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DOI
10.57262/die036-0708-537

Publication date
2023

Document Version
Final published version

Published in
Differential and Integral Equations

Citation (APA)
Chu, J., Marynets, K., & Wang, Z. (2023). Existence and approximate solutions of a nonlinear model for the antarctic circumpolar current. *Differential and Integral Equations*, *36*(7-8), 537-558. https://doi.org/10.57262/die036-0708-537

Differential and Integral Equations
https://doi.org/10.57262/die036-0708-537
Volume 36, Numbers 7-8 (2023), 537–558

EXISTENCE AND APPROXIMATE SOLUTIONS OF A NONLINEAR MODEL FOR THE ANTARCTIC CIRCUMPOLAR CURRENT

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Abstract. We consider a nonlinear Neumann boundary value problem which is derived for the Antarctic Circumpolar Current. By the theory of topological degree, we prove the existence results for the problem with semilinear oceanic vorticity term. We also construct the approximate solutions for such a nonlinear model.

1. Introduction

Circulating ocean currents are driven by the combined forces of gravity and Coriolis, and triggered by the wind stress (see the discussion in [24, 25]). Since the Coriolis effect deflects winds to the right in the Northern Hemisphere and to the left in the Southern Hemisphere, the ocean currents in the Northern Hemisphere deflect clockwise rotation and in the Southern Hemisphere – counter-clockwise. These slow flows are a dominant factor in the circulation of ocean water around the entire planet. There are seven major ocean currents (North Atlantic, South Atlantic, Indian, North Pacific,

AMS Subject Classifications: 45G99, 58J3, 76B03.
Accepted for publication: August 2022.
Jifeng Chu was supported by the National Natural Science Foundation of China (Grant No. 12071296) and by Science and Technology Innovation Plan of Shanghai (Grant No. 20JC1414200).

South Pacific, Arctic, Antarctic), and each of them presents some specific flow pattern (see the discussions in [12, 13]).

In a recent paper [14], a model of the general motion of ocean currents was obtained in the setting of spherical coordinates, and such a model can be described as an elliptic boundary value problem in terms of a stream function. When the model is used to the Arctic gyre, where permanent thick ice floats on the ocean, the elliptic boundary value problems can be transformed into a second-order differential equation on a semi-infinite interval, constrained by some asymptotic conditions [4, 8]. After the works [4, 8], the existence and related dynamical behaviors for the motion of the Arctic gyres have been studied in a series of works [5, 6, 7, 19, 21, 35], including the explicit solutions for the linear vorticity, nontrivial solutions or monotone solutions for the nonlinear vorticity and their stability in the sense of Lyapunov. We remark that the short-wavelength method has been used in [22, 23] to study the stability for some geophysical flows in rotating spherical coordinates.

We point out that the analysis in studying the Arctic gyres cannot be used to study the Antarctic Circumpolar Current (ACC) because there are some essential differences between the Arctic ocean and the Antarctic ocean. For example, the North Pole is located in the middle of the Arctic ocean and the South Pole is located in Antarctica, which is completely encircled by the eastward moving ACC (see the discussion in [13]). Compared with the Arctic gyres, the ACC is very important in studying the global ocean circulation and climate because the ACC is linked to Atlantic, Indian, and Pacific oceans [1]. Besides, the ACC is one of the most significant ocean currents and the only current that completely encircles the polar axis, flowing eastward through the southern regions of the Atlantic, Indian and Pacific oceans [13]. We refer to [9, 17, 20, 22, 26, 27, 28, 29, 31, 32] for recent results on the ACC.

The aim of the present paper is to investigate a nonlinear Neumann boundary value problem which models the ACC. In Section 2, we will derive the mathematical model of the ACC. In Section 3, based on an application of topological degree theory, we prove the existence results when the vorticity function satisfies the semilinear conditions, which are characterized by the eigenvalues of the corresponding linear problem. Similar approaches have been used to study the existence of second order ordinary differential equations with Dirichlet conditions or periodic conditions. See [16, 18, 33, 34] and the references therein. Because it is very important but also even impossible to obtain the explicit solutions for the problem due to the presence of the nonlinear vorticity function, we construct the approximate solutions for the problem in Section 3. Some numerical computations for a concrete example are also given.

2. Mathematical model of the ACC

Let us introduce the spherical coordinates (r, θ, φ) with $\theta \in [0, \pi)$ being the polar angle, such that $\theta = 0$ corresponds to the North Pole, and $\varphi \in [0, 2\pi)$ being the azimuthal angle (see Fig. 1).

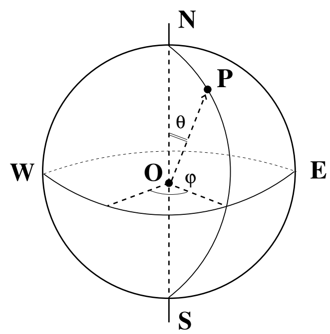

FIGURE 1. The azimuthal and polar spherical coordinates φ and θ of a point P on the spherical surface of the Earth: $\theta = 0$ and $\theta = \pi$ correspond to the North and South Pole, respectively, while $\theta = \pi/2$ corresponds to the Equator.

In terms of the stream function $\psi(\varphi, \theta)$, the horizontal gyre flow on the spherical Earth has azimuthal and polar velocity components given by

$$v = \frac{1}{\sin\theta}\psi_\varphi, \quad w = -\psi_\theta. \tag{2.1}$$

Then, if $\Psi(\varphi, \theta)$ is the stream function associated with the vorticity of the ocean motion (not accounting for the effects of the Earth's rotation), defined as

$$\Psi(\varphi, \theta) = \omega\cos\theta + \psi(\varphi, \theta),$$

where $\omega > 0$ is the non-dimensional form of the Coriolis parameter, the vorticity equation of the gyre flow is given by

$$\frac{1}{\sin^2\theta}\,\Psi_{\varphi\varphi} + \Psi_\theta\cot\theta + \Psi_{\theta\theta} = F(\Psi - \omega\,\cos\theta)\,, \tag{2.2}$$

where $F(\Psi - \omega\,\cos\theta)$ is the oceanic vorticity and $2\omega\cos\theta$ is the planetary vorticity generated by the Earth's rotation (see the discussions in [14]).

By applying the stereographic projection of the unit sphere centered at origin from the North Pole to the equatorial plane (see Fig. 2), the model (2.2) in spherical coordinates is thus transformed into an equivalent planar elliptic partial differential equation.

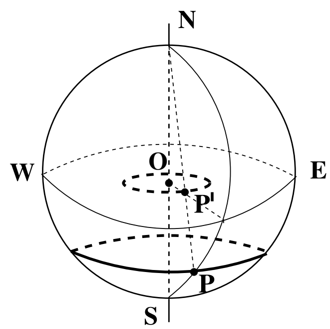

FIGURE 2. The stereographic projection $P \mapsto P'$ from the North Pole to the equatorial plane: for any point P in the Southern Hemisphere, the straight line connecting it to the North Pole intersects the equatorial plane in a point P' belonging to the interior of the circular region delimited by the Equator. The depicted thick band, delimited by two parallels of latitude, represents one of the jets of the Antarctic Circumpolar Current and is mapped bijectively into an annular planar region concentric with the Equator.

In our coordinates the stereographic projection, defined by

$$\xi = r\, e^{i\,\phi} \quad \text{with} \quad r = \cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1-\cos\theta}\,, \tag{2.3}$$

with (r, ϕ) being the polar coordinates in the equatorial plane, transforms (2.2) into

$$\psi_{\xi\bar{\xi}} + 2\omega\,\frac{1-\xi\bar{\xi}}{(1+\xi\bar{\xi})^3} - \frac{F(\psi)}{(1+\xi\bar{\xi})^2} = 0\,.$$

The above equation is equivalent, using the Cartesian coordinates (x, y) in the complex ξ-plane, to the following semilinear elliptic partial differential equation

$$\Delta\psi + 8\omega\,\frac{1-(x^2+y^2)}{(1+x^2+y^2)^3} - \frac{4F(\psi)}{(1+x^2+y^2)^2} = 0. \tag{2.4}$$

See the discussions in [11, 14, 15], where the stereographic projections are used to investigate flows in rotating spherical coordinates. The ACC presents a considerable uniformity in the azimuthal direction and this feature is helpful to simplify the problem further. Indeed, gyres with no variation in the azimuthal direction correspond to radially symmetric solutions $\psi = \psi(r)$ of (2.4). In this setting, the change of variables $\psi(r) = U(s)$, $s_1 < s < s_2$ with

$$r = e^{-s/2} \quad \text{for} \quad 0 < s_1 = -2\ln(r_+) < s_2 = -2\ln(r_-),$$

for $0 < r_- < r_+ < 1$, transforms the equation (2.4) to the second-order ordinary differential equation

$$U''(s) - \frac{e^s}{(1+e^s)^2}\, F(U(s)) + \frac{2\omega e^s(1-e^s)}{(1+e^s)^3} = 0\,, \qquad s_1 < s < s_2\,. \tag{2.5}$$

Finally let us introduce the change of variables

$$u(t) = U(s) \quad \text{with} \quad t = \frac{s - s_1}{s_2 - s_1}.$$

Then (2.5) is equivalent to the following equation

$$u'' = a(t)F(u) + b(t)\,, \qquad 0 < t < 1\,, \tag{2.6}$$

where $a(t)$ and $b(t)$ are two positive continuous functions given as

$$\begin{cases} a(t) = \dfrac{(s_2 - s_1)^2\, e^{(s_2-s_1)t+s_1}}{(1+e^{(s_2-s_1)t+s_1})^2}, \\ b(t) = -\dfrac{2\omega(s_2-s_1)^2 e^{(s_2-s_1)t+s_1}(1-e^{(s_2-s_1)t+s_1})}{(1+e^{(s_2-s_1)t+s_1})^3}. \end{cases} \tag{2.7}$$

It follows from (2.3) that

$$u'(t) = -\frac{1}{2}\, r\, \psi_r = -\frac{1}{2}\, \psi_\theta\, \sin\theta.$$

Note that throughout the Southern ocean, $\sin\theta$ is always positive. Then (2.1) shows that the flow in a jet component of the ACC, between the parallels of latitude defined by an appropriate choice of $r_\pm \in (0,1)$, is modeled by coupling the differential equation (2.6) with the Neumann boundary conditions

$$u'(0) = u'(1) = 0, \tag{2.8}$$

which express the fact that there is no flow across the boundary of the jet. The boundary value problem (2.6)-(2.8) is therefore a model for a jet component of the ACC.

3. Existence results

In order to prove the existence results for the problem (2.6)-(2.8), we need some preliminary facts on the weighted eigenvalue problem

$$\begin{cases} u''(t) + \lambda a(t)u(t) = 0, \\ u'(0) = 0, \quad u'(1) = 0. \end{cases} \tag{3.1}$$

See [2, 10] and a recent monograph [3] for the discussions on weighted Sturm-Liouville problems. It is obvious that $\lambda_0 = 0$ is an eigenvalue of (3.1) with the eigenfunction being constant. Moreover, it is easy to check that if λ is an eigenvalue of (3.1) and u is a corresponding eigenfunction, then

$$\lambda = \frac{\int_0^1 (u'(t))^2 dt}{\int_0^1 a(t)u^2(t)dt}.$$

Therefore, all nonzero eigenvalues of (3.1) are positive since $a(t)$ given as in (2.7) is positive. It is well-known that all nontrivial eigenvalues admit variational characterizations, and can be ordered in a non-decreasing sequence, that is, problem (3.1) has a sequence of eigenvalues:

$$0 = \lambda_0(a) \le \lambda_1(a) \le \cdots \le \lambda_k(a) \le \lambda_{k+1}(a) \le \cdots,$$

and $\lim\limits_{k\to\infty} \lambda_k(a) = +\infty$.

Now, we first establish the existence results for the nonlinear problem

$$\begin{cases} u''(t) + g(t,u) = b(t), \\ u'(0) = 0, \quad u'(1) = 0, \end{cases} \tag{3.2}$$

where $g(t,u) : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $b(t)$ is given as in (2.7). Note that b is also non-negative.

Theorem 3.1. *Assume that the function $g(t,u)$ is continuous and satisfies the inequality*

$$\limsup_{|u|\to\infty} \frac{g(t,u)}{u} \le \Phi(t), \quad \textit{uniformly in } t, \tag{3.3}$$

and

$$\lambda_1(\Phi) > 1. \tag{3.4}$$

Then the problem (3.2) has at least one solution.

Proof. We shall work in the space

$$\mathbb{C}_N = \left\{ u : [0,1] \to \mathbb{R} \text{ is continuous and } u'(0) = u'(1) = 0 \right\}.$$

Let us consider the following homotopic problems

$$\begin{cases} u''(t) + \tau g(t,u) - \tau b(t) + (1-\tau)\Phi(t)u = 0, \\ u'(0) = 0, \quad u'(1) = 0. \end{cases} \tag{3.5}$$

It is obvious that one can transform the problem (3.5) into a fixed point problem in $\mathbb{C}_N$. In fact, u is a solution of (3.5) if and only if u is a fixed point in $\mathbb{C}_N$ of the following operator

$$\mathcal{T}_\tau u(t) := u(0) - \int_0^t (t-s)\Big[\tau g(s,u(s)) - \tau b(s) + (1-\tau)\Phi(s)u(s)\Big]ds.$$

To apply the topological degree theory, we will find a priori bounds for all solutions of the homoptopic problem (3.5). Let ε_0 be small enough. By condition (3.3), we know that there exists $\psi \in L^1[0,1]$ with $\psi \geq 0$ such that

$$ug(t,u) \leq (\Phi(t) + \varepsilon_0)u^2 + \psi(t)$$

for all $u \in \mathbb{R}$ and $t \in [0,1]$. Thus, for all $u \in \mathbb{R}$ and $t \in [0,1]$,

$$u\Big[\tau g(t,u) - \tau b(t) + (1-\tau)\Phi(t)u\Big] \leq \Big[\Phi(t) + \varepsilon_0\Big]u^2 + b(t)u + \psi(t).$$

Let $u(t)$ be a solution of (3.5) for some $\tau \in [0,1]$. Then

$$\int_0^1 u''(t)u(t)dt = -\int_0^1 \Big[\tau g(t,u(t))u(t) - \tau b(t)u(t) + (1-\tau)\Phi(t)u^2(t)\Big]dt,$$

and thus

$$\|u'\|_2^2 = \int_0^1 \Big[\tau g(t,u(t))u(t) - \tau b(t)u(t) + (1-\tau)\Phi(t)u^2(t)\Big]dt.$$

Note that $\lambda_1(\Phi)$ has the following variational characterization

$$\lambda_1(\Phi) = \inf_{u\in\mathcal{H}} \frac{\int_0^1 |u'|^2 dt}{\int_0^1 \Phi(t)|u|^2 dt}, \tag{3.6}$$

where $\mathcal{H} = \{u \in \mathcal{W}^{2,1}(0,1) | u'(0) = u'(1) = 0, u(t) \text{ is not constant}\}$. Thus, we obtain that

$$\begin{aligned} &\int_0^1 \Big[\tau g(t,u(t))u(t) - \tau b(t)u(t) + (1-\tau)\Phi(t)u^2(t)\Big]dt \\ \leq &\int_0^1 \Big[(\Phi(t)+\varepsilon_0)u^2(t) + b(t)u(t) + \psi(t)\Big]dt \\ \leq &\Big[\frac{1}{\lambda_1(\Phi)} + \frac{\varepsilon_0}{\lambda_1(1)}\Big]\|u'\|_2^2 + \int_0^1 b(t)u(t)dt + \|\psi\|_1. \end{aligned}$$

Using the Hölder inequality and (3.6), it is easy to obtain

$$\int_0^1 b(t)u(t)dt \le \|b\|_2\|u'\|_2/\lambda_1(1).$$

Therefore,

$$\|u'\|_2^2 \le \Big[\frac{1}{\lambda_1(\Phi)} + \frac{\varepsilon_0}{\lambda_1(1)}\Big]\|u'\|_2^2 + \frac{\|b\|_2}{\lambda_1(1)}\|u'\|_2 + \|\psi\|_1.$$

Due to the condition (3.4), the above equality implies that if ε_0 is small enough such that

$$\varepsilon_0 < \lambda_1(1)\Big[1 - \frac{1}{\lambda_1(\Phi)}\Big],$$

then there exists a constant $K_1 > 0$ such that $\|u'\|_2 \le K_1$.

Using the Hölder inequality, we obtain that

$$\begin{aligned}|u(t)| &= \Big|\int_0^t u'(s)ds + u(0)\Big| \le \int_0^t |u'(s)|ds + |u(0)| \\ &\le \|u'\|_2 + |u(0)| \le K_1 + |u(0)| =: K_2.\end{aligned}$$

Thus,

$$\|u\|_\infty \le K_2. \tag{3.7}$$

It follows from (3.7) that there exists some $\varphi \in L^1[0,1]$ with $\varphi \ge 0$ such that

$$|\tau g(t,u(t)) - \tau b(t) + (1-\tau)\Phi(t)u(t)| \le \varphi(t)$$

for all $t \in [0,1]$, $\tau \in [0,1]$ and all solutions u of problem (3.5).

Now,

$$\begin{aligned}|u'(t)| &= \Big|\int_0^t u''(s)ds\Big| = \Big|\int_0^t \Big[\tau g(s,u(s)) - \tau b(s) + (1-\tau)\Phi(s)u(s)\Big]ds\Big| \\ &\le \int_0^1 |\tau g(t,u(t)) - \tau b(t) + (1-\tau)\Phi(t)u(t)|dt \le \|\varphi\|_1 =: K_3.\end{aligned}$$

Up to now, we have proved that all possible solutions of problem (3.5) admit a priori bounds in $\mathbb{C}_N$, that is, we are able to find a positive constant r large enough such that $r > \max\{K_2, K_3\}$ and all possible solutions of (3.5) belong to the set

$$\Omega = \{u \in \mathbb{C}_N : \|u\|_\infty < r,\ \|u'\|_\infty < r\}.$$

By the invariance of Leray-Schauder degree under homotopies [30], we know

$$\deg(I - \mathcal{T}_1, \Omega, 0) = \deg(I - \mathcal{T}_0, \Omega, 0).$$

It follows from the condition (3.4) that

$$\begin{cases} u''(t) + \Phi(t)u = 0, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases}$$

has only the trivial solution $u = 0$, and thus $\deg(I - \mathcal{T}_0, \Omega, 0) \neq 0$. Therefore,

$$\deg(I - \mathcal{T}_1, \Omega, 0) \neq 0,$$

which implies that problem (3.2) has at least one solution in Ω. □

In the literature, the conditions (3.3)-(3.4) are called the first nonuniform non-resonance conditions. The next result deals with the higher nonuniform non-resonance conditions.

Theorem 3.2. *Assume that the function $g(t, u)$ is continuous and there exist two continuous functions ϕ, Φ such that*

$$\phi(t) \leq \liminf_{|u| \to \infty} \frac{g(t,u)}{u} \leq \limsup_{|u| \to \infty} \frac{g(t,u)}{u} \leq \Phi(t), \quad \textit{uniformly in } t,$$

and there exists $k \in \mathbb{N}$ with $k \geq 2$ such that

$$\lambda_{k-1}(\phi) < 1, \quad \lambda_k(\Phi) > 1. \tag{3.8}$$

Then the problem (3.2) has at least one solution.

Proof. For any $\gamma \in L^1[0, 1]$ satisfying

$$\phi(t) \leq \gamma(t) \leq \Phi(t), \quad t \in [0, 1].$$

The condition (3.8) ensures that the problem

$$\begin{cases} u''(t) + \gamma(t)u = 0, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases}$$

has only the trivial solution $u = 0$. Thus, the results can be proved by the similar methods in the proof of [18, Theorem 2.4]. Here, we omit the details. □

Now, we return to the original problem (2.6)-(2.8), which corresponds to

$$g(t, u) = -a(t)F(u).$$

Therefore,

$$\liminf_{|u| \to \infty} \frac{g(t,u)}{u} = -a(t) \limsup_{|u| \to \infty} \frac{F(u)}{u},$$

$$\limsup_{|u|\to\infty} \frac{g(t,u)}{u} = -a(t) \liminf_{|u|\to\infty} \frac{F(u)}{u}.$$

The following two results are direct consequences of Theorem 3.1 and Theorem 3.2, respectively.

Theorem 3.3. *Assume that the function $F(u)$ is continuous and there exists a constant $\alpha < 0$ such that*

$$\liminf_{|u|\to\infty} \frac{F(u)}{u} \geq \alpha \quad \text{and} \quad \lambda_1(a) < -\frac{1}{\alpha}. \tag{3.9}$$

Then the problem (2.6)-(2.8) has at least one solution.

Theorem 3.4. *Assume that the function $F(u)$ is continuous and there exist two negative constants α, β such that*

$$\alpha \leq \liminf_{|u|\to\infty} \frac{F(u)}{u} \leq \limsup_{|u|\to\infty} \frac{F(u)}{u} \leq \beta. \tag{3.10}$$

Suppose further that there exists $k \in \mathbb{N}$ with $k \geq 2$ such that

$$\lambda_{k-1}(a) < -\frac{1}{\alpha}, \quad \lambda_k(a) > -\frac{1}{\beta}. \tag{3.11}$$

Then the problem (2.6)-(2.8) has at least one solution.

Remark 3.5. If we consider the linear vorticity function $F(u) = -\kappa u$ with $\kappa > 0$, then condition (3.10) becomes $\alpha = \beta = -\kappa$. Thus, condition (3.9) becomes $\lambda_1(a) < 1/\kappa$ and (3.11) reads as

$$\lambda_{k-1}(a) < \frac{1}{\kappa} < \lambda_k(a), \quad k \geq 2.$$

It is easy to see that no eigenvalues of (3.1) are rational. Therefore problem (2.6)-(2.8) always has at least one solution when κ is rational. In next section, we will present the numerical approximate solutions for a concrete example with $F(u) = -2u$.

4. Approximate solutions

We present a numerical-analytic technique to obtain the approximate solutions of problem (2.6)-(2.8). Although Neumann boundary conditions give us no information about the initial state of the flow or the values of solution at $t = 0$ and $t = 1$, the technique we derive below allows us to achieve this. Indeed, we construct a sequence of functions that approximates the solution of (2.6)-(2.8) and show that its limit is in fact the exact solution.

4.1. **Equivalent parametrized system.** Let $x_1(t) = u(t)$ and $x_2(t) = u'(t)$. Then we can rewrite the differential equation (2.6)-(2.8) as the following problem

$$\begin{cases} x_1'(t) = x_2(t), \\ x_2'(t) = a(t)F(x_1) + b(t), \\ x_2(0) = x_2(1) = 0. \end{cases} \tag{4.1}$$

Let us introduce two parameters $x_1(0) = z_1$ and $x_1(1) = \eta_1$. Then the boundary condition in (4.1) can be written as

$$Ax(0) + Bx(1) = d(\eta), \tag{4.2}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$d(\eta) = [\eta_1, 0]^T, \quad \eta = [x_1(1), x_2(1)]^T = [\eta_1, 0]^T.$$

Let $x(t) = [x_1(t), x_2(t)]^T \in D \subset \mathbb{R}^2$ and $f(t, x(t)) = [x_2(t), a(t)F(x_1) + b(t)]^T$, where D is a closed and bounded domain. Then (4.1) is equivalent to the vector form

$$x'(t) = f(t, x(t)), \quad x \in \mathbb{R}^2. \tag{4.3}$$

We shall study the equivalent parametrized problem (4.3) with (4.2).

4.2. **Approximate solutions.** Let us introduce a vector

$$\delta_D(f) := \frac{1}{2}\Big[\max_{(t,x)\in[0,1]\times D} f(t,x) - \min_{(t,x)\in[0,1]\times D} f(t,x)\Big].$$

It is obvious that

$$\delta_D(f) \le \max_{(t,x)\in[0,1]\times D} |f(t,x)|.$$

We make the following assumptions with respect to the original problem (4.1).

(A1) the subset

$$D_\beta := \Big\{ z \in D : B\Big(z, \max_{t\in[0,T]} |t[d(\eta) - (A + I_2)z]|\Big) \subset D, \forall \eta \in D \Big\},$$

is non-empty, with $z = [x_1(0), x_2(0)]^T = [z_1, 0]^T \in D$ corresponding to the initial values of solution of (4.3) under the constraints (4.2).

(A2) the function $f(t, x)$ in (4.3) satisfies the Lipschitz condition

$$|f(t,u) - f(t,v)| \le K\,|u - v|, \quad t \in [0,1], \ \{u, v\} \subset D,$$

with a non-negative constant matrix $K = (k_{ij})_{i,j=1}^2$.

(A3) $r(Q) < 1$, where $r(Q)$ is the spectral radius of the matrix $Q := \frac{3T}{10}K$.

Now, we consider the following sequence of functions:

$$\begin{cases} x_m(t,z,\eta) = z + \displaystyle\int_0^t f(s, x_{m-1}(s,z,\eta))ds \\ \qquad -t\displaystyle\int_0^1 f(s, x_{m-1}(s,z,\eta))ds + t[d(\eta) - (A+I_2)z], \quad m \geq 1, \\ x_0(t,z,\eta) = z + t[d(\eta) - (A+I_2)z] \in D, \end{cases} \tag{4.4}$$

where $t \in [0,1]$, $z = [x_1(0), x_2(0)]^T = [z_1, 0]^T$ and $\eta = [x_1(1), x_2(1)]^T = [\eta_1, 0]^T$.

Theorem 4.1. *Assume that the function $f : [0,1] \times D \to \mathbb{R}^2$ and the parametrized boundary constraints* (4.2) *satisfy* (A1)–(A3). *Then for all fixed $z \in D_\beta$, $\eta \in D$:*

(1) *The sequences* (4.4) *are continuously differentiable and satisfy the parametrized boundary conditions*

$$Ax_m(0,z,\eta) + Bx_m(1,z,\eta) = d(\eta), \quad m \geq 0.$$

(2) *The sequences* (4.4) *converge uniformly to the limit function*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta).$$

(3) *The limit function $x_\infty(t,z,\eta)$ satisfies the parametrized boundary conditions*

$$Ax_\infty(0,z,\eta) + Bx_\infty(1,z,\eta) = d(\eta).$$

(4) *$x_\infty(t,z,\eta)$ is the unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_0^t f(s,x(s))ds - t\int_0^1 f(s,x(s))ds + t[d(\eta) - (A+I_2)z], \tag{4.5}$$

and thus is also a solution of the Cauchy problem for a modified system

$$\begin{cases} x'(t) = f(t,x(t)) + \Delta(z,\eta), \\ x(0) = z, \end{cases} \tag{4.6}$$

where

$$\Delta(z,\eta) := [d(\eta) - (A+I_2)z] - \int_0^1 f(s,x(s))ds. \tag{4.7}$$

(5) *The following error estimation holds:*

$$|x_\infty(t,z,\eta) - x_m(t,z,\eta)| \leq \frac{20}{9} t\,(1-t)\,Q^m (I_2 - Q)^{-1}\delta_D(f).$$

Proof. We prove that the sequence (4.4) is a Cauchy sequence in the Banach space $C([0,1],\mathbb{R}^2)$. First, we show that $x_m(t,z,\eta) \in D$, for all $(t,z,\eta) \in [0,T] \times D_\beta \times D$, $m \in \mathbb{N}$.

Applying the estimate

$$\Big| \int_0^t \Big[f(\tau) - \frac{1}{T}\int_0^T f(s)ds \Big] d\tau \Big| \leq \frac{1}{2}\alpha_1(t) \Big[\max_{t\in[0.T]} f(t) - \min_{t\in[0,T]} f(t) \Big], \quad (4.8)$$

where

$$\alpha_1(t) = 2t\Big(1 - \frac{t}{T}\Big),\ |\alpha_1(t)| \leq \frac{T}{2},\ t \in [0,T]\,, \quad (4.9)$$

on the sequence (4.4) for $T = 1$, we obtain that

$$\begin{aligned} &|x_m\,(t,z,\eta) - x_0\,(t,z,\eta)| \\ &\leq \Big| \int_0^t \Big[f(s, x_{m-1}\,(s,z,\eta)) - \int_0^1 f(s, x_{m-1}\,(s,z,\eta)) ds \Big] dt \Big| \\ &\leq \alpha_1(t)\delta_D(f) \leq \frac{1}{2}\delta_D(f), \quad m \in \mathbb{N}, \end{aligned}$$

which implies that $x_m(t,z,\eta) \in D$ whenever $(t,z,\eta) \in [0,1] \times D_\beta \times D$.

Let

$$\begin{aligned} &x_{m+1}(t,z,\eta) - x_m(t,z,\eta) \\ &\int_0^t [f(s, x_m(s,z,\eta)) - f(s, x_{m-1}(s,z,\eta))]\,ds \\ &\qquad - t\int_0^1 [\ f\,(s, x_m(s,z,\eta)) - f\,(s, x_{m-1}(s,z,\eta))\ ]\,ds, \quad m \in \mathbb{N}, \\ &\qquad\quad r_m(t,z,\eta) := |x_m(t,z,\eta) - x_{m-1}(t,z,\eta)|\,, \quad m \in \mathbb{N}. \end{aligned}$$

By virtue of the estimate (4.8) and the Lipschitz condition (A2), we have:

$$r_{m+1}(t,z,\eta) \leq K\Big[\,(1-t)\int_0^t r_m(s,z,\eta)ds + t\int_t^1 r_m(s,z,\eta)ds\Big], \quad m \geq 0.$$

By the fact

$$r_1(t,z,\eta) = |x_1(t,z,\eta) - x_0(t,z,\eta)| \leq \alpha_1(t)\delta_D(f),$$

and using the inequality

$$\alpha_{m+1}(t) \leq \frac{10}{9}\Big(\frac{3}{10}T\Big)^m \alpha_1(t), \quad m \geq 0, \quad (4.10)$$

which was obtained from

$$\alpha_{m+1}(t) = \Big(1 - \frac{t}{T}\Big) \int_0^t \alpha_m(s)ds + \frac{t}{T} \int_t^T \alpha_m(s)ds, \qquad m \geq 0, \tag{4.11}$$

$$\alpha_0(t) = 1,\ \alpha_1(t) = 2t\Big(1 - \frac{t}{T}\Big),$$

we know that

$$r_{m+1}(t, z, \eta) \leq K^m \alpha_{m+1}(t)\delta_D(f), \quad m \geq 0.$$

Therefore,

$$r_{m+1}(t, z, \eta) \leq \frac{10}{9}\alpha_1(t)\Big[Q^m \delta_D(f) + KQ^{m-1}\big|d(\eta) - (A + I_2)z\big|\Big], \quad m \geq 0,$$

and thus

$$\begin{aligned} |x_{m+j}(t, z, \eta) - x_m(t, z, \eta)| &\leq \frac{10}{9}\alpha_1(t) \sum_{i=1}^{j} Q^{m+i}\delta_D(f) \\ &= \frac{10}{9}\alpha_1(t)Q^m \sum_{i=0}^{j-1} Q^i \delta_D(f). \end{aligned} \tag{4.12}$$

It follows from the condition (A3) that

$$\sum_{i=0}^{j-1} Q^i \leq (I_n - Q)^{-1}, \qquad \lim_{m\to\infty} Q^m = O_2,$$

where O_2 is a zero 2×2 matrix.

Now, we conclude from (4.12) that the sequence $\{x_m(t, z, \eta)\}$ uniformly converges in the domain $(t, z, \eta) \in [0, 1] \times D_\beta \times D$ to the limit function $x_\infty(t, z, \eta)$. Since each function of such sequence satisfies the boundary conditions (4.2) for all values of z_1 and η_1, the limit function $x_\infty(t, z, \eta)$ also satisfies the boundary conditions. Let $m \to \infty$ in equality (4.4), we know that the limit function satisfies (4.5) and (4.6), where $\Delta(z, \eta)$ is given as (4.7). □

4.3. **Relation between the limit function and the solution of** (4.1)**.** Let us consider the problem with a *control parameter*

$$\begin{cases} x'(t) = f(t, x(t)) + \mu, & t \in (0, 1), \\ x(0) = z, \end{cases} \tag{4.13}$$

where $\mu = [\mu_1, \mu_2]^T$ is the control parameter.

Theorem 4.2. *Assume that* (A1)-(A3) *hold. Then* $x = x(\cdot, z, \eta, \mu)$ *is the solution of problem* (4.13) *with the boundary conditions* (4.2) *if and only if* $x = x(\cdot, z, \eta, \mu)$ *coincides with the limit function* $x_\infty(\cdot, z, \eta, \mu)$. *Moreover,*

$$\mu = \mu_{z,\eta} = [d(\eta) - (A + I_2)z] - \int_0^1 f(s, x_\infty(s, z, \eta, \mu))ds. \tag{4.14}$$

Proof. *Sufficiency.* Suppose that μ is given as (4.14). By Theorem 4.1, the limit function $x_\infty(\cdot, z, \eta, \mu)$ is the unique solution of problem (4.13) with (4.2) for the fixed values of parameters z and η when $\mu = \mu_{z,\eta}$. Besides, it is obvious that $x_\infty(t, z, \eta, \mu)$ satisfies initial condition $x_\infty(0, z, \eta, \mu) = z$. Therefore, $x_\infty(t, z, \eta, \mu)$ is a solution of problem (4.13) with $\mu = \mu_{z,\eta}$.

Necessity. Fix an arbitrary $\bar{\mu} \in \mathbb{R}^2$. We assume that the initial value problem (4.13) with $\mu = \overline{\mu}$ has a solution $\bar{x}(t)$ satisfying the boundary conditions (4.2). Then $\bar{x}(t)$ satisfies the integral equation

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds + \bar{\mu}t, \quad t \in [0, 1],$$

from which we know

$$\bar{\mu} = \bar{x}(1) - z - \int_0^1 f(s, \bar{x}(s))ds.$$

Using the the boundary restrictions (4.2), we get:

$$\bar{\mu} = [d(\eta) - (A + I_2)z] - \frac{1}{T}z - \int_0^1 f(s, \bar{x}(s))ds. \tag{4.15}$$

Similarly,

$$\begin{aligned} x_\infty(t, z, \eta, \mu) &= z + \int_0^t f(s, x_\infty(s, z, \eta, \mu))ds + \mu_{z,\eta}t, \\ \mu_{z,\eta} &= [d(\eta) - (A + I_2)z] - \int_0^1 f(s, x_\infty(s, z, \eta, \mu))ds. \end{aligned} \tag{4.16}$$

Thus, for all $t \in [0, 1]$,

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds - t\int_0^1 f(s, \bar{x}(s))ds + t[d(\eta) - (A + I_2)z],$$

$$\begin{aligned} x_\infty(t, z, \eta, \mu) = z + &\int_0^t f(s, x_\infty(s, z, \eta, \mu))ds \\ &- t\int_0^T f(s, x_\infty(s, z, \eta, \mu))ds + t[d(\eta) - (A + I_2)z]. \end{aligned}$$

Now, we conclude that $\bar{x}(t) \in D$ and $x_\infty(t, z, \eta, \mu) \in D$, and

$$x_\infty(t, z, \eta, \mu) - \bar{x}(t) = \int_0^t [f(s, x_\infty(s, z, \eta, \mu)) - f(s, \bar{x}(s))]ds \\ - \quad t\int_0^1 [f(s, x_\infty(s, z, \eta, \mu)) - f(s, \bar{x}(s))]ds.$$

Let $\sigma(t) = |x_\infty(t, z, \eta, \mu) - \bar{x}(t)|$. Then by the condition (A2) and the above equality, we obtain that

$$\sigma(t) \leq K\Big[\int_0^t \sigma(s)ds + t\int_0^1 \sigma(s)ds\Big] \leq K\alpha_1(t) \max_{s\in[0,1]} \omega(s), \quad t \in [0, 1], \tag{4.17}$$

where $\alpha_1(t)$ is given by (4.9).

Using (4.17) recursively, we arrive at an inequality:

$$\sigma(t) \leq K^m \alpha_m(t) \max_{s\in[0,1]} \omega(s), \quad t \in [0, 1], \quad m \in \mathbb{N}$$

where $\alpha_m(t)$ are given by (4.11). Taking into account (4.10), we obtain that

$$\sigma(t) \leq K\alpha_1(t)\frac{10}{9}\Big(\frac{3K}{10}\Big)^{m-1} \cdot \max_{s\in[0,1]} \sigma(s), \quad t \in [0, 1].$$

By the condition (A3) and taking the limit $m \to \infty$, it holds that

$$\max_{s\in[0,1]} \sigma(s) \leq Q^m \max_{s\in[0,1]} \sigma(s) \to 0,$$

which means that $\bar{x}(t)$ coincides with $x_\infty(t, z, \eta, \mu)$. Besides, it follows from (4.15) and (4.16) that $\bar{\mu} = \mu_{z,\eta}$. □

As a consequence of Theorem 4.2, the following result presents the relation between the limit function $x = x_\infty(t, z, \eta)$ and the solution of the parametrized problem (4.3).

Theorem 4.3. *Assume that conditions* (A1)-(A3) *hold. Then the pair* $(x_\infty(\cdot, z^*, \eta^*), \eta^*)$ *is a solution of the parametrized problem* (4.3) *with* (4.2) *if and only if* $z^* = (z_1^*, 0)$, $\eta^* = (\eta_1^*, 0)$ *satisfy the determining equation*

$$\Delta(z, \eta) = [d(\eta) - (A + I_2)z] - \int_0^1 f(s, x_\infty(s, z, \eta))ds = 0. \tag{4.18}$$

Next, we show that the determining equation (4.18) defines all possible solutions of problem (4.3).

Theorem 4.4. *Assume that conditions* (A1)-(A3) *hold. Suppose further that there exist vectors $z \in D_\beta$ and $\eta \in D$ satisfying the determining equation* (4.18)*. Then problem* (4.3) *has the solution $x(\cdot)$ such that $x(0) = z$, $x(1) = \eta$, and this solution is given as*

$$x(t) = x_\infty(t, z, \eta), \quad t = [0, 1], \tag{4.19}$$

where $x_\infty(t, z, \eta)$ is the limit function of the sequence (4.4)*. On the other hand, if problem* (4.3) *has a solution $x(\cdot)$, then $x(\cdot)$ is given by* (4.19) *and satisfies the determining equation* (4.18) *when $z = x(0)$, $\eta = x(1)$.*

Proof. Assume that there exist such $z \in D_\beta$ and $\eta \in D$ satisfying the determining system (4.18). Then by Theorem 4.3, the function (4.19) is a solution of problem (4.3).

On the other hand, if $x(\cdot)$ is the solution of problem (4.3), then this function is the solution of problem (4.13) for $\mu = 0$, $z = x(0)$. Since $x(\cdot)$ satisfies the boundary restrictions (4.2), we know that the equality (4.19) hold by Theorem 4.2. Besides, $\mu = \mu_{z,\eta} = 0$, $z = x(0)$, where the vector η is given as $x_1(0) = z_1$, $x_1(1) = \eta_1$. But $\mu_{z,\eta}$ is given by formula (4.14), which ensures that the first equation (4.18) of the determining system is satisfied if $z = x(0)$, $\eta = x(1)$, that is, $\Delta(z, \eta) = 0$. Therefore, we have specified such pairs $(z, \eta) = (x(0), x(1))$, which satisfy (4.18). □

4.4. **Example.** Now, we give a concrete example to illustrate the derived algorithm for approximation of solutions. For this purpose, we take a physically relevant oceanic linear vorticity $F(u) = -2u$. The original problem (2.6)-(2.8) is decomposed as

$$\begin{cases} x_1'(t) = x_2(t)(:= f_1(t, x_2)), \\ x_2'(t) = a(t)F(x_1(t)) + b(t)(:= f_2(t, x_1)), \\ x_2(0) = x_2(1) = 0. \end{cases} \tag{4.20}$$

Additionally, we introduce two parameters

$$z_1 = x_1(0) \text{ and } \eta_1 = x_1(1)$$

and modify the Neumann boundary conditions as follows

$$Ax(0) + I_2 x(1) = \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix},$$

where $x(\cdot) = [x_1(\cdot), x_2(\cdot)]^T$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $d(\eta) = \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix}$.

It is easy to check that the functions f_i, $i = 1, 2$ satisfy (A2) with a matrix

$$K = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{bmatrix}.$$

Moreover, the maximal eigenvalue of matrix $Q = \begin{bmatrix} 0 & \frac{3}{10} \\ \frac{3}{40} & 0 \end{bmatrix}$ defined in (A3) satisfies

$$r(Q) = 0.15 < 1.$$

Thus, all conditions of Theorem 4.1 are satisfied, and we can construct a sequence of functions $\{x_m(t, z, \eta)\}$ that approximates solutions to (4.20). This sequence is given by

$$x_m(t, z, \eta) = \begin{pmatrix} z_1 + \int_0^t x_{1,m}(s, z, \eta)ds - t\int_0^1 x_{1,m}(s, z, \eta)ds + t(\eta_1 - z_1) \\ \int_0^t(-2a(s)x_{2,m}(s, z, \eta) + b(s))ds - t\int_0^1(-2a(s)x_{2,m}(s, z, \eta) + b(s))ds \end{pmatrix},$$

for $m \in \mathbb{N}$, with the zeroth approximation defined by

$$x_0(t, z, \eta) = \begin{pmatrix} z_1 + t(\eta_1 - z_1) \\ 0 \end{pmatrix}. \tag{4.21}$$

The numerical values of the artificially introduced parameters z_1, η_1 are found as solutions to the determining system

$$\Delta_m(z, \eta) = \begin{pmatrix} -\int_0^1 x_{1,m}(s, z, \eta)ds + (\eta_1 - z_1) \\ -\int_0^1(-2a(s)x_{2,m}(s, z, \eta) + b(s))ds \end{pmatrix} = 0, \quad m \in \mathbb{N}_0. \tag{4.22}$$

As mentioned before, the zeroth approximation to the exact solution of (4.20) depending on parameters, is given by (4.21). After solving the determining system (4.22) for $m = 0$ we define the zeroth approximation to values of the unknown parameters $z_{1,0}$, $\eta_{1,0}$ which are

$$z_{1,0} = 1074.320725, \quad \eta_{1,0} = 1074.320725.$$

Then substituting these values into (4.21), we get the zeroth approximation to the exact solution of problem (4.20) as

$$X_{1,0}(t) = 1074.320725, \quad X_{2,0}(t) = 0.$$

Now, we proceed with the approximation process in the case of $m = 1$. Computations show that the first approximation to the exact solution of (4.20) is

$$X_{1,1}(t) = 1115.212769 - 85.16415t,$$

$$
\begin{aligned}
X_{1,2}(t) = \frac{1}{(1+e^t)^2}&(0.02(-8516.415e^{2t}\ln(1+e^t) + 66381.4906e^{2t} \\
&+ 8516.415te^{2t} - 17032.83e^t\ln(1+e^t) - 220671.7419e^t \\
&+ 8516.415te^t - 8516.415\ln(1+e^t) + 177902.7676)),
\end{aligned}
$$

which corresponds to the following values of parameters:

$$z_{1,1} = 1115.212769, \quad \eta_{1,1} = 1030.048619.$$

In Fig. 3, we give a componentwise comparison between the zeroth and the first approximations to the exact solution of (4.20).

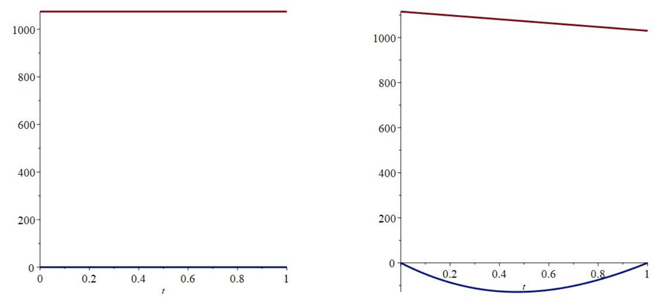

FIGURE 3. The first (red line) and the second (blue line) components of the approximate solution in the zeroth (to the left) and the first (to the right) approximations

These calculations can be continued in order to obtain an even more precise approximation. In particular, in the second approximation we obtain the following values of the unknown parameters:

$$z_{2,1} = 1115.891767, \quad \eta_{2,1} = 1030.721175,$$

while the third iteration leads to

$$z_{3,1} = 1116.219099, \quad \eta_{3,1} = 1030.367006.$$

The componentwise comparison of the graphs of the approximate solutions in the second and the third approximations are given in Fig. 4. Summarizing the obtained results, one can depict the components to the exact solution obtained in all 4 iteration steps (including the zeroth approximation). These graphs are given in Fig. 5.

In addition, the graphs of the error functions

$$\epsilon_m(t) = X_m(t) - f(t, X_m(t)), \quad t \in [0,1]$$

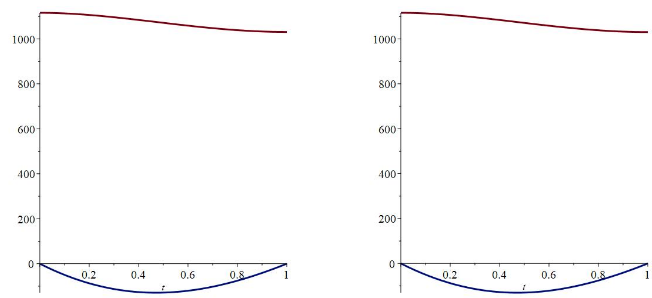

FIGURE 4. The first (red line) and the second (blue line) components of the approximate solution in the second (to the left) and the third (to the right) approximations

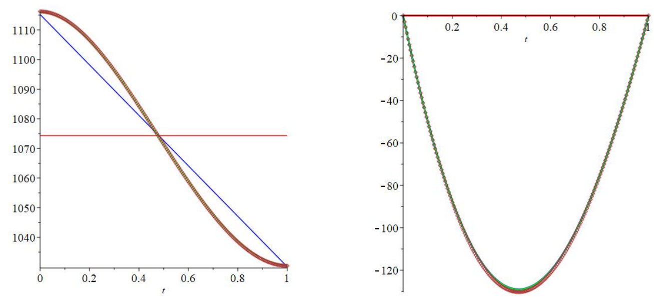

FIGURE 5. The first (to the left) and the second (to the right) components of the approximate solution and their 0-th (red line), first (blue line), second (green line) and third (brown points) approximations

for the first, second and third approximations are given in Fig. 6.

Thus, by this example, we have verified the applicability and effectiveness of the developed algorithm in the case of the linear oceanic vorticity. Moreover, we have constructed graphs of these approximations to give a more clear picture of the profile of the ACC flow.

Acknowledgements. The authors are grateful to the anonymous referee for the careful reading and valuable suggestions.

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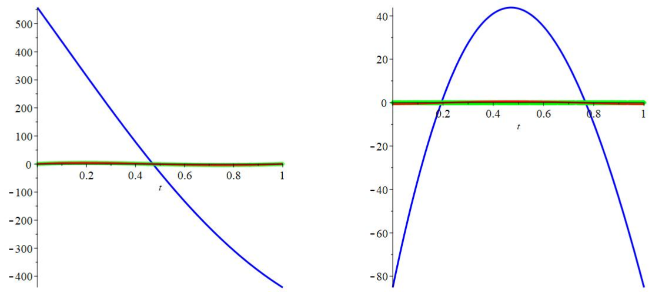

FIGURE 6. The first (to the left) and the second (to the right) components of the error functions in the first (blue line), second (green point) and third (red line) approximations

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