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Norming and dense sets of extreme points of the unit ball in spaces of bounded Lipschitz functions

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A R T I C L E   I N F O

A B S T R A C T

On spaces of finite signed Borel measures on a metric space one has introduced the Fortet-Mourier and Dudley norms, by embedding the measures into the dual space of the Banach space of bounded Lipschitz functions, equipped with different – but equivalent – norms: the FM-norm and the BL-norm, respectively. The norm of such a measure is then obtained by maximising the value of the measure when applied by integration to extremal functions of the unit ball. We introduce Lipschitz extension operators, essentially based on those defined by McShane, and investigate their properties. A remarkable one is that non-trivial extreme points are mapped to non-trivial extreme points of FM- and BL-norm unit balls. Using these extension operators, we define suitable 'small' subsets of extremal functions that are weak-star dense in the full set of extreme points of the unit ball, for any underlying metric space. For connected metric spaces, we additionally find a larger set of extremal functions for the BL-norm, similar to such a set that was defined previously by J. Johnson for the FM-norm. This set is then also weak-star dense in the extremal functions. These results may open an avenue to obtaining computational approaches for the Dudley norm on signed Borel measures.

1. Introduction

Consider a metric space (S, d) and the vector space $\mathrm{Lip}(S, d)$ of Lipschitz functions for the metric d. We shall fix the metric throughout this study, so it will be omitted in notation. The Lipschitz constant of $f \in \mathrm{Lip}(S)$ is denoted by $|f|_L$. Banach spaces of Lipschitz functions are of interest, for example because they induce norms on finite signed measures on S, by viewing measures as functionals on theses Banach spaces by integration (see e.g. [15,16,20]). The norm is then the restriction of the dual norm to the embedded space of measures.

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Here, we are interested in the Banach space of bounded Lipschitz functions $\mathrm{BL}(S)$, with the norm

$$\|f\|_{\mathrm{BL}} := \|f\|_\infty + |f|_L. \tag{1}$$

In particular, subsets of extreme points of its closed unit ball B_{BL}^S will be of interest that are norming for measures in $\mathcal{M}(S)$, the vector space of finite signed Borel measures on S, for the associated dual norm. That is, if (S, d) is a complete separable metric space, then embedding of $\mu \in \mathcal{M}(S)$ in the dual space $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{BL}})^*$ results in the *Dudley norm* or *dual bounded Lipschitz norm* on $\mathcal{M}(S)$:

$$\|\mu\|_{\mathrm{BL}}^* := \sup\Big\{\int_S f \, d\mu \colon f \in B_{\mathrm{BL}}^S\Big\} \tag{2}$$

The associated metric, also called the *flat metric* [16], metrizes the weak topology on the finite positive measures $\mathcal{M}^+(S)$, which is defined by pairing with bounded continuous functions by integration (cf. e.g. [11,12,15] and also [16]). It is an alternative for metrics from the Wasserstein family that also have this property (see [33]), but which are originally defined as distance between measures with equal mass only and with finite first moments if the metric space has infinite diameter. Generalisations of these metrics have been introduced (see e.g. [21,25,26,31] and also [6]). Recently, interesting applications in image processing and analysis have been examined based on such generalized Wasserstein metrics, enabled by their interpretation in terms of optimal transport (see [21,31]). In a setting of measure-valued differential equations, flat metric and Wasserstein-1-type metric have been combined [1].

Sums of 'dipoles', $T = \sum_i(\delta_{p_i} - \delta_{n_i})$ (possibly infinitely many), taken in the space of distributions on the boundary Ω of a domain in $\mathbb{R}^3$, play a role in Ginzburg-Landau theory. Such T are connected to the location and topological degree of singularities of function in $H^{1/2}(\Omega, S^1)$ (see [5]). A Kantorovich-Rubinstein-like norm of T – which relates to the Wasserstein-1 metric, but generally differs from the Dudley norm – turns out to be equal to the length of the minimal connection associated to the point configuration defined by (p_i) and (n_i) (see [5], Lemma 2, p.3). Generalisations have been made to compact convex subsets of $\mathbb{R}^n$, e.g. in [3], or – even more general – to metrics spaces (S, d), where T is viewed as a continuous linear functional on the Banach space $(\mathrm{Lip}(S, d)/\mathbb{R}, |\cdot|_L)$ consisting of classes of Lipschitz functions that differ by a constant [27].

In mathematical theory, norm estimates are often sufficient. In various applications though, e.g. numerical analysis, one wants to be able to compute arbitrarily good approximations of the selected norm to estimate error. For measure-valued equations in which total mass is not conserved, one often prefers the use of the Dudley norm $\|\cdot\|_{\mathrm{BL}}^*$, because of its dual description. The Wasserstein-1 metric shares this property through the Kantorovich-Rubinstein formula [33]. Not all generalized metrics have such nice dual expression. However, see [25] for a relation to the so-called *Fortet-Mourier norm* on $\mathcal{M}(S)$ (see e.g. [20]):

$$\|\mu\|_{\mathrm{FM}}^* := \sup\Big\{\int_S f \, d\mu \colon f \in \mathrm{BL}(S), \ \|f\|_\infty \le 1, \ |f|_L \le 1\Big\}, \tag{3}$$

for any Polish space S.

To that end of computing norms, one can show that the supremum in (2) is actually attained at an extreme point of B_{BL}^S (see Proposition 2.2). Therefore, it is of interest to determine a set of extreme points of B_{BL}^S to which the suppremum in (2) can be restricted for determining the $\|\cdot\|_{\mathrm{BL}}^*$ norm of measures $\mu \in \mathcal{M}(S)$. We call such a set '*norming*'.

In [14], we achieved the objective of deriving particular explicit expressions and computational approaches for the Fortet-Mourier norm on $\mathcal{M}(S)$. This norm is associated to the norm

$$\|f\|_{\mathrm{FM}} := \max\big(\|f\|_\infty, |f|_L\big),$$

on BL(S), which is equivalent to $\|\cdot\|_{\mathrm{BL}}$. Hence, $\|\cdot\|_{\mathrm{FM}}^*$ is equivalent to $\|\cdot\|_{\mathrm{BL}}^*$. Previously, a computational approach for determining $\|\mu\|_{\mathrm{FM}}^*$ was available only for very particular cases, e.g. $S = [0,1]$ and μ finitely supported, see [17].

We could do so, because a characterisation of the extreme points of the unit ball B_{FM}^S of $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{FM}})$ had been provided by Farmer [13] in general (though for the seemingly different metric dual space $(\mathrm{Lip}_0(S), |\cdot|_L)$ introduced by Lindenstrauss [22], that turned out to be isometrically isomorphic to $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{FM}})$, see Appendix B), while a few particular cases had been covered before by Roy [30], Rao & Roy [28], and Rolewicz [29], in particular $S = [0,1]$. See also [7,32]. Moreover, Johnson [18] had described a dense subset of the extreme points of B_{FM}^S for compact and connected S that suffices to compute the norm (3). In particular the latter provided the inspiration for the results obtained in [14]. The techniques of proof that were employed do not work for $\|\cdot\|_{\mathrm{BL}}$ though, essentially because one cannot change the $\|\cdot\|_\infty$-norm of a function in the boundary of B_{BL}^S while staying within B_{BL}^S, without changing also its Lipschitz constant. For B_{FM}^S these can be changed independently.

To our knowledge, no results existed in the literature regarding the characterisation of the extreme points of B_{BL}^S or the description of a dense subset of such points, similar to Johnson's, suitable for computing the norm $\|\cdot\|_{\mathrm{BL}}^*$. Here we provide two dense and norming sets: a 'small' set E_{BL}^S, constructed through Lipschitz extension operators and a 'Johnson-like' set J_{BL}^S that consists of extreme points when S is connected. Our main results are Theorem 5.1 and its corollaries, Corollary 5.1 and Corollary 5.2, all for general metrics spaces (S, d). No connectedness, nor compactness condition needs to be imposed on S for the 'small' set E_{BL}^S to consist of extreme points. The set J_{FM}^S and the new set J_{BL}^S do consist of extreme points if S is connected. No compactness of S is required (cf. Proposition 5.2).

The description of E_{BL}^S and the proof of the theorem requires revisiting the classical McShane extension operator (cf. [23], Theorem 1) that extends Lipschitz functions defined on a subset P of S (with induced metric) to the whole space S without increasing its Lipschitz constant. Upon further inspection, this non-linear operator, denoted by $\mathcal{E}_P^{S,0}$, turns out to preserve the Lipschitz constant. We give an overview of its properties and that of a slightly modified Lipschitz extension operator $\mathcal{E}_P^S$ that plays a pivotal role in the Metric Tietze Extension Theorem (Theorem 4.1), which is central in the argumentation in this paper. Because of this centrality and because a full proof of this results is not readily available in the literature (although reference and a sketch of proof is found in [34], Theorem 1.5.6), we provide a full proof in Appendix A.

A particular result that is worth noting in this context, is Theorem 4.2, that states that the extension operator $\mathcal{E}_P^S$ maps non-trivial extreme points of B_{BL}^P into non-trivial extreme points of B_{BL}^S (non-triviality will be specified in Section 3). This was known for $(\mathrm{Lip}_0(S), |\cdot|_L)$ – hence, in disguise for $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{FM}})$ – by Farmer [13] (see Proposition 3). His proof relied on his characterisation of extreme points. Our proof for $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{BL}})$ can be modified slightly, then covering also the case of $\|\cdot\|_{\mathrm{FM}}$, without the need of a characterisation of the extreme points. This proof is sketched in Appendix C.

Moreover, on our way towards the main result, we established various auxiliary results on extreme points of B_{BL}^S and B_{FM}^S. We found several of these of sufficiently general interest to collect them in Section 3.

2. Notation and preliminary results

Throughout, (S, d) will be a metric space. The main object of interest in this paper are the unit balls in BL(S) and their convex structure, mainly for the norm $\|\cdot\|_{\mathrm{BL}}$, but occasionally also for $\|\cdot\|_{\mathrm{FM}}$, which will be denoted by

$$B_\bullet^S := \{ f \in \mathrm{BL}(S) : \|f\|_\bullet \leq 1 \} \qquad \text{with } \bullet = \mathrm{BL}, \mathrm{FM}.$$

A measure $\mu \in \mathcal{M}(S)$ is called (strictly) *separable* if there exists a separable set $S_0 \subset S$, such that μ is concentrated on S_0, i.e. $|\mu|(S_0) = |\mu|(S)$, where $|\mu| = \mu^+ + \mu^-$ is the total variation measure associated to μ. We denote the linear subspace of separable measures in $\mathcal{M}(S)$ by $\mathcal{M}_s(S)$. If S is separable, then $\mathcal{M}(S) = \mathcal{M}_s(S)$. The map $\mu \mapsto I_\mu : \mathcal{M}(S) \to \mathrm{BL}(S)^*$, defined by

$$I_\mu(f) := \langle \mu, f \rangle := \int_S f \, d\mu, \qquad f \in \mathrm{BL}(S), \tag{4}$$

is injective, because on a metric space, any $\mu \in \mathcal{M}(S)$ is regular (cf. [2], Theorem 7.7.1, p.70). This embedding makes it natural to equip $\mathcal{M}(S)$ with the restriction of the dual norm $\|\cdot\|_{\mathrm{FM}}^*$ or $\|\cdot\|_{\mathrm{BL}}^*$ on $\mathrm{BL}(S)^*$. This is the idea behind the introduction of the Dudley norm (2) or the Fortet-Mourier norm (3) on measures. Thus,

$$\|\mu\|_\bullet^* := \sup_{f \in B_\bullet^S} \left| \langle \mu, f \rangle \right| = \sup_{f \in B_\bullet^S} \langle \mu, f \rangle, \qquad \bullet = \mathrm{FM}, \mathrm{BL}. \tag{5}$$

These two norms on measures are equivalent. $\mathcal{M}(S)_{\mathrm{BL}}$ denotes the space $\mathcal{M}(S)$ equipped with the $\|\cdot\|_\bullet^*$-norm topology. $\overline{\mathcal{M}}(S)_{\mathrm{BL}}$ and $\overline{\mathcal{M}}_s(S)_{\mathrm{BL}}$ are the completions of $\mathcal{M}(S)$ and $\mathcal{M}_s(S)$, viewed as closure in $(\mathrm{BL}(S)^*, \|\cdot\|_{\mathrm{BL}}^*)$.

Following Pachl [24], any finite real linear combination of Dirac measures is called a *molecular measure*:

$$\mathcal{M}ol(S) := \mathrm{span}_{\mathbb{R}}\big\{ \delta_x : x \in S \big\}. \tag{6}$$

Clearly, $\mathcal{M}ol(S) \subset \mathcal{M}_s(S)$. It is dense for $\|\cdot\|_\bullet^*$ ($\bullet = \mathrm{FM}, \mathrm{BL}$).

Proposition 2.1. *The map* $\phi \mapsto f_\phi : \big(\mathcal{M}ol(S), \|\cdot\|_\bullet^*\big)^* \to \big(\mathrm{BL}(S), \|\cdot\|_\bullet\big)$, *with* $f_\phi(x) := \phi(\delta_x)$, $x \in S$, *is a linear isometric isomorphism for* $\bullet = \mathrm{FM}, \mathrm{BL}$.

$\mathrm{BL}(S)$ is a Banach algebra for the norm $\|\cdot\|_{\mathrm{BL}}$ ([11], Lemma 3). We denote by $\mathbb{1}$ its multiplicative unit, i.e. the constant function 1. Moreover, $\mathrm{BL}(S)$ is a vector lattice (or Riesz space) for the usual point-wise partial ordering '$\leq$' of functions. The lattice operations are given pointwise:

$$\begin{aligned} \sup(f, g)(x) &= (f \vee g)(x) &= \max\big(f(x), g(x)\big), \\ \inf(f, g)(x) &= (f \wedge g)(x) &= \min\big(f(x), g(x)\big), \end{aligned}$$

for any $f, g \in \mathrm{BL}(S)$. In particular, for $f \in \mathrm{BL}(S)$, $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$ are in $\mathrm{BL}(S)$. One has

$$|f \vee g|_L, \ |f \wedge g|_L \ \leq \ \max\big(|f|_L, |g|_L\big)$$

(cf. [11]), Lemma 4). $\|\cdot\|_{\mathrm{BL}}$ is not a Riesz norm.

Let C be a convex set in a vector space V. $e \in C$ is an extreme point if there is no open line segment that contains e and lies entirely in C. Equivalently, (see [10], p.101), $e \in C$ is an extreme point if and only if:

$$e = \frac{1}{2}(x + y), \ x, y \in C \qquad \text{if and only if} \qquad x = y = e. \tag{7}$$

The set of extreme points of C is denoted by $\mathrm{ext}(C)$.

Proposition 2.1 has the following consequence:

Proposition 2.2. *Let (S,d) be a metric space and $\mu \in \mathcal{M}_s(S)$. Let $\bullet =$ FM or BL. Then there exists $f_\mu^\bullet \in \text{ext}(B_\bullet^S)$ such that $\|\mu\|_\bullet^* = \langle \mu, f_\mu^\bullet \rangle$.*

Proof. According to Proposition 2.1 and the Banach-Alaoglu Theorem (cf. [9], Theorem V.3.1, p.130), $B_\bullet^S$ is compact for the $\sigma(\text{BL}(S), \overline{\mathcal{M}ol}(S)_{\text{BL}})$-topology. [4], Proposition 1, p. II.54 yields that the convex function $f \mapsto \langle \mu, f \rangle$ attains a maximum value at some $f_\mu^\bullet \in \text{ext}(B_\bullet^S)$ of the weak*-compact convex set $B_\bullet^S$. Expression (5) completes the argument. □

The following alternative characterisation of extreme points will be the 'workhorse' in our argumentation in proofs later on:

Lemma 2.1. *Let C be a convex set in a vector space V. $e \in \text{ext}(C)$ if and only if*

$$\{x \in V : e + x \in C \text{ and } e - x \in C\} = \{0\}. \tag{8}$$

Proof. If $e \in C$ is not an extreme point, then $e = \frac{1}{2}(x + y)$ for some $x, y \in C$, $x \neq y$. Put $z = \frac{1}{2}(x - y)$. Then $z \neq 0$, while $e + z = x \in C$ and $e - z = y \in C$. On the other hand, if (8) does not hold, then there exists $z \neq 0$ in V such that $x := e + z$ and $y := e - z$ are in C. Then $x \neq y$, while $e = \frac{1}{2}(x + y)$. Therefore, e cannot be an extreme point of C. □

Two immediate consequences of this lemma can be observed.

Corollary 2.1. *Let $(X, \|\cdot\|)$ be a normed space, $R > 0$ and put $B_R := \{x \in X : \|x\| \leq R\}$. Then $\text{ext}(B_R) \subset \{x \in X : \|x\| = R\}$.*

Proof. If $e \in \text{ext}(B_R)$ and $\|e\| < R$, take $x \in X$ with $0 < \|x\| \leq R - \|e\|$. Then $x \neq 0$, while $e + x$ and $e - x$ are in B_R. This contradicts Lemma 2.1. □

A set C in a vector space V is called symmetric around 0, if $x \in C$ implies $-x \in C$. That is, $C = -C$.

Corollary 2.2. *Let C be a convex set in a vector space V. If C is symmetric around 0, then $\text{ext}(C)$ is symmetric around 0.*

For various parts of the overall result, we will need the following two functional analytic results, of which the second is called the $\text{K}^2\text{-M}^3\text{-R}$ Theorem in [10] for the contributions of Krein, Klee, Milman, Mazur, Minkowski and Rutman, see [10], p.104 and also [19], Theorem 1.1. It summarizes various characterisations on a subset of the extreme points of a compact convex set, such that this subset still 'generates' this compact set as its closed convex hull.

Lemma 2.2. *Let X be a Banach space and F a dense subset of X. Let B^* be the closed unit ball in X^* and $A \subset B^*$. If $\|x\|_X = \sup\{\phi(x) : \phi \in A\}$ for all $x \in F$, then the weak*-closed convex hull of A is B^*.*

Proof. See Lemma 1.1 in [18]. □

Theorem 2.1. *($\text{K}^2\text{-M}^3\text{-R}$ Theorem) Let K be a convex compact set in a locally convex Hausdorff topological vector space X. Put $E = \text{ext}(K)$ and $A \subset K$. Then the following are equivalent:*

(i) The closed convex hull of A is K;
(ii) The closure of A contains E;

(*iii*) *The closure of A contains at least one point of each minimal facet of K;*
(*iv*) *For each continuous linear functional ϕ on X, $\sup \phi(A) = \sup \phi(K)$.*

Proof. See [10], Section V.1. □

3. Basic properties of extreme points of balls of Lipschitz functions

Let (S, d) be a metric space, without further assumptions (like completeness or separability). The geometry of the unit ball in $\mathrm{BL}(S)$ depends on the chosen norm. Hence, so does the set of extreme points. The FM-norm and BL-norm, though equivalent, introduce quite a different geometry. The argumentation required to exhibit extreme points in their unit balls differs substantially at points. However, some basic properties of the extreme points of the balls B_{FM}^S and B_{BL}^S are worth exhibiting now, as these partially direct the lines of argumentation later.

We start with an observation (cf. [28], Proposition 2.2, for complex-valued functions on $[0,1]$). The function constantly equal to 1 is denoted by $\mathbb{1}$.

Lemma 3.1. *Let $\bullet = \mathrm{FM}, \mathrm{BL}$. If $f \in B_{\bullet}^S$ is such that $|f| = \mathbb{1}$, then $f \in \mathrm{ext}(B_{\bullet}^S)$. In particular, if $\bullet = \mathrm{BL}$, then $f = \pm \mathbb{1}$.*

Proof. Define $S^{\pm} := \{x \in S : f(x) = \pm 1\}$. Let $g \in \mathrm{BL}(S)$ such that both $f + g$ and $f - g$ are in $B_{\bullet}^S$. Suppose $g \neq 0$ (i.e. f were not extreme). Let $x \in S$ such that $g(x) \neq 0$. If $g(x) > 0$ and $x \in S^+$, then $f(x) + g(x) > 1$; if $x \in S^-$, then $f(x) - g(x) < -1$. Similar reasoning applies to the case $g(x) < 0$, resulting in the conclusion that either $\|f + g\|_{\infty} > 1$ or $\|f - g\|_{\infty} > 1$, contradicting that both $f + g$ and $f - g$ are in $B_{\bullet}^S$. So, f is extreme.
If $\bullet = \mathrm{BL}$ and $|f| = \mathbb{1}$, then $\|f\|_{\infty} = 1$ and $|f|_L = 0$. So, $f = \pm \mathbb{1}$. □

A continuous function f on S that satisfies $|f| = \mathbb{1}$ must be constant on each connected component of S, taking the value either $+1$ or -1. That is, such f is locally constant. These constitute what we will call *trivial extreme points* of $B_{\bullet}^S$. We shall now be concerned with the *non-trivial extreme points*:

$$\mathrm{ext}_*(B_{\bullet}^S) := \mathrm{ext}(B_{\bullet}^S) \setminus \{f \in B_{\bullet}^S : |f| = \mathbb{1}\}, \qquad (\bullet = \mathrm{FM}, \mathrm{BL}). \tag{9}$$

Note that $\mathrm{ext}_*(B_{\bullet}^S) = \emptyset$ if S is a trivial space consisting of a singleton. We shall assume that the space S contains at least two elements.

For the FM-norm the conclusion of Corollary 2.1 (i.e., for an extreme point f of B_{FM}^S one has $\max(\|f\|_{\infty}, |f|_L) = 1$) can be strengthened:

Lemma 3.2. *If $f \in \mathrm{ext}(B_{\mathrm{FM}}^S)$, then $\|f\|_{\infty} = 1$. If $f \in \mathrm{ext}_*(B_{\mathrm{FM}}^S)$, then also $|f|_L = 1$.*

Proof. In view of Corollary 2.1, $\|f\|_{\infty} \leq 1$. Suppose, that $\|f\|_{\infty} < 1$. Let $\delta := 1 - \|f\|_{\infty} > 0$. Then

$$\|f \pm \delta \mathbb{1}\|_{\infty} \leq 1 \qquad \text{and} \qquad |f \pm \delta \mathbb{1}|_L \leq |f|_L + \delta |\mathbb{1}|_L = |f|_L \leq 1.$$

Thus, both $f + \delta \mathbb{1}$ and $f - \delta \mathbb{1} \in B_{\mathrm{FM}}^S$. According to Lemma 2.1, this contradicts that f is an extreme point. Hence, $\|f\|_{\infty} = 1$.

According to the first part, either $|f(x)| = 1$ for all x, or there exists $x_0 \in S$ such that $|f(x_0)| < 1$. In the latter case, suppose $|f|_L < 1$. Pick L such that $0 < L \leq 1 - |f|_L$. Let $\delta > 0$ such that $|f(x_0)| + \delta \leq 1$. Take $h > 0$ sufficiently small, such that $(2 + \frac{|f|_L}{L})h < \delta$. Define

$$g(x) := \big[h - Ld(x, x_0)\big]^+.$$

Then $g \in \mathrm{BL}(S)$, $g \neq 0$ and $|g|_L \leq L$. Moreover, $g(x) = 0$ when $d(x, x_0) \geq h/L$ and for $x \in S$ such that $d(x, x_0) < h/L$ one has

$$
\begin{aligned}
|f(x) \pm g(x)| &\leq |f(x) - f(x_0)| \; + \; |f(x_0) \pm (h - Ld(x, x_0))| \\
&\leq |f|_L d(x, x_0) + |f(x_0)| + h + Ld(x, x_0) \\
&< |f|_L \frac{h}{L} + |f(x_0)| + 2h < |f(x_0)| + \delta \leq 1.
\end{aligned}
$$

Thus,

$$\|f \pm g\|_\infty \leq 1 \qquad \text{and} \qquad |f \pm g|_L \leq |f|_L + |g|_L \leq |f|_L + L \leq 1.$$

So, both $f + g$ and $f - g$ are in B_{FM}^S, contradicting that f is an extreme point according to Lemma 2.1. $\quad\square$

Remark 3.1. In [30], Theorem 3.1, it has been shown that extreme functions f in the unit ball of $\mathrm{Lip}([0,1])$ for the FM-norm must have $|f'(x)| = 1$ almost everywhere with respect to Lebesgue measure, whereas for extreme functions f in the unit ball of $\mathrm{Lip}([0,1] \times [0,1])$ for the FM-norm this no longer holds: the set on which $|f'(x)| = 1$ can have arbitrary (but positive) Lebesque measure (less than 1; cf. [29], Theorem 1). This does not violate Lemma 3.2, since $|f|_L = \mathrm{ess\,sup}_{x \in S} |f'(x)|$.

For the BL-norm, the situation is different. If f is an extreme point of B_{BL}^S, then Corollary 2.1 immediate yields that $|f|_L = 1 - \|f\|_\infty$. Interestingly, the following result holds, for which the main inspiration came from [28], Lemma 2.7. It will be used in the proof of Proposition 5.2, which itself is auxiliary to a main result in this paper, namely the generalisation to the BL-norm setting of Johnson's description of a weak-star dense subset of the set of extreme points for the FM-norm, see Corollary 5.2.

Lemma 3.3. *If* $f \in \mathrm{ext}_*(B_{\mathrm{BL}}^S)$, *then* $\inf_{x \in S} f(x) = -\sup_{x \in S} f(x)$. *In particular, if S is compact, then f attains both* $\|f\|_\infty$ *and* $-\|f\|_\infty$.

Proof. Put $M := \sup_{x \in S} f(x)$, $m := \inf_{x \in S} f(x)$ and let $\mu := |M + m| \geq 0$. Since f is non-trivial, i.e. $|f| \neq \mathbb{1}$, $|f|_L > 0$ and $0 < \gamma := \|f\|_\infty = 1 - |f|_L < 1$. Consequently, $-1 < m \leq M < 1$ and $\mu < 2$. Consider, for $-1 < a < 1$ and $b \in \mathbb{R}$ the functions

$$g_{a,b} := af + b\mathbb{1} \in \mathrm{BL}(S).$$

We shall show that if $\mu > 0$, then one can find $(a, b) \neq 0$ such that both $f + g_{a,b}$ and $f - g_{a,b}$ are in B_{BL}^S, while $g_{a,b} \neq 0$. Then Lemma 2.1 yields that f cannot be an extreme point of B_{BL}^S, contradicting the assumption. Hence one must have $M = -m$.

To that end, assume $\mu > 0$ and consider the region of interest

$$R_\mu := \Big\{ (a, b) \in \mathbb{R}^2 \, : \, -1 < a < 1, \; |b| < \frac{1}{2}\mu \min(1 + a, 1 - a) \Big\} \tag{10}$$

for the parameters a and b. R_μ is defined in such a way that $-|M + m| \leq \frac{2b}{1 \pm a} \leq |M + m|$ whenever $(a, b) \in R_\mu$. It is an open set containing $(0, 0)$. In fact, it is the interior of the closed polyhedral set that is the convex hull of the four points $\pm(1, 0)$, $\pm(0, \frac{1}{2}\mu)$.

Write g instead of $g_{a,b}$ for notational convenience. Then

$$(1+a)m + b \leq f + g \leq (1+a)M + b,$$
$$(1-a)m - b \leq f - g \leq (1-a)M - b.$$

Hence, given the particular form of g,

$$\|f + g\|_\infty = \max\big[(1+a)M + b, -(1+a)m - b\big],$$
$$\|f - g\|_\infty = \max\big[(1-a)M - b, -(1-a)m + b\big],$$
$$|f \pm g|_L = (1 \pm a)|f|_L = (1 \pm a)(1 - \|f\|_\infty) = (1 \pm a)\big(1 - \max(M, -m)\big).$$

Then

$$\|f + g\|_{\mathrm{BL}} = 1 + a + (1 + a)\left(\max\left[M + \frac{b}{1+a}, -m - \frac{b}{1+a}\right] - \max(M, -m)\right)$$

$$= 1 + a + (1 + a)\left(\max\left[\frac{2b}{1+a}, -m - M\right] - \frac{b}{1+a} + M - \max(M, -m)\right)$$

$$= 1 + a - b + (1 + a)\left(\max\left[\frac{2b}{1+a}, -(M+m)\right] - \max(-(M+m), 0)\right). \tag{11}$$

Similarly, one computes that

$$\|f - g\|_{\mathrm{BL}} = (1 - a) + (1 - a)\left(\max\left[\frac{2b}{1-a}, M + m\right] - \frac{b}{1-a} - m - \max(M, -m)\right)$$

$$= 1 - a - b + (1 - a)\left(\max\left[\frac{2b}{1-a}, M + m\right] - \max(M + m, 0)\right). \tag{12}$$

R_μ has been chosen such that if $(a, b) \in R_\mu$, then

$$\|f - g\|_{\mathrm{BL}} = \begin{cases} 1 - a - b, & \text{if } M + m > 0, \\ 1 - a + b, & \text{if } M + m < 0 \end{cases} \tag{13}$$

and

$$\|f + g\|_{\mathrm{BL}} = \begin{cases} 1 + a + b, & \text{if } M + m > 0, \\ 1 + a - b, & \text{if } M + m < 0 \end{cases}. \tag{14}$$

Thus, if $M + m > 0$ and we require that both $\|f + g\|_{\mathrm{BL}} \leq 1$ and $\|f - g\|_{\mathrm{BL}} \leq 1$, then $a = -b$. If $M + m < 0$, the same conditions yield $a = b$. Since R_μ is open and contains $(0, 0)$, in either case R_μ contains an $(a, -a) \neq 0$ or $(a, a) \neq 0$.

The statement for S compact immediately follows from the general case, since the continuous function f attains its supremum and infimum on the compact set S. $\quad \square$

4. Extension of general Lipschitz functions and extreme points

The following (elementary) extension result allows limiting our attention to complete spaces (S, d). Let $(\hat{S}, \hat{d})$ be the completion of (S, d) and view S as a dense subset of $\hat{S}$.

Proposition 4.1. *Every $f \in \text{Lip}(S, d)$ has a unique extension $\hat{f} \in \text{Lip}(\hat{S}, \hat{d})$, such that $|\hat{f}|_L = |f|_L$. If f is bounded, then $\hat{f}$ is bounded and $\|\hat{f}\|_\infty = \|f\|_\infty$.*

The proof is left to the reader. According to this result, $\text{BL}(S, d)$ is linearly isometrically isomorphic to $\text{BL}(\hat{S}, \hat{d})$. Their respective unit balls and sets of extreme points can hence be identified.

Central in our description of subsets of extreme points B_{BL}^S (and B_{FM}^S) will be the extension to S of a bounded Lipschitz function on a (finite) subset P – with the restricted metric. Extension operators for real and vector valued Lipschitz functions have been extensively studied (cf. overviews provided in e.g. [8,34], among others). Crucial is what we call the '*Metric Tietze's Extension Theorem*', which is stronger than the well-known result by McShane [23], Theorem 1, that states that any Lipschitz function f on P has an extension F to S that has Lipschitz constant that is not greater than that of f.

Theorem 4.1 *(Metric Tietze's Extension Theorem). Let (S, d) be a metric space and $P \subset S$ a non-empty subset, equipped with the restriction of d to P. If $f \in \text{BL}(P)$, then there exists $F \in \text{BL}(S)$ such that $F|_P = f$, $\|F\|_\infty = \|f\|_\infty$ and $|F|_L = |f|_L$.*

For a full proof, see Appendix A. One should be aware that extension with preservation of Lipschitz constant is delicate for vector-valued Lipschitz functions. Theorem 4.1 then does not hold for a general codomain (e.g. see [8], Section 4.2): there exists spaces S and subsets P, such that a Lipschitz function $f : P \to \mathbb{R}^2$ cannot be extended to S with preservation of the Lipschitz constant. However, in specific cases it may, see e.g. the Kirszbraun-Valentine Theorem, (cf. [8], Theorem 4.2.3, p.221), but then by a different extension method. The McShane-type of Lipschitz extension is given by the operator

$$\mathcal{E}_P^{S,0} f(x) := \sup_{p \in P} \left[f(p) - |f|_L \, d(p, x) \right], \qquad x \in S. \tag{15}$$

We take the convention that for P a singleton, $|f|_L = 0$ for any $f \in \text{BL}(P)$, for the Lipschitz constant is then the supremum in $\mathbb{R}_+$ of the empty set. Thus, extension from singletons $P = \{x_0\}$ leads to constant functions on S.

The extension operator involved in Theorem 4.1 has particularly nice properties with regard to extreme points as we shall show below. It is defined by

$$\mathcal{E}_P^S : \text{BL}(P) \to \text{BL}(S) : f \mapsto F := \max\left(\mathcal{E}_P^{S,0}(f), -\|f\|_\infty\right). \tag{16}$$

We start with exhibiting basic properties of the McShane extension operator $\mathcal{E}_P^{S,0}$.

Proposition 4.2. *Let P be a non-empty subset of S, with the metric induced by S, and let $f, g \in \text{BL}(P)$, then:*

(i) $\mathcal{E}_P^{S,0}(f) \leq \|f\|_\infty \mathbb{1}$.
(ii) $\mathcal{E}_P^{S,0}(c\mathbb{1}) = c\mathbb{1}$ *for every $c \in \mathbb{R}$.*
(iii) *If $f \leq g$ and $|f|_L \geq |g|_L$, then $\mathcal{E}_P^{S,0}(f) \leq \mathcal{E}_P^{S,0}(g)$.*
(iv) *If $|f \vee g|_L \geq |f|_L$ and $|f \vee g|_L \geq |g|_L$, then $\mathcal{E}_P^{S,0}(f \vee g) \leq \mathcal{E}_P^{S,0}(f) \vee \mathcal{E}_P^{S,0}(g)$.*

Proof. Statements (i), (ii) and (iii) can be derived easily from the definition of the extension operator. For (iv), let $x \in S$. Then

$$\mathcal{E}_P^{S,0}(f \vee g)(x)$$

$$= \max \left(\sup_{p \in P : f(p) \geq g(p)} \left[f(p) - |f \vee g|_L d(x, p) \right], \sup_{p \in P : f(p) < g(p)} \left[g(p) - |f \vee g|_L d(x, p) \right] \right)$$

$$\leq \max \left(\sup_{p \in P : f(p) \geq g(p)} \big[f(p) - |f|_L d(x, p) \big], \sup_{p \in P : f(p) < g(p)} \big[g(p) - |g|_L d(x, p) \big] \right)$$

$$\leq \max \big(\mathcal{E}_P^{S,0}(f)(x), \mathcal{E}_P^{S,0}(g)(x) \big). \quad \square$$

Note that if the condition in Proposition 4.2 (*iv*) holds then equality of Lipschitz constants must hold in at least one of the two inequalities, because in general $|f \vee g|_L \leq \max(|f|_L, |g|_L)$. This observation yields the following

Corollary 4.1. *Let P be a non-empty subset of S, with the metric induced by S. If $f \in \mathrm{BL}(P)$ and $g \in \mathrm{BL}(S)$ are such that $\big| f \vee g|_P \big|_L = |f|_L$ and $\big| f \vee g|_P \big|_L \geq |g|_L$, then*

$$\mathcal{E}_P^{S,0}\big(f \vee g|_P \big) \vee g \ = \ \mathcal{E}_P^{S,0}(f) \vee g. \tag{17}$$

Proof. Since $\mathcal{E}_P^{S,0}(g|_P) \leq g$, we derive from Proposition 4.2 (*iv*) that

$$\mathcal{E}_P^{S,0}\big(f \vee g|_P \big) \vee g \leq \mathcal{E}_P^{S,0}(f) \vee g. \tag{18}$$

Because $\big| f \vee g|_P \big|_L \leq |f|_L$ and $f \vee g|_P \geq f$, obviously, Proposition 4.2 (*iii*) yields

$$\mathcal{E}_P^{S,0}\big(f \vee g|_P \big) \geq \mathcal{E}_P^{S,0}(f).$$

Taking the maximum with g on both sides in the last inequality gives the inequality opposite to (18). $\quad \square$

Note the following special case of this corollary that is relevant to the extension operator $\mathcal{E}_P^S$:

Corollary 4.2. *Let P be a non-empty subset of S, with the metric induced by S. If $f \in \mathrm{BL}(P)$ and $c \in \mathbb{R}$ are such that $|f \vee c\mathbb{1}|_L = |f|_L$, then*

$$\mathcal{E}_P^{S,0}\big(f \vee c\mathbb{1} \big) \vee c\mathbb{1} = \mathcal{E}_P^{S,0}(f) \vee c\mathbb{1}.$$

Remark 4.1. If $\overline{P}$ is the closure of P in S (not necessarily complete), then $\mathcal{E}_P^{\overline{P}}(f)$ is the restriction to $\overline{P}$ of the extension $\hat{f}$ to the completion $\hat{P}$ from Proposition 4.1.

The extension operators behave as expected under composition:

Proposition 4.3. *Let $P' \subset S$ and $P \subset P'$ be non-empty subsets, each equipped with the metric induced by S. Then $\mathcal{E}_{P'}^{S,0} \circ \mathcal{E}_P^{P',0} = \mathcal{E}_P^{S,0}$ and also $\mathcal{E}_{P'}^S \circ \mathcal{E}_P^{P'} = \mathcal{E}_P^S$.*

Proof. First consider the extension without truncation, defined by $\mathcal{E}_P^{S,0}$. Let $f \in \mathrm{BL}(P)$ and $x \in S$. Then, using that $\big| \mathcal{E}_P^{P',0}(f) \big|_L = |f|_L$ according to Theorem 4.1 and the triangle inequality, one obtains:

$$\big(\mathcal{E}_{P'}^{S,0} \circ \mathcal{E}_P^{P',0} \big)(f)(x) = \sup_{p' \in P'} \left[\left(\sup_{p \in P} \big[f(p) - |f|_L d(p', p) \big] \right) - |f|_L d(x, p') \right]$$

$$= \sup_{p' \in P'} \sup_{p \in P} \big[f(p) - |f|_L \big(d(p', p) + d(x, p') \big) \big] \tag{19}$$

$$\leq \sup_{p' \in P} \sup_{p \in P} \big[f(p) - |f|_L d(x, p) \big] \ = \ \mathcal{E}_P^{S,0}(f)(x).$$

On the other hand, equality (19) can be reformulated, and using $P \subset P'$ one arrives at

$$\left(\mathcal{E}_{P'}^{S,0} \circ \mathcal{E}_{P}^{P',0}\right)(f)(x) = \sup_{(p,p')\in P\times P'} \left[f(p) - |f|_L\big(d(p',p) + d(x,p')\big)\right]$$

$$\geq \sup_{(p,p'):\, p'=p} \left[f(p) - |f|_L\big(d(p',p) + d(x,p')\big)\right] = \mathcal{E}_{P}^{S,0}(f)(x).$$

According to the Tietze Extension Theorem, $\|\mathcal{E}_{P}^{P'}(f)\|_\infty = \|f\|_\infty$. If P is not a singleton, because $|\mathcal{E}_{P}^{P'}(f)|_L = |f|_L$, there exists a sequence $\big((x_n,y_n)\big)_{n\in\mathbb{N}} \in P \times P$, with $x_n \neq y_n$ and $|f(x_n) - f(y_n)|/d(x_n,y_n) \to |f|_L$ as $n \to \infty$. Since $\mathcal{E}_{P}^{P',0}(f)|_P = f$ and $f \geq -\|f\|_\infty \mathbb{1}$ on P, one obtains by McShane's Theorem that

$$|f|_L \leq \left|\mathcal{E}_{P}^{P',0}(f) \vee \big(-\|f\|_\infty\mathbb{1}\big)\right|_L \leq \left|\mathcal{E}_{P}^{P',0}(f)\right|_L \leq |f|_L. \tag{20}$$

Thus, equality of Lipschitz constants must hold everywhere in (20). If $P = \{x_0\}$ is a singleton, then $\mathcal{E}_{P}^{P',0}(f) = f(x_0)\mathbb{1}$ and equality holds in (20) trivially. Application of Corollary 4.2 and the first part of the proof now yields

$$\begin{aligned}
\left(\mathcal{E}_{P'}^{S} \circ \mathcal{E}_{P}^{P'}\right)(f) &= \left[\mathcal{E}_{P'}^{S,0}\big(\mathcal{E}_{P}^{P'}(f)\big)\right] \vee \left[-\big\|\mathcal{E}_{P}^{P'}(f)\big\|_\infty \mathbb{1}\right] \\
&= \left[\mathcal{E}_{P'}^{S,0}\big(\mathcal{E}_{P}^{P',0}(f) \vee (-\|f\|_\infty\mathbb{1})\big)\right] \vee \left[-\|f\|_\infty\mathbb{1}\right] \\
&= \left[\mathcal{E}_{P'}^{S,0}\big(\mathcal{E}_{P}^{P',0}(f)\big)\right] \vee \left[-\|f\|_\infty\mathbb{1}\right] \\
&= \mathcal{E}_{P}^{S,0}(f) \vee \left[-\|f\|_\infty\mathbb{1}\right] = \mathcal{E}_{P}^{S}(f). \quad \square
\end{aligned}$$

Farmer [13] considered extreme points of the Lipschitz dual $S^\#$, i.e. the vector space $\mathrm{Lip}_0(S)$ of Lipschitz functions that vanish at a fixed distinguished point $e \in S$, equipped with the $|\cdot|_L$ seminorm, that becomes a norm on this subspace of $\mathrm{Lip}(S)$. He showed that the McShane extension operator $\mathcal{E}_{P}^{S,0}$ maps extreme points of the unit ball in $P^\#$ to extreme points of the unit ball in $S^\#$, for closed subsets P of S (cf. [13], Lemma 2, p. 810). Hidden in the techniques of proof in [18] for the case of compact and connected S, a similar result could be observed for $\mathrm{BL}(S)$ with the FM-norm (stated precisely in Appendix C). The following result establishes this result for the BL-norm. Its novel method of proof is applicable to the FM-norm as well with small modification (see Appendix C) without restricting conditions of connectedness nor compactness. Hence, through the identification of $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{FM}})$ with $(\mathrm{Lip}_0(S), |\cdot|_L)$, detailed in Appendix B, it provides a novel proof for that result as well.

Theorem 4.2. *Let P be a subset of S with at least two points, equipped with the restriction of d as metric. Then $\mathcal{E}_{P}^{S}$ maps $\mathrm{ext}_*(B_{\mathrm{BL}}^{P})$ into $\mathrm{ext}_*(B_{\mathrm{BL}}^{S})$.*

Proof. Let $\overline{P}$ be the closure of P in S. According to Remark 4.1, $\mathcal{E}_{P}^{\overline{P}}$ identifies B_{BL}^{P} with $B_{\mathrm{BL}}^{\overline{P}}$ and $\mathrm{ext}_*(B_{\mathrm{BL}}^{P})$ with $\mathrm{ext}_*(B_{\mathrm{BL}}^{\overline{P}})$. In view of Proposition 4.3 it then suffice to show that $\mathcal{E}_{\overline{P}}^{S}$ maps $\mathrm{ext}_*(B_{\mathrm{BL}}^{\overline{P}})$ into $\mathrm{ext}_*(B_{\mathrm{BL}}^{S})$. That is, without loss of generality we can assume that P is closed.

Let $f \in \mathrm{ext}_*(B_{\mathrm{BL}}^{P})$ and $F := \mathcal{E}_{P}^{S}(f)$. Theorem 4.1 yields $F|_P = f$, $\|F\|_\infty = \|f\|_\infty$ and $|F|_L = |f|_L$. Since $f \neq \pm\mathbb{1}$, $F \neq \pm\mathbb{1}$. So, it suffices to show that F is an extreme point of B_{BL}^{S}. Suppose that $G \in \mathrm{BL}(S)$ is such that both $F + G$ and $F - G$ are in B_{BL}^{S}. We have to show that $G = 0$.

We have $f \pm G|_P = (F \pm G)|_P \in B_{\mathrm{BL}}^{P}$ and $f \in \mathrm{ext}(B_{\mathrm{BL}}^{P})$, hence by Lemma 2.1: $G|_P = 0$. Define $M_F^- := \{x \in S : F(x) = -\|F\|_\infty\}$ and $\tilde{H}(s,p) := \frac{H(p)-H(s)}{d(s,p)}$ for $H \in \mathrm{BL}(S)$, $s \in S \setminus P$, $p \in P$. Note that $|\tilde{H}(s,p)| \leq |H|_L$.

Let $x \in (M_F^- \cup P)^c$. Then $F(x) > -\|F\|_\infty = -\|f\|_\infty$, hence, by definition of $\mathcal{E}_{P}^{S}$: $F(x) = \sup_{p\in P}[f(p) - |f|_L d(x,p)]$. Pick $(p_n) \subset P$ with

$$F(x) = \lim_{n \to \infty} [f(p_n) - |f|_L d(x, p_n)]. \tag{21}$$

Note that the sequence $(f(p_n))$ is bounded. Then, the sequence $(d(x, p_n))$ is bounded by (21). Thus, there exists a subsequence (p_{n_k}) of (p_n) such that $(d(x, p_{n_k}))$ converges, and we have $\lim_k d(x, p_{n_k}) \geq d(x, P) > 0$, since P could be assumed closed. Together with (21), this implies (recall $F|_P = f$):

$$\lim_{k \to \infty} \tilde{F}(x, p_{n_k}) = \lim_{k \to \infty} \frac{F(p_{n_k}) - F(x)}{d(x, p_{n_k})} = |f|_L. \tag{22}$$

We have $(F|_P = f, G|_P = 0)$

$$\|F \pm G\|_\infty \geq \|(F \pm G)|_P\|_\infty = \|f\|_\infty$$

Also, $f \in \mathrm{ext}(B_{\mathrm{BL}}^P)$, so $\|f\|_{\mathrm{BL}} = 1$ (otherwise $\|f\|_{\mathrm{BL}} < 1$ so $f \pm \varepsilon \in B_{\mathrm{BL}}^P$ for some $\varepsilon > 0$). So for all $p \in P$, we have

$$|f|_L = 1 - \|f\|_\infty \geq 1 - \|F \pm G\|_\infty \geq |F \pm G|_L \geq |\hat{F}(x, p) \pm \hat{G}(x, p)|,$$

thus $|\hat{F}(x, p) + |\hat{G}(x, p)|| \leq |f|_L$. In particular, (recall (22) and $G|_P = 0$)

$$\frac{|G(x)|}{d(x, p_{n_k})} = |\hat{G}(x, p_{n_k})| \leq |f|_L - \hat{F}(x, p_{n_k}) \quad \longrightarrow \quad 0 \quad \text{as } k \to \infty.$$

Since $(d(x, p_{n_k}))$ is bounded, we conclude that $G(x) = 0$.

It remains to show that we have $G|_{M_F^- \cap P^c} = 0$. Let $x \in M_F^-$, i.e., $F(x) = -\|F\|_\infty$. We have

$$|F \pm G|_L \geq |(F \pm G)|_P|_L = |f|_L,$$

so (recall $\|f\|_{\mathrm{BL}} = 1$):

$$\|f\|_\infty = 1 - |f|_L \geq 1 - |F \pm G|_L \geq \|F \pm G\|_\infty \geq |F(x) \pm G(x)|. \tag{23}$$

Now, if $G(x) > 0$, then $|F(x) - G(x)| = \|F\|_\infty + G(x) > \|f\|_\infty$, while if $G(x) < 0$, we have $|F(x) + G(x)| = \|F\|_\infty + G(x) > \|f\|_\infty$, in both cases contradicting (23). Thus $G(x) = 0$. $\quad\square$

5. Norming and dense sets of extreme points

Our interest in the extreme points of the unit ball of Banach spaces of Lipschitz functions originates from the definition of useful norms on $\mathcal{M}(S)$ by means of embedding measures in the dual of such Banach spaces and Proposition 2.2. The latter reduces the task of computing $\|\mu\|_\bullet^*$, $\bullet = \mathrm{FM}$ or BL, to determining $\sup\{\langle \mu, f \rangle : f \in \mathrm{ext}(B_\bullet^S)\}$, provided μ is separable.

The full set of extreme points of $B_\bullet^S$ cannot be conveniently described for this purpose. The K^2-M^3-R Theorem, Theorem 2.1, indicates what type of sets to look for in order to take the supremum over a smaller set of extreme points. Recall the natural pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{M}(S)$ and $\mathrm{BL}(S)$, defined by integration, as in (4). Proposition 2.1, which asserts that $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{BL}})$ is the dual space of $\overline{\mathcal{Mol}}(S)_{\mathrm{BL}} = \mathcal{M}_s(S)$, yields that the $\sigma(\mathrm{BL}(S), \overline{\mathcal{Mol}}(S)_{\mathrm{BL}})$-weak topology defined by the above pairing is the weak*-topology on $\mathrm{BL}(S)$. According to the Banach-Alaoglu Theorem ([9], Theorem V.3.1), $B_\bullet^S$ is compact in this locally convex Hausdorff vector space topology. Thus, according to Theorem 2.1 (iv), a subset A of $\mathrm{ext}(B_\bullet^*)$ is norming for $\|\cdot\|_\bullet^*$ if and only if any of the conditions (i)-(iii) in the K^2-M^3-R Theorem holds.

We provide sets $E_\bullet^S$ of extreme points of which the weak*-closed convex hull is $B_\bullet^S$, thus proving property (*i*). Lemma 2.2 provides the technique to do so. As a consequence one gets the weak*-density in the extreme points (which is property (*ii*)). As 'by-product' of this approach, it is possible to reduce attention to the dense subset $\mathcal{M}ol(S)$ of $\mathcal{M}(S)$ of measures with *finite support*. This will allow to define 'small' norming sets $E_\bullet^S$, defined by means of the Lipschitz extension operators $\mathcal{E}_P^S$. Johnson, in the setting of connected and compact metric spaces in [18], aimed at providing a set J_{FM}^S of extreme points in B_{FM}^S that was 'as large as possible', possibly having a different objective in mind.

For $f \in \mathrm{BL}(S)$, define

$$M_f := \{x \in S \colon |f(x)| = \|f\|_\infty\},$$

which is non-empty in the case that S is compact. Johnson defined

$$J_{\mathrm{FM}}^S := \big\{ f \in B_{\mathrm{FM}}^S \colon \|f\|_\infty = 1, \ \exists P_f \subset S \text{ finite}, \ P_f \neq \emptyset : \tag{24}$$
$$\big(\ x \in S \setminus M_f \ \Rightarrow \ \exists p \in P_f : |f(x) - f(p)| = d(x, p) \ \big) \ \big\}$$

(cf. the set $\mathcal{A}$ defined in [18], Proposition 1.1). Below we shall now define 'our' small sets $E_\bullet^S$, but also provide for the case of the BL-norm a novel 'Johnson-like' large set J_{BL}^S of extreme points of B_{BL}^S. Motivation for the definition of J_{BL}^S comes from Lemma 3.3.

Thus, we introduce the subset of $B_\bullet^S$:

$$\hat{E}_\bullet^S := \bigcup_{P \subset S, \text{ finite}} \mathcal{E}_P^S\big(\mathrm{ext}_*(B_\bullet^P)\big), \qquad \bullet = \mathrm{FM}, \mathrm{BL}. \tag{25}$$

According to Theorem 4.2 (for BL-norm) and Theorem C.1 (for FM-norm), $\hat{E}_\bullet^S$ consists of non-trivial extreme points of $B_\bullet^S$, obtained by McShane-type Lipschitz extension. Note that $\mathrm{ext}_*(B_\bullet^P) = \emptyset$ when P is a singleton. So, the union runs over all finite sets P with at least two elements. Expand the set $\hat{E}_{\mathrm{BL}}^S$ by adding the trivial extreme points:

$$E_{\mathrm{BL}}^S := \hat{E}_{\mathrm{BL}}^S \ \cup \ \{\pm \mathbb{1}\}. \tag{26}$$

For the FM-norm, there are non-trivial extreme functions that cannot be reached by the Lipschitz extension operators $\mathcal{E}_P^S$, see Appendix C, but which must be included for the norming property to hold. These need to be added separately, apart from the trivial extreme points. So, in this case:

$$E_{\mathrm{FM}}^S := \hat{E}_{\mathrm{FM}}^S \ \cup \ \{f \in B_{\mathrm{FM}}^S \colon |f| = \mathbb{1}\} \ \cup \ \{h_P \colon P \subset S \text{ finite}, \ P \neq \emptyset\}, \tag{27}$$

where

$$h_P(x) := (-\mathbb{1}) \vee \sup_{p \in P}\big[1 - d(x, p)\big], \qquad x \in S. \tag{28}$$

Lemma C.1 shows that the functions h_P are extreme points of B_{FM}^S. Note that these functions h_P cannot be reached by extension from a Lipschitz function on P, because $(h_P)|_P = \mathbb{1}$ and $\mathcal{E}_P^S(\mathbb{1}) = \mathbb{1}$ (cf. Proposition 4.2 (*ii*)).

The Johnson-like set J_{BL}^S is defined similar to (24):

$$\hat{J}_{\mathrm{BL}}^S := \big\{ f \in B_{\mathrm{BL}}^S \colon \|f\|_{\mathrm{BL}} = 1, \ f(M_f) = \{\|f\|_\infty, -\|f\|_\infty\}, \tag{29}$$
$$\exists P_f \subset S \text{ finite}, \ P_f \neq \emptyset :$$

$$(\ x \in S \setminus M_f \ \Rightarrow \ \exists p \in P_f : |f(x) - f(p)| = (1 - \|f\|_\infty)d(x,p) \ ) \ \}$$
$$J_{\mathrm{BL}}^S := \hat{J}_{\mathrm{BL}}^S \ \cup \ \{\pm\mathbb{1}\} \tag{30}$$

Note that in the case of the FM-norm, the trivial extreme points, satisfying $|f| = \mathbb{1}$, are contained in J_{FM}^S as formulated in (24), because $M_f = S$ for such f. Hence, the condition with P_f is trivially satisfied. For the BL-norm, the trivial extreme points $f = \pm\mathbb{1}$ also have $M_f = S$, but $f(M_f)$ does not consist of both two values $\pm\|f\|_\infty$. It is necessary though, to have the trivial extreme points 'on board'. Note too, that the functions h_P, which were added to $\hat{E}_{\mathrm{FM}}^S$ to get E_{FM}^S, are included in J_{FM}^S.

Proposition 5.1. $E_\bullet^S \subset J_\bullet^S$ *for* $\bullet = \mathrm{BL}$ *or* FM.

Proof. *Case of FM-norm.* If $f \in E_\bullet^S$ is trivial, $|f| = \mathbb{1}$ and hence $M_f = S$. Thus, we need to check only the conditions not involving P_f. One has $\|f\|_\infty = 1$, so $f \in J_{\mathrm{FM}}^S$.
If $f = h_P$ for some $\emptyset \neq P \subset S$ finite, then $\|f\|_\infty = 1$ and $M_f = P \cup \{x \in S : f(x) = -1\}$. One may take $P_f = P$ and then the second condition in the definition is satisfied.
If $f \in \hat{E}_{\mathrm{FM}}^S$ is non-trivial, then $\|f\|_\infty$ and $|f|_L = 1$, according to Lemma 3.2. Moreover, $f = \mathcal{E}_P^S(f^*)$ for some $f^* \in \mathrm{ext}_*(B_{\mathrm{FM}}^P)$ and $P \subset S$ finite, with $|P| \geq 2$. By the Tietze Extension Theorem, $|f^*|_L = |f|_L = 1$ and it is clear from (15) and (16) that f satisfies the second condition in the definition, using $P_f = P$.
Case of BL-norm. If $f \in E_{\mathrm{BL}}^S$ is trivial, $|f| = \mathbb{1}$ and hence $M_f = S$. Thus, again we need to check only the conditions not involving P_f. In this case, $f = \pm\mathbb{1}$, according to Lemma 3.1, which are included in J_{BL}^S by construction.
If $f \in \hat{E}_{\mathrm{BL}}^S$, then $f = \mathcal{E}_P^S(f^*)$ for some $f^* \in \mathrm{ext}_*(B_{\mathrm{BL}}^P)$ with P finite. From (15) and (16) it is immediate that $\mathcal{E}_P^S(f^*)$ satisfies the last condition appearing in the definition of $\hat{J}_{\mathrm{BL}}^S$ with $P_f := P$. The other two conditions follow from Corollary 2.1 and Lemma 3.3 applied to $f^* \in \mathrm{ext}_*(B_{\mathrm{BL}}^P)$. Hence, $f = \mathcal{E}_P^S(f^*) \in \hat{J}_{\mathrm{BL}}^S \subset J_{\mathrm{BL}}^S$. $\square$

The next result will clarify the relationship between J_{BL}^S and $\mathrm{ext}(B_{\mathrm{BL}}^S)$. To that end we need the following lemma, which is a reformulation of [28], Lemma 2.1 for real numbers.

Lemma 5.1. *Let* $x, y \in \mathbb{R}$ *and let* $|x| + |y| = 1$. *If* $\alpha, \beta \in \mathbb{R}$ *are such that* $|x \pm \alpha| + |y \pm \beta| \leq 1$, *then either* $xy = \alpha = \beta = 0$, *or*

$$xy \neq 0, \quad |\alpha| \leq \min\{|x|, |y|\} \quad and \quad \alpha\frac{|x|}{x} + \beta\frac{|y|}{y} = 0.$$

Proof. See [28], Lemma 2.1. $\square$

We can only prove that J_{BL}^S (and J_{FM}^S, see Appendix C) consist of extreme points if S is connected. The proof for the BL-norm is remarkably more complicated than that for the FM-norm. It is unclear whether $J_\bullet^S$ always contains non-extremal points when S is not connected. Example 6.2 gives an example of a non-extremal point when S is finite.

Proposition 5.2. *If* S *is connected, then* $J_\bullet^S \subset \mathrm{ext}(B_\bullet^S)$ *for* $\bullet = \mathrm{FM}$ *or* BL.

Proof. Assume that S is connected.
Case of FM-norm: The argument is essentially that in [18], proof of Proposition 1.1, with removal of the compactness condition. Suppose that $f \in J_{\mathrm{FM}}^S$ and let P_f be the corresponding set P. Suppose that $g \in \mathrm{BL}(S)$ is such that both $f + g$ and $f - g$ are in B_{FM}^S. We shall show that $g = 0$ and conclude by means of Lemma 8 that f is extreme.
If $x \in M_f$, then $f(x) \in \{\pm 1\}$ and $1 \geq \|f \pm g\|_\infty \geq |f(x) \pm g(x)|$. Therefore $g(x) = 0$. If $x \in S \setminus (M_f \cup P_f)$,

then there exists $p_x \in P_f$ such that $\tilde{f}(x, p_x) := \frac{f(p_x) - f(x)}{d(x, p_x)} = \pm 1$. Define $\tilde{g}(x, p_x)$ similarly. Then for both choices of sign,

$$1 \geq \|f \pm g\|_{\mathrm{FM}} \geq |f \pm g|_L \geq |\tilde{f}(x, x_p) \pm \tilde{g}(x, x_p)| = |1 \pm \tilde{g}(x, x_p)|.$$

So, $\tilde{g}(x, x_p) = 0$ and $g(x) = g(x_p)$. Therefore,

$$g(S) = g(M_f) \cup g(S \setminus (M_f \cup P_f)) \cup g(P_f) = g(P_f).$$

In particular, $g(S)$ is finite. Since g is continuous and S is connected, $g(S)$ consists of a singleton. If $M_f \neq \emptyset$ (e.g. if S is compact), then $0 \in g(M_f) \subset g(S)$. So $g = 0$. If $M_f = \emptyset$, then also $\{0\} \subset g(S)$, by contradiction: suppose that $0 \notin g(S)$. Pick a sequence $(x_n) \subset S$ such that $|f(x_n)| \to \|f\|_\infty = 1$. Then

$$|f| + |g| = \max(|f + g|, |f - g|) \leq \max(\|f + g\|_\infty, \|f - g\|_\infty) \leq 1$$

Consequently,

$$0 \leq |g(x_n)| \leq 1 - |f(x_n)| \to 0 \qquad \text{as } n \to \infty.$$

So, $|g(x_n)| \to 0$. However, $g(S)$ is finite. Thus, $g(x_n) = 0$ must hold for sufficiently large n, which implies that $0 \in g(S)$. We arrived at a contradiction. Hence, $g = 0$ and we conclude that f is extreme.

Case of BL-norm: Suppose that $f \in J_{\mathrm{BL}}^S$ and $f \pm g \in B_{\mathrm{BL}}^S$ for some $g \in \mathrm{BL}(S)$. We show that $g = 0$, i.e. $f \in \mathrm{ext}(B_{\mathrm{BL}}^S)$. Recall that $J_{\mathrm{BL}}^S = \hat{J}_{\mathrm{BL}}^S \cup \{\pm 1\}$. If $f = \pm 1$, it is immediate that g must be zero. In the other case, we have $f \in \hat{J}_{\mathrm{BL}}^S$. For $s, t \in S$, $s \neq t$, define $\tilde{f}(s, t) := \frac{f(s) - f(t)}{d(s, t)}$ and $\tilde{g}(s, t) := \frac{g(s) - g(t)}{d(s, t)}$. Let P_f be a finite subset as in the definition of $\hat{J}_{\mathrm{BL}}^S$, and let $s \in M_f^c \cap P_f^c$. Set $\gamma := \|f\|_\infty$ and note that $\gamma \in (0, 1)$. Now, let $p_s \in P_f$ be such that $|\tilde{f}(s, p_s)| = 1 - \gamma$. Such p_s exists according to the definition of J_{BL}^S. Then, for all $x \in M_f$ we have

$$\begin{cases} |f(x)| + |\tilde{f}(s, p_s)| = \gamma + 1 - \gamma = 1, \\ |f(x) \pm g(x)| + |\tilde{f}(s, p_s) \pm \tilde{g}(s, p_s)| \leq \|f \pm g\|_\infty + |f \pm g|_L = \|f \pm g\|_{\mathrm{BL}} \leq 1. \end{cases}$$

Note that $f(x) \neq 0 \neq \tilde{f}(s, p_s)$ since $\gamma \in (0, 1)$, so Lemma 5.1 yields

$$g(x) \frac{|f(x)|}{f(x)} + \tilde{g}(s, p_s) \frac{|\tilde{f}(s, p_s)|}{\tilde{f}(s, p_s)} = 0, \qquad \text{for all } x \in M_f. \tag{31}$$

Keeping s (and p_s) fixed and varying x, this implies

$$g(x) = -\frac{\tilde{g}(s, p_s)|\tilde{f}(s, p_s)|}{\gamma \tilde{f}(s, p_s)} f(x) = c f(x), \qquad \text{for all } x \in M_f, \tag{32}$$

where $c := -\frac{\tilde{g}(s, p_s)|\tilde{f}(s, p_s)|}{\gamma \tilde{f}(s, p_s)} = \frac{g(x)}{f(x)}$ is constant. Now, (31) becomes

$$c\gamma + \tilde{g}(s, p_s) \frac{|\tilde{f}(s, p_s)|}{\tilde{f}(s, p_s)} = 0,$$

so $\tilde{g}(s, p_s) = \frac{-c\gamma}{1 - \gamma} \tilde{f}(s, p_s)$ (recall that $|\tilde{f}(s, p_s)| = 1 - \gamma$). Therefore,

$$g(s) - g(p_s) = \frac{-c\gamma}{1-\gamma}(f(s) - f(p_s)). \tag{33}$$

Since $f \in J_{\mathrm{BL}}^S$, there exist $x_\pm \in M_f$ with $f(x_\pm) = \pm\|f\|_\infty = \pm\gamma$. For the moment, assume the following:

$$\begin{cases} \text{There exist } n \in \mathbb{N} \text{ and } \{x_1, \ldots, x_n\} \subset S, \text{ such that } x_1 = x_-, x_n = x_+ \\ \text{and } g(x_{i+1}) - g(x_i) = \frac{-c\gamma}{1-\gamma}(f(x_{i+1}) - f(x_i)) \text{ for } i = 1, \ldots, n-1. \end{cases} \tag{34}$$

We will prove (34) below, but we finish the proof first, using this assumption. We have

$$\begin{aligned} 2c\gamma = cf(x_n) - cf(x_1) &\overset{(32)}{=} g(x_n) - g(x_1) \\ &= \sum_{i=1}^{n-1}\big(g(x_{i+1}) - g(x_i)\big) = \sum_{i=1}^{n-1}\frac{-c\gamma}{1-\gamma}\big(f(x_{i+1}) - f(x_i)\big) \\ &= \frac{-c\gamma}{1-\gamma}(f(x_n) - f(x_1)) \\ &= \frac{-c\gamma}{1-\gamma}2\gamma. \end{aligned}$$

So either $c = 0$ or $\gamma = 0$ or $1 = \frac{-\gamma}{1-\gamma}$. The last two options cannot hold since $\gamma \in (0,1)$. We conclude that $c = 0$. Now, (32) implies that $g = cf = 0$ on $M_f \supset \{x_+, x_-\}$ and from (33), we derive that $g(s) \in g(P_f)$ for all $s \in S \setminus P_f$. Thus, $\{0\} \subset g(S) \subset \{0\} \cup g(P_f)$, so $g(S)$ is finite and contains zero. Since S is connected and g is continuous, $g(S)$ is connected. Also, it is finite and $0 \in g(S)$, so $g(S) = \{0\}$, i.e. $g = 0$. We conclude that $f \in \mathrm{ext}(B_{\mathrm{BL}}^S)$.

It remains to prove (34). To this end, define for $x \in S$:

$$\begin{aligned} M_x := \{s \in S : &\exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in S \text{ such that } x_1 = x, x_n = s, \\ &g(x_{i+1}) - g(x_i) = \frac{-c\gamma}{1-\gamma}(f(x_{i+1}) - f(x_i)) \text{ for } 1 \leq i \leq n-1\}. \end{aligned} \tag{35}$$

We write $x \sim y$ if and only if $y \in M_x$. Clearly, if $x_+ \in M_{x_-}$, we have proved (34).

We first show that $\sim$ is an equivalence relation. By taking $n = 1$ in the definition of M_x we see that trivially $x \in M_x$, proving reflexivity. For symmetry, suppose that $y \in M_x$, with corresponding $\{x_1, \ldots, x_n\}$ as in the definition of M_x. Define $\bar{x}_i := x_{n-i+1}$ for $i = 1, \ldots, n$. Then $\bar{x}_1 = x_n = y$, $\bar{x}_n = x_1 = x$ and

$$\begin{aligned} g(\bar{x}_{i+1}) - g(\bar{x}_i) = -\Big(g(x_{n-i+1}) - g(x_{n-i})\Big) &= -\Big(\frac{-c\gamma}{1-\gamma}(f(x_{n-i+1}) - f(x_{n-i}))\Big) \\ &= \frac{-c\gamma}{1-\gamma}(f(\bar{x}_{i+1}) - f(\bar{x}_i)), \end{aligned}$$

so $x \in M_y$. For transitivity, suppose that $x \in M_y$ with corresponding $\{x_1, \ldots, x_n\}$ and $y \in M_z$ with corresponding $y_1, \ldots, y_m$. Define $N := n + m$ and $\bar{x}_i := x_i$ for $1 \leq i \leq n$, $\bar{x}_i := y_{i-n}$ for $n+1 \leq i \leq N$. Note that $\bar{x}_n = \bar{x}_{n+1} = y$ and note that all the properties are satisfied to conclude that $x \in M_z$. We conclude that $\sim$ is an equivalence relation. Consequently, any two equivalence classes M_x, M_y are either equal or disjoint.

We show that each M_x is a closed subset of S. Suppose that $(s_n) \subset M_x$ and $s_n \to s \in S$. By continuity of f, we have $f(s_n) \to f(s)$. Since $f \in J_{\mathrm{BL}}^S$, we have for each $n \in \mathbb{N}$ either $s_n \in M_f$, so $|f(s_n)| = \gamma$, or $|f(s_n) - f(p)| = (1-\gamma)d(s,p)$ for some $p \in P_f$. Write $P_f = \{p_1, \ldots, p_k\}$ with $p_i \in S$ and $k \in \mathbb{N}$. Define

$$I_i := \{n \in \mathbb{N} : |f(s_n) - f(p_i)| = (1-\gamma)d(s_n, p_i)\},$$

$$I^\gamma := \{n \in \mathbb{N} : f(s_n) = \gamma\}, \quad I^{-\gamma} := \{n \in \mathbb{N} : f(s_n) = -\gamma\}.$$

It holds that $\mathbb{N} = \cup_{i=1}^{k} I_i \cup I^\gamma \cup I^{-\gamma}$, so at least one of the sets on the right-hand side must be infinite. Let I be such an infinite set on the right-hand side and write $I = \{n_1, n_2, \ldots\}$ with $n_j < n_{j+1}$ for all $j \in \mathbb{N}$, so that (s_{n_j}) is a subsequence of (s_n).

If $I = I_i$ for some $i \in \{1, \ldots, k\}$, then

$$|f(s) - f(p_i)| = \lim_{j \to \infty} |f(s_{n_j}) - f(p_i)| = \lim_{j \to \infty} (1 - \gamma) d(s_{n_j}, p_i) = (1 - \gamma) d(s, p_i),$$

using that $f(s_{n_j}) \to f(s)$ and $d(s_{n_j}, s) \to 0$. By (33), it now follows that $s \in M_{p_i}$ (take $n = 2$, $x_1 = p_i$, $x_2 = s$ in the definition (35), note that $s \in M_{p_i}$ trivially holds when $s = p_i$). Moreover, $s_{n_j} \in M_{p_i}$ by (33), so $p_i \sim s$ and $p_i \sim s_{n_j}$ for all $j \in \mathbb{N}$, implying $s_{n_j} \sim s$. Also, $s_{n_j} \in M_x$ for all $j \in \mathbb{N}$, so $x \sim s_{n_j}$. Consequently, $x \sim s$, i.e. $s \in M_x$.

The other cases are $I = I^{\pm\gamma}$. In these cases, we have $f(s_{n_j}) = \pm\gamma$ for all $j \in \mathbb{N}$, so $f(s) = \lim_{j \to \infty} f(s_{n_j}) = \pm\gamma = f(s_{n_j})$. Therefore, $s, s_{n_j} \in M_f$, and (32) yields $g(s_{n_j}) = g(s) = \pm c\gamma$. Now, trivially we have $g(s_{n_j}) - g(s) = \frac{-c\gamma}{1-\gamma}(f(s_{n_j}) - f(s))$ as both sides are zero, and it follows that $s_{n_j} \in M_s$. Thus, $s \sim s_{n_j}$ and $x \sim s_{n_j}$ for all $j \in \mathbb{N}$, implying $x \sim s$, i.e. $s \in M_x$. We conclude that M_x is closed.

Next, observe that

$$S = \cup_{i=1}^{k} M_{p_i} \cup M_{x_+} \cup M_{x_-}. \tag{36}$$

For any $s \in S \setminus M_f$, (33) holds for some $p_s \in \{p_1, \ldots, p_k\} = P_f$, hence $s \in M_{p_s}$. For $s \in M_f$ we have $f(s) = \pm\gamma = f(x_\pm)$ and $g(s) = \pm c\gamma = g(x_\pm)$, so taking $n = 2, x_1 = x_\pm, x_2 = s$ in the definition of $M_{x_\pm}$, we see that $s \in M_{x_\pm}$. So indeed, (36) holds.

Our goal was to show that $x_+ \in M_{x_-}$. Suppose that the latter is not true. Then $M_{x_+} \cap M_{x_-} = \emptyset$. Let $J := \{i \in 1, \ldots, k\} : p_i \notin M_{x_+}\}$ and let $M := \cup_{i \in J} M_{p_i} \cup M_{x_-}$. Then M is a finite union of closed sets, hence closed. Moreover, $M \cap M_{x_+} = \emptyset$ and $x_- \in M_{x_-} \subset M$, so $M \neq \emptyset$. Also, M_{x_+} is closed, $x_+ \in M_{x_+}$ and $S = M \dot\cup M_{x_+}$, contradicting the connectedness of S. We conclude that $x_+ \in M_{x_-}$, justifying (34). $\square$

The following result is essential in establishing the weak*-density of $E_\bullet^S$ in $\mathrm{ext}(B_\bullet^S)$, ($\bullet = $ FM or BL).

Proposition 5.3. *Let (S, d) be a metric space and $\bullet = $ FM or BL. For every $\mu \in \mathcal{M}ol(S)$, there exists $f_\bullet^\mu \in E_\bullet^S$ such that $\|\mu\|_\bullet^* = \langle \mu, f_\bullet^\mu \rangle$.*

Proof. Let $\mu \in \mathcal{M}ol(S)$ and put $P := \mathrm{supp}(\mu) = \{s_1, \ldots, s_n\}$ with all s_i distinct. So, $\mu = \sum_{j=1}^{n} a_j \delta_{s_j}$, with $0 \neq a_j \in \mathbb{R}$. Let P inherit the metric d from S. View μ as an element of $\mathrm{BL}(S)^*$, or as an element $\mu|_P$ of $\mathrm{BL}(P)^*$, given by the same expression.

For any $f \in \mathrm{BL}(S)$ the restriction $f|_P \in \mathrm{BL}(P)$ and $\|f|_P\|_\infty \leq \|f\|_\infty$, $|f|_P|_L \leq |f|_L$. So, $\|f|_P\|_\bullet \leq \|f\|_\bullet$ and consequently, $\{f|_P : P \in B_\bullet^S\} \subset B_\bullet^P$. Moreover, $\langle \mu|_P, f|_P \rangle = \langle \mu, f \rangle$. Hence,

$$\|\mu\|_\bullet^* = \sup\{\langle \mu|_P, f|_P \rangle : f \in B_\bullet^S\} \leq \|\mu|_P\|_\bullet^* \tag{37}$$

Since (P, d) is a finite metric space, it is Polish and Proposition 2.2 yields the existence of $f^* \in \mathrm{ext}(B_\bullet^P)$ such that $\|\mu|_P\|_\bullet^* = \langle \mu|_P, f^* \rangle$.

If f^* is non-trivial, then $F^* := \mathcal{E}_P^S(f^*) \in E_\bullet^S$ is such that $\langle \mu, F^* \rangle = \langle \mu|_P, F^*|_P \rangle = \langle \mu|_P, f^* \rangle = \|\mu|_P\|_\bullet^*$. Since $\|\mu\|_\bullet^* \geq \langle \mu, F^* \rangle$, equation (37) yields that $\langle \mu, F^* \rangle = \|\mu\|_\bullet^*$, so we may take $f_\bullet^\mu := F^*$.

If f^* is trivial, i.e. $|f^*| = 1$, then in case $\bullet = $ BL, $f^* = \pm\mathbb{1}$ and one can take $f_{\mathrm{BL}}^\mu := \pm\mathbb{1} \in E_{\mathrm{BL}}^S$. The case $\bullet = $ FM is slightly more involved. Consider $P^+ := \{p \in P : f^*(p) = 1\}$ and $h := h_{P^+} \in \mathrm{ext}(B_{\mathrm{FM}}^S)$ as defined in Lemma C.1 as function on S. Since for any $p \in P \setminus P^+$, $f^*(p) = -1$ and $|f^*|_L \leq 1$, one has for any $p \in P \setminus P^+$ and $p' \in P^+$

$$\frac{|f^*(p) - f^*(p')|}{d(p, p')} = \frac{2}{d(p, p')} \leq 1.$$

So $d(p, P^+) \geq 2$. Lemma C.1, applied to extension from P^+ to P, shows that $h|_P = f^*$. h is included in E_{FM}^S by construction. $\quad\square$

We can now prove the main result and its immediate consequences.

Theorem 5.1. *Let (S, d) be a metric space and $\bullet = \mathrm{FM}$ or BL. Then $E_\bullet^S$ is a weak*-dense subset of $\mathrm{ext}(B_\bullet^S)$. If, additionally, S is compact, then $E_\bullet^S$ is $\|\cdot\|_\infty$-dense in $\mathrm{ext}(B_\bullet^S)$.*

Proof. We shall employ Lemma 2.2 and Theorem 2.1 to obtain the desired result. First we apply Lemma 2.2 to the situation where $X = \overline{\mathcal{M}ol}(S)_\bullet$ and $F = \mathcal{M}ol(S)$, which is dense in X for the $\|\cdot\|_\bullet^*$-norm. Proposition 2.1 yield that $X^* = \mathrm{BL}(S)$. Proposition 5.3 yields that $E_\bullet^S$ is determining the $\|\cdot\|_\bullet^*$-norm on F. Lemma 2.2 allows to conclude that $B_\bullet^S$ is the weak*-closed convex hull of $E_\bullet^S$.

Moreover, by the Banach-Alaoglu Theorem (cf. [9]), $B_\bullet^S$ is compact for the weak*-topology. Apply Theorem 2.1 to the locally convex space $\mathrm{BL}(S)$, equipped with the weak*-topology. Statement (i) has been proven above. So, (ii) holds. That is, $E_\bullet^S$ is weak*-dense in $\mathrm{ext}(B_\bullet^S)$.

Assume now that S is compact. Let $f \in \mathrm{ext}(B_\bullet^S)$. By the first part, there exists a net $(f_\alpha)_{\alpha \in A}$ in $E_\bullet^S$ such that $f_\alpha \to f$ weak*. Thus, $f_\alpha(x) \to f(x)$ for every $x \in S$, by applying $\delta_x \in \mathcal{M}ol(S)$. Since S is metric and compact, it is totally bounded. Let $\varepsilon > 0$. Then there exist $x_1, \ldots x_n$ such that $S = \bigcup_{i=1}^n B(x_i, \varepsilon/3)$. For each i one can find $\alpha_i \in A$ such that for all $\alpha \succeq \alpha_i$, $|f_\alpha(x_i) - f(x)| < \varepsilon/3$. Then there also exist $\alpha_0 \in A$ such that the same inequality holds for all $1 \leq i \leq n$ and all $a \succeq \alpha_0$. Pick $x \in S$. There exists x_i such that $x \in B(x_i, \varepsilon/3)$. Then for all $\alpha \succeq \alpha_0$,

$$|f_\alpha(x) - f(x)| \leq |f_\alpha(x) - f_\alpha(x_i)| + |f_\alpha(x_i) - f(x_i)| + |f(x_i) - f(x)|$$

$$\leq |f_\alpha|_L d(x, x_i) + \frac{1}{3}\varepsilon + |f|_L d(x, x_i) < \varepsilon.$$

We conclude that $\|f_\alpha - f\|_\infty \to 0$. $\quad\square$

In view of the arguments at the start of this section, we immediately obtain from the K^2-M^3-R Theorem that the sets $E_\bullet^S$ are norming on $\mathcal{M}_s(S)$:

Corollary 5.1. *Let $\mu \in \mathcal{M}_s(S)$. Then $\|\mu\|_\bullet^* = \sup\{\langle \mu, f \rangle : f \in E_\bullet^S\}$ ($\bullet = \mathrm{FM}, \mathrm{BL}$).*

Moreover, we obtain weak*-density of the large set J_{BL}^S in the extreme points, provided S is connected.

Corollary 5.2. *If (S, d) is a connected metric space, then the set $J_\bullet^S$ consists of extreme points $J_\bullet^S$ of $B_\bullet^S$ and is weak*-dense in $\mathrm{ext}(B_\bullet^S)$ ($\bullet = \mathrm{FM}, \mathrm{BL}$).*

Proof. Combine Proposition 5.2 and Theorem 5.1. $\quad\square$

6. Considerations for the case of a finite space

In this section we take a closer look at the interpretation of the results of the previous sections for the case when S is a metric space with finitely many points. This seems a highly specific setting, but it is relevant for multiple reasons. Firstly, for a general metric space the relevant norming and dense set $E_\bullet^S$ is obtained mostly by applying the Lipschitz extension operator to (non-trivial) extreme points of the unit ball of functions on a finite metric space P. It is then a natural question to see to what extent the latter are determined by the

extreme points of a ball for a space $P' \subset P$. Secondly, highly relevant in view of the question of computing FM- and BL-norms of measures, the norm of a molecular measure μ with support $\mathrm{supp}(\mu) = P$, which is finite, is determined by the integrals against μ of the functions in the finite set $\mathrm{ext}(B_\bullet^P)$ (see [14], Section 6). Below we limit attention to the relatively novel case of the BL-norm. Although S is finite, we shall keep talking about functions on S.

Theorem 5.1 provides density of

$$E_{\mathrm{BL}}^S = \bigcup_{P \subset S, \text{ finite}} \mathcal{E}_P^S\big(\mathrm{ext}_*(B_{\mathrm{BL}}^P)\big) \cup \{f \in B_{\mathrm{BL}}^S : |f| = 1\} \tag{38}$$

in $\mathrm{ext}(B_{\mathrm{BL}}^S)$. When S is finite, we can take $P = S$ on the right-hand side and (38) becomes trivial (knowing Theorem 4.2). Still, let us study (38) a bit further. One may wonder, whether it is possible to reach any non-trivial extremal function by applying the extension operator to a suitable non-trivial extremal function defined on a *proper* subset P of S.

One can quickly see that this is not the case. Since $B_\bullet^S$ is symmetric around 0, also $\mathrm{ext}(B_\bullet^S)$ is symmetric around 0 (cf. Corollary 2.2). If $f = \mathcal{E}_P^S(F)$ for some $F \in \mathrm{ext}_*(B_\bullet^P)$, then $-f$ is not of this form, generally. This motivates to study yet another extension operator:

$$\bar{\mathcal{E}}_P^S(f) := -\mathcal{E}_P^S(-f) = \min\left(\inf_{p \in P}[f(p) + |f|_L d(p, \cdot)], \|f\|_\infty\right).$$

It is easily seen that $\bar{\mathcal{E}}_P^S$ enjoys many of the same properties as $\mathcal{E}_P^S$. In particular, $\bar{\mathcal{E}}_P^S$ maps $\mathrm{BL}(P)$ into $\mathrm{BL}(S)$ and preserves supremum norms and Lipschitz constants.

The properties of $\mathcal{E}_P^S$ expressed in Proposition 4.3 and Theorem 4.2 indicate that for any $P \subset P' \subset S$:

$$\mathcal{E}_{P'}^S \circ \mathcal{E}_P^{P'}\big(\mathrm{ext}_*(B_\bullet^P)\big) = \mathcal{E}_P^S\big(\mathrm{ext}_*(B_\bullet^P) \subset \mathrm{ext}_*(B_\bullet^S). \tag{39}$$

Since $\mathrm{ext}_*(B_\bullet^S) = -\mathrm{ext}^*(B_\bullet^S)$, (39) also holds with $\mathcal{E}_P^S$ replaced by $\bar{\mathcal{E}}_P^S$. Thus for any finite S containing at least two points we have

$$\bigcup_{x \in S} \mathcal{E}_{S \setminus \{x\}}^S\big(\mathrm{ext}_*(B_{\mathrm{BL}}^{S \setminus \{x\}})\big) \cup \bar{\mathcal{E}}_{S \setminus \{x\}}^S\big(\mathrm{ext}_*(B_{\mathrm{BL}}^{S \setminus \{x\}})\big) \subset \mathrm{ext}_*(B_{\mathrm{BL}}^S). \tag{40}$$

A natural question is then whether equality holds in (40).

For the moment, assume that indeed we have equality in (40) and suppose that $n := |S| \in \mathbb{N}_{\geq 3}$. By induction, we would then have

$$\mathrm{ext}_*(B_{\mathrm{BL}}^S) = \bigcup_{\substack{x,y \in S, \\ x \neq y}} \bigcup \{T_n \circ \ldots \circ T_3(\mathrm{ext}_*(B_{\mathrm{BL}}^{\{x,y\}})) : T_i = \mathcal{E}_{P_{i-1}}^{P_i} \text{ or } T_i = \bar{\mathcal{E}}_{P_{i-1}}^{P_i},$$

$$\{x,y\} \subset P_i \subset P_{i+1} \subset S \text{ and } |P_i| = i\}. \tag{41}$$

Now, we could construct all extreme points explicitly, since we 'only' have to take all combinations of extensions of the extreme points for $P = \{x, y\}$, which are explicitly given by

$$\mathrm{ext}_*(B_{\mathrm{BL}}^{\{x,y\}}) = \{(x, y) \mapsto \pm(\frac{d(x,y)}{d(x,y)+2}, \frac{-d(x,y)}{d(x,y)+2})\}. \tag{42}$$

This can be seen easily by identifying $\mathrm{BL}(P) \cong \mathbb{R}^2$ and drawing B_{BL}^P. However, the next example shows that in general the other inclusion in (40) does NOT hold and (41) fails. Thus, one cannot obtain all extreme points via the inductive procedure described. In addition, the example shows that '$P \subset S$' could not have been replaced by '$P \subsetneq S$' in (38).

Example 6.1. Let S be finite with $|S| \geq 3$. If equality in (40), hence (41), were true, then for all $f \in \mathrm{ext}_*(B_{\mathrm{BL}}^S)$ there would exist $x, y \in S$ with $f(x) = \|f\|_\infty, f(y) = -\|f\|_\infty$ and $2\|f\|_\infty = d(x,y)(1 - \|f\|_\infty)$. Indeed, the property holds for $S = P$, $|P| = 2$ by (42) and obviously, it is preserved by $\mathcal{E}_P^S$ and $\bar{\mathcal{E}}_P^S$.

Now let $S := \{0, 1.5, 2.5, 4\} \subset \mathbb{R}$ and identify $f \in \mathrm{BL}(S)$ with $(f(0), f(1.5), f(2.5), f(4)) \in \mathbb{R}^4$. Using the expression for $\mathrm{ext}(B_{\mathrm{BL}}^S)$ in terms of linear constraints in [14], one can (numerically) verify that $f := (0.5, -0.25, 0.25, -0.5)$ is an extreme point of B_{BL}^S. However, $2\|f\|_\infty = 1 \neq 2 = d(0,4)(1 - 0.5)$, so it does not satisfy the aforementioned property.

We conclude that equality in (40) does not hold and (41) is false in general.

Now we turn our attention to the 'Johnson-like' set J_{BL}^S. For finite S, the property that defines J_{BL}^S holds for all extreme points, not only for those in E_{BL}^S (cf. Proposition 5.1):

Lemma 6.1. *Let S be finite, $f \in \mathrm{ext}(B_{\mathrm{BL}}^S)$ and $x \in S \setminus M_f$. Then there exists $y \in S \setminus \{x\}$ such that $|f(x) - f(y)| = |f|_L d(x,y)$.*

Proof. Suppose the claim is false. Then we have $|f(x) - f(y)| < |f|_L d(x,y)$ for all $y \in S \setminus \{x\}$. Also, $|f(x)| < \|f\|_\infty$, so

$$\alpha := (\|f\|_\infty - |f(x)|) \wedge \min_{y \in S \setminus \{x\}} [|f|_L d(x,y) - |f(x) - f(y)|] > 0.$$

Put $g := \mathbb{1}_{\{x\}} \alpha \in \mathrm{BL}(S)$ (since S is finite). We show that $\|f \pm g\|_{\mathrm{BL}} \leq 1$, contradicting the fact that $f \in \mathrm{ext}(B_{\mathrm{BL}}^S)$. We have

$$\|f \pm g\|_\infty = \max\{\|f\|_\infty, |f(x) \pm \alpha|\} \leq \max\{\|f\|_\infty, |f(x)| + \alpha\}$$
$$\leq \max\{\|f\|_\infty, |f(x)| + \|f\|_\infty - |f(x)|\}$$
$$= \|f\|_\infty.$$

Furthermore,

$$|f \pm g|_L = |f|_L \vee \max_{y \in S \setminus \{x\}} \frac{|(f \pm g)(x) - (f \pm g)(y)|}{d(x,y)}$$
$$= |f|_L \vee \max_{y \in S \setminus \{x\}} \frac{|f(x) - f(y) \pm \alpha|}{d(x,y)}$$
$$\leq |f|_L \vee \max_{y \in S \setminus \{x\}} \frac{|f(x) - f(y)| + |f|_L d(x,y) - |f(x) - f(y)|}{d(x,y)}$$
$$= |f|_L.$$

Thus $\|f \pm g\|_{\mathrm{BL}} \leq \|f\|_\infty + |f|_L = 1$, finishing the proof. $\square$

If S is a finite metric space then it is disconnected, so the proof of $J_{\mathrm{BL}}^S \subset \mathrm{ext}(B_{\mathrm{BL}}^S)$ does not apply. Also, the definition of J_{BL}^S now boils down to $J_{\mathrm{BL}}^S = \{f \in B_B^S : \|f\|_{\mathrm{BL}} = 1, \ f(M_f) = \{\|f\|_\infty, -\|f\|_\infty\}\}$, since one can take $P_f = S$ for the finite set P_f in the definition. However, Lemma 6.1 makes one wonder whether there is a variation of the defining property of J_{BL}^S that does characterize $\mathrm{ext}(B_{\mathrm{BL}}^S)$. E.g. is the set

$$\{f \in \mathrm{BL}(S) : \|f\|_{\mathrm{BL}} = 1, f(M_f) = \{\pm \|f\|_\infty\}, \forall s \in M_f^c \ \exists s \in S \setminus \{s\}$$
$$\text{s.t. } |f(p) - f(s)| = (1 - \|f\|_\infty)d(p,s)\} \tag{43}$$

contained in $\text{ext}(B_{\text{BL}}^S)$? Then we would have equality by Lemma 6.1, Lemma 3.3 and Corollary 2.1, i.e. a full characterization of $\text{ext}(B_{\text{BL}}^S)$. Unfortunately, the next example proves the contrary.

Example 6.2. Let $S := \{0, 1.5, 2, 4\}$ and identify $f \in \text{BL}(S)$ with $(f(0), f(1.5), f(2), f(4)) \in \mathbb{R}^4$. Then $f := (0.5, -0.25, 0, -0.5)$ is an element of the set (43). However, one can (numerically) verify that it is not an extreme point of B_{BL}^S (e.g. as in Example 6.1 using [14]).

We conclude that for finite S, the properties from Lemma 6.1, Lemma 3.3 and Corollary 2.1 are necessary conditions to be in $\text{ext}(B_{\text{BL}}^S)$, but not sufficient conditions.

Appendix A. Proof of metric Tietze extension theorem

We could not find a reference for the Metric Tietze Extension Theorem in the literature that provides a full proof. See [34], Theorem 2.5.6 though, which does not provide full details. The result is fundamental for our results. Therefore, we include a full detailed proof here.

Proof. *(Metric Tietze Extension Theorem, Theorem 4.1).*
Let $F_0 = \mathcal{E}_P^{S,0}(f)$ and $F = \mathcal{E}_P^S(f)$ be as defined in (15) and (16). First, we show that $F|_P = F_0|_P = f$. Let $p_0 \in P$. For all $p \in P$, we have

$$f(p) - |f|_L d(p, p_0) \le f(p) - (f(p) - f(p_0)) = f(p_0) = f(p_0) - |f|_L d(p_0, p_0),$$

thus $F_0(p_0) = f(p_0)$. Moreover, $f(p_0) \ge -\|f\|_\infty$ yields $F(p_0) = F_0(p_0) = f(p_0)$, proving that $F|_P = F_0|_P = f$.

As a consequence, we have $\|F\|_\infty \ge \|f\|_\infty$. For the other inequality, note that $f(p) - |f|_L d(p, x) \le \|f\|_\infty$ for any $x \in S$ and $p \in P$, so $F_0 \le \|f\|_\infty$. From the definition of F, it follows that $-\|f\|_\infty \le F \le \|f\|_\infty$, i.e. $\|F\|_\infty \le \|f\|_\infty$. Hence, $\|F\|_\infty = \|f\|_\infty$.

We have $|F|_L \ge |F|_P|_L = |f|_L$, provided that F is Lipschitz continuous. It remains to show that the latter holds and $|F|_L \le |f|_L$. Let $x, y \in S$ and $p \in P$. Applying the triangle inequality twice yields

$$f(p) - |f|_L d(p, y) - |f|_L d(x, y) \le f(p) - |f|_L d(p, x) \le f(p) - |f|_L d(p, y) + |f|_L d(x, y).$$

Taking the supremum over $p \in P$, we obtain

$$F_0(y) - |f|_L d(x, y) \le F_0(x) \le F_0(y) + |f|_L d(x, y).$$

Therefore, $|F_0(x) - F_0(y)| \le |f|_L d(x, y)$ and F_0 is Lipschitz continuous. It follows that $F = \max(F_0, -\|f\|_\infty)$ is Lipschitz continuous with $|F|_L \le |F_0|_L \le |f|_L$ ([11], Lemma 4). We conclude that $|F|_L = |f|_L$. $\square$

Appendix B. Isomorphism between $(\text{BL}(S), \|\cdot\|_{\text{FM}})$ and the metric dual space

Farmer [13] considered the Lipschitz dual $S^\#$ of the metric space (S, d), which was introduced by Lindenstrauss [22]. It consists of the vector space $\text{Lip}_0(S)$ of all functions in $\text{Lip}(S)$ that vanish at a fixed chosen 'distinguished point' $e \in S$, equipped with $|\cdot|_L$ as norm. We exhibit here a sequence of isomorphisms that identify $(\text{BL}(S, d), \|\cdot\|_{\text{FM},d})$ with such a space $(\text{Lip}_0(S^+, d^+), |\cdot|_{L,d^+})$, linearly isometrically, order theoretically and algebraically. These isomorphisms can be found in [34], Section 1.7.

Let (S, d) be a metric space. Define

$$d'(x, y) := d(x, y) \wedge 2.$$

Lemma B.1. *The identity map is a linear, isometric, lattice and algebraic isomorphism between* $(\mathrm{BL}(S,d), \|\cdot\|_{\mathrm{FM},d})$ *and* $(\mathrm{BL}(S,d'), \|\cdot\|_{\mathrm{FM},d'})$.

Proof. We leave the elementary check to the reader. See e.g. [34], Proposition 1.7.1. $\quad\square$

For a metric space (S,d') of diameter at most two, i.e.

$$\mathrm{diam}(S,d') := \sup\{d'(x,y) : , x,y \in S\} \le 2,$$

like the space in Lemma B.1, one can add a remote distinguished point e, not in S, to obtain $S^+ := S \cup \{e\}$ with metric

$$d^+(x,y) := \begin{cases} d(x,y), & \text{if } x,y \in S, \\ 1, & \text{if } x = e, \text{ or } y = e, \ x \ne y, \\ 0, & \text{if } x = e = y. \end{cases}$$

$Lip_0(S^+, d^+)$ is then the space of Lipschitz functions on S^+ for d^+ that vanish at e.

Lemma B.2. *If* $\mathrm{diam}(S,d') \le 2$, *then* $(\mathrm{BL}(S), \|\cdot\|_{\mathrm{FM},d'})$ *is linearly isometrically, order theoretically and algebraically isomorphic to* $(\mathrm{Lip}_0(S^+, d^+), |\cdot|_{L,d^+})$.

Proof. The isomorphism is given by extending $f \in \mathrm{BL}(S)$ by 0 at e. See [34], Theorem 1.7.2. $\quad\square$

The two results show that the set B_{FM}^S remains the same set of functions in its image in $\mathrm{Lip}_0(S^+, d^+)$, except that each function has been trivially extended to the added distinguished point e. This unit ball becomes the unit ball in $\mathrm{Lip}_0(S^+, d^+)$. Its convex structure does not change, so extreme points correspond.

Appendix C. An alternative proof and auxiliary result for the FM-norm

Through the sequence of isomorphisms provided above in Appendix B, the case of the FM-norm reduces essentially to already established results on the Lipschitz dual space, e.g. those by Farmer [13]. In his proofs he uses the characterisation of extreme points, [13], Theorem 1. An equivalent of this is not available for the BL-norm. However, the equivalent of Theorem 4.2, which is central in the construction of the small sets of extreme points $E_\bullet^S$, does hold for the FM-norm, see [13], Lemma 2. Below we shall present a proof of this result, without using Farmer's characterisation theorem, but instead follow the line of reasoning in the proof of Theorem 4.2. It requires only minor modifications.

Recall that $\mathrm{ext}_*(B_\bullet^P) = \emptyset$ if P is a singleton.

Theorem C.1. *Let P be a subset of S with at least two elements, equipped with the restriction of d as metric. Then, $\mathcal{E}_P^S$ maps $\mathrm{ext}_*(B_{\mathrm{FM}}^P)$ into $\mathrm{ext}_*(B_{\mathrm{FM}}^S)$.*

Proof. A similar argument as at the start of the proof of Theorem 4.2 shows that without loss of generality we may assume P to be closed.

Theorem 4.1 yields that $\mathcal{E}_P^S$ maps B_{FM}^P isometrically into B_{FM}^S. Let $f \in \mathrm{ext}_*(B_{\mathrm{FM}}^P)$ and put $F := \mathcal{E}_P^S(f)$. According to Theorem 4.1 and Lemma 3.2 one has $\|F\|_\infty = \|f\|_\infty = 1$ and $|F|_L = |f|_L = 1$. Since $|f| \ne \mathbb{1}$ on P and $F|_P = f$, $|F| \ne \mathbb{1}$ on S. Therefore, it remains to show that F is extreme. It suffices to show that if $F + G$ and $F - G$ are both in B_{FM}^S for some $G \in \mathrm{BL}(S)$, then $G = 0$ (cf. Lemma 2.1).

First, $(F \pm G)|_P = f \pm G|_P \in B_{\mathrm{FM}}^P$, with $G|_P \in \mathrm{BL}(P)$. f is an extreme point of B_{FM}^P, so $G|_P = 0$. Define $M_F^- := \{x \in S : F(x) = -1\}$. For $H \in B_{\mathrm{FM}}^S$, define

$$\hat{H}(s,p) := \frac{H(p) - H(s)}{d(s,p)}, \qquad s \in S \setminus P, \ p \in P. \tag{44}$$

Then $|\hat{H}(s,p)| \leq 1$. Let $x \in (M_F^- \cup P)^c$. There exist $p_n \in P$, $n \in \mathbb{N}$, such that

$$f(p_n) - d(x,p_n) \uparrow F_0(x) = F(x) \quad \text{as } n \to \infty. \tag{45}$$

Because f is bounded and (45) holds, the sequence $(d(x,p_n))_n$ must be bounded. Since P is assumed to be closed and $x \notin P$, $\sup_n d(x,p_n) \geq \inf_n d(x,p_n) > 0$. Let (p_{n_k}) be a subsequence such that $(d(x,p_{n_k}))$ converges (necessarily to a non-zero limit). Then, (45) yields

$$\hat{F}(x,p_{n_k}) \to 1 \quad \text{as } k \to \infty.$$

Since both $F + G$ and $F - G$ are in B_{FM}^S, we get

$$1 \geq \|F \pm G\|_{\mathrm{FM}} \geq |F \pm G|_L \geq |\hat{F}(x,p) \pm \hat{G}(x,p)| \qquad \text{for all } p \in P,$$

which implies $\left|\hat{F}(x,p) + |\hat{G}(x,p)|\right| \leq 1$ for all p. Since $G|_P = 0$, we find that for sufficiently large k,

$$\frac{|G(x)|}{d(x,p_{n_k})} = |\hat{G}(x,p_{n_k})| \leq 1 - \hat{F}(x,p_{n_k}) \to 0 \quad \text{as } k \to \infty.$$

Because the sequence $(d(x,p_{n_k}))_k$ has non-zero limit, $G(x) = 0$. If $x \in M_f^-$, then $F(x) = -1$ and

$$1 \geq \|F \pm G\|_{\mathrm{FM}} \geq \|F \pm G\|_\infty \geq |F(x) \pm G(x)| = |1 \mp G(x)|.$$

We conclude that $G(x) = 0$ in this case too. So $G = 0$ and f is an extreme point. $\quad\square$

Recall the definition (28) of the functions h_P. It was remarked that these cannot be reached by Lipschitz extension operators. The following lemma summarizes fundamental properties of these functions h_P.

Lemma C.1. *Let P be a closed, proper and non-empty subset of S. Then $h_P \in \mathrm{ext}(B_{\mathrm{FM}}^S)$. If there exists $x \in S \setminus P$ such that $d(x,P) < 2$, then $h_P \in \mathrm{ext}_*(B_{\mathrm{FM}}^S)$. Otherwise, h_P is a trivial extreme point.*

Proof. By construction, $h_P \in B_{\mathrm{FM}}^S$ and $h_P(p) = 1$ for all $p \in P$. Let $g \in \mathrm{BL}(S)$ be such that $h_P + g$ and $h_P - g$ are both in B_{FM}^S. Then $g(p) = 0$ for all $p \in P$. Put $M_h^- := \{x \in S : h_P(x) = -1\}$. Then $g = 0$ on M_h^- too, since both $h_P(x) + g(x) \geq -1$ and $h_P(x) - g(x) \geq -1$ for $x \in M_h^-$. If $S \setminus (M_h^- \cup P) = \emptyset$, then $|h_P| = \mathbb{1}$, so h_P is a trivial extreme point of B_{FM}^S.

Suppose $S \setminus (M_h^- \cup P) \neq \emptyset$ and pick $x \in (M_h^- \cup P)^c$. Since P is closed, $d(x,P) > 0$. Moreover, there exist $p_n \in P$, $n \in \mathbb{N}$, such that $h_P(x) = \lim_{n \to \infty} 1 - d(x,p_n)$. Since for any $p \in P$ one has $g(p) = 0$ and $h_P(p) = 1$, and $h_P \pm g \in B_{\mathrm{FM}}^S$,

$$\frac{|h_P(x) \pm g(x) - 1|}{d(x,p)} \leq |h_P \pm g|_L \leq 1 \qquad \text{for all } p \in P. \tag{46}$$

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $|h_P(x) - 1 + d(x,p_n)| < \varepsilon$ for all $n \geq N$. From (46) one derives that

$$\left|d(x,p_n) \pm g(x)\right| \leq \left|h_P(x) - 1 + d(x,p_n)\right| + \left|h_P(x) \mp g(x) - 1\right| < \varepsilon + d(x,p)$$

for all $n \geq N$ and $p \in P$. Taking $p = p_n$, we obtain $|g(x)| < \varepsilon$. Since $\varepsilon > 0$ was taken arbitrarily, $g(x) = 0$. Hence $g = 0$ on S and $h_P \in \text{ext}(B_{\text{FM}}^S)$, according to Lemma 2.1.

Finally, assume that there exists $x_0 \in S \setminus P$ such that $d(x_0, P) < 2$. Then $-1 < h_P(x_0) < 1$, so h_P cannot be a trivial extreme point. If for every $x \in S \setminus P$, $d(x, p) \geq 2$ for all $p \in P$. Then $h_P(x) = -1$ for $x \in S \setminus P$, while $h_P(p) = 1$ for every $p \in P$. Hence, h_P is a trivial extreme point. □

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