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DOI
10.1016/j.ifacol.2023.10.1787

Publication date
2023

Published in
IFAC-PapersOnLine

Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 56-2 (2023) 1357–1362

Data-driven control of input-affine systems via approximate nonlinearity cancellation

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Abstract: We consider data-driven control of input-affine systems via approximate nonlinearity cancellation. Data-dependent semi-definite program is developed to characterize the stabilizer such that the linear dynamics of the closed-loop systems is stabilized and the influence of the nonlinear dynamics is decreased. Because of the additional nonlinearity brought by the state-dependent input vector field, nonlinearity cancellation is more difficult to achieve. We show that under some assumptions on the nonlinearity, the nonlinearity cancellation control approach can render the equilibrium locally asymptotically stable even if the additional nonlinearity is neglected. Data-based estimation of the region of the attraction is also presented.

Keywords: Data-driven control, nonlinear control, region of attraction estimation, robust control, learning control

1. INTRODUCTION

Control design for complex dynamical systems with limited knowledge on the dynamics has always been challenging in control engineering. Data-driven control is one of the efforts devoted to overcoming this challenge. Using input-output data and some preknowledge on the dynamics, data-driven control synthesizes controllers to achieve various design objectives. On the contrary to the *indirect* data-driven control method which identifies the system model first, the *direct* data-driven control method characterizes controllers without explicit system identification. Inspired by the fundamental lemma by Willems et al. (2005), which shows that the behavior of a linear time-invariant system can be represented by a single input-output trajectory, notable results on direct data-driven control have been reported. For instance, Coulson et al. (2019) and Huang et al. (2022) developed data-enabled predictive control using data-based representation, and De Persis and Tesi (2020) characterized feedback stabilizers and linear quadratic regulators via data-dependent linear matrix inequalities and semi-definite programs.

For nonlinear dynamical systems, direct data-driven controllers have been developed via various approaches and techniques, such as the virtual reference feedback tuning (VRFT) by Campi and Savaresi (2006), the online data-driven control by Tanaskovic et al. (2017), and the Koopman based data-driven predictive control by Lian et al. (2021). The work of van Waarde and Camlibel (2021) applied a matrix Finsler's lemma to data-driven control, Fraile et al. (2021) designed controllers by point-wise linear approximation of feedback linearizable systems, and Alsalti et al. (2023) developed data-based nonlinear predictive control scheme for feedback linearizable systems. Based on the results of De Persis and Tesi (2020) and considering polynomial systems, Guo et al. (2022a) designed data-driven stabilizers via sum of squares (SOS) relaxation, Luppi et al. (2022) investigated the case where the nonlinearities satisfying quadratic constraints, and Bisoffi et al. (2022) developed data-driven control designs using Petersen's lemma. For non-polynomial systems, when the collected data is close to the equilibrium to be stabilized, Guo et al. (2022b) obtained local stabilization results by approximating the nonlinear dynamics via Taylor's expansion. Using Taylor polynomials, Martin et al. (2022) designed state-feedback controllers with data corrupted by Gaussian noise. If the basis functions of the unknown dynamics are available, De Persis et al. (2023) derived data-dependent semi-definite programs whose solution gives data-driven controllers that (approximately) cancel out the nonlinearity. Data-driven safety controllers have been developed for polynomial systems by Luppi et al. (2021) and Nejati et al. (2022), respectively.

The recent result by De Persis et al. (2023) has established data-driven control via (approximate) nonlinearity cancellation. In particular, when both the state and the input enter the dynamics nonlinearly, a *dynamic* controller is designed by dynamic extension such that the equilibrium is locally asymptotically stable. This work considers a class of input-affine systems and shows that the nonlinearity cancellation approach by De Persis et al. (2023) can be applied to design data-driven *static* feedback controllers for input-affine systems with state-dependent input vector field. Following the idea of approximate nonlinearity cancellation, we also stabilize the linear part of the closed-loop dynamics and decrease the influence of the nonlinear part. However, because of the state-dependent input vector field,

10.1016/j.ifacol.2023.10.1787

the designed control input brings additional nonlinearity into the dynamics and nonlinearity cancellation becomes more difficult.

Outline of the contribution. We consider the discrete-time input affine systems whose input vector field can be written as the sum of a state-independent part and a state-dependent part. Using the prior information on the dynamics, we express the dynamics in the linear-like form and design a feedback controller based on the basis functions. The collected data is assumed to be perturbed by noise with a known bound. Data-based conditions are then derived to characterize stabilizers for the linear closed-loop dynamics, and suitable costs are chosen to decrease the effect of the nonlinear dynamics. Specifically, we use the control input's nonlinear component entering through the state-independent part of input vector field to approximately cancel out the system nonlinearity. The additional nonlinearity brought through the state-dependent part of the input vector field is not especially handled. Under mild assumptions on the nonlinear basis functions, local asymptotic stability of the equilibrium is guaranteed and the RoA is estimated based on data. The performance of the controller can be improved if the influence of the additional nonlinearity is also decreased in the semi-definite programs. It is of importance to investigate how to further decrease the impact of the nonlinearities in our future works.

The rest of the work is organized as follows. The data-driven stabilization problem is formulated in Section 2. The stabilizer design and RoA estimation are presented in Sections 3 and 4, respectively. A numerical example is illustrated in Section 5. Some conclusive remarks are drawn in Section 6.

2. PROBLEM FORMULATION

Consider the discrete-time system

$$x^+ = f(x) + g(x)u$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input. Without loss of generality, assume that the uncontrolled system has an equilibrium at the origin, *i.e.*, $f(0) = 0$. One can always write the system into the linear-like form

$$x^+ = AZ(x) + (B_0 + B_1 W(x))u \tag{1}$$

where the function $W(x) \in \mathbb{R}^{p\times m}$ contains basis functions dependent on x, and

$$Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}$$

with $Q(x) \in \mathbb{R}^{q\times 1}$ containing nonlinear basis functions. The constant matrices $A \in \mathbb{R}^{n\times(n+q)}$, $B_0 \in \mathbb{R}^{n\times m}$ and $B_1 \in \mathbb{R}^{n\times p}$ are unknown.

Assumption 1. The functions $Z(x)$ and $W(x)$ are known. There is at least one nonzero element in B_0. ■

Remark 1. We assume $Z(x)$ and $W(x)$ are known for nonlinearity cancellation. The information on the basis functions can be obtained via pre-knowledge on the dynamics. To ensure that the designed controller contains a linear feedback component, we assume that B_0 has at least one non-zero element. This implies that $g(x)$ contains 1 as a basis function, or at least one of the non-polynomial basis functions $g_{ij}(x)$, $i = 1, \dots, n$, $j = 1, \dots, m$, has value 1 at the origin, so that it can be approximate at the origin by Taylor's expansion with the constant term $g_{ij}(0) = 1$. ■

Following the idea of nonlinearity cancellation developed by De Persis et al. (2023), we aim at designing a static state feedback controller $u = KZ(x)$, such that the linear component of the closed-loop system is stable and the influence of the nonlinear component is minimized.

As the matrices A and B are not available for controller design, we use data to represent the system dynamics. Denote the dataset collected in the experiment as $\mathcal{DS} := \{x(k); u(k)\}_{k=0}^T$. Arrange the collected data as

$$\begin{aligned}
X_0 &:= [x(0)\ x(1)\ \cdots\ x(T-1)] \in \mathbb{R}^{n\times T},\\
X_1 &:= [x(1)\ x(2)\ \cdots\ x(T)] \in \mathbb{R}^{n\times T},\\
U_0 &:= [u(0)\ u(1)\ \cdots\ u(T-1)] \in \mathbb{R}^{m\times T},\\
\overline{U}_0 &:= [W(x(0))u(0)\ W(x(1))u(1)\\
&\qquad \cdots\ W(x(T-1))u(T-1))] \in \mathbb{R}^{p\times T},\\
Z_0 &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \\ Q(x(0)) & Q(x(1)) & \cdots & Q(x(T-1)) \end{bmatrix} \in \mathbb{R}^{(n+q)\times T}.
\end{aligned}$$

Assume that the collected data is affected by noise $d(k)$ during the data acquisition phase for $k = 0, \dots, T-1$, and the data satisfies that

$$X_1 = AZ_0 + B_0 U_0 + B_1 \overline{U}_0 + D_0 \tag{2}$$

where $D_0 := [d(0)\ d(1)\ \cdots\ d(T-1)] \in \mathbb{R}^{n\times T}$. For the sake of simplicity, this work does not consider the disturbances affecting the execution of the control input, which could be handled by following a similar approach as shown in Section IV of the work by De Persis et al. (2023) and will be studied in our future work.

The data-driven control problem considered in this work is formulated as follows.

Problem 1. For the discrete-time system (1) satisfying Assumption 1, design a feedback controller $u = KZ(x)$ via the dataset $\mathcal{DS}$, such that the origin is an asymptotically stable equilibrium for the closed-loop system.

3. CONTROL DESIGN

In this section, we first derive the data-based closed-loop representation that is composed of a linear part and a nonlinear part. Then, an optimization problem is developed to solve Problem 1.

Consider the feedback controller $u = KZ(x)$ where $K \in \mathbb{R}^{m\times(n+q)}$ is the control gain. Let the matrices K, $Y \in \mathbb{R}^{T\times(n+q)}$ and $P \in \mathbb{R}^{(n+q)\times(n+q)}$ satisfies that $P \succ 0$,

$$K = U_0 Y P^{-1}, \quad Z_0 Y = P. \tag{3}$$

Similarly as presented in Guo et al. (2022a), the closed-loop dynamics can be written as

$$\begin{aligned}
x^+ &= AZ(x) + (B_0 + B_1 W(x))KZ(x)\\
&\overset{(3)}{=} AZ_0 Y P^{-1} Z(x) + (B_0 + B_1 W(x))U_0 Y P^{-1} Z(x)\\
&= (AZ_0 + B_0 U_0 + B_1 W(x) U_0) Y P^{-1} Z(x)\\
&\overset{(2)}{=} (X_1 - B_1 \overline{U}_0 - D_0 + B_1 W(x) U_0) Y P^{-1} Z(x).
\end{aligned}$$

We define that $G = [G_1\ G_2] = YP^{-1} \in \mathbb{R}^{T\times(n+q)}$ with $G_1 \in \mathbb{R}^{T\times n}$ and $G_2 \in \mathbb{R}^{T\times q}$. The product $YP^{-1}Z(x)$ can then be found as

$$YP^{-1}Z(x) = [G_1 \; G_2]\begin{bmatrix} x \\ Q(x)\end{bmatrix} = G_1 x + G_2 Q(x). \qquad (4)$$

Substituting (4) into the closed-loop dynamics, we write the dynamics into the sum of a linear part and a nonlinear part, *i.e.*,

$$x^+ = \mathcal{A}x + \Psi(x) \qquad (5)$$

where $\mathcal{A} = (X_1 - E\widehat{U}_0)G_1$ with $E = [B_1 \; D_0]$, $\widehat{U}_0 = \begin{bmatrix}\overline{U}_0^\top & I_T\end{bmatrix}^\top$ and

$$\begin{aligned}\Psi(x) &= (X_1 - E\widehat{U}_0)G_2 Q(x) \\ &\quad + B_1 W(x) U_0 G_1 x + B_1 W(x) U_0 G_2 Q(x). \qquad (6)\end{aligned}$$

To solve Problem 1, we will find the matrices Y and P such that the feedback controller $u = YP^{-1}Z(x)$ stabilizes the matrix $\mathcal{A}$ and decreases the influence of the nonlinear component $\Psi(x)$.

First, we characterize the feedback controller that renders $\mathcal{A}$ stable. As $\mathcal{A}$ depends on the unknown matrix E, the following assumption is posed.

Assumption 2. There exists a known matrix $R_E \succ 0$ such that $EE^\top \preceq R_E$. ■

Remark 2. The unknown matrix E consists of the system parameter B_1 and the noise data D_0. It is reasonable to assume that the latter is bounded as $D_0 D_0^\top \preceq R_D$ for some known $R_D \succ 0$. A similar bound on B_1, *i.e.*, $B_1 B_1^\top \preceq R_B$ for some known $R_B \succ 0$, can be obtained from data if $\begin{bmatrix}\overline{U}_0 \\ Z_0\end{bmatrix}$ has full row rank (Guo et al., 2022a, Remark 2). Then, R_E takes the form of $R_E = R_B + R_D$ as

$$EE^\top = [B_1 \; D_0]\begin{bmatrix} B_1^\top \\ D_0^\top\end{bmatrix} = B_1 B_1^\top + D_0 D_0^\top \preceq R_B + R_D.$$

The bound on B_0 is not needed. Note that it is possible to estimate the matrix $[A \; B_0 \; B_1]$ via (2) using least-square estimate. ■

Let matrices Y and P take the form

$$Y = [Y_1 \; Y_2], \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2\end{bmatrix} \qquad (7)$$

with $Y_1 \in \mathbb{R}^{T\times n}$, $Y_2 \in \mathbb{R}^{T\times q}$, $P_1 \in \mathbb{R}^{n\times n}$, and $P_2 \in \mathbb{R}^{q\times q}$. By the relation $G = YP^{-1}$, one has that $G_1 = Y_1 P_1^{-1}$ and $G_2 = Y_2 P_2^{-1}$. We give the following lemma that characterizes Y_1 and P_1 such that the resulting G_1 makes $\mathcal{A}$ stable.

Lemma 1. Under Assumption 2, for any fixed $\Omega \succ 0$, if there exist matrices Y_1, $P_1 \succ 0$, $P_1 = P_1^\top$, and constant ϵ such that

$$\begin{bmatrix} P_1 - \Omega & Y_1^\top \widehat{U}_0^\top & Y_1^\top X_1^\top \\ \widehat{U}_0 Y_1 & \epsilon I_{(p+T)} & 0_{(p+T)\times n} \\ X_1 Y_1 & 0_{n\times(p+T)} & P_1 - \epsilon R_E\end{bmatrix} \succeq 0 \qquad (8a)$$

$$\epsilon > 0 \qquad (8b)$$

then it holds that

$$-P_1 + Y_1^\top (X_1 - E\widehat{U}_0)^\top P_1^{-1}(X_1 - E\widehat{U}_0)Y_1 \preceq -\Omega. \qquad (9)$$

■

Proof. For any $\epsilon > 0$, by Schur complement, condition (8a) is equivalent to

$$-P_1 + \epsilon^{-1} Y_1^\top \widehat{U}_0^\top \widehat{U}_0 Y_1 - Y_1^\top X_1^\top(-P_1 + \epsilon R_E)^{-1} X_1 Y_1 \preceq -\Omega$$

which can be written as

$$\begin{aligned}\begin{bmatrix} -P_1 + \Omega & Y_1^\top X_1^\top \\ X_1 Y_1 & -P_1\end{bmatrix} + \epsilon^{-1}\begin{bmatrix} Y_1^\top \widehat{U}_0^\top \\ 0\end{bmatrix}[\widehat{U}_0 Y_1 \; 0] \\ + \epsilon\begin{bmatrix} 0 \\ -I_n\end{bmatrix} R_E [0 \; -I_n] \preceq 0. \qquad (10)\end{aligned}$$

Recalling Assumption 2, and using Petersen's lemma (Bisoffi et al., 2022, Fact 2), condition (10) is equivalent to

$$\begin{aligned}\begin{bmatrix} -P_1 + \Omega & Y_1^\top X_1^\top \\ X_1 Y_1 & -P_1\end{bmatrix} + \begin{bmatrix} Y_1^\top \widehat{U}_0^\top \\ 0\end{bmatrix} E^\top [0 \; -I_n] \\ + \begin{bmatrix} 0 \\ -I_n\end{bmatrix} E\,[\widehat{U}_0 Y_1 \; 0] \preceq 0. \qquad (11)\end{aligned}$$

Rearranging (11) gives

$$\begin{bmatrix} -P_1 + \Omega & Y_1^\top (X_1 - E\widehat{U}_0)^\top \\ (X_1 - E\widehat{U}_0)Y_1 & -P_1\end{bmatrix} \preceq 0 \qquad (12)$$

which, by Schur complement, is equivalent to (9). □

A direct result following Lemma 1 is that the origin is a globally exponentially stable equilibrium of the linear dynamics $x^+ = \mathcal{A}x = (X_1 - E\widehat{U}_0)G_1 x$. Define the Lyapunov function $V(x) = x^\top P_1^{-1} x$. Recalling that $G_1 = Y_1 P_1^{-1}$, the difference between $V(x^+)$ and $V(x)$ is

$$\begin{aligned}& V(x^+) - V(x) \\ &= x^\top G_1^\top (X_1 - E\widehat{U}_0)^\top P_1^{-1}(X_1 - E\widehat{U}_0)G_1 x - x^\top P_1 x \\ &= x^\top P_1^{-1}\Big[Y_1^\top (X_1 - E\widehat{U}_0)^\top P_1^{-1}(X_1 - E\widehat{U}_0)Y_1 \\ &\qquad - P_1\Big] P_1^{-1} x.\end{aligned}$$

By Lemma 1, the solution to (8a) and (8b) leads to

$$V(x^+) - V(x) \le -x^\top P_1^{-1}\cdot \Omega \cdot P_1^{-1} x.$$

As $\Omega \succ 0$, Lemma 1 ensures that the origin is a globally exponentially stable equilibrium of the linear dynamics $x^+ = \mathcal{A}x$, and the decay rate is enforced by the choice of Ω.

We know that between the two components of G, G_1 is chosen to stabilize $\mathcal{A}$ and G_2 is chosen to decrease the influence of the nonlinear term $\Psi(x)$. We recall that the nonlinear term $\Psi(x)$ takes the form

$$\begin{aligned}\Psi(x) = \underbrace{(X_1 - E\widehat{U}_0)G_2 Q(x)}_{=:\psi_1(x)} + \underbrace{B_1 W(x) U_0 G_2 Q(x)}_{=:\psi_2(x)} \\ + B_1 W(x) U_0 G_1 x. \qquad (13)\end{aligned}$$

If the input vector field is state-independent ($W(x) = 0$), then $\psi_1(x)$ is the closed-loop nonlinearity. To decrease the effect of $\psi_1(x)$, we minimize $\|(X_1 - E\widehat{U}_0)G_2\|$. However, as the input vector field is state-dependent, there are additional closed-loop nonlinearities. In this work, we show that by choosing G_2 such that it decreases the effect of $\psi_1(x)$, the designed controller can render the origin locally asymptotically stable. Meanwhile, one may also penalize $\|G_2\|$ in the cost to decrease the influence of $\psi_2(x)$. If we neglect the unknown matrix E, we set the cost as $\|X_1 G_2\| + \|\widehat{U}_0 G_2\| + \lambda\|G_2\|$ for any fixed weight $\lambda \ge 0$ and establish the following minimization problem

$$\min_{P_1, Y_1, G_2, \epsilon} \quad \|X_1 G_2\| + \|\widehat{U}_0 G_2\| + \lambda \|G_2\| \tag{14a}$$
$$\text{s.t.} \quad \text{(8a), (8b)}$$
$$Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{q\times n} \end{bmatrix} \tag{14b}$$
$$Z_0 G_2 = \begin{bmatrix} 0_{n\times q} \\ I_q \end{bmatrix}, \tag{14c}$$

where (14b) and (14c) ensure that $Z_0 Y = P$.

Remark 3. If we do not neglect the unknown matrix E, we can change the cost term $\|\widehat{U}_0 G_2\|$ into $\lambda_{max}(R_E)^{1/2}\|\widehat{U}_0 G_2\|$ where $\lambda_{max}(R_E)$ represents the largest eigenvalue of R_E. Alternatively, we may minimize the norm of

$$\Xi := (X_1 - E\widehat{U}_0)G_2 \tag{15}$$

which requires additional conditions to handle E under Assumption 2. By the nonstrict Petersen's lemma (Bisoffi et al., 2022, Fact 2) and Schur complement, if for any $\delta > 0$ it holds that

$$\begin{bmatrix} \gamma^2 I_q & G_2^\top \widehat{U}_0^\top & G_2^\top X_1^\top \\ \widehat{U}_0 G_2 & \delta I_{p+T} & 0_{(p+T)\times n} \\ X_1 G_2 & 0_{n\times(p+T)} & I_n - \delta R_E \end{bmatrix} \succeq 0, \tag{16}$$

then $G_2^\top (X_1 - E\widehat{U}_0)^\top (X_1 - E\widehat{U}_0) G_2 - \gamma^2 I_q \preceq 0$, *i.e.*, $\Xi^\top \Xi \preceq \gamma^2 I_q$. Hence, to minimize $\|\Xi\|$, we can choose γ as the cost. In this case, the optimization problem becomes

$$\min_{P_1, Y_1, G_2, \epsilon, \gamma, \delta} \quad \gamma + \lambda \|G_2\| \tag{17a}$$
$$\text{s.t.} \quad \text{(8a), (8b), (14b), (14c), (16)}$$
$$\delta > 0. \tag{17b}$$

■

Now we analyze the stability of the overall closed-loop system $x^+ = \mathcal{A}x + \Psi(x)$. The stabilization result is given as follows.

Theorem 1. Consider the nonlinear system (1). If the optimization problem (14) is feasible and it holds that

$$\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0, \quad \lim_{|x|\to 0} |W(x)| = 0, \tag{18}$$

then the control input $u = U_0 Y P^{-1} Z(x)$ renders the origin asymptotically stable for the closed-loop system. ■

Proof. Under conditions (14b) and (14c) and by the choice of the control gain $K = U_0 Y P^{-1}$, the system (1) under the feedback $u = KZ(x)$ results in the closed-loop dynamics $x^+ = \mathcal{A}x + \Psi(x)$. Using Lemma 1, we have shown that the condition (8a) guarantees that the linear system $x^+ = \mathcal{A}x$ is stable. For the overall closed-loop system $x^+ = \mathcal{A}x + \Psi(x)$, if the stable linear part dominates the nonlinear part near the origin, then the origin is asymptotically stable. To ensure that the linear part dominates the nonlinear part, we examine the function $\Psi(x)$ in (13). For $\psi_1(x)$, we need $\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$, which also ensures that $\psi_2(x)$ converges to the origin faster than the linear part in a neighborhood of the origin. Similarly, for the term $B_1 W(x) U_0 G_1 x$, we need

$$\lim_{|x|\to 0} \frac{|W(x) U_0 G_1 x|}{|x|} = 0. \tag{19}$$

Note that for any $x \neq 0$, it holds that

$$0 \le \frac{|W(x) U_0 G_1 x|}{|x|} \le \frac{|W(x)||U_0 G_1||x|}{|x|} = |W(x)||U_0 G_1|. \tag{20}$$

We also note that if $\lim_{|x|\to 0} |W(x)| = 0$, then

$$\lim_{|x|\to 0} |W(x)||U_0 G_1| = 0.$$

Hence, by the squeeze theorem, as (20) holds true, (19) is guaranteed by $\lim_{|x|\to 0} |W(x)| = 0$. Therefore, (18) makes sure that the linear dynamics of the closed-loop system dominates the nonlinear dynamics near the origin, and hence the origin is asymptotically stable. □

Remark 4. The conditions in (18) can be easily checked as $Q(x)$ and $W(x)$ contain known basis functions. As pointed out in Section IV.A by De Persis et al. (2023), these conditions are satisfied for any polynomial systems, as well as the systems with functions $f(0) = 0$, $g_{ij}(0) = 1$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $f(\cdot), g_{ij}(\cdot)$ differentiable at the origin so that the functions can be written as the sum of a linear part and a Taylor's remainder. ■

Remark 5. In the optimization problem, the cost is chosen to minimize the influence of nonlinearity $\psi_1(x)$, which aims at cancelling the system nonlinearity using the control effort $U_0 G_2 Q(x)$. Due to the state-dependent input vector field $W(x)$, the control input brings additional nonlinearity into the closed-loop dynamics. We try to decrease the effect of the additional nonlinearity by penalizing the term $\|G_2\|$. As shown in the simulation results in Section 5, penalizing $\|G_2\|$ does not improve the controller performance. How to design the cost and conditions of the optimization problem so that $\Psi(x)$ has minimum influence on the dynamics is an interesting topic to be further investigated in our future work. ■

4. ROA ESTIMATION

In this section, we analyze the RoA of the closed-loop system under the designed controllers using data. First, the definition of the RoA is given as follows.

Definition 1. A set $\mathcal{S}$ is *positively invariant* for the system $x^+ = f(x)$ if for every $x(0) \in \mathcal{S}$ the solution $x(t)$ for $t > 0$ satisfies $x(t) \in \mathcal{S}$. If for every initial condition $x(0) \in \mathcal{R}$, it holds that $\lim_{k\to\infty} x(k) = 0$, then $\mathcal{R}$ is a *region of attraction* of the system with respect to the origin. ■

Recall that (8a) in Theorem 1 guarantees that for any given $\Omega \succ 0$,

$$Y_1^\top (X_1 - E\widehat{U}_0)^\top P_1^{-1} (X_1 - E\widehat{U}_0) Y_1 - P_1 \preceq -\Omega. \tag{21}$$

Using the definitions $G_1 = Y_1 P_1^{-1}$ and $\mathcal{A} = (X_1 - E\widehat{U}_0) G_1$, the above inequality implies

$$\mathcal{A}^\top P_1^{-1} \mathcal{A} - P_1^{-1} \preceq -P_1^{-1} \Omega P_1^{-1}. \tag{22}$$

Defining $\Phi = P_1^{-1}\Omega P_1^{-1}$ gives $\mathcal{A}^\top P_1^{-1} \mathcal{A} - P_1^{-1} \prec -\Phi$.

For the closed-loop dynamics $x^+ = \mathcal{A}x + \Psi(x)$, the difference between $V(x^+)$ and $V(x)$ is

$$V(x^+) - V(x) = x^\top \left(\mathcal{A}^\top P_1^{-1} \mathcal{A} - P_1^{-1}\right) x + 2x^\top \mathcal{A}^\top P_1^{-1} \Psi(x) + \Psi(x)^\top P_1^{-1} \Psi(x).$$

Bear in mind that

$$
\begin{aligned}
&\Psi(x) \\
&= B_1 W(x) U_0 G_1 x + (X_1 - E\widehat{U}_0 + B_1 W(x) U_0) G_2 Q(x) \\
&= B_1 \left(W(x) U_0 G Z(x) - \overline{U}_0 G_2 Q(x)\right) - D_0 G_2 Q(x) \\
&\quad + X_1 G_2 Q(x).
\end{aligned}
$$

We rewrite $\Psi(x)$ as $\Psi(x) = E\Upsilon(x) + X_1 G_2 Q(x)$ where

$$
\Upsilon(x) = \begin{bmatrix} W(x) U_0 G Z(x) - \overline{U}_0 G_2 Q(x) \\ -G_2 Q(x) \end{bmatrix}.
$$

Next, by simple calculation, we obtain that

$$
\begin{aligned}
&x^\top \mathcal{A}^\top P_1^{-1} \Psi(x) \\
&= (X_1 G_2 Q(x))^\top P_1^{-1} X_1 G_2 Q(x) + \Upsilon(x)^\top E^\top P^{-1} E \Upsilon(x) \\
&\quad + 2(X_1 G_2 Q(x))^\top P_1^{-1} E \Upsilon(x).
\end{aligned}
$$

Using the above expressions and recalling that under Assumption 2, it holds that $\|E\|_2 \leq \|R_E\|_2$, we can write

$$
V(x^+) - V(x) \leq \underbrace{-x^\top \Phi x + l_1(x) + l_2(x) + l_3(x) + l_4(x)}_{=:l(x)} \tag{23}
$$

where

$$
\begin{aligned}
l_1(x) &:= \left(2X_1 G_1 x + X_1 G_2 Q(x)\right)^\top P_1^{-1} X_1 G_2 Q(x), \\
l_2(x) &:= 2\|R_E\|_2 \left|\widehat{U}_0 G_1 x\right| \left|P_1^{-1} X_1 G_2 Q(x)\right|, \\
l_3(x) &:= 2\|R_E\|_2 \left|(X_1 G_1 x)^\top P_1^{-1}\right. \\
&\qquad \left. + (X_1 G_2 Q(x))^\top P_1^{-1}\right| |\Upsilon(x)|, \\
l_4(x) &:= \|R_E\|_2^2 \|P_1^{-1}\| \left|2\widehat{U}_0 G_1 x + \Upsilon(x)\right| |\Upsilon(x)|.
\end{aligned}
$$

Note that the functions $l_i(x)$, $i = 1, \ldots, 4$, can all be computed from data and the known basis functions. Now we are ready to present the data-based estimation of the RoA.

Proposition 1. Consider system (1) with the control input designed by Theorem 1. Let $\mathcal{L} := l(x) < 0$ with $l(x)$ defined in (23). Then, any sublevel set $\mathcal{R}_c := \{x : V(x) \leq c\}$ of $V(x) = x^\top P_1^{-1} x$ contained in $\mathcal{L} \cup \{0\}$ is a positively invariant set for the closed-loop system and defines an estimate of the RoA relative to $x = 0$. ■

5. NUMERICAL EXAMPLE

Consider the polynomial system

$$
\begin{aligned}
x_1^+ &= 0.5 x_2 \\
x_2^+ &= x_1 + x_2^3 + (1 + x_2) u.
\end{aligned} \tag{24}
$$

Let $Z(x) = \begin{bmatrix} x_1 & x_2 & x_2^2 & x_2^3 \end{bmatrix}^\top$ and $W(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$. We collect data by conducting an experiment with input uniformly distributed within the interval $[-0.5, 0.5]$ and with an initial state within the same interval. The noise is uniformly distributed within the interval $[-0.1, 0.1]$. The length of the data collected is set as $T = 30$.

First, we solve the optimization problem (14) by setting $\Omega = 1 \cdot 10^{-8}$. When $\|G_2\|$ is not penalized ($\lambda = 0$), the solution of the optimization problem gives

$$
P_1 = 10^{-7} \cdot \begin{bmatrix} 0.1187 & 0.0026 \\ 0.0026 & 0.1262 \end{bmatrix},
$$

$$
K = \begin{bmatrix} -0.1600 & -0.0537 & 0.1539 & -0.7047 \end{bmatrix}.
$$

We observe that the first two components in K render the linear part of the dynamics stable, while the rest of the components decrease the influence of the nonlinear part. In particular, the term $-0.7047 x_2^3$ is the approximate cancellation performed on the nonlinearity x_2^3 in the dynamics. Only approximate cancellation is achieved because of the matrix E containing unknown noise D_0 and system parameters B_1.

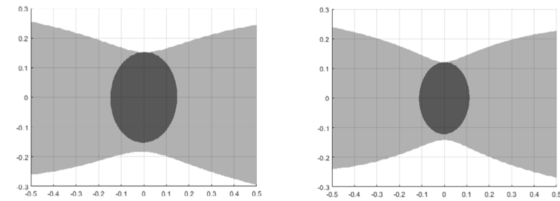

Fig. 1. Estimation of the RoA using the cost $\|X_1 G_2\| + \|\widehat{U}_0 G_2\| + \lambda \|G_2\|$ (light grey) and the largest sub-level set of $V = x^\top P_1^{-1} x$ contained in it (dark grey) with $\lambda = 0$ (left) and $\lambda = 1$ (right).

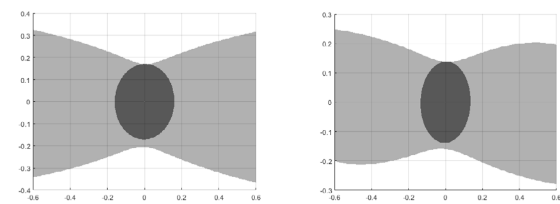

Fig. 2. Estimation of the RoA using the cost $\|X_1 G_2\| + \lambda_{max}(R_E)^{1/2} \|\widehat{U}_0 G_2\| + \lambda \|G_2\|$ (light grey) and the largest sub-level set of $V = x^\top P_1^{-1} x$ contained in it (dark grey) with $\lambda = 0$ (left) and $\lambda = 1$ (right).

From the solution we calculate that $\|\Xi\| = 0.3330$ and

$$
\Psi(x) = [0; -0.7047 x_2^4 + 0.4491 x_2^3 + 0.1002 x_2^2 - 0.1600 x_1 x_2].
$$

The estimated RoA is illustrated in Fig. 1 (left) , where the dark ellipse is $\{x : x^\top P_1^{-1} x \leq 1.86 \cdot 10^6\}$.

When we penalize $\|G_2\|$ ($\lambda = 1$), the solution to (14) is

$$
P_1 = 10^{-7} \cdot \begin{bmatrix} 0.1150 & 0.0019 \\ 0.0019 & 0.1350 \end{bmatrix},
$$

$$
K = \begin{bmatrix} -0.1734 & -0.0517 & 0.0192 & -0.5855 \end{bmatrix}.
$$

We calculate from the solution that $\|\Xi\| = 0.4149$ and

$$
\Psi(x) = [0; -0.5855 x_2^4 + 0.4337 x_2^3 - 0.0325 x_2^2 - 0.1734 x_1 x_2].
$$

The estimated RoA is illustrated in Figure 1 (right) where the dark ellipse is $\{x : x^\top P_1^{-1} x \leq 1.1 \cdot 10^6\}$. When the cost is set as $\|X_1 G_2\| + \lambda_{max}(R_E)^{1/2} \|\widehat{U}_0 G_2\| + \lambda \|G_2\|$, the simulation results are illustrated in Fig. 2. Due to space limit, details of the solutions are omitted here.

Next, we consider the optimization problem (17) that minimizes the norm of Ξ. When $\|G_2\|$ not penalized ($\lambda = 0$), the solution is

$$
P_1 = 10^{-7} \cdot \begin{bmatrix} 0.0902 & 0.0020 \\ 0.0020 & 0.1116 \end{bmatrix},
$$

$$
K = \begin{bmatrix} -0.1893 & -0.0539 & 0.1466 & -0.6839 \end{bmatrix}.
$$

It can be calculated that $\|\Xi\| = 0.3484$ and

$$
\Psi(x) = [0; -0.6839 x_2^4 + 0.4627 x_2^3 + 0.0927 x_2^2 - 0.1893 x_1 x_2].
$$

The estimated RoA is illustrated in Fig. 3 (left) where the dark ellipse is $\{x : x^\top P_1^{-1} x \leq 1.58 \cdot 10^6\}$.

When $\|G_2\|$ is penalized ($\lambda = 1$), solving (17) gives

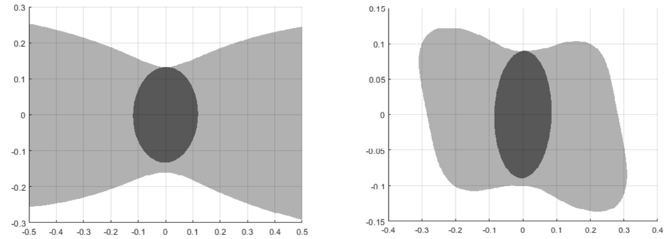

Fig. 3. Estimation of the RoA using the cost $\gamma + \lambda\|G_2\|$ (light grey) and the largest sub-level set of $V = x^\top P_1^{-1} x$ contained in it (dark grey) with $\lambda = 0$ (left) and $\lambda = 1$ (right).

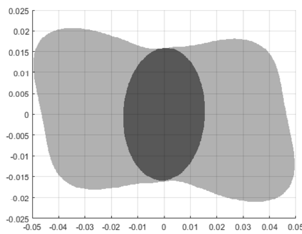

Fig. 4. Estimation of the RoA using the cost $\|X_1 G_2\| + \|\widehat{U}_0 G_2\| + \|U_0 G_2\|$.

$$P_1 = 10^{-7} \cdot \begin{bmatrix} 0.1125 & 0.0054 \\ 0.0054 & 0.1245 \end{bmatrix},$$
$$K = [-0.3475 \ -0.0955 \ 0.0376 \ -0.6633] .$$

It can be calculated that $\|\Xi\| = 0.3388$ and

$$\Psi(x) = [0; -0.6633x_2^4 + 0.3743x_2^3 - 0.0580x_2^2 - 0.3475x_1x_2].$$

Using this solution, the estimated RoA is illustrated in Figure 3 (right), where the dark ellipse is $\{x : x^\top P_1^{-1} x \leq 0.65 \cdot 10^6\}$.

The simulation results show that for all variations of the cost, the designed controllers are closer to cancel the nonlinear term x_2^3 in the dynamics when $\|G_2\|$ is not penalized, and also result in larger estimated RoA. When $\|G_2\|$ is not penalized, the optimization problem (14) returns a smaller $\|\Xi\|$ and a larger estimated RoA compared with (17). As pointed out in Remark 5 and verified by the simulation results, the designed controller can only approximately cancel the nonlinearity and brings additional nonlinearity into the dynamics. Future investigations will be focused on decreasing the norm of matrix by which the additional nonlinearity enters the dynamics.

In comparison, we change the penalized term from $\lambda\|G_2\|$ to $\|U_0 G_2\|$, and the optimization problem returns a linear controller as $U_0 G_2$ is the gain of the nonlinear part of the controller. The control gain returned is $K = [-0.3283 \ -0.0543 \ -6.3815 \cdot 10^{-8} \ -7.7083 \cdot 10^{-7}]$. The remaining nonlinearity in x_2-subsystem is $-7.70831x_2^4 + x_2^3 - 0.05434x_2^2 - 0.3283x_1x_2$, where x_2^3 is the nonlinear term from the system dynamics and the rest is brought by $B_1 W(x) U_0 G_1 x$. The estimated RoA is given in Figure 4, which is much smaller than the one obtained from the nonlinear controllers.

6. CONCLUSION

This work applies the data-driven control approach via nonlinearity cancellation developed by De Persis et al. (2023) to input-affine nonlinear systems with state-dependent input vector field. Data-based optimization problems are developed to design static state feedback stabilizers. Local asymptotical stability is guaranteed under mild assumptions and data-based RoA estimation is achieved. Due to the complexity of the input vector field, the additional part of the nonlinearity is difficult to cancel and is not explicitly handled in this work. Future works involve considering more general disturbances and designing controllers that further decreases the impact of the closed-loop nonlinearities.

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