}\right\}\right] = [3, 4.5],$$

$$\tilde{a}_{45} \in \left[\max\left\{\max\left\{a_{45}-\xi^*,1\right\}, \frac{\tilde{a}_{25}}{a_{24}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{45}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{24}-\xi^*,1\right\}}\right\}\right] = [2.25, 4].$$

Therefore, we can obtain the optimal feasible solutions of the modified OW and BO vectors such that

$$\tilde{a}_{15} \in [3, 4.5], \tilde{a}_{25} = 9, \tilde{a}_{35} = 3, \tilde{a}_{45} \in [2.25, 4], \tilde{a}_{55} = 1 \text{ with } \tilde{a}_{2j} = \frac{\tilde{a}_{25}}{\tilde{a}_{j5}}, \ \forall j.$$

Choosing any two $\tilde{a}_{15}$ and $\tilde{a}_{45}$ values in their optimal feasible ranges and using Eq. (12), we can obtain a group of optimal weights for the criteria.

Table 3
Comparison vectors for the car purchase.

Criteria	C_1	C_2	C_3	C_4	C_5
Best criterion: C_2	3	1	2	2	8
Worst criterion: C_5	2	8	2	4	1

Example 3. Suppose the comparison vectors for the car purchase are as shown in Table 3.

First, the inconsistent criteria are C_1 and C_3, and they are all downside criteria. Based on Eq. (16), we can compute their separate optimal ξ^* values such that

$$\xi_1^* = -\frac{3+2+1-\sqrt{(3+2+1)^2-4(3\times 2-8)}}{2} = 0.3166,$$

$$\xi_3^* = -\frac{2+2+1-\sqrt{(2+2+1)^2-4(2\times 2-8)}}{2} = 0.7016.$$

Based on Proposition 2, the unique optimal ξ^* of the BWM is $\xi^* = \xi_3^* = 0.7016$; then, the unique optimal modified values $\tilde{a}_{23}$ and $\tilde{a}_{35}$ for criterion C_3 and $\tilde{a}_{25}$ ($\tilde{a}_{BW}$) are

$$\tilde{a}_{23} = a_{23} + \xi^* = 2.7016, \tilde{a}_{35} = a_{35} + \xi^* = 2.7016, \tilde{a}_{25} = a_{25} - \xi^* = 7.2984.$$

Based on Eq. (22), the ranges of the optimal feasible solutions for $\tilde{a}_{15}$ and $\tilde{a}_{45}$ are

$$\tilde{a}_{15} \in \left[\max\left\{\max\left\{a_{15}-\xi^*,1\right\}, \frac{\tilde{a}_{25}}{a_{21}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{15}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{21}-\xi^*,1\right\}}\right\}\right] = [1.9717, 2.7016],$$

$$\tilde{a}_{45} \in \left[\max\left\{\max\left\{a_{45}-\xi^*,1\right\}, \frac{\tilde{a}_{25}}{a_{24}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{45}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{24}-\xi^*,1\right\}}\right\}\right] = [3.2984, 4.7016].$$

Therefore, we can obtain the optimal feasible solutions of the modified OW and BO vectors such that

$$\tilde{a}_{15} \in [1.9717, 2.7016], \tilde{a}_{25} = 7.2984, \tilde{a}_{35} = 2.7016,$$
$$\tilde{a}_{45} \in [3.2984, 4.7016], \tilde{a}_{55} = 1 \text{ with } \tilde{a}_{2j} = \frac{\tilde{a}_{25}}{\tilde{a}_{j5}}, \ \forall j.$$

Choosing any two $\tilde{a}_{15}$ and $\tilde{a}_{45}$ values in their optimal feasible ranges and using Eq. (12), we can obtain a group of optimal weights for the criteria.

Example 4. Suppose the comparison vectors for the car purchase are as shown in Table 4.

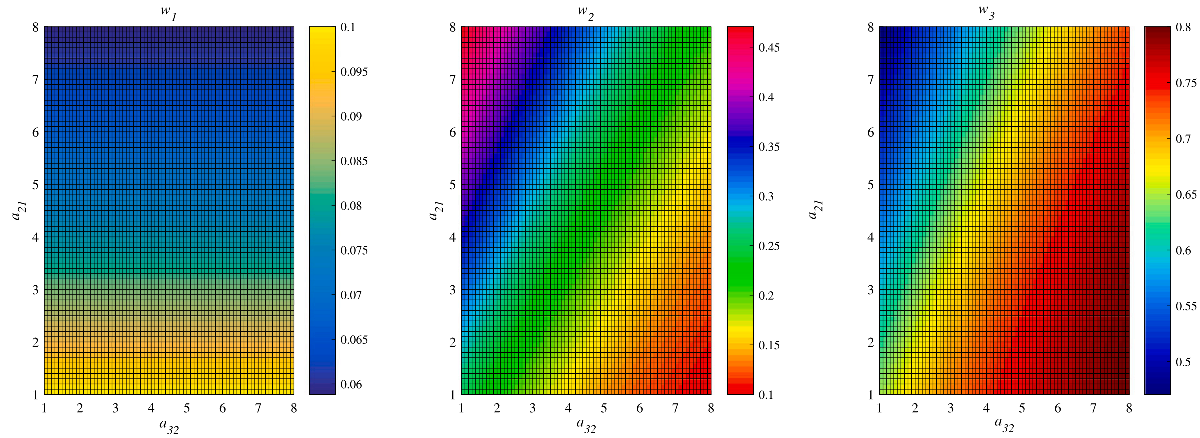

Fig. 2. The variation of criteria weights with different values for a_{32} and a_{21}.

Table 4
Comparison vectors for the car purchase.

Criteria	C_1	C_2	C_3	C_4	C_5
Best criterion: C_2	3	1	2	3	8
Worst criterion: C_5	4	8	2	2	1

Table 5
Comparison vectors for the car purchase.

Criteria	C_1	C_2	C_3	C_4	C_5
Best criterion: C_2	2	1	4	3	8
Worst criterion: C_5	4	8	2	3	1

First, the inconsistent criteria are C_1, C_3, and C_4, and they include both upside and downside criteria. Based on Eq. (16), we can compute their separate optimal ξ^* values such that

$$\xi_1^* = \frac{3+4+1-\sqrt{(3+4+1)^2-4(3\times 4-8)}}{2} = 0.5359,$$

$$\xi_3^* = -\frac{2+2+1-\sqrt{(2+2+1)^2-4(2\times 2-8)}}{2} = 0.7016,$$

$$\xi_4^* = -\frac{3+2+1-\sqrt{(3+2+1)^2-4(3\times 2-8)}}{2} = 0.3166.$$

As $(a_{21}-\xi_3^*)\times(a_{15}-\xi_3^*) = 2.2984\times 3.2984 = 7.5810 > a_{25}-\xi_3^* = 7.2984$, ξ_3^* is not the feasible solution for criterion C_1. Based on Proposition 3, the unique optimal ξ^* of the BWM is

$$\xi^* = \frac{3\times 4-2\times 2}{3+4+2+2} = 0.7273.$$

Then, the unique optimal modified $\tilde{a}_{21}$ and $\tilde{a}_{15}$ for criterion C_1, $\tilde{a}_{23}$ and $\tilde{a}_{35}$ for criterion C_3, and $\tilde{a}_{25}$ are

$$\tilde{a}_{21} = a_{21}-\xi^* = 2.2727, \tilde{a}_{15} = a_{15}-\xi^* = 3.2727, \tilde{a}_{23} = a_{23}+\xi^* = 2.7273,$$
$$\tilde{a}_{35} = a_{35}+\xi^* = 2.7273, \tilde{a}_{25} = \tilde{a}_{21}\times\tilde{a}_{15} = (a_{21}-\xi^*)\times(a_{15}-\xi^*) = 7.4380.$$

Based on Eq. (22), the feasible optimal solution for $\tilde{a}_{45}$ is

$$\tilde{a}_{45} \in \left[\max\left\{\max\left\{a_{45}-\xi^*, 1\right\}, \frac{\tilde{a}_{25}}{a_{24}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{45}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{24}-\xi^*, 1\right\}}\right\}\right] = [1.9956, 2.7273].$$

Therefore, we can obtain the optimal feasible solutions of the modified BO and OW vectors such that

$$\tilde{a}_{15} = 3.2727, \tilde{a}_{25} = 7.4380, \tilde{a}_{35} = 2.7273, \tilde{a}_{45} \in [1.9956, 2.7273],$$
$$\tilde{a}_{55} = 1 \text{ with } \tilde{a}_{2j} = \frac{\tilde{a}_{25}}{\tilde{a}_{j5}}, \forall j.$$

Finally, we can obtain a group of optimal weights for the criteria.

The calculations of Example 2–4 on three typical inconsistent comparison systems correspond to the results of Proposition 1–3 and Theorem 3. They yield the unique optimal objective function value ξ^*, the unique optimal modified best to worst value $\tilde{a}_{BW}$ and the feasible optimal modified comparison vectors for the equivalent BWM using the analytical functions directly. This method takes advantage of the inherent strengths of the analytical solutions to make the whole computation process more accessible or more convenient.

Third, to calculate the lower and upper bounds of the criteria weights when multiple optimal solutions exist, we use the analytical formulas in Theorem 4 to calculate the interval weights for the criteria, which can then be used to rank the criteria.

Example 5. Consider the data in [36] for the car purchase problem, as shown in Table 5.

First, the inconsistent criterion is C_4. Based on Eq. (16), we can compute its optimal ξ^* value such that

$$\xi_4^* = \frac{3+3+1-\sqrt{(3+3+1)^2-4(3\times 3-8)}}{2} = 0.1459.$$

The unique optimal ξ^* of the BWM is $\xi^* = 0.1459$, and then the unique optimal modified values $\tilde{a}_{24}$ and $\tilde{a}_{45}$ for criterion C_4 and $\tilde{a}_{25}$ ($\tilde{a}_{BW}$) are

$$\tilde{a}_{24} = a_{24}-\xi_4^* = 2.8541, \tilde{a}_{45} = a_{45}-\xi_4^* = 2.8541, \tilde{a}_{25} = a_{25}+\xi_4^* = 8.1459.$$

Based on Eq. (22), the feasible optimal solutions for $\tilde{a}_{15}$ and $\tilde{a}_{35}$ are

$$\tilde{a}_{15} \in \left[\max\left\{\max\left\{a_{15}-\xi^*, 1\right\}, \frac{\tilde{a}_{25}}{a_{21}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{15}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{21}-\xi^*, 1\right\}}\right\}\right] = [3.8541, 4.1459],$$

$$\tilde{a}_{35} \in \left[\max\left\{\max\left\{a_{35}-\xi^*, 1\right\}, \frac{\tilde{a}_{25}}{a_{23}+\xi^*}\right\},\right.$$
$$\left.\min\left\{a_{35}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{23}-\xi^*, 1\right\}}\right\}\right] = [1.9648, 2.1136].$$

Therefore, we can obtain the lower and upper bounds of the optimal modified BO and OW vectors. Specifically, the ranges of the optimal modified OW vector are shown in the first two lines of Table 6.

Finally, based on Eq. (24) and the ranges of the optimal modified OW vector, we can calculate the lower and upper bounds of the optimal

Table 6
Ranges of the optimal modified OW vector and the optimal criteria weights.

Optimal solutions	Ranges	$\tilde{a}_{15}(w_1)$	$\tilde{a}_{25}(w_2)$	$\tilde{a}_{35}(w_3)$	$\tilde{a}_{45}(w_4)$	$\tilde{a}_{55}(w_5)$
Optimal modified values	Lower bounds	3.8541	8.1459	1.9648	2.8541	1
	Upper bounds	4.1459	8.1459	2.1136	2.8541	1
Optimal criteria weights	Lower bounds	0.2145	0.4461	0.1085	0.1563	0.0545
	Upper bounds	0.2289	0.4571	0.1176	0.1602	0.0561

Table 7
BWM comparison vectors for the car purchase.

Criteria	C_1	C_2	C_3	C_4	C_5
Best criterion: C_2	2	1	4	3	8
Worst criterion: C_5	4	8	4	2	1

weights of the criteria. Taking the weight of criterion C_1, for instance, the computations are

$$\underline{w}_1 = \frac{\inf\{\tilde{a}_{15}\}}{\inf\{\tilde{a}_{15}\} + \sum_{j\neq 1}\sup\{\tilde{a}_{15}\}} = \frac{3.8541}{3.8541 + 8.1459 + 2.1136 + 2.8541 + 1} = 0.2145,$$

$$\bar{w}_1 = \frac{\sup\{\tilde{a}_{15}\}}{\sup\{\tilde{a}_{15}\} + \sum_{j\neq 1}\inf\{\tilde{a}_{j5}\}} = \frac{4.1459}{4.1459 + 8.1459 + 1.9648 + 2.8541 + 1} = 0.2289.$$

Similarly, we can compute the ranges of the optimal weights of the other criteria, and the results are summarized in the last two lines of Table 6.

Example 6. Consider the data in [36] for the car purchase problem, as shown in Table 7.

First, the inconsistent criteria are C_3 and C_4. Based on Eq. (16), their optimal ξ^* values are

$$\xi_3^* = \frac{4+4+1-\sqrt{(4+4+1)^2 - 4(4\times 4 - 8)}}{2} = 1,$$

$$\xi_4^* = -\frac{3+2+1-\sqrt{(3+2+1)^2 - 4(3\times 2 - 8)}}{2} = 0.3166.$$

Because $(a_{24}+\xi_3^*)\times(a_{45}+\xi_3^*) = 12 > a_{25}+\xi_3^* = 9$, the unique optimal ξ^* of the BWM model is $\xi^* = \xi_3^* = 1$; then, the unique optimal modified values $\tilde{a}_{23}$ and $\tilde{a}_{35}$ for criterion C_3 and $\tilde{a}_{25}$ ($\tilde{a}_{BW}$) are

$$\tilde{a}_{23} = a_{23} - \xi_3^* = 3, \tilde{a}_{35} = a_{35} - \xi_3^* = 3, \tilde{a}_{25} = a_{25} + \xi_3^* = 9.$$

Based on Eq. (22), the feasible optimal solutions for $\tilde{a}_{15}$ and $\tilde{a}_{45}$ are

$$\tilde{a}_{15} \in \left[\max\left\{\max\left\{a_{15}-\xi^*, 1\right\}, \frac{\tilde{a}_{25}}{a_{21}+\xi^*}\right\}, \min\left\{a_{15}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{21}-\xi^*, 1\right\}}\right\}\right] = [3, 5],$$

$$\tilde{a}_{45} \in \left[\max\left\{\max\left\{a_{45}-\xi^*, 1\right\}, \frac{\tilde{a}_{25}}{a_{24}+\xi^*}\right\}, \min\left\{a_{45}+\xi^*, \frac{\tilde{a}_{25}}{\max\left\{a_{24}-\xi^*, 1\right\}}\right\}\right] = [2.25, 3].$$

Therefore, the optimal modified OW vector ranges are as shown in the first two lines of Table 8. Based on Eq. (24) and the ranges of the optimal modified OW vector, we can calculate the optimal interval criteria weights, and the results are summarized in the last two lines of Table 8. Specifically, the computations for criterion C_1 are

$$\underline{w}_1 = \frac{\inf\{\tilde{a}_{15}\}}{\inf\{\tilde{a}_{15}\} + \sum_{j\neq 1}\sup\{\tilde{a}_{15}\}} = \frac{3}{3+9+3+3+1} = 0.1579,$$

$$\bar{w}_1 = \frac{\sup\{\tilde{a}_{15}\}}{\sup\{\tilde{a}_{15}\} + \sum_{j\neq 1}\inf\{\tilde{a}_{j5}\}} = \frac{5}{5+9+3+2.25+1} = 0.2469.$$

Examples 5 and 6 utilize the same data and yield the same results as in [36]. The derived results confirm the effectiveness and precision of the method. The interval weights contribute to weighting and ranking the criteria for the case of multiple optimal solutions.

Finally, to determine a unique optimal solution among the multiple optimal solutions of Examples 5 and 6, we use the developed secondary objective function in Section 3.3 and the analytical formulas in Theorem 6 to obtain the unique optimal criteria weights.

Example 7. We continue with the data in Example 5 for buying a car. Because there are multiple optimal solutions regarding the criteria weights, we use the proposed secondary goal function to determine a unique optimal solution. As there are multiple optimal modified values of criteria C_1 and C_3, we can construct the following two optimization models

$$\min\max\left\{\left|\tilde{a}_{21}-a_{21}\right|, \left|\tilde{a}_{15}-a_{15}\right|\right\} \quad s.t. \begin{cases}\left|\tilde{a}_{21}-a_{21}\right| \leqslant \xi^*,\\ \left|\tilde{a}_{15}-a_{15}\right| \leqslant \xi^*,\\ \tilde{a}_{21}\times\tilde{a}_{15} = \tilde{a}_{25},\\ \tilde{a}_{21}, \tilde{a}_{15} \geqslant 1,\end{cases}$$

and

$$\min\max\left\{\left|\tilde{a}_{23}-a_{23}\right|, \left|\tilde{a}_{35}-a_{35}\right|\right\} \quad s.t. \begin{cases}\left|\tilde{a}_{23}-a_{23}\right| \leqslant \xi^*,\\ \left|\tilde{a}_{35}-a_{35}\right| \leqslant \xi^*,\\ \tilde{a}_{23}\times\tilde{a}_{35} = \tilde{a}_{25},\\ \tilde{a}_{23}, \tilde{a}_{35} \geqslant 1.\end{cases}$$

Based on Theorem 5, the above two models are equivalent to the following two optimization models

$$\min\ \eta_1 \quad s.t. \begin{cases}(2+\eta_{21})\times(4+\eta_{15}) = 8.1459,\\ 0 \leqslant \eta_{21} \leqslant \eta_1,\\ 0 \leqslant \eta_{15} \leqslant \eta_1,\end{cases}$$

and

$$\min\ \eta_3 \quad s.t. \begin{cases}(4+\eta_{23})\times(2+\eta_{35}) = 8.1459,\\ 0 \leqslant \eta_{23} \leqslant \eta_3,\\ 0 \leqslant \eta_{35} \leqslant \eta_3.\end{cases}$$

According to Theorem 6, as $a_{21}\times a_{15} = a_{23}\times a_{25} = 8 < \tilde{a}_{25} = 8.1459$, the optimal solutions for η_1 and η_3 are

$$\eta_1^* = \eta_3^* = \frac{-(2+4)+\sqrt{(2+4)^2 - 4(2\times 4 - 8.1459)}}{2} = 0.0242.$$

Therefore, $\eta_{15}^* = \eta_{21}^* = \eta_{35}^* = \eta_{23}^* = 0.0242$, leading to $\tilde{a}_{15} = a_{15}+\eta_{15}^* = 4.0242, \tilde{a}_{21} = 2.0242, \tilde{a}_{35} = 2.0242, \tilde{a}_{23} = 4.0242$. Consequently, the unique optimal modified OW vector is

$$\tilde{a}_{15} = 4.0242, \tilde{a}_{25} = 8.1459, \tilde{a}_{35} = 2.0242, \tilde{a}_{45} = 2.8541, \tilde{a}_{55} = 1.$$

Using Eq. (12) based on only the modified OW vector, we can derive the following unique criteria weights.

$$w_1 = 0.2230, w_2 = 0.4513, w_3 = 0.1122, w_4 = 0.1581, w_5 = 0.0554.$$

Example 8. We use the same data as in Example 6 for buying a car. There are multiple optimal modified values of criteria C_1 and C_4, and we obtain the following two independent optimization models

$$\min\ \eta_1 \quad s.t. \begin{cases}(2+\eta_{21})\times(4+\eta_{15}) = 9,\\ 0 \leqslant \eta_{21} \leqslant \eta_1,\\ 0 \leqslant \eta_{15} \leqslant \eta_1,\end{cases}$$

and

$$\min\ \eta_4 \quad s.t. \begin{cases}(3+\eta_{24})\times(2+\eta_{45}) = 9,\\ 0 \leqslant \eta_{24} \leqslant \eta_4,\\ 0 \leqslant \eta_{45} \leqslant \eta_4.\end{cases}$$

According to Theorem 6, as $a_{21}\times a_{15} = 8 < \tilde{a}_{25} = 9$ and $a_{24}\times a_{45} = 6 < \tilde{a}_{25} = 9$, the optimal solutions for η_1 and η_4 are

$$\eta_1^* = \frac{-(2+4)+\sqrt{(2+4)^2 - 4(2\times 4 - 9)}}{2} = 0.1623,$$

$$\eta_4^* = \frac{-(2+3)+\sqrt{(2+3)^2 - 4(2\times 3 - 9)}}{2} = 0.5414.$$

Table 8
Ranges of the optimal modified OW vector and the optimal criteria weights.

Optimal solutions	Ranges	$\tilde{a}_{15}(w_1)$	$\tilde{a}_{25}(w_2)$	$\tilde{a}_{35}(w_3)$	$\tilde{a}_{45}(w_4)$	$\tilde{a}_{55}(w_5)$
Optimal modified values	Lower bounds	3	9	3	2.25	1
	Upper bounds	5	9	3	3	1
Optimal criteria weights	Lower bounds	0.1579	0.4286	0.1429	0.1111	0.0476
	Upper bounds	0.2469	0.4932	0.1644	0.1579	0.0548

Table 9
The comparative analysis among different BWM models.

Methods	Optimization model	Optimal solutions	Interval weights	Analytic solutions	Solution tools	Preference modification suggestion
Nonlinear BWM [9]	NLP	Multiple	No	No	Software	No
Interval weights and a linear BWM [36]	LP	Unique	Yes	No	Software	No
Multiplicative BWM [37]	LP	Multiple	Yes	No	Software	No
The proposed method	NLP	Unique	Yes	Yes	Analytical	Yes

Therefore, $\eta^*_{15} = \eta^*_{21} = 0.1623$ and $\eta^*_{24} = \eta^*_{45} = 0.5414$, leading to $\tilde{a}_{15} = a_{15} + \eta^*_{15} = 4.1623, \tilde{a}_{21} = 2.1623, \tilde{a}_{45} = 2.5414, \tilde{a}_{24} = 3.5414$. Consequently, the unique optimal modified OW vector is

$$\tilde{a}_{15} = 4.1623, \tilde{a}_{25} = 9, \tilde{a}_{35} = 3, \tilde{a}_{45} = 2.5414, \tilde{a}_{55} = 1.$$

Using Eq. (12) based on only the modified OW vector, we can derive the following unique criteria weights.

$$w_1 = 0.2112, w_2 = 0.4568, w_3 = 0.1523, w_4 = 0.1290, w_5 = 0.0508.$$

The results in Examples 7 and 8 continue the work of Examples 5 and 6, aiming to select a unique solution from the multiple optimal solutions of the nonlinear BWM. The unique solutions are easy to calculate, and we find that they are very close to the center of the intervals we derived before, so they yield great convenience and reasonable criteria weights for the decision-making problem.

4.2. Comparative analysis and discussion with existing BWM models

The nonlinear BWM and linear BWM models in the two pioneering papers [9,36] connect logically, echo each other, and form a complete BWM framework. The multiplicative BWM [37] uses a modified objective function with an algebraic approach, making the entire optimization problem easy to linearize and able to be solved quickly and without distortions. It is a solid supplement and extension of the BWM from a more mathematical perspective. By introducing alternative linear objective functions and interval analysis, both the methods in [36,37] can handle the nonlinearity and multi-optimality of the original BWM. Instead of developing a new metric in the framework of the BWM, we propose an equivalent BWM, obtain some optimal analytical solutions of the nonlinear BWM, and finally construct a secondary objective function to derive a unique optimal solution for the BWM. Specifically, we compare our method with the existing BWM models based on some theoretical (see Table 9) and practical aspects (see Table 10).

Based on Table 9, the following three theoretical points are compared: (1) multiple optimal solutions based on the criteria weights; (2) interval analysis for multiple optimal solutions; and (3) the unique optimal solution.

- First, both the initial nonlinear BWM [9] and the multiplicative BWM [37] have multiple optimal solutions. We might obtain different results if we use different optimization software to solve them. Instead of solving them with the help of optimization software, using the analytical form for the multiple optimal solutions is mathematically rigorous and more general, guiding us to determine the optimal feasible criteria weights more conveniently. As a result, we can arbitrarily select the optimal modified values within their optimal feasible ranges and then obtain a group of optimal weights.
- Second, regarding the lower and upper bounds of the optimal criteria weights, the methods in [36,37] construct two optimization models and solve them one by one for all criteria using optimization software. However, the weights derived in the multiplicative BWM [37] are additive weights, and we need to transform them into multiplicative weights using a normalization process, which might lead to information distortions. Our results on the interval weights are the same as those in [36], but our analytical results are more straightforward, without the need to solve too many optimization models.
- Third, neither of the methods in [9,37] yields the unique optimal solution to the criteria weights. The linear BWM [36] can yield a unique solution. However, it might be different from the nonlinear BWM, and thus, it leads to different feasible regions and results. In contrast, our developed secondary objective function maintains all the nonlinear BWM characteristics and offers an analytical form of the unique optimal solution, which is more mathematically sound while retaining the philosophy of the nonlinear BWM.

Furthermore, as all the existing BWM models are optimization models, the computation time when using optimization software to solve these models is an important indicator for comparing the performance merits of these methods. To measure the performance of these models in terms of computation time, we randomly generate 10000 best–worst pairwise comparison vectors with different numbers of criteria. The best to worst value a_{BW} is drawn independently from the scale $\{2, 3, 4, 5, 6, 7, 8, 9\}$, and the remaining BO values $a_{Bj}(j \neq B, W)$ and OW values $a_{jW}(j \neq B, W)$ are drawn independently from the scale $\{2, \dots, a_{BW}-1\}$. To ensure the reliability of the experimental results, we use the fmincon and linprog functions in MATLAB to solve the nonlinear BWM model and multiplicative/linear BWM models, respectively, with simulated data to derive the criteria weights. All the tests are conducted in MATLAB 2020b on a computer with an Intel(R) Core(TM) i5-8265U CPU @ 1.60 GHz and a memory of 8 GB. Table 10 shows the simulated computation time (seconds) of different BWM models with different numbers of criteria.

According to the average computation time of the 10000 simulation examples shown in Table 10, the proposed analytical framework is the fastest computational method, almost 300 times faster than the linear BWM and multiplicative BWM. This is because the derived optimal analytical solutions are simple arithmetic calculations. The slowest is the nonlinear BWM, which needs an average of 1 s to solve a nonlinear optimization model and takes approximately 25 000 times longer than the proposed method. Although there is a significant difference in computation time between the fastest and slowest methods, all methods can yield the optimal solution quickly. However, in the context of a large-scale MCDM or group decision-making (GDM) problem, the differences are noticeable.

Table 10
The computation time (seconds) of different BWM models with different numbers of criteria.

Methods	$n = 5$	$n = 7$	$n = 9$	$n = 11$	$n = 13$	$n = 15$
Nonlinear BWM [9]	0.5533	0.6256	0.8718	0.9430	1.0644	1.1028
Linear BWM [36]	0.0095	0.0092	0.0103	0.0107	0.0106	0.0117
Multiplicative BWM [37]	0.0097	0.0106	0.0112	0.0104	0.0114	0.0130
The proposed method	2.6253E−05	2.8474E−05	3.0250E−05	3.3161E−05	3.4699E−05	3.7256E−05

Practically speaking, we discuss the characteristics of our approach regarding the following three points: (1) practical model interpretability; (2) practical preference-modification suggestions; and (3) practical computational convenience.

- First, the proposed optimization model of the BWM is driven by the preference-modification process, and the criteria weights are calculated based on the optimally modified comparison vectors. However, the criteria weights in [9,36,37] are based directly upon the optimization models. From a practical perspective, the decision results derived from the optimally modified preference information are more interpretable.
- Second, the proposed equivalent BWM offers practical preference-modification suggestions to improve the consistency level of the pairwise comparison systems. However, the models in [9,36,37] cannot directly embody the preference-modification process. By using the proposed analytical framework, obtaining the optimal interval modified comparison vectors or unique optimal modified comparison vectors, and finally providing DMs with the adjusted preference information, the results in this paper would be more flexible and reliable.
- Third, the proposed analytical framework gives several optimal closed-form solutions, and the analytical solution saves the most time. Although one can also use optimization software to solve the optimization models in [9,36,37] quickly for a single DM's preference information, it is time-consuming to handle large-scale group preference information one by one using these methods in practical applications. In such situations, the advantage of the proposed analytical results is more prominent.

In summary, the comparison results indicate that the proposed method can be successfully applied to the solutions of the BWM. Our approach can accurately and comprehensively determine different optimal criteria weights for inconsistent BWM comparison systems. The analytical results can provide DMs with both multiple flexible optimal solutions and unique precise optimal solutions, all in the form of mathematical formulas. Hence, the proposed analytical framework for the BWM offers a solid theoretical foundation and is more convenient than solving these optimization models using optimization software.

5. Conclusions and future studies

The BWM, as a novel MCDM method that has emerged in recent years, has been widely and successfully applied in many fields and has received increasing attention. This study proposes an analytical framework for the optimal solutions of the BWM based on an equivalent BWM. Our approach comprehensively determines the analytical forms of the criteria weights in terms of optimal solutions, optimal interval weights, and a unique optimal solution. The numerical examples reveal that we can directly utilize the analytical solutions to calculate the related optimal solutions without the help of optimization software. Our research makes significant contributions to the theory of the BWM. First, introducing an equivalent BWM and a secondary objective function makes the proposed method complete and effective in addressing both multiple and unique optimal solutions involved in the original BWM. Second, the analytical solutions can simplify the solution process of these programming models while maintaining good performance in terms of computational accuracy, and they can be easily adapted to a different number of criteria without increasing the computational complexity, so the proposed method is more convenient and practical than solving these optimization models using optimization software. Finally, the secondary objective function that determines the unique solution inherits all the features of the nonlinear BWM, and the unique solution is also close to the interval center found in the nonlinear BWM.

For future studies, we will consider the following research directions: First, most existing criteria weighting methods are prone to several cognitive biases, such as splitting bias, anchoring bias, and equalizing bias. The BWM was tested as an effective debiasing solution to reduce the anchoring bias [14] and equalizing bias [15] in MCDM based on an experimental study. Thus, it will be of theoretical and practical significance to examine how the BWM helps remedy these cognitive biases with its analytical solutions. Second, the BWM has been widely used in current evaluation and MCDM problems, but research on GDM is still limited. The Bayesian BWM [19] has been introduced to find the optimal weights of a set of criteria based on the input preferences of multiple DMs using the best–worst framework. Based on our derived analytical optimal criteria weights, it is also interesting to investigate the aggregation of these output optimal criteria weights of DMs from a probabilistic perspective. Finally, the Bayesian BWM helps find the aggregated weights of criteria for a group of DMs by using the Bayesian statistical method. Given the existence of interactive behaviors in GDM, it may be unrealistic to assume that all DMs are fluent in the decision-making problem and can reach an agreement. Hence, it is necessary to conduct further research on group consensus [49–51] for the BWM in GDM in the future.

CRediT authorship contribution statement

Qun Wu: Methodology and model design, Conceptualization, Formal analysis, Writing & editing. **Xinwang Liu:** Validation, Funding acquisition, Review & editing. **Ligang Zhou:** Validation, Funding acquisition, Review & editing, Supervision. **Jindong Qin:** Validation, Funding acquisition, Review & editing. **Jafar Rezaei:** Validation, Methodology, Conceptualization, Review & editing, Supervision.

Declaration of competing interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript.

Data availability

No data was used for the research described in the article.

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