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Extended observer form with vector fields

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ABSTRACT

This paper addresses the problem of transforming a single-input single-output discrete-time system into the extended observer form which comprises a linear observable component and a nonlinear injection term depending on the input, output and their forward shifts up to a finite order. The necessary and sufficient conditions for the existence of the extended observer form are provided in terms of vector fields. The algorithm is presented to find a parametrised state transformation necessary to transform the system into the extended observer form. The obtained results are applicable also in case of non-reversible systems.

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1. Introduction

State feedback is a standard tool in control engineering. To utilise this tool, one needs to either measure or estimate the state variables. The state estimation is relatively easy for nonlinear systems if the state equations are in the so-called observer form, see, for example Krener Respondek (1985). The latter means that the equations are linear up to the injection terms, which depend on system measurable variables, i.e. output and input variables. The problem of transforming nonlinear state equations into the observer form by state transformation has been extensively studied for continuous-time systems, see Conte et al. (2007), Krener and Isidori (1983), Krener and Respondek (1985), Xia and Gao (1989) and discrete-time systems, see Lee and Hong (2011), Lee and Nam (1991), Mullari and Kotta (2021b). However, such transformations are often not feasible, which is why extensions of the observer form have been addressed.

The concept 'extended observer form' stands for the case when the injection terms depend, besides the input and output variables, also on their time derivatives (in the continuous-time case, Conte et al., 2007; Glumineau et al., 1996; Plestan & Glumineau, 1997) or backward shifts (in the discrete-time case, Huijberts, 1999; Huijberts et al., 1999; Kaparin & Kotta, 2018, 2019). For systems in the extended observer form one can, like in case of the classical observer form, construct an observer with liner error dynamics that converges asymptotically to zero. The results of previous papers raise a number of issues that need to be addressed. First, the results that are based on the original state equations and are coordinate independent assume that the state equations are reversible (Califano et al., 2003). Second, the linear part of the observer form suggested is allowed to be input independent that complicates the observer design (Califano et al., 2003). Third, some approaches require to find an input–output equation (Huijberts, 1999; Huijberts et al., 1999; Kaparin & Kotta, 2018, 2019) or the observable form (Lee & Hong, 2011) of the system equations as an intermediate step. These issues motivate to study the problem further.

In this paper, we study the problem of transforming a nonlinear single-input single-output (SISO) discrete-time control system into the extended observer form, where the nonlinear injection terms as well as the parametrised state transformation depend on the system input, its forward shifts up to the order r, the output and its forward shifts up to the order s. Since the problem is always solvable (unlike the one in Califano et al. (2003)) for $s = r = n-1$, where n is the dimension of the state-space, the challenge is to find minimal values of s and r for which the problem is still solvable. Although the future values of inputs and outputs are unknown, we will prove that such form can always be transformed (like in the input-free case, Simha et al., 2018) into the form with injection terms depending on the input and output values and a finite number of their past values (backward shifts). The reason for not addressing directly the form with injection terms depending on past measurements is that solvability conditions are easier to obtain for the case where the injection terms depend on future values of measurements since the state equations are defined in terms of forward shifts. Mixing backward and forward shifts require to take into account the functional dependence of variables which adds difficulties. This situation does not happen when future values of the input and output are used in the injection terms. Necessary and sufficient conditions are given for the transformability of the state equations of a given nonlinear discrete-time SISO system, using the parametrised state transformation, into an extended observer form. These conditions are stated in terms of a vector field Ξ, defined uniquely (in case of observable system) from state equations or more precisely, from forward shifts of the output. Note that checking the conditions does not require

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finding the input–output equation of the system. Moreover, the proposed conditions are constructive in a sense that one can easily determine the minimal values of s and r for which the conditions are satisfied and the transformation components can be found as invariants and canonical parameters of the projections of the backward shifts of Ξ. This requires solving an underdetermined set of partial differential equations (the need which is typical in many nonlinear control problems), the solutions of which are not unique, because there are less equations than unknowns. The latter just means that there is a freedom in the choice of the injection terms in the extended observer form.

In the special case, when $r = s = 0$, one recovers the existing solvability conditions of the problem on transformability the equations into the classical observer form. Compared to the classical case, the solvability conditions presented in the paper are weaker in the sense that the commutativity of smaller number of vector fields is required. Moreover, the computation of the corresponding transformation becomes easier, since a smaller set of partial differential equations must be solved.

The algebraic approach based on the vector space of vector fields is used to derive the results, see Kaldmäe et al. (2022) and Mullari et al. (2017). An important concept in this approach is the backward shift of a vector field, which is utilised in the paper to derive the results. It should be also noted that the algebraic approach used in the paper is developed to study generic properties of a control system. This means that the transformation is valid on some open subspace of the space, where the transformation is defined, instead being valid only locally in a neighbourhood of an equilibrium point. Note that higher generality of results has a price – we have to require that the state equations are analytic compared to smoothness requirement when one works in a neighbourhood of the equilibrium point.

In conclusion, a novel approach is proposed in this paper to transform an SISO discrete-time system into the extended observer form. The main advantages of the proposed approach are:

- The necessary and sufficient solvability conditions are given directly in terms of the original state equations unlike the earlier results that require to find the i/o equation (Kaparin & Kotta, 2018) or the observable form (Lee & Hong, 2011) of the system as an intermediate step. Our results do not require that the state equations are reversible; the weaker property of submersivity is assumed instead. The parametrised state transformation that transforms the extended observer form where the i/o injection terms depend on future input and output values to the form where the injection terms depend on their past values is also given.
- The results hold almost everywhere in an open set and not around the equilibrium point of the system as in Califano et al. (2003) and Lee and Hong (2011).
- A geometric interpretation for coordinate transformation as common invariants and canonical parameters of the set of vector fields Ξ and its backward shifts is provided. This makes the results more transparent and allows easier comparison with the classical case. Note that the solvability conditions are direct, but not simple extensions of those for the classical observer form (Mullari & Kotta, 2021b).

This paper is organised as follows. In Section 2, the methodology used in the paper is introduced. The problem statement is given in Section 3 and its solution in Section 4. Section 5 is devoted to transforming a system in an extended observer form with injection terms depending on future values of the input and output to the extended observer form where the injection terms depend on the past values of the input and output instead. Two examples are given in Section 6 and conclusions are drawn in Section 7.

2. Preliminaries

In this section, we recall briefly some preliminary results that originate mostly from Mullari et al. (2017); for more information, see also Mullari and Kotta (2021b). Consider the discrete-time single-input nonlinear control system

$$x^{\langle 1\rangle}(t) = \overline{\Phi}\,(x(t), u(t))\,, \tag{1}$$

where $x^{\langle 1\rangle}(t) := x(t+1), t \in \mathbb{Z}$, the state variable $x(t) \in \bar{X} \subset \mathbb{R}^n$, the control variable $u(t) \in U \subset \mathbb{R}$, and the state transition map $\overline{\Phi} : \bar{X} \times U \to \bar{X}$ is supposed to be analytic. Both $\bar{X}$ and U are assumed to be open sets. We assume that the map $\overline{\Phi}$ can be extended to the map $\Phi = [\overline{\Phi}^T, \chi^T]^T : \bar{X} \times U \to \bar{X} \times \mathbb{R}$ so that Φ has the global analytic inverse $[\Lambda^T, \lambda^T]^T : \Phi(\bar{X} \times U) \to \bar{X} \times U$. Introduce the additional variable at time instant t, $z(t) \in \mathbb{R}$, by

$$z(t) = \chi(x(t), u(t)). \tag{2}$$

The systems (1) and (2) define the inversive difference field $\mathcal{K}$ of meromorphic functions in a finite number of variables from the set $\mathcal{C} = \{x, u^{\langle k\rangle}, z^{\langle -l\rangle}, k \geq 0, l \geq 1\}$. Here $u^{\langle k\rangle}$ denotes the kth-order forward shift of x and $z^{\langle -l\rangle}$ the lth-order backward shift of z. The first-order forward shift of variable x is defined by Equation (1) and the first-order backward shifts of x and u by

$$x^{\langle -1\rangle} = \Lambda(x, z^{\langle -1\rangle}), \quad u^{\langle -1\rangle} = \lambda(x, z^{\langle -1\rangle}). \tag{3}$$

The forward and backward shifts of a function $\varphi(x, u, u^{\langle 1\rangle}, \ldots, u^{\langle k\rangle}, z^{\langle -1\rangle}, \ldots, z^{\langle -l\rangle}) \in \mathcal{K}$ are defined as the compositions $\varphi^{\langle 1\rangle} := \varphi(\overline{\Phi}(x, u), u^{\langle 1\rangle}, \quad u^{\langle 2\rangle}, \ldots, u^{\langle k+1\rangle}, \chi(x, u), \ldots, z^{\langle -l+1\rangle})$, and $\varphi^{\langle -1\rangle} := \varphi(\Lambda(x, z^{\langle -1\rangle}), \lambda(x, z^{\langle -1\rangle}), \quad u, \ldots, u^{\langle k-1\rangle}, z^{\langle -2\rangle}, \ldots, z^{\langle -l-1\rangle})$, respectively. The higher order forward and backward shifts of x are defined recursively as

$$\begin{aligned} x^{\langle k\rangle} &= \overline{\Phi}^k(x, u, u^{\langle 1\rangle}, \ldots, u^{\langle k-1\rangle}), \\ x^{\langle -k\rangle} &= \Lambda^k(x, z^{\langle -1\rangle}, z^{\langle -2\rangle}, \ldots, z^{\langle -k\rangle}), \end{aligned} \tag{4}$$

see more in Mullari Kotta (2021b). Due to (2) and (4), the higher order forward shifts of z can be computed as $z^{\langle k\rangle} = \chi\left(\overline{\Phi}^k(x, u, \ldots, u^{\langle k-1\rangle}), u^{\langle k\rangle}\right)$, and due to (3) and (4), the higher order backward shifts of u as $u^{\langle -k\rangle} = \lambda\left(\Lambda^{k-1}(x, z^{\langle -1\rangle}, \ldots, z^{\langle -k+1\rangle}), z^{\langle -k\rangle}\right)$.

Consider the infinite set of symbols $\mathrm{d}\mathcal{C} = \{\mathrm{d}x, \mathrm{d}u^{\langle k\rangle}, \mathrm{d}z^{\langle -l\rangle}, k \geq 0, l \geq 1\}$ and let $\mathcal{E} := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\mathcal{C}\}$ be the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}\mathcal{C}$, called the 1-form

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k\geq 0} B_k \mathrm{d}u^{\langle k\rangle} + \sum_{l\geq 1} C_l \mathrm{d}z^{\langle -l\rangle},$$

where only the finite number of coefficients differ from zero (Aranda-Bricaire et al., 1996). Define the space $\mathcal{E}^* = \mathrm{span}_{\mathcal{K}}\{\partial/\partial x,\quad \partial/\partial u^{\langle k\rangle},\ k \geq 0,\ \partial/\partial z^{\langle -l\rangle},\ l \geq 1\}$, dual to $\mathcal{E}$, whose elements are the vector fields

$$\overline{\Xi} = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \sum_{k\geq 0} \overline{\xi}_k \frac{\partial}{\partial u^{\langle k\rangle}} + \sum_{l\geq 1} \widetilde{\xi}_l \frac{\partial}{\partial z^{\langle -l\rangle}}. \tag{5}$$

By duality between $\mathcal{E}$ and $\mathcal{E}^*$ the scalar products of 1-form and vector fields satisfy the relations:

$$\left\langle \mathrm{d}x_i, \overline{\Xi}\right\rangle = \xi_i, \quad \langle \mathrm{d}u^{\langle k\rangle}, \overline{\Xi}\rangle = \overline{\xi}_k, \quad \langle \mathrm{d}z^{\langle -l\rangle}, \overline{\Xi}\rangle = \widetilde{\xi}_l.$$

The backward shift of the vector field $\overline{\Xi}$ in (5) is the vector field

$$\overline{\Xi}^{\langle -1\rangle} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{k\geq 0} b_k \frac{\partial}{\partial u^{\langle k\rangle}} + \sum_{l\geq 1} c_l \frac{\partial}{\partial z^{\langle -l\rangle}}, \tag{6}$$

where

$$a_i = \left\langle \mathrm{d}x_i^{\langle 1\rangle}, \overline{\Xi}\right\rangle^{\langle -1\rangle} = \langle \mathrm{d}\overline{\Phi}_i, \overline{\Xi}\rangle^{\langle -1\rangle}, \quad b_k = \left\langle \mathrm{d}u^{\langle k+1\rangle}, \overline{\Xi}\right\rangle^{\langle -1\rangle},$$
$$c_l = \left\langle \mathrm{d}z^{\langle -l+1\rangle}, \overline{\Xi}\right\rangle^{\langle -1\rangle}. \tag{7}$$

Note that the forward and backward shift operators commute with the scalar product, i.e. for an arbitrary 1-form $\omega \in \mathcal{E}$ and a vector field $\Xi \in \mathcal{E}^*$

$$\left\langle \omega, \overline{\Xi}\right\rangle^{\langle 1\rangle} = \left\langle \omega^{\langle 1\rangle}, \overline{\Xi}^{\langle 1\rangle}\right\rangle, \quad \left\langle \omega, \overline{\Xi}\right\rangle^{\langle -1\rangle} = \left\langle \omega^{\langle -1\rangle}, \overline{\Xi}^{\langle -1\rangle}\right\rangle. \tag{8}$$

The projection of $\overline{\Xi}$ in (5) is the vector field

$$\overline{\Xi}^{\pi} = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}. \tag{9}$$

Note that the shift and projection operators do not commute. From (6), (7) and (9) follows the computation formula of the projection of the backward shift of a vector field:

$$\overline{\Xi}^{\langle -1\rangle\pi} = \sum_{i=1}^{n} \left\langle \mathrm{d}x_i^{\langle 1\rangle}, \overline{\Xi}\right\rangle^{\langle -1\rangle} \frac{\partial}{\partial x_i}. \tag{10}$$

Lemma 2.1 (Mullari & Kotta, 2021b): *If* $\overline{\Xi} \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$, *then for* $k = 1, \ldots, n, \overline{\Xi}^{\langle -k\rangle} \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial x,\ \partial/\partial z^{\langle -1\rangle}, \ldots, \partial/\partial z^{\langle -k\rangle}\}$.

Lemma 2.2: *If*

$$\overline{\Xi} = \sum_{i=1}^{n} \xi_i(x, u, \ldots, u^{\langle k\rangle}) \frac{\partial}{\partial x_i},$$

then

$$\overline{\Xi}^{\langle -1\rangle\pi} = \sum_{i=1}^{n} a_i(x, z^{\langle -1\rangle}, u, \ldots, u^{\langle k-1\rangle}) \frac{\partial}{\partial x_i}$$

for some functions a_i.

Proof: Follows directly from (10). ∎

Lemma 2.3 below, necessary for the proof of the main theorem, is the extension of Lemma 3 in Mullari and Kotta (2021a) for the case when the number of independent commuting vector fields is less than the dimension of the space where they are defined. Note that Lemma 2.3 as well as Lemma 3 from Mullari and Kotta (2021a) are the analogues of Theorem 2.36 from Nijmeijer and Van Der Schaft (1990), which proves the local validity of the result around a fixed point on a manifold. The proof, as commented in Mullari and Kotta (2021a), is not easy to extend for the generic case, since it relies on shifting the point under the flows of the vector fields. The coordinate transformation can, under commutativity assumption, be constructed as the composition of these flows. Moreover, the approach in this paper is algebraic, not differential geometric. For the above reasons, the alternative proof is given.

Denote $\bar{n} := n + r$, where $r \in \{0, \ldots, n-1\}$, the meaning of which will be explained in the next section. In Lemma 2.3, we assume that in an open set $\overline{\mathcal{C}}$ of an $\bar{n}$-dimensional subspace $\mathbb{R}^{\bar{n}}$ with coordinates $\bar{x} := \{x, u, \ldots, u^{\langle r-1\rangle}\}$ are defined $\bar{m}$ vector fields $\Xi_l = \sum_{q=1}^{\bar{n}} \xi_{ql}(\bar{x})\,\partial/\partial\bar{x}_q$, $l = 1, \ldots, \bar{m}$, $\bar{m} < \bar{n}$. One says that a coordinate transformation $X = \Psi(\bar{x})$, defined in $\bar{\mathcal{C}}$, is adapted to vector fields Ξ_l if in the new coordinates the vector fields Ξ_l are the partial derivative operators with respect to the corresponding coordinates, i.e. $\Psi_*\Xi_l = \partial/\partial X_l, l = 1, \ldots, \bar{m}$.

Lemma 2.3: *Assume that the vector fields* $\Xi_l = \sum_{q=1}^{\bar{n}} \xi_{ql}(\bar{x})\partial/\partial\bar{x}_q$, $l = 1, \ldots, \bar{m}$,

(a) *are linearly independent,*

$$\dim_{\mathcal{K}}\left(\mathrm{span}_{\mathcal{K}}\{\Xi_1, \ldots, \Xi_{\bar{m}}\}\right) = \bar{m},$$

(b) *and commute:*

$$\left[\Xi_l, \Xi_j\right] \equiv 0, \quad l, j = 1, \ldots, \bar{m}. \tag{11}$$

Then there exists in $\overline{\mathcal{C}}$ *the coordinate transformation* $X = \Psi(\bar{x})$, *adapted to these vector fields, such that* $\bar{m}$ *independent functions* $\Psi_i(\bar{x})$ *(as the* canonical parameters *of* Ξ_l*) satisfy*

$$\langle \mathrm{d}\Psi_i(\bar{x}), \Xi_l(\bar{x})\rangle \equiv \delta_{i,l}, \quad i, l = 1, \ldots, \bar{m}, \tag{12}$$

and $\bar{n} - \bar{m}$ *independent functions* $\Psi_i(\bar{x})$ *(as the* common invariants *of* Ξ_l*) satisfy*

$$\langle \mathrm{d}\Psi_i(\bar{x}), \Xi_l(\bar{x})\rangle \equiv 0, \quad i = \bar{m}+1, \ldots, \bar{n}, \quad l = 1, \ldots, \bar{m}. \tag{13}$$

$l = 1, \ldots, \bar{m}$, *are the partial derivative operators with respect to the corresponding coordinates, i.e.*

Proof: See Appendix A. ∎

3. Problem statement

In this section, we first define the extended observer form, into which we want to transform the original state equations. Second, we introduce the parametrised state transformation, necessary for this purpose. Third, we introduce a set of vector fields, defined by the system equations, in terms of which we formulate in the next section the necessary and sufficient solvability conditions. Finally, some additional results on these vector fields will be proven to be applied in the proof of the main result.

Consider the system (1) together with the output function

$$x^{\langle 1\rangle} = \overline{\Phi}(x,u), \quad y = h(x), \tag{14}$$

where the output $y \in Y \subseteq \mathbb{R}$ and Y is an open set.

Denote $y^{\langle 0\rangle} = h(x)$. Compute, using (4),

$$y^{\langle l\rangle} = h(\overline{\Phi}^l(x,u,\dots,u^{\langle l-1\rangle})), \quad l = 1,\dots,n-1. \tag{15}$$

Recall from Kotta et al. (2015), the spaces $\mathcal{Y} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{\langle l\rangle}, l \geq 0\}$, $\mathcal{U} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u^{\langle j\rangle}, j \geq 0\}$, $\mathcal{X} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$. The subspace $\mathcal{O} = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$ is called the *observable space of system* (14).

Assumption 3.1: *The system* (14) *satisfies the generic observability condition* $\dim_{\mathcal{K}} \mathcal{O} = n$.

Define the set of 1-form:

$$\omega_l := \sum_{i=1}^{n} \frac{\partial y^{\langle l\rangle}}{\partial x_i} \mathrm{d}x_i, \quad l = 0,\dots,n-1. \tag{16}$$

Assumption 3.1 is equivalent to the condition that the 1-form ω_l, $l = 0,\dots,n-1$, are linearly independent:

$$\dim_{\mathcal{K}}\left(\mathrm{span}_{\mathcal{K}}\{\omega_l, l = 0,\dots,n-1\}\right) = n.$$

The goal is to find the parametrised state transformation $X = \Psi(x,u,\dots,u^{\langle r-1\rangle})$, such that Equation (14) in the new coordinates is in the extended observer form with the degrees (s,r), where s and r take the minimal possible values[1] from the set $\{0,\dots,n-1\}$:

$$\begin{aligned}
X_i^{\langle 1\rangle} &= X_{i+1}, \quad i = 1,\dots,s,\\
X_i^{\langle 1\rangle} &= X_{i+1} + \varphi_i(y,\dots,y^{\langle s\rangle},u,\dots,u^{\langle r\rangle}),\\
&\quad i = s+1,\dots,n-1,\\
X_n^{\langle 1\rangle} &= \varphi_n(y,\dots,y^{\langle s\rangle},u,\dots,u^{\langle r\rangle}), \quad y = X_1.
\end{aligned} \tag{17}$$

If s and r are both zero, then (17) coincides with the classical observer form. In case $s = 0$, the first set of equations in (17) is obviously missing.

Under Assumption 3.1, define the vector field $\Xi \in \mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$ such that

$$\langle \omega_l, \Xi\rangle \equiv \delta_{l,n-1}, l = 0,\dots,n-1, \tag{18}$$

and compute the projections $\Xi^{\langle -l\rangle\pi}$ of its backward shifts up to the order $n-1$.

Remark 3.1: Observe that, due to (15), the lth-order forward shift of y depends, in general, on u and its forward shifts up to the order $l-1$, as do, by (16), the coefficients of $\omega_l, l = 1,\dots,n-1$. That is, the system of Equations (18) depends, in general, on u and its forward shifts up to the order $n-2$. The latter means that the coefficients of Ξ are, in general, the functions of $x, u, \dots, u^{\langle n-2\rangle}$.

Lemma 3.2 (Mullari & Kotta, 2021b): *The vector field Ξ and the projections of its backward shift up to the order $n-1$ are linearly independent:*

$$\dim_{\mathcal{K}}(\mathrm{span}_{\mathcal{K}}\{\Xi^{\langle -l\rangle\pi}, l = 0,\dots,n-1\}) = n.$$

All new state coordinates in the *classical* observer form case can be defined as the canonical parameters of commutable vector fields $\Xi^{\langle -l\rangle\pi}$, $l = 0,\dots,n-1$, see Lemma 5 in Mullari and Kotta (2021b). If (14) is transformable into the *extended* observer form, then only a set of $n-s$ vector fields are commutable (see Theorem 4.1) and thus only $n-s$ new state coordinates can be defined as canonical parameters of the vector fields. The rest of the new coordinates can be defined as common invariants of the corresponding vector fields given in Lemma 3.3.

Lemma 3.3: *The output y and its forward shifts up to the order $s-1$ are the common invariants of the vector fields $\Xi^{\langle -l\rangle\pi}$, $l = 0,\dots,n-s-1$, i.e.*

$$\left\langle \mathrm{d}y^{\langle i\rangle}, \Xi^{\langle -l\rangle\pi}\right\rangle \equiv 0, \quad i = 0,\dots,s-1, \tag{19}$$

and the sth-order forward shift is the canonical parameter of $\Xi^{\langle -n+s+1\rangle\pi}$, i.e.

$$\left\langle \mathrm{d}y^{\langle s\rangle}, \Xi^{\langle -l\rangle\pi}\right\rangle \equiv \delta_{l,n-s-1}. \tag{20}$$

Proof: Note that according to (16)

$$\mathrm{d}y = \omega_0, \quad \mathrm{d}y^{\langle i\rangle} = \omega_i + \sum_{k=0}^{i-1} \frac{\partial y^{\langle i\rangle}}{\partial u^{\langle k\rangle}} \quad \partial u^{\langle k\rangle}, \quad i = 1,\dots,n-1. \tag{21}$$

Using (18) and (21), one may verify that $\langle \mathrm{d}y^{\langle i+l\rangle}, \Xi\rangle \equiv \delta_{i+l,n-1}$, $i = 0,\dots,s$, $l = 0,\dots,n-s-1$, whose lth-order backward shift reads $\langle \mathrm{d}y^{\langle i\rangle}, \Xi^{\langle -l\rangle}\rangle \equiv \delta_{i+l,n-1}$. Using Lemmas 2.1 and (9) is easy to conclude $\langle \mathrm{d}y^{\langle i\rangle}, \Xi^{\langle -l\rangle\pi}\rangle \equiv \delta_{i+l,n-1}$. Next, subtracting l from the both indices of Kronecker delta yields $\langle \mathrm{d}y^{\langle i\rangle}, \Xi^{\langle -l\rangle\pi}\rangle \equiv \delta_{i,n-l-1}$ for $i = 0,\dots,s, l = 0,\dots,n-s-1$. From this formula follow directly (19) and (20). ∎

4. The main result

Theorem 4.1 gives the main result of this paper, that is the necessary and sufficient solvability conditions for possibility to transform the state equations into the extended observer form with the degrees (s,r). Moreover, the algorithm will be given for finding the parametrised state transformation. Finally we will demonstrate that in the case $s = r = 0$ the conditions of Theorem 4.1 recover the existing conditions related to the classical observer form.

Theorem 4.1 (Main result): *Under Assumption* 3.1, *Equation* (14) *can be transformed, via a parametrised state transformation $X = \Psi(x, u, \dots, u^{\langle r-1\rangle})$, into the extended observer form* (17) *with the degrees (s, r) if and only if the following conditions are satisfied, where Ξ below is defined by* (18)*:*

(i)

$$\left[\frac{\partial}{\partial z^{\langle -q\rangle}}, \Xi^{\langle -l\rangle\pi}\right] \equiv 0, \quad q, l = 1, \dots, n-s-1, \quad (22)$$

(ii)

$$\left[\Xi^{\langle -l\rangle\pi}, \Xi^{\langle -j\rangle\pi}\right] \equiv 0 \quad l, j = 0, \dots, n-s-1, \quad (23)$$

(iii)

$$\frac{\partial y^{\langle s\rangle}}{\partial u^{\langle k\rangle}} \equiv 0, \quad k = r, \dots, s-1,$$

(iv)

$$\left[\frac{\partial}{\partial u^{\langle k\rangle}}, \Xi\right] \equiv 0, \quad k = r, \dots, n-2.$$

Proof: *Sufficiency.* The proof relies on Lemmas 2.2, 2.3, 3.2 and 3.3 and consists of three steps. In the first step, the parametrised state transformation is defined. In the second step, we will show that the forward shifts of these coordinates cannot depend on forward shifts of u higher than the order r. In the third step, we will prove that in the new coordinates the state equations take the extended observer form (17).

The parametrised state transformation can be interpreted as a coordinate transformation in the $(n+r)$-dimensional subspace $\overline{\mathcal{C}}$ of $\mathcal{C}$ with coordinates $\bar{x} = (x, u, \dots, u^{\langle r-1\rangle})^T$ such that the coordinates $u, \dots, u^{\langle r-1\rangle}$ remain unchanged. Before giving a formal proof, we will briefly sketch its idea. Note that one has to define the coordinate transformation in the space $\bar{\mathcal{C}}$, since the vector field Ξ and the projections of its backward shifts depend, in general, also on variables $u, \dots, u^{\langle r-1\rangle}$. From Lemma 2.3, one may conclude that if in the $(n+r)$-dimensional space with coordinates $\bar{x}$ exist $n-s$ independent commuting vector fields whose coordinates depend only on variables $\bar{x}$ (this aspect is important), then one can define in this space the following coordinate transformation, where $X = \Psi(x, u, \dots, u^{\langle r-1\rangle})$, but the input and its forward shifts up to the order $r-1$ remain unchanged. The new coordinates include $r+s$ independent invariants of these vector fields and $n-s$ canonical parameters of the vector fields. These invariants are the output y and its forward shifts up to the order $s-1$ (see Lemma 3.3), and also $u, \dots, u^{\langle r-1\rangle}$, because these vector fields belong to $\mathrm{span}_{\mathcal{K}}\{\partial/\partial x\}$. Lemma 2.3 (taking $\bar{n} = n+r$, $\bar{m} = n-s$, $\Xi_1 = \Xi^{\langle -n+s+1\rangle\pi}, \dots, \Xi_{\bar{m}-1} = \Xi^{\langle -1\rangle\pi}$, $\Xi_{\bar{m}} = \Xi$) points how to find the other set of new coordinates as canonical parameters of the vector fields.

If (iv) holds, then by Remark 3.1, the coefficients of Ξ do not depend on higher order forward shifts of u than $r-1$. Then, by Lemma 2.2, also the coefficients of $\Xi^{\langle -l\rangle\pi}$, $l = 1, \dots, n-s-1$, do not depend on $u^{\langle k\rangle}$, $k > r-1$. If additionally (i) is satisfied, then the coefficients of $\Xi^{\langle -l\rangle\pi}$, $l = 0, \dots, n-s-1$, depend only on $\bar{x}$. If also (ii) is valid, then by Lemma 2.3 there exists an adapted coordinate transformation which will be defined below at the first step of the proof.

Step 1. Due to Lemma 3.2 and (ii), in $\overline{\mathcal{C}}$ there exist $n-s$ independent commuting vector fields $\Xi^{\langle -l\rangle\pi}$, $l = 0, \dots, n-s-1$. Because $\dim_{\mathcal{K}} \overline{\mathcal{C}} = n+r$, and because the coefficients of $\Xi^{\langle -l\rangle\pi}$ depend only on $\bar{x}$, then, by Lemma 2.3, there exist $r+s$ independent functions as the invariants of these vector fields, and as well $n-s$ independent functions as the canonical parameters of these vector fields. These functions depend on $x, u, u^{\langle 1\rangle}, \dots, u^{\langle r-1\rangle}$ and they can be taken as the components of the parametrised state transformation.

If $s > 0$, then find the first set of new state coordinates as the independent invariants of $\Xi^{\langle -l\rangle\pi}$, $l = 0, \dots, n-s-1$. According to Lemma 2.3, in $(n+r)$-dimensional $\overline{\mathcal{C}}$ exist $r+s$ common independent invariants of these vector fields. As mentioned above, the variables $u, \dots, u^{\langle r-1\rangle}$ belong to the set of the invariants of $\Xi^{\langle -l\rangle\pi}$, $l = 0, \dots, n-s-1$, but obviously, they do not count as the new *state* coordinates. The remaining s independent (by Assumption 3.1) invariants can be found from Lemma 3.3 as

$$\left\langle \mathrm{d}y^{\langle i\rangle}, \Xi^{\langle -l\rangle\pi}\right\rangle \equiv 0, \quad i = 0, \dots, s-1, \quad l = 0, \dots, n-s-1, \quad (24)$$

and so one can define the first s new state coordinates as

$$X_i = y^{\langle i-1\rangle}, \quad i = 1, \dots, s. \quad (25)$$

The new coordinates $X_{s+1}, \dots, X_n$ are defined by system of equations

$$\left\langle \mathrm{d}\Psi_i, \Xi^{\langle -l\rangle\pi}\right\rangle \equiv \delta_{i,n-l}, \quad i = s+1, \dots, n,$$
$$l = 0, \dots, n-s-1, \quad (26)$$

as the canonical parameters of Ξ and the projections of its backward shifts up to the order $n-s-1$. When $s > 0$, then the number of these vector fields is less than n and therefore, $X_{s+1}, \dots, X_n$ are, in principle, defined from (26) uniquely up to some additive functions f of system invariants $y^{\langle j\rangle}$, $j = 0, \dots, s-1$ (see Lemma 3.3) and $u^{\langle k\rangle}$, $k = 0, \dots, r-1$. The reason is that if the function $\Psi_i(x, u, \dots, u^{\langle r-1\rangle})$, $i = s+1, \dots, n$, satisfies Equation (26), then also a function $\Psi_i(x, u, \dots, u^{\langle r-1\rangle}) + f(y, \dots, y^{\langle s-1\rangle}, u, \dots, u^{\langle r-1\rangle})$ does where $f(0) = 0$. The exception is X_{s+1}, that, by the form of the extended observer form (17), has to be defined uniquely by

$$X_{s+1} = \Psi_{s+1}(x, u, \dots, u^{\langle r-1\rangle}) = y^{\langle s\rangle}, \quad (27)$$

since otherwise the sth equation in (17) is not satisfied.

To define the coordinates $X_{s+2}, \dots, X_n$, construct an $[(n-s) \times n]$-matrix $\Theta = [\theta_{s+1}^T \; \dots \; \theta_n^T]^T$, whose rows can be interpreted as the 1-forms, and an $[n \times (n-s)]$-matrix[2]

$$M := \left[\Xi^{\langle -n+s+1\rangle\pi} \; \dots \; \Xi^{\langle -1\rangle\pi} \;\; \Xi\right], \quad (28)$$

whose columns are, according to Lemma 3.2, linearly independent. Therefore, $\operatorname{rank}_{\mathcal{K}} M = n - s$ and there exists the left inverse of M as the $[(n-s) \times n]$-matrix

$$M_L^{-1} = (M^T M)^{-1} M^T = \Theta : \quad \Theta M \equiv I_{n-s}, \tag{29}$$

whereby $\operatorname{rank}_{\mathcal{K}} \Theta = n - s$. The rows θ_i of Θ are linearly independent as the rows of a matrix of full rank and, due to (28), satisfy the condition

$$\left\langle \theta_i, \Xi^{\langle -l \rangle \pi} \right\rangle \equiv \delta_{i,n-l}, \quad i = s+2, \dots, n,$$
$$l = 0, \dots, n-s-1. \tag{30}$$

The 1-form θ_i, defined by (29), is not necessarily the total differentials. However, we will show that there exist coefficients β_{iq} and $\tilde{\beta}_{ik}$ such that

$$\mathrm{d}\Psi_i = \theta_i + \sum_{q=1}^{s} \beta_{iq}(\bar{x}) \mathrm{d}\Psi_q + \sum_{k=0}^{r-1} \tilde{\beta}_{ik}(\bar{x}) \mathrm{d}u^{\langle k \rangle},$$
$$i = s+2, \dots, n. \tag{31}$$

Really, from (26), (30), and Lemma 2.3 follows that the 1-form $\mathrm{d}X_i = \mathrm{d}\Psi_i(x, u, \dots, u^{\langle r-1 \rangle}), i = 1, \dots, s, \theta_i, i = s+1, \dots, n$, and $\mathrm{d}u^{\langle k \rangle}, k = 0, \dots, r-1$, is linearly independent and therefore they span the entire space $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{\langle r-1 \rangle}\}$. Consequently, the vector space $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}\Psi_1, \dots, \mathrm{d}\Psi_s, \theta_{s+1}, \dots, \theta_n, \mathrm{d}u, \dots, \mathrm{d}u^{\langle r-1 \rangle}\}$ is integrable and it has exact bases, i.e. there exist linear combinations as in (31), being the total differentials.

Step 2. Will show now that in the new coordinates X, defined at the previous step, under conditions (iii) and (iv), the forward shifts $X^{\langle 1 \rangle}$ do not depend on $u^{\langle k \rangle}, k > r$. If (iii) holds, then $X_1, \dots, X_{s+1}$ do not depend on $u^{\langle k \rangle}, k \geq r$, meaning that their forward shifts do not depend on $u^{\langle k \rangle}, k > r$. If (iv) holds, then the coefficients of Ξ do not depend on $u^{\langle k \rangle}, k \geq r$, and by Lemma 2.2 this is also true for the coefficients of $\Xi^{\langle -1 \rangle \pi}, \dots, \Xi^{\langle -n+s+1 \rangle \pi}$. From (26) follows then that $X_{s+2}, \dots, X_n$ can be defined so that they do not depend on $u^{\langle k \rangle}, k \geq r$ and so $X_{s+2}^{\langle 1 \rangle}, \dots, X_n^{\langle 1 \rangle}$ do not depend on $u^{\langle k \rangle}, k > r$.

Step 3. We will prove now that in coordinates X Equation (14) takes the form (17). Obviously, from (25) and (27) one gets, as in (17),

$$X_i^{\langle 1 \rangle} = X_{i+1}, \quad i = 1, \dots, s.$$

Since $\mathrm{d}\Psi_i \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{\langle r-1 \rangle}\}, i = 1, \dots, n,$ then $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{\langle r \rangle}\} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\Psi, \mathrm{d}u, \dots, \mathrm{d}u^{\langle r \rangle}\}$, and because $\mathrm{d}\Psi_i^{\langle 1 \rangle}, i = 1, \dots, n$, belong to $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{\langle r \rangle}\}$, then one may write $\mathrm{d}\Psi^{\langle 1 \rangle}$ as the linear combination

$$\mathrm{d}\Psi_i^{\langle 1 \rangle} = \sum_{q=1}^{n} \alpha_{iq}(x, u, \dots, u^{\langle r \rangle}) \mathrm{d}\Psi_q$$
$$+ \sum_{k=0}^{r} \bar{\alpha}_{ik}(x, u, \dots, u^{\langle r \rangle}) \mathrm{d}u^{\langle k \rangle},$$
$$i = s+1, \dots, n. \tag{32}$$

Because the left-hand side of (32) is a total differential, then there exist the functions $\phi_i(X, u, \dots, u^{\langle r \rangle}) = \Psi_i \circ \overline{\Phi} \circ \Psi^{-1}, i = s+1, \dots, n$, such that

$$\alpha_{iq} = \left. \frac{\partial \phi_i(X, u, \dots, u^{\langle r \rangle})}{\partial X_q} \right|_{X = \Psi(x, u, \dots, u^{\langle r-1 \rangle})},$$
$$q = 1, \dots, n,$$
$$\bar{\alpha}_{ik} = \left. \frac{\partial \phi_i(X, u, \dots, u^{\langle r \rangle})}{\partial u^{\langle k \rangle}} \right|_{X = \Psi(x, u, \dots, u^{\langle r-1 \rangle})}, \quad k = 0, \dots, r.$$

On the other hand, computing the total differentials from both sides of the second and third formulae of (17) and taking into account (25) and (27), we obtain

$$\mathrm{d}X_i^{\langle 1 \rangle} = \mathrm{d}X_{i+1} + \sum_{j=1}^{s+1} \frac{\partial \varphi_i(X_1, \dots, X_{s+1}, u, \dots, u^{\langle r \rangle})}{\partial X_j} \mathrm{d}X_j +$$
$$+ \sum_{k=0}^{r} \frac{\partial \varphi_i(X_1, \dots, X_{s+1}, u, \dots, u^{\langle r \rangle})}{\partial u^{\langle k \rangle}} \mathrm{d}u^{\langle k \rangle},$$
$$i = s+1, \dots, n-1,$$
$$\mathrm{d}X_n^{\langle 1 \rangle} = \sum_{j=1}^{s+1} \frac{\partial \varphi_n(X_1, \dots, X_{s+1}, u, \dots, u^{\langle r \rangle})}{\partial X_j} \mathrm{d}X_j$$
$$+ \sum_{k=0}^{r} \frac{\partial \varphi_n(X_1, \dots, X_{s+1}, u, \dots, u^{\langle r \rangle})}{\partial u^{\langle k \rangle}} \mathrm{d}u^{\langle k \rangle}. \tag{33}$$

From (32) and (33) follows that to show the validity of the second and third formulae of (17), we need to prove that

$$\alpha_{iq} = \delta_{i+1,q}, \quad i = s+1, \dots, n, \quad q = s+2, \dots, n. \tag{34}$$

From (24), (25) and (26) follows $\left\langle \mathrm{d}\Psi_i, \Xi^{\langle -l \rangle \pi} \right\rangle \equiv \delta_{i,n-l}, i = 1, \dots, n, l = 0, \dots, n-s-1$. Denoting $q := n - l$, the last equality takes the form

$$\left\langle \mathrm{d}\Psi_i, \Xi^{\langle -n+q \rangle \pi} \right\rangle \equiv \delta_{i,q}, \quad i = 1, \dots, n, \quad q = s+1, \dots, n. \tag{35}$$

Using (35), we can prove (34), taking the scalar product of both sides of (32) with $\Xi^{\langle -n+q \rangle \pi}, q = s+1, \dots, n$. This gives, due to (35) and the definition of the Kronecker delta, $\alpha_{iq} = \langle \mathrm{d}\Psi_i^{\langle 1 \rangle}, \Xi^{\langle -n+q \rangle \pi} \rangle, i, q = s+1, \dots, n$. The last equality is identical to $\alpha_{iq} = \langle \mathrm{d}\Psi_i^{\langle 1 \rangle}, \Xi^{\langle -n+q \rangle} \rangle$, according to $\Xi^{\langle -n+q \rangle} \in \operatorname{span}_{\mathcal{K}}\{\partial/\partial x, \partial/\partial z^{\langle -1 \rangle}, \dots, \partial/\partial z^{\langle -n+q \rangle}\}$ and $\mathrm{d}\Psi^{\langle 1 \rangle} \in \{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{\langle r \rangle}\}$. Due to (8), the last equality can be rewritten in the form $\alpha_{iq} = \langle \mathrm{d}\Psi_i, \Xi^{\langle -n+q-1 \rangle} \rangle^{\langle 1 \rangle} = \langle \mathrm{d}\Psi_i, \Xi^{\langle -n+q-1 \rangle \pi} \rangle^{\langle 1 \rangle}$. Taking into account (35) and the fact that the value of a constant is invariant with respect to the forward shift, we get from the last equality $\alpha_{iq} = \delta_{i,q-1}$, which is equivalent to $\alpha_{i,q} = \delta_{i+1,q}$, i.e. (34) holds.

Necessity. To prove (i), (ii) and (iv), we will show by induction that in the new coordinates

$$\Xi^{\langle -j \rangle \pi} = \frac{\partial}{\partial X_{n-j}}, \quad j = 0, \dots, n-s-1. \tag{36}$$

Prove first that (36) is true for $j = 0$, i.e.

$$\Xi = \frac{\partial}{\partial X_n}. \tag{37}$$

Note that, according to (17),

$$y^{\langle l\rangle} = X_{l+1} + \begin{cases} 0, & l = 0, \dots, s, \\ \rho_l\left(X_1, \dots, X_l, u, \dots, u^{\langle r+l-s-1\rangle}\right), & l = s+1, \dots, n-1, \end{cases} \tag{38}$$

where ρ_l are certain sums of $\varphi_{s+1}, \dots, \varphi_{n-1}$ and their forward shifts. Using (16) and (38), one obtains

$$\omega_l = \mathrm{d}X_{l+1} + \begin{cases} 0, & l = 0, \dots, s, \\ \displaystyle\sum_{i=1}^{l} \frac{\partial \rho_l}{\partial X_i}\mathrm{d}X_i, & l = s+1, \dots, n-1, \end{cases}$$

which, according to (18), yields (37).

Next, we prove that if

$$\Xi^{\langle -j\rangle} = \frac{\partial}{\partial X_{n-j}} \tag{39}$$

holds for some $j = 0, \dots, n-s-2$ then it holds for $j+1$ too. Taking into account (6), (7) and (39), one gets

$$\Xi^{\langle -j-1\rangle} = \left(\Xi^{\langle -j\rangle}\right)^{\langle -1\rangle} = \sum_{i=1}^{n} \left\langle \mathrm{d}X_i^{\langle 1\rangle}, \frac{\partial}{\partial X_{n-j}} \right\rangle^{\langle -1\rangle} \frac{\partial}{\partial X_i}. \tag{40}$$

Since (17) yields

$$\mathrm{d}X_i^{\langle 1\rangle} = \begin{cases} \mathrm{d}X_{i+1}, & i = 1, \dots, s, \\ \mathrm{d}X_{i+1} + \sum_{k=1}^{s+1} \frac{\partial \varphi_i}{\partial X_k}\mathrm{d}X_k \\ \quad + \sum_{p=0}^{r} \frac{\partial \varphi_i}{\partial u_p}\mathrm{d}u_p, & i = s+1, \dots, n-1, \\ \sum_{k=1}^{s+1} \frac{\partial \varphi_n}{\partial X_k}\mathrm{d}X_k + \sum_{p=0}^{r} \frac{\partial \varphi_n}{\partial u_p}\mathrm{d}u_p, & i = n, \end{cases}$$

one may conclude that $\langle \mathrm{d}X_i^{\langle 1\rangle}, \partial/\partial X_{n-j}\rangle \equiv \delta_{i+1,n-j}$, which implies that (40) can be rewritten as $\Xi^{\langle -j-1\rangle} = \partial/\partial X_{n-j-1}$. Now the validity of (36) follows directly from the definition of the projection (9). ■

Remark 4.1: Under Assumption 3.1, one can always transform the state Equation (14) into the extended observer form (17) with the degree $s = n-1$, because for this degree the conditions (i) and (ii) of Theorem 4.1 are satisfied automatically. The coordinate transformation $X_i = y^{\langle i-1\rangle}, i = 1, \dots, n$, leads to the extended observer form

$$X_i^{\langle 1\rangle} = X_{i+1}, \quad i = 1, \dots, n-1,$$
$$X_n^{\langle 1\rangle} = \varphi_n(y, \dots, y^{\langle n-1\rangle}, u, \dots, u^{\langle r\rangle}),$$

where r is determined by the conditions (iii) and (iv).

Algorithm for finding the minimal (s, r) and the respective parametrised state transformation.

Step 1. Compute the 1-form ω_l, $l = 0, \dots, n-1$, as in (16). Check if they are linearly independent. If not, then stop.

Step 2. Find the vector field Ξ as a solution of Equation (18). Find the projections of its backward shifts up to the order $n-1$.

Step 3. Compute the Lie brackets (22) and (23) for $l, j = 0, \dots, n-1$. If all these Lie brackets satisfy the conditions (i) and (ii), take $s = 0$. If not, then find the greatest integer $\bar{l} \in [0, \dots, n-1]$ such that (i) and (ii) hold for all $\Xi^{\langle -l\rangle\pi}, \Xi^{\langle -j\rangle\pi}, l, j = 0, \dots, \bar{l}$. Take $s = n - \bar{l} - 1$.

Step 4. Define the first set of new coordinates $X_i = y^{\langle i-1\rangle}$, $i = 1, \dots, s+1$.

Step 5. Compute the Lie derivatives of $y^{\langle s\rangle}$ and Ξ with respect to $\partial/\partial u^{\langle k\rangle}$, $k = 0, \dots, n-1$. If all Lie derivatives with respect to $\partial/\partial u^{\langle k\rangle}$ in (iii) and (iv) identically equal to zero, take $r = 0$. If not, take $r \in \{1, \dots, n-1\}$ as the smallest integer such that (iii) and (iv) are satisfied.

Step 6. Construct the matrix M as in (28) and compute its left inverse (29). Its rows are the 1-form $\theta_{s+1}, \dots, \theta_n$.

Step 7. Find the remaining state coordinates $X_i = \Psi_i(x, u, \dots, u^{\langle r-1\rangle})$, $i = s+2, \dots, n$, as in (31). Below we will demonstrate how to find the coefficients β_{iq} and $\tilde{\beta}_{ik}$. Take the exterior derivative from both sides of (31) to get

$$\mathrm{d}\theta_i + \sum_{q=1}^{s} \mathrm{d}\beta_{iq}(\bar{x}) \wedge \mathrm{d}\Psi_q + \sum_{k=0}^{r-1} \mathrm{d}\tilde{\beta}_{ik}(\bar{x}) \wedge \mathrm{d}u^{\langle k\rangle} \equiv 0,$$
$$i = s+2, \dots, n.$$

This results in a system of partial differential equations for finding β_{iq} and $\tilde{\beta}_{ik}$ as the functions of $x, u, \dots, u^{\langle r-1\rangle}$. This system is, in general, underdetermined, so that one has certain freedom in the choice of β_{iq} and $\tilde{\beta}_{ik}$. The goal is to keep the 1-form (31) as simple as possible. The new coordinates $X_i = \Psi_i(x, u, \dots, u^{\langle r+1\rangle})$ will, in general, also depend on arbitrary additive functions of variables $X_1 = y, \dots, X_s = y^{\langle s-1\rangle}, u, \dots, u^{\langle r-1\rangle}$.

In the limiting case $s = r = 0$, one recovers from Theorem 4.1 the result for the classical observer form when the injection terms do not depend on the shifts of input and output, as in Mullari and Kotta (2023).

Corollary 4.2: *Under Assumption* 3.1, *the system* (14) *can be transformed via a state transformation $X = \Psi(x)$ into the classical observer form*

$$X_i^{\langle 1\rangle} = X_{i+1} + \varphi_i(y, u), \quad i = 1, \dots, n-1,$$
$$X_n^{\langle 1\rangle} = \varphi_n(y, u), \quad X_1 = y,$$

if and only if the following conditions are satisfied:

(a)

$$\left[\frac{\partial}{\partial z^{\langle -q\rangle}}, \Xi^{\langle -l\rangle\pi}\right] \equiv 0, \quad q, l = 1, \dots, n-1,$$

(b)

$$\left[\Xi^{\langle -l\rangle\pi}, \Xi^{\langle -j\rangle\pi}\right] \equiv 0, \quad l,j = 0,\ldots,n-1,$$

(c)

$$\left[\frac{\partial}{\partial u^{\langle k\rangle}}, \Xi\right] \equiv 0, \quad k = 0,\ldots,n-2.$$

Moreover, the new coordinates $X = \Psi(x)$ can be computed from the system of differential equations:

$$\left\langle \mathrm{d}X_i, \Xi^{\langle -n+l\rangle\pi}\right\rangle \equiv \delta_{i,l}, \quad i,l = 1,\ldots,n.$$

Proof: The conditions (a), (b), (c) follow directly from conditions (i) –(iv) of Theorem 4.1 in case $s = 0$ and $r = 0$. ■

5. Extended observer form with past values of inputs and outputs

Note that the future values of input and output may not be available. Therefore in this section we show that the form (17) can always be transformed into the alternative extended observer form where the injection terms do not depend on the future values of input and output, but instead on their past values. We will show how to define the new state coordinates necessary for such transformation.

Consider a more specific representation of the form (17), defined as

$$\begin{aligned} X_i^{\langle 1\rangle} &= X_{i+1}, \qquad i = 1,\ldots,s,\\ X_i^{\langle 1\rangle} &= X_{i+1} + \varphi_i(y,\ldots,y^{\langle s\rangle},u,\ldots,u^{\langle r_i\rangle}),\\ &\quad i = s+1,\ldots,n-1, \\ X_n^{\langle 1\rangle} &= \varphi_n(y,\ldots,y^{\langle s\rangle},u,\ldots,u^{\langle r_n\rangle}),\\ y &= X_1. \end{aligned} \tag{41}$$

If one takes $r = \max_i\{r_i,\ i = s+1,\ldots,n\}$, then Equation (41) is in the form (17). The form (41) allows us to specify the maximum values of r_i. Since the i/o equation of (14), and therefore also of (41), is in the form

$$y^{\langle n\rangle} = \sum_{i=s+1}^{n} \varphi_i^{\langle n-i\rangle},$$

and the i/o equation may depend on the forward shifts of u up to the order $n-1$, then one must have $r_i + n - i \leq n-1$, which leads to $r_i \leq i-1$, $i = s+1,\ldots,n$. In this section, we show that Equation (41) can always be transformed into the form

$$\begin{aligned} Z_i^{\langle 1\rangle} &= Z_{i+1} + \tilde{\varphi}_i(y,\ldots,y^{\langle -N\rangle},u,\ldots,u^{\langle -N\rangle}),\\ &\quad i = 1,\ldots,n-N-1,\\ Z_{n-N}^{\langle 1\rangle} &= Z_{n-N+1} + \tilde{\varphi}_{n-N}(y,\ldots,y^{\langle -N\rangle},u,\ldots,u^{\langle -N\rangle})\\ &\quad - \psi(y,u)^{\langle -N\rangle}, \\ Z_i^{\langle 1\rangle} &= Z_{i+1}, \quad i = n-N+1,\ldots,n-1,\\ Z_n^{\langle 1\rangle} &= \psi(y,u),\\ y &= Z_1, \end{aligned} \tag{42}$$

where $N = \max\{s,r\}$, $\psi(y,u) = y$ if $z = u$ and $\psi(y,u) = u$ if $z = y$, by a parametrised state transformation of the form $Z = \tilde{\Psi}(X; z^{\langle -1\rangle},\ldots,z^{\langle -N\rangle},u,\ldots,u^{\langle r-1\rangle})$.

The latter form is obviously preferable from the practical point of view since the future values of inputs and outputs are not, in general, available. The reason why in this paper we have transformed the state equation (14) into the form (17), is twofold. First, this route is, in general, more simple to use and is directly suggested by observability assumption (Assumption 3.1). Second, the results of this paper generalise those from Mullari and Kotta (2021b) on possible transformation of Equation (14) into the classical observer form where in (17) $s = r = 0$. Note that the approach in this paper follows the one of Mullari and Kotta (2021b). Transformation of the state equations into the form (42) would require to use the analogues of the 1-form (16) that rely on backward shifts of the output and construct the parametrised state transformations using the analogue of Ξ and its forward shifts.

The idea of transforming Equation (41) into the form (42) is based on the fact that one can move the injection terms φ_i, $i = s+1,\ldots,n$, to different locations in the state equations. By this we mean that there exists a parametrised state transformation such that when φ_i affects the forward shift of ith original state variable, then $\varphi_i^{\langle -1\rangle}$ affects the forward shift of the $(i-1)$th new state variable. For example, consider a parametrised state transformation

$$\begin{aligned} \tilde{X}_i &= X_i, \quad i = 1,\ldots,s,\\ \tilde{X}_j &= X_j - \varphi_j^{\langle -1\rangle}, \quad j = s+1,\ldots,n-1,\\ \tilde{X}_n &= X_n - \varphi_n^{\langle -1\rangle} + \psi(y,u)^{\langle -1\rangle}. \end{aligned}$$

Straightforward computations show that the transformed equations are in the form

$$\begin{aligned} \tilde{X}_i^{\langle 1\rangle} &= \tilde{X}_{i+1}, \quad i = 1,\ldots,s-1,\\ \tilde{X}_i^{\langle 1\rangle} &= \tilde{X}_{i+1} + \varphi_{i+1}(y^{\langle -1\rangle},\ldots,y^{\langle s-1\rangle},u^{\langle -1\rangle},\ldots,u^{\langle r_i-1\rangle}),\\ &\quad i = s,\ldots,n-2,\\ \tilde{X}_{n-1}^{\langle 1\rangle} &= \tilde{X}_n + \varphi_n(y^{\langle -1\rangle},\ldots,y^{\langle s-1\rangle},u^{\langle -1\rangle},\ldots,u^{\langle r_{n-1}-1\rangle})\\ &\quad - \psi(y,u)^{\langle -1\rangle},\\ \tilde{X}_n^{\langle 1\rangle} &= \psi(y,u),\\ y &= \tilde{X}_1. \end{aligned} \tag{43}$$

Note that in (43) all the injection terms are shifted back once. Thus Equation (43) depends on y, u, their one step backward shifts and forward shifts up to the order $N-1$. One can repeat similar transformation until there are no more forward shifts of u and y affecting the state equations. Note that the latter can always be done since $r_i \leq i-1$. Finally, one ends up with equations of the form (42). Note also that the term $\psi(y,u)$ is added to the transformations and Equation (42) just to make the parametrised state transformation invertible. Otherwise the variable $\tilde{X}_n = X_n - \varphi_n^{\langle -1\rangle} = 0$ in the above transformation.

Altogether we get the following result.

Lemma 5.1: *The system in the form* (41) *is always transformable into the form* (42) *by means of the parametrised state* *transformation*

$$Z_i = X_i - \Gamma_i + \Upsilon_i, \quad i = 1, \dots, n, \tag{44}$$

where

$$\Gamma_i := \begin{cases} 0, & \text{for } i = 1, \\ \sum_{k=\max(i,s+1)}^{\min(i+N-1,n)} \left(\varphi_k \left(y, \dots, y^{\langle s \rangle}, u, \dots, u^{\langle r_k \rangle}\right)\right)^{\langle i-k-1 \rangle}, & \text{for } i = 2, \dots, n, \end{cases} \tag{45}$$

and

$$\Upsilon_i := \begin{cases} 0, & \text{for } i = 1, \dots, n-N, \\ \psi(y,u)^{\langle i-n-1 \rangle}, & \text{for } i = n-N+1, \dots, n. \end{cases} \tag{46}$$

Proof: Note that the output equations of (42) and (41) are identical according to (44) for $i = 1$. Next, shift forward (44), substitute the right-hand side of (41) for $X_i^{\langle 1 \rangle}$, and then replace X_{i+1} in accordance with (44). This results in

$$\begin{aligned} Z_i^{\langle 1 \rangle} &= Z_{i+1} + \Omega_i + \Gamma_{i+1} - \Gamma_i^{\langle 1 \rangle} + \Upsilon_i^{\langle 1 \rangle} - \Upsilon_{i+1}, \\ & \quad i = 1, \dots, n-1, \\ Z_n^{\langle 1 \rangle} &= \Omega_n - \Gamma_n^{\langle 1 \rangle} + \Upsilon_n^{\langle 1 \rangle}, \end{aligned} \tag{47}$$

where

$$\Omega_i := \begin{cases} 0, & \text{for } i = 1, \dots, s, \\ \varphi_i \left(y, \dots, y^{\langle s \rangle}, u, \dots, u^{\langle r_i \rangle}\right), & \text{for } i = s+1, \dots, n. \end{cases}$$

Taking into account that

$$\begin{aligned} &\Gamma_2 - \Gamma_1^{\langle 1 \rangle} \\ &= \sum_{k=\max(2,s+1)}^{\min(N+1,n)} \varphi_k \left(y^{\langle -k+1 \rangle}, \dots, y^{\langle s-k+1 \rangle}, \right. \\ &\qquad \left. u^{\langle -k+1 \rangle}, \dots, u^{\langle r_k-k+1 \rangle}\right), \end{aligned}$$

$$\Gamma_{i+1} - \Gamma_i^{\langle 1 \rangle} = \begin{cases} \varphi_{i+N}\left(y^{\langle -N \rangle}, \dots, y^{\langle s-N \rangle}, u^{\langle -N \rangle}, \dots, u^{\langle r_{i+N}-N \rangle}\right), & \text{if } i \le s,\ i \le n-N, \\ 0 & \text{if } i \le s,\ i > n-N, \\ \varphi_{i+N}\left(y^{\langle -N \rangle}, \dots, y^{\langle s-N \rangle}, u^{\langle -N \rangle}, \dots, u^{\langle r_{i+N}-N \rangle}\right) - \\ \quad -\varphi_i\left(y, \dots, y^{\langle s \rangle}, u, \dots, u^{\langle r_i \rangle}\right), & \text{if } i > s,\ i \le n-N, \\ -\varphi_i\left(y, \dots, y^{\langle s \rangle}, u, \dots, u^{\langle r_i \rangle}\right), & \text{if } i > s,\ i > n-N, \end{cases}$$

for $i = 2, \dots, n-1$,

$$\Upsilon_i^{\langle 1 \rangle} - \Upsilon_{i+1} = \begin{cases} -\psi(y,u)^{\langle -N \rangle}, & \text{if } i = n-N, \\ 0, & \text{otherwise}, \end{cases}$$

$\Gamma_n^{\langle 1 \rangle} = \varphi_n(y, \dots, y^{\langle s \rangle}, u, \dots, u^{\langle r_n \rangle})$, and $\Upsilon_n^{\langle 1 \rangle} = \psi(y,u)$, one may verify that (47) leads to (42), where

$$\begin{aligned} \tilde{\varphi}_1 &:= \sum_{k=\max(2,s+1)}^{\min(N+1,n)} \varphi_k \left(y^{\langle -k+1 \rangle}, \dots, y^{\langle s-k+1 \rangle}, \right. \\ &\qquad \left. u^{\langle -k+1 \rangle}, \dots, u^{\langle r_k-k+1 \rangle}\right), \\ \tilde{\varphi}_i &:= \varphi_{i+N}\left(y^{\langle -N \rangle}, \dots, y^{\langle s-N \rangle}, u^{\langle -N \rangle}, \dots, u^{\langle r_{i+N}-N \rangle}\right) \end{aligned}$$

for $i = 2, \dots, n-N$. ■

6. Example

Example 6.1: Consider the reversible system:

$$x_1^{\langle 1 \rangle} = x_3, \quad x_2^{\langle 1 \rangle} = x_2 u + x_3 + x_4, \quad x_3^{\langle 1 \rangle} = x_2 - x_1,$$
$$x_4^{\langle 1 \rangle} = x_1 x_3, \quad y = x_3.$$

Taking $z = u$, one obtains

$$x_1^{\langle -1 \rangle} = \frac{x_4}{x_1}, \quad x_2^{\langle -1 \rangle} = \frac{x_4 + x_1 x_3}{x_1}, \quad x_3^{\langle -1 \rangle} = x_1,$$
$$x_4^{\langle -1 \rangle} = \frac{x_1(x_2 - x_1) - (x_4 + x_1 x_3) z^{\langle -1 \rangle}}{x_1}.$$

To check the validity of Assumption 3.1, compute

$$y = x_3, \quad y^{\langle 1 \rangle} = x_2 - x_1, \quad y^{\langle 2 \rangle} = x_2 u + x_4,$$
$$x^{\langle 3 \rangle} = (x_2 u + x_4 + x_3) u^{\langle 1 \rangle} + x_1 x_3,$$

and

$$\omega_0 = \mathrm{d}x_3, \quad \omega_1 = -\mathrm{d}x_1 + \mathrm{d}x_2, \quad \omega_2 = u\mathrm{d}x_2 + \mathrm{d}x_4,$$
$$\omega_3 = x_3 \mathrm{d}x_1 + u u^{\langle 1 \rangle} \mathrm{d}x_2 + (u^{\langle 1 \rangle} + x_1)\mathrm{d}x_3 + u^{\langle 1 \rangle} \mathrm{d}x_4.$$

The obtained 1-forms ω_k are linearly independent. Then (18) takes the form

$$\langle \mathrm{d}x_3, \Xi \rangle \equiv 0, \quad \langle -\mathrm{d}x_1 + \mathrm{d}x_2, \Xi \rangle \equiv 0, \quad \langle u\mathrm{d}x_2 + \mathrm{d}x_4, \Xi \rangle \equiv 0,$$
$$\langle x_3 \mathrm{d}x_1 + u u^{\langle 1 \rangle} \mathrm{d}x_2 + (u^{\langle 1 \rangle} + x_1)\mathrm{d}x_3 + u^{\langle 1 \rangle} \mathrm{d}x_4, \Xi \rangle \equiv 1,$$

yielding

$$\Xi = \frac{1}{x_3}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - u\frac{\partial}{\partial x_4}\right). \tag{48}$$

Since now $n-1 = 3$, to check the conditions of Theorem 4.1, we have to compute the projections of the backward shift of Ξ up to the order 3. Compute first

$$\mathrm{d}x_1^{\langle 1 \rangle} = \mathrm{d}x_3, \quad \mathrm{d}x_2^{\langle 1 \rangle} = u\mathrm{d}x_2 + \mathrm{d}x_3 + \mathrm{d}x_4 + x_2 \mathrm{d}u,$$
$$\mathrm{d}x_3^{\langle 1 \rangle} = -\mathrm{d}x_1 + \mathrm{d}x_2,$$
$$\mathrm{d}x_4^{\langle 1 \rangle} = x_3 \mathrm{d}x_1 + x_1 \mathrm{d}x_3,$$

to get

$$\Xi^{\langle -1 \rangle \pi} = \frac{\partial}{\partial x_4}, \quad \Xi^{\langle -2 \rangle \pi} = \frac{\partial}{\partial x_2},$$

$$\Xi^{\langle -3\rangle\pi} = z^{\langle -1\rangle}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}. \tag{49}$$

One can easily see from (48) and (49) that the vector fields $\Xi, \Xi^{\langle -1\rangle\pi}$ and $\Xi^{\langle -2\rangle\pi}$ commute and their coefficients do not depend on the backward shifts of z, i.e. the conditions (i) and (ii) of Theorem 4.1 are satisfied for $l, j = 0, \ldots, 2$, suggesting $s = 1$. Then, according to (25) and (27), one can take $X_1 = y = x_3, X_2 = y^{\langle 1\rangle} = x_2 - x_1$.

Next determine the value of r from (iii) and (iv). Checking (iii) shows that y and $y^{\langle 1\rangle}$ do not depend on u and its forward shifts, but since the coefficients of Ξ depend on u, the condition (iv) points to $r = 1$. Consequently, the remaining new state coordinates X_3 and X_4 must be the functions of x and u. Find them from (26), which now takes the form

$$\left\langle \mathrm{d}\Psi_i, \Xi^{\langle -l\rangle\pi}\right\rangle \equiv \delta_{i,n-l}, \quad i = 3,4, \quad l = 0,1,2.$$

For this purpose construct, according to (28), the matrix

$$M = \left[\Xi^{\langle -2\rangle\pi} \;\; \Xi^{\langle -1\rangle\pi} \;\; \Xi\right] = \begin{pmatrix} 0 & 0 & \frac{1}{x_3} \\ 1 & 0 & \frac{1}{x_3} \\ 0 & 0 & 0 \\ 0 & 1 & -\frac{u}{x_3}\end{pmatrix}$$

and compute its left inverse

$$\Theta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ u & 0 & 0 & 1 \\ x_3 & 0 & 0 & 0\end{pmatrix}.$$

From above we get

$$\theta_3 = u\mathrm{d}x_1 + \mathrm{d}x_4, \quad \theta_4 = x_3\mathrm{d}x_1.$$

Next find the total differentials $\mathrm{d}\Psi_3$ and $\mathrm{d}\Psi_4$ as the linear combinations of $\theta_3, \theta_4, \mathrm{d}\Psi_1$ and $\mathrm{d}u$ as in (31):

$$\begin{aligned}\mathrm{d}\Psi_3 &= \theta_3 + \beta_{31}\mathrm{d}\Psi_1 + \tilde{\beta}_{30}\mathrm{d}u \\ &= u\mathrm{d}x_1 + \mathrm{d}x_4 + \beta_{31}\mathrm{d}x_3 + \tilde{\beta}_{30}\mathrm{d}u, \\ \mathrm{d}\Psi_4 &= \theta_4 + \beta_{41}\mathrm{d}\Psi_1 + \tilde{\beta}_{30}\mathrm{d}u = x_3\mathrm{d}x_1 + \beta_{41}\mathrm{d}x_3 + \tilde{\beta}_{40}\mathrm{d}u.\end{aligned} \tag{50}$$

Computing the exterior derivatives of the 1-form (50) yields

$$\begin{aligned}&\frac{\partial\beta_{31}}{\partial x_1}\mathrm{d}x_1\wedge\mathrm{d}x_3 + \left(\frac{\partial\tilde{\beta}_{30}}{\partial x_1} - 1\right)\mathrm{d}x_1\wedge\mathrm{d}u \\ &\quad + \frac{\partial\beta_{31}}{\partial x_2}\mathrm{d}x_2\wedge\mathrm{d}x_3 \\ &\quad + \frac{\partial\tilde{\beta}_{30}}{\partial x_2}\mathrm{d}x_2\wedge\mathrm{d}u + \frac{\partial\beta_{31}}{\partial x_4}\mathrm{d}x_3\wedge\mathrm{d}x_4 \\ &\quad + \left(\frac{\partial\tilde{\beta}_{30}}{\partial x_3} - \frac{\partial\beta_{31}}{\partial u}\right)\mathrm{d}x_3\wedge\mathrm{d}u \\ &\quad + \frac{\partial\tilde{\beta}_{30}}{\partial x_4}\mathrm{d}x_4\wedge\mathrm{d}u \equiv 0, \\ &\left(\frac{\partial\beta_{41}}{\partial x_1} - 1\right)\mathrm{d}x_1\wedge\mathrm{d}x_3 + \frac{\partial\tilde{\beta}_{40}}{\partial x_1}\mathrm{d}x_1\wedge\mathrm{d}u \\ &\quad + \frac{\partial\beta_{41}}{\partial x_2}\mathrm{d}x_2\wedge\mathrm{d}x_3 + \frac{\partial\tilde{\beta}_{40}}{\partial x_2}\mathrm{d}x_2\wedge\mathrm{d}u \\ &\quad - \frac{\partial\beta_{41}}{\partial x_4}\mathrm{d}x_3\wedge\mathrm{d}x_4 + \left(\frac{\partial\tilde{\beta}_{40}}{\partial x_3} - \frac{\partial\beta_{41}}{\partial u}\right)\mathrm{d}x_3\wedge\mathrm{d}u \\ &\quad + \frac{\partial\tilde{\beta}_{40}}{\partial x_4}\mathrm{d}x_4\wedge\mathrm{d}u \equiv 0.\end{aligned} \tag{51}$$

The exterior derivatives (51) can identically equal to zero only if all their coefficients vanish. This results in a system of partial differential equations to solve for $\beta_{\hat{i}q}$ and $\tilde{\beta}_{\hat{i}0}$. Since the goal is to get the simplest 1-form (50), we require that majority of the coefficients are equal to zero, so that system of equations is still satisfied. The simplest non-zero choice is $\tilde{\beta}_{30} = \beta_{41} = x_1$, resulting in

$$\mathrm{d}\Psi_3 = \theta_3 + x_1\mathrm{d}u, \quad \mathrm{d}\Psi_4 = \theta_4 + x_1\mathrm{d}x_3.$$

That is, the parametrised coordinate transformation is

$$X_1 = x_3, \quad X_2 = x_2 - x_1, \quad X_3 = x_1u + x_4, \quad X_4 = x_1x_3,$$

with the inverse

$$\begin{aligned}x_1 &= \frac{X_4}{X_1}, \quad x_2 = \frac{X_4 + X_1X_2}{X_1}, \quad x_3 = X_1, \\ x_4 &= \frac{X_1X_3 - X_4u}{X_1}.\end{aligned}$$

The state equations in the new coordinates are

$$\begin{aligned}&X_1^{\langle 1\rangle} = X_2, \quad X_2^{\langle 1\rangle} = X_3 + y^{\langle 1\rangle}u, \quad X_3^{\langle 1\rangle} = X_4 + yu, \\ &X_4^{\langle 1\rangle} = yy^{\langle 1\rangle}, \quad y = X_1.\end{aligned} \tag{52}$$

Below we will show how to go from the extended observer form (17) to the alternative form (42) where the input–output injection terms do not depend on future values of inputs and outputs. Using the parametrised state transformation (44), transform Equations (52) into the form (42), taking $N = 1$. From (52) we have

$$\begin{aligned}&\varphi_2(y, y^{\langle 1\rangle}, u, u^{\langle 1\rangle}) = y^{\langle 1\rangle}u, \quad \varphi_3(y, y^{\langle 1\rangle}, u, u^{\langle 1\rangle}) = yu, \\ &\varphi_4(y, y^{\langle 1\rangle}, u, u^{\langle 1\rangle}) = yy^{\langle 1\rangle}.\end{aligned}$$

Compute now the coordinates $Z_i, i = 1, \ldots, 4$ with the help of (44).

For $i = 1$, from (45) and (46) follows respectively that $\Gamma_1 = 0$ and $\Upsilon_1 = 0$; therefore $Z_1 = X_1$.

For $i = 2$, in (45) $\min(i + N - 1, n) = 2, \max(i, s + 1) = 2$, resulting in $\Gamma_2 = (\varphi_2(y, y^{\langle 1\rangle}, u, u^{\langle 1\rangle}))^{\langle -1\rangle} = (y^{\langle 1\rangle}u)^{\langle -1\rangle} = yu^{\langle -1\rangle}$. Form (46), one additionally gets $\Upsilon_2 = 0$. Therefore $Z_2 = X_2 - yu^{\langle -1\rangle}$.

For $i = 3$, in (45) $\min(i + N - 1, n) = 3, \max(i, s + 1) = 3$, and so $\Gamma_3 = (\varphi_2(y, y^{\langle 1\rangle}, u, u^{\langle 1\rangle}))^{\langle -1\rangle} = (yu)^{\langle -1\rangle} = y^{\langle -1\rangle}u^{\langle -1\rangle}$. Form (46), one additionally gets $\Upsilon_3 = 0$. Therefore $Z_3 = X_3 - y^{\langle -1\rangle}u^{\langle -1\rangle}$.

For $i = 4$, in (45) $\min(i + N - 1, n) = 4$, $\max(i, s + 1) = 4$, and so $\Gamma_4 = (\varphi_4(y, y^{\langle 1\rangle}, u, u^{\langle 1\rangle}))^{\langle -1\rangle} = (yy^{\langle 1\rangle})^{\langle -1\rangle} = y^{\langle -1\rangle}y$.

Form (46), one additionally gets $\Upsilon_4 = y^{\langle -1\rangle}$, and so $Z_4 = X_4 - y^{\langle -1\rangle}(y-1)$.

The inverse of the parametrised state transformation

$$Z_1 = X_1, \quad Z_2 = X_2 - u^{\langle -1\rangle}y, \quad Z_3 = X_3 - u^{\langle -1\rangle}y^{\langle -1\rangle},$$
$$Z_4 = X_4 - y^{\langle -1\rangle}(y-1)$$

is

$$X_1 = Z_1, \quad X_2 = Z_2 + u^{\langle -1\rangle}y, \quad X_3 = Z_3 + u^{\langle -1\rangle}y^{\langle -1\rangle},$$
$$X_4 = Z_4 + y^{\langle -1\rangle}(y-1).$$

The state equations in the transformed coordinates are

$$Z_1^{\langle 1\rangle} = Z_2 + u^{\langle -1\rangle}y, \quad Z_2^{\langle 1\rangle} = Z_3 + y^{\langle -1\rangle}u^{\langle -1\rangle},$$
$$Z_3^{\langle 1\rangle} = Z_4 + y^{\langle -1\rangle}(y-1), \quad Z_4^{\langle 1\rangle} = y.$$

Example 6.2: Examine a hydraulic press with the vertical cylinder, lifting the load upwards via pumping oil into the chamber under the piston. The state equations of this system can be described by

$$\begin{aligned} x_1^{\langle 1\rangle} &= x_1 + x_2T, \\ x_2^{\langle 1\rangle} &= x_2 + \frac{\left[S(x_3-x_4) - mg - \mu x_2\right]T}{M}, \\ x_3^{\langle 1\rangle} &= x_3 + \frac{\beta(u-x_2)T}{(l_0+x_1)}, \\ x_4^{\langle 1\rangle} &= C - x_3, \quad y = x_1, \end{aligned} \tag{53}$$

see Equation (14), adapted to upward movement, in Mullari and Schlacher (2014). In (53), x_1 and x_2 are the position of the piston and its velocity, respectively, x_3 and x_4 are the pressures under and above the piston, respectively, and T is the sampling time. The constants have the following meaning: m is the mass loaded on the piston, S – the effective piston area, μ – the damping coefficient, l_0 – the height of the chamber under the piston, and β – the isothermal bulk modulus of the oil and C – the sum of the pressures under and above the piston. We consider here the so-called *saving circuit*, where C is kept constant. The input u in (53) is the volume of oil, pumped into the chamber under the piston during time T, divided by S. It can be computed by formula $u = US/\bar{S}$, where $\bar{S}$ is the effective area of the piston of the pump between the supply tank and the chamber under the piston of the hydraulic press. U is the distance, by which the piston of the pump moves downward during time T, i.e. U is the variable by which the required value of u is achieved.

Taking $z = u$, compute

$$\begin{aligned} x_1^{\langle -1\rangle} &= -\frac{(Tu^{\langle -1\rangle} - x_1)\beta + (C-x_4-x_3)l_0}{\beta + C - x_4 - x_3}, \\ x_2^{\langle -2\rangle} &= \frac{u^{\langle -1\rangle}\beta}{\beta + C - x_3 - x_4} + \frac{(C-x_3-x_4)(l_0+x_1)}{T(\beta+C-x_3-x_4)}, \\ x_3^{\langle -1\rangle} &= C - x_4, \\ x_4^{\langle -1\rangle} &= \frac{(C-x_3-x_4)(l_0+x_1)m}{T^2S(\beta+C-x_3-x_4)} + \frac{mx_2}{ST} \\ &\quad + \frac{u^{\langle -1\rangle}m\beta - \mu(C-x_3-x_4)(l_0+x_1)}{ST(\beta+C-x_3-x_4)} \\ &\quad - \frac{(mg + S(x_4-C))(\beta+C-x_3-x_3) + \mu u^{\langle -1\rangle}\beta}{S(\beta+C-x_3-x_4)}. \end{aligned}$$

Compute, using the symbolic software, the forward shifts of y up to the order 3, to get

$$y^{\langle 1\rangle} = x_1 + x_2T,$$
$$y^{\langle 2\rangle} = x_1 + x_2T + \frac{\left[S(x_3-x_4) - Mg - \mu x_2\right]T^2}{M},$$

whereas the expression of $y^{\langle 3\rangle}$ is too lengthy to present here. Note that Assumption 3.1 is satisfied. Next we find the 1-form $\omega_l, l = 0, \ldots, 3$, of (16), getting

$$\omega_0 = \mathrm{d}x_1, \quad \omega_1 = \mathrm{d}x_1 + T\mathrm{d}x_2,$$
$$\omega_2 = \mathrm{d}x_1 - \frac{(\mu T - 2m)T}{M} + \frac{ST^2}{m}(\mathrm{d}x_3 - \mathrm{d}x_4),$$

and omitting again ω_3. Using symbolic software, compute from Equations (18) the vector field

$$\Xi = \frac{m}{2ST^2}\left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right). \tag{54}$$

Since $n-1 = 3$, to check the conditions of Theorem 4.1, we have to compute the projections of the backward shifts of Ξ up to the order 3. Find first

$$\begin{aligned} \mathrm{d}x_1^{\langle 1\rangle} &= \mathrm{d}x_1 + T\mathrm{d}x_2, \\ \mathrm{d}x_2^{\langle 1\rangle} &= \mathrm{d}x_1 - \frac{(\mu T - 2m)T}{M}\mathrm{d}x_2 + \frac{ST^2}{m}(\mathrm{d}x_3 - \mathrm{d}x_4), \\ \mathrm{d}x_3^{\langle 1\rangle} &= \frac{(x_2-u)\beta T}{(l_0+x_1)^2}\mathrm{d}x_1 - \frac{\beta T}{l_0+x_1}\mathrm{d}x_2 + \mathrm{d}x_3, \quad \mathrm{d}x_4 = -\mathrm{d}x_3 \end{aligned}$$

to get

$$\begin{aligned} \Xi^{\langle -1\rangle\pi} &= \frac{m}{2ST^2}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right), \\ \Xi^{\langle -2\rangle\pi} &= \frac{1}{T}\frac{\partial}{\partial x_2} + \frac{m}{2ST^2}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right), \\ \Xi^{\langle -3\rangle\pi} &= \frac{\partial}{\partial x_1} + \frac{2m - \mu T}{mT}\frac{\partial}{\partial x_2} \\ &\quad + \left(\frac{\beta + C - x_3 - x_4}{Tu^{\langle -1\rangle} - l_0 - x_1} + \frac{m}{2ST^2}\right)\frac{\partial}{\partial x_3} \\ &\quad - \frac{m}{2ST^2}\frac{\partial}{\partial x_4}. \end{aligned} \tag{55}$$

From (54) and (55), it is obvious that the vector fields Ξ, $\Xi^{\langle -1\rangle\pi}$ and $\Xi^{\langle -2\rangle\pi}$ commute and their coefficients do not depend on the backward shifts of z, which is not the case for $\Xi^{\langle -3\rangle\pi}$. That is, the conditions (i) and (ii) of Theorem 4.1 are satisfied for l, $j = 0, 1, 2$, suggesting $s = 1$. Then, according to (25) and (27), one can take $X_1 = y = x_1$, $X_2 = y^{\langle 1\rangle} = x_1 + x_2T$.

Checking conditions (iii) and (iv) shows that since neither y, $y^{\langle 1\rangle}$ nor the coefficients of Ξ do not depend on u and its forward shifts, we get $r = 0$. Consequently, the coordinates X_3 and X_4 must be the functions of x only. Equation (26) to find them takes now the form

$$\left\langle \mathrm{d}\Psi_i, \Xi^{\langle -l\rangle\pi} \right\rangle \equiv \delta_{i,n-l}, \quad i = 3,4, \quad l = 0,1,2.$$

To solve the above equations, construct as in (28), the matrix

$$M = \left[\Xi^{\langle -2\rangle\pi}\ \Xi^{\langle -1\rangle\pi}\ \Xi\right] = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{T} & 0 & 0 \\ \frac{m}{2ST^2} & \frac{m}{2ST^2} & \frac{m}{2ST^2} \\ -\frac{m}{2ST^2} & -\frac{m}{2ST^2} & \frac{m}{2ST^2} \end{pmatrix}$$

and compute its left inverse

$$\Theta = \begin{pmatrix} 0 & T & 0 & 0 \\ 0 & -T & \frac{ST^2}{m} & -\frac{ST^2}{m} \\ 0 & 0 & \frac{ST^2}{m} & \frac{ST^2}{m} \end{pmatrix}.$$

The rows of the matrix M_L^{-1} are the total differentials and so

$$\mathrm{d}\Psi_3 = -T\mathrm{d}x_2 + \frac{ST^2}{m}(\mathrm{d}x_3 - \mathrm{d}x_4),$$

$$\mathrm{d}\Psi_4 = \frac{ST^2}{m}(\mathrm{d}x_3 + \mathrm{d}x_4).$$

To conclude, the parametrised coordinate transformation is

$$X_1 = x_1, \quad X_2 = x_1 + x_2T,$$

$$X_3 = -x_2T + (x_3 - x_4)\frac{ST^2}{m}, \quad X_4 = \frac{ST^2}{m}(x_3 + x_4)$$

with the inverse

$$x_1 = X_2, \quad x_2 = \frac{X_2 - X_1}{T},$$

$$x_3 = \frac{(X_4 + X_3 + X_2 - X_1)m}{2ST^2},$$

$$x_4 = \frac{(X_4 - X_3 - X_2 + X_1)m}{2ST^2}.$$

The state equations in the new coordinates are, taking into account that $X_1 = y$ and $X_2 = y^{\langle 1\rangle}$,

$$X_1^{\langle 1\rangle} = X_2, \quad X_2^{\langle 1\rangle} = X_3 + 3y^{\langle 1\rangle} - 2y - gT^2 + \frac{(y - y^{\langle 1\rangle})\mu T}{m},$$

$$X_3^{\langle 1\rangle} = X_4 + \frac{ST^3u\beta - T^2\left[-(y + l_0)(gm - CS) + (y^{\langle 1\rangle} - y)S\beta\right]}{m(l_0 + y)} + \frac{(y - y^{\langle 1\rangle})(m - \mu T)}{m},$$

$$X_4^{\langle 1\rangle} = \frac{CST^2}{m} + \frac{ST^3u\beta + (y - y^{\langle 1\rangle})ST^2\beta}{(y + l_0)m}.$$

7. Conclusion

The problem of transforming the discrete-time state equations into the extended observer form has been studied. The necessary and sufficient conditions for the existence of the state transformation have been formulated in terms of the backward shifts of the vector fields, defined by the system dynamics, for the case when the state equations are not necessarily reversible (with respect to the state variable). The method to find the required state transformation has been given. The earlier results on the classical observer form follow directly from the main theorem. Finally, a method has been suggested to transform the obtained extended observer form into an alternative form, where the injection terms do not depend on the (non-available) future values of inputs and outputs but instead on their past values.

Although all the necessary computations to find the required transformation are given in the paper, for some systems these computations may become difficult to do. In particular, the computation of backward shifts of system variables or partial differential equations may not have an analytic (or even closed form) solution. However, similar problem occurs in previous solutions relying on the i/o equation, too.

Notes

1. Note that under Assumption 3.1 the problem is always solvable for $s = r = n-1$.
2. Though X_{s+1} is already given by (27), we have to include $\theta_{s+1} = \mathrm{d}\Psi_{s+1}$ into the matrix Θ to satisfy the conditions $\langle \theta_i, \Xi^{\langle -n+s+1\rangle\pi}\rangle \equiv 0$, $i = s+2, \ldots, n$.

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Appendix 1. Proof of Lemma 2.3

Note that if (a) and (b) hold, then by Lemma 9 in Mullari et al. (2017), (13) holds. Next we will show that from (a) and (b) follows also the existence of $\bar{m}$ additional independent functions $\Psi_i(\bar{x})$, $i = 1, \ldots, \bar{m}$, satisfying (12).

If (a) holds, i.e. the vector fields $\Xi_l, l = 1, \ldots, \bar{m}$, are linearly independent, then we can define $\bar{m}$ linearly independent 1-forms $\omega_i = \sum_{q=1}^{\bar{n}} A_{iq}(\bar{x})\mathrm{d}\bar{x}_q$ such that

$$\langle \omega_i, \Xi_l \rangle \equiv \delta_{i,l}, \quad i, l = 1, \ldots, \bar{m}. \tag{A1}$$

This is possible, since (A1) gives $\bar{m}^2$ constraints on $\bar{m} \cdot \bar{n}$ coefficients A_{iq}. So, there is $\bar{m}(\bar{n} - \bar{m})$ degrees of freedom to define the 1-form ω_i.

Show next that the 1-form satisfying the conditions (13) and (A1) are also jointly linearly independent, i.e.

$$\dim_{\mathcal{K}} \left(\operatorname{span}_{\mathcal{K}} \left\{ \omega_i, \quad i = 1, \ldots, \bar{m}, \, \mathrm{d}\Psi_j, \, j = \bar{m}+1, \ldots, \bar{n} \right\} \right) = \bar{n}. \tag{A2}$$

Suppose by contradiction that this is not the case, i.e. there exists a 1-form $\mathrm{d}\Psi_j = \sum_{i=1}^{\bar{m}} \gamma_{ji}(\bar{x})\omega_i$, where at least one coefficient γ_{ji} is non-zero. Then, by (A1), $\langle \mathrm{d}\Psi_j, \Xi_l \rangle = \gamma_{jl}$, which contradicts (13). Consequently, (A2) holds.

It remains to be proven that the 1-form ω_i can be chosen to be the total differentials. That is, their exterior derivatives

$$\mathrm{d}\omega_i = \sum_{q=2}^{\bar{n}} \sum_{k=1}^{q-1} \left(\frac{\partial A_{ik}}{\partial \bar{x}_q} - \frac{\partial A_{iq}}{\partial \bar{x}_k} \right) \mathrm{d}\bar{x}_q \wedge \mathrm{d}\bar{x}_k$$

identically equal to zero. This is possible, if $\partial A_{ik}/\partial \bar{x}_q - \partial A_{iq}/\partial \bar{x}_k \equiv 0$ for all $q = 2, \ldots, \bar{n}, k = 1, \ldots, q-1$. Because this expression is obviously anti-symmetric with respect to changing indices q and k, then also $\partial A_{iq}/\partial \bar{x}_k - \partial A_{ik}/\partial \bar{x}_q \equiv 0$ for all $k = 2, \ldots, \bar{n}, q = 1, \ldots, k-1$. The last two formulae result in

$$\frac{\partial A_{ik}}{\partial \bar{x}_q} - \frac{\partial A_{iq}}{\partial \bar{x}_k} \equiv 0, \quad i = 1, \ldots, \bar{m}, \quad k, q = 1, \ldots, \bar{n}. \tag{A3}$$

Consequently, the fact that the 1-form ω_i are the total differentials, is equivalent to (A3). To prove (A3), write (A1) componentwise,

$$\langle \omega_i, \Xi_l \rangle = \sum_{q=1}^{\bar{n}} A_{iq}(\bar{x})\xi_{ql}(\bar{x}) \equiv \delta_{i,l}, \quad i, l = 1, \ldots, \bar{m}, \tag{A4}$$

and compute their partial derivatives with respect to $\bar{x}$. Taking into account the product derivative formula and the fact that the partial derivative of Kronecker delta identically equals to zero, we obtain $\bar{n} \cdot \bar{m}^2$ equalities

$$\sum_{q=1}^{\bar{n}} \frac{\partial A_{iq}}{\partial \bar{x}_k} \xi_{ql} = - \sum_{q=1}^{\bar{n}} A_{iq} \frac{\partial \xi_{ql}}{\partial \bar{x}_k}, \quad k = 1, \ldots, \bar{n}. \tag{A5}$$

Apply the Lie derivative to both sides of (A4), taking into account that the Lie derivative of $\delta_{i,l}$ as a constant identically equals to zero, to get, after adding two last terms whose sum equals to zero,

$$L_{\Xi_j} \langle \omega_i, \Xi_l \rangle = \sum_{k,q=1}^{\bar{n}} \left(\frac{\partial A_{iq}}{\partial \bar{x}_k} \xi_{ql}\xi_{kj} + A_{iq} \frac{\partial \xi_{ql}}{\partial \bar{x}_k} \xi_{kj} + A_{iq} \frac{\partial \xi_{qj}}{\partial \bar{x}_k} \xi_{kl} - A_{iq} \frac{\partial \xi_{qj}}{\partial \bar{x}_k} \xi_{kl} \right) \equiv 0. \tag{A6}$$

Condition (b), i.e. the equality (11), written componentwise, is

$$\sum_{q=1}^{\bar{n}} \left[\sum_{k=1}^{\bar{n}} \left(\frac{\partial \xi_{ql}}{\partial \bar{x}_k} \xi_{kj} - \frac{\partial \xi_{qj}}{\partial \bar{x}_k} \xi_{kl} \right) \right] \frac{\partial}{\partial \bar{x}_q} \equiv 0. \tag{A7}$$

The left-hand side of (A7) is the zero vector field, if all its coefficients identically equal to zero, i.e.

$$\sum_{k=1}^{\bar{n}} \left(\frac{\partial \xi_{ql}}{\partial \bar{x}_k} \xi_{kj} - \frac{\partial \xi_{qj}}{\partial \bar{x}_k} \xi_{kl} \right) \equiv 0, \quad k = 1, \ldots, \bar{n}. \tag{A8}$$

Rewrite (A6) as

$$\sum_{k,q=1}^{\bar{n}} \left(\frac{\partial A_{iq}}{\partial \bar{x}_k} \xi_{ql}\xi_{kj} + A_{iq} \frac{\partial \xi_{qj}}{\partial \bar{x}_k} \xi_{kl} \right) + \sum_{q=1}^{\bar{n}} A_{iq} \sum_{k=1}^{\bar{n}} \left(\frac{\partial \xi_{ql}}{\partial \bar{x}_k} \xi_{kj} - \frac{\partial \xi_{qj}}{\partial \bar{x}_k} \xi_{kl} \right) \equiv 0.$$

Due to (A8), the second sum on the left-hand side identically equals to zero and we obtain

$$\sum_{k,q=1}^{\bar{n}} \left(\frac{\partial A_{iq}}{\partial \bar{x}_k} \xi_{ql}\xi_{kj} + A_{iq} \frac{\partial \xi_{qj}}{\partial \bar{x}_k} \xi_{kl} \right) \equiv 0,$$

which, by (A5) results in

$$\sum_{k,q=1}^{\bar{n}} \left(\frac{\partial A_{iq}}{\partial \bar{x}_k} \xi_{ql}\xi_{kj} - \frac{\partial A_{iq}}{\partial \bar{x}_k} \xi_{qj}\xi_{kl} \right) \equiv 0.$$

After changing the summation indices k and q in the second term, we obtain

$$\sum_{k,q=1}^{\bar{n}} \left(\frac{\partial A_{iq}}{\partial \bar{x}_k} - \frac{\partial A_{ik}}{\partial \bar{x}_q} \right) \xi_{ql}\xi_{kj} \equiv 0, \quad i, j, l = 1, \ldots, \bar{m}. \tag{A9}$$

To complete the proof, it remains to be shown that if (A9) is valid, then one can choose the 1-form ω_i to be the total differentials of certain functions $\Psi_i(\bar{x})$:

$$A_{iq} = \frac{\partial \Psi_i}{\partial \bar{x}_q}, \quad i = 1, \dots, \bar{m}, \quad q = 1, \dots, \bar{n},$$

or equivalently, that (A3) is valid.

Define the $(\bar{n} \times \bar{m})$-matrix $\overline{M} = [\Xi_1 \ \dots \ \Xi_{\bar{m}}]$ with columns being the vector fields Ξ_l. By assumption (a) of the lemma, $\mathrm{rank}_{\mathcal{K}} \overline{M} = \bar{m}$. Obviously, one can reorder the coordinates $\bar{x}$ so that the upper $(\bar{m} \times \bar{m})$-block P of matrix $\overline{M}$, whose elements are $P_{ql} = \xi_{ql}, q, l = 1, \dots, \bar{m}$, has the generic rank $\bar{m}$.

Recall that A_{iq} must be chosen so that (A4) holds. Because the number of 1-form ω_i is $\bar{m}$, one has to determine $\bar{m} \cdot \bar{n}$ coefficients, whereas the system of equations (A4) contains only $\bar{m}^2$ independent equations. Therefore, $\bar{m} \cdot (\bar{n} - \bar{m})$ coefficients A_{iq} can be chosen freely as follows. Define $\bar{m}$ independent functions $\overline{\Psi}_i(\bar{x}_{\bar{m}+1}, \dots, \bar{x}_{\bar{n}}), i = 1, \dots, \bar{m}$, and, for each ω_i, choose its coefficients $A_{iq}, q = \bar{m}+1, \dots, \bar{n}$, as the partial derivatives below

$$A_{iq} = \frac{\partial \overline{\Psi}_i}{\partial \bar{x}_q}, \quad i = 1, \dots, \bar{m}, \quad q = \bar{m}+1, \dots, \bar{n}. \tag{A10}$$

The remaining $A_{iq}, q = 1, \dots, \bar{m}$, as $\bar{m}^2$ unknown functions, can be uniquely found from (A4).

Since the partial derivative operators commute, then due to (A10)

$$\frac{\partial A_{iq}}{\partial \bar{x}_k} - \frac{\partial A_{ik}}{\partial \bar{x}_q} \equiv 0, \quad i = 1, \dots, \bar{m}, \quad k, q = \bar{m}+1, \dots, \bar{n}. \tag{A11}$$

By (A10) and the choice of the functions $\overline{\Psi}_i(\bar{x}_{\bar{m}+1}, \dots, \bar{x}_{\bar{n}}), i = 1, \dots, \bar{m}$,

$$\frac{\partial A_{iq}}{\partial \bar{x}_k} \equiv 0, \quad q = \bar{m}+1, \dots, \bar{n}, \quad k = 1, \dots, \bar{m}. \tag{A12}$$

Then, due to (A11) and (A12), all expressions in the parentheses of (A9), where at least one index k or q is greater than $\bar{m}$, identically equal to zero and (A9) reduces to

$$\sum_{k,q=1}^{\bar{m}} \left(\frac{\partial A_{iq}}{\partial \bar{x}_k} - \frac{\partial A_{ik}}{\partial \bar{x}_q} \right) \xi_{ql} \xi_{kj} \equiv 0, \quad i, j, l = 1, \dots, \bar{m}. \tag{A13}$$

For each fixed i, (A13) can be interpreted as the product of three matrices:

$$P^T Q_i P \equiv 0_{\bar{m} \times \bar{m}}, \tag{A14}$$

where $0_{\bar{m} \times \bar{m}}$ is the $(\bar{m} \times \bar{m})$ zero matrix, and the elements of the $(\bar{m} \times \bar{m})$-matrix Q_i are

$$Q_{i,kq} = \frac{\partial A_{iq}}{\partial \bar{x}_k} - \frac{\partial A_{ik}}{\partial \bar{x}_q}, \quad i, q, k = 1, \dots, \bar{m}.$$

We need to show that Q_i is the zero matrix. Recall that the generic rank of P (and also of P^T) is $\bar{m}$ and, because $P^T Q_i P$ is, due to (A14), the zero matrix, then $\mathrm{rank}_{\mathcal{K}} P^T Q_i P \equiv 0$. According to the Sylvester formula, $\mathrm{rank}_{\mathcal{K}} (P^T Q_i P) = \min(\mathrm{rank}_{\mathcal{K}} P, \mathrm{rank}_{\mathcal{K}} Q_i)$, therefore $\mathrm{rank}_{\mathcal{K}} Q_i = 0$, which is possible if Q_i is the zero matrix and (A13) holds.

So, by (A2) one can define $\bar{n}$ independent functions $\Psi_i(\bar{x}), i = 1, \dots, \bar{n}$, as the new coordinates $X = \Psi(\bar{x})$ such that (12) and (13) hold. $\square$