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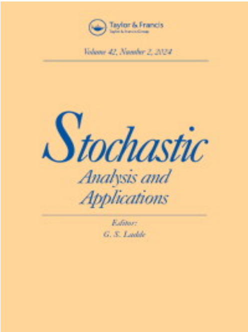

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ABSTRACT

The transition kernel of an $\mathbb{R}^n$-valued diffusion or jump diffusion process $\{X_t\}$ is known to satisfy the Feller property if $\{X_t\}$ is the solution of an SDE whose coefficients are Lipschitz continuous. This Lipschitz route to Feller falls short if $\{X_t\}$ is the solution of an SDE whose coefficients depend on a state-dependent regime-switching process $\{\theta_t\}$. In this paper it is shown that pathwise uniqueness and the Feller property are satisfied under mild conditions for a regime-switching jump diffusion process $\{X_t, \theta_t\}$ with hybrid jumps, i.e. jumps in $\{X_t\}$ that occur simultaneously with $\{\theta_t\}$ switching.

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1. Introduction

Continuous-time Markov processes with hybrid state space have proven their great value in modeling complex stochastic systems. Applications range from finance [1] to air traffic [2, 3], and to biology [4]. Lygeros and Prandini [5] review literature on processes that involve an $\mathbb{R}^n$-valued component $\{X_t\}$ and a discrete-valued component $\{\theta_t\}$, and where both components may interact with each other, i.e. $\{\theta_t\}$ influences the dynamics of $\{X_t\}$ and vice versa. Foundations for the study of hybrid state Markov processes with a two-sided interaction between the dynamics of $\{X_t\}$and $\{\theta_t\}$ have been laid by Davis [6, 7] for Piecewise Deterministic Markov Processes (PDMP) and by Skorohod [8] and Ghosh et al. [9] for state-dependent regime-switching diffusions. In contrast to the latter, PDMP involves hybrid jumps, i.e. a jump in $\{X_t\}$ that occurs simultaneously with $\{\theta_t\}$ switching. Processes that involve Brownian motion and hybrid jumps have been studied from a Markov automaton perspective [10–15] and as pathwise unique solutions of Itô-Skorohod SDEs on a hybrid state space [16 –18].

A natural condition of a Markov process is that the transition kernel satisfies the *Feller property*, i.e. the transition kernel transforms a continuous function into a continuous function. For diffusion and jump-diffusion processes, the Feller property has shown to be satisfied if the coefficients of the corresponding SDEs satisfy Lipschitz conditions [19–21]. However, this Lipschitz route to the Feller property does not exist for the Markov transition

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Table 1. Conditions imposed by [22–27] on key elements in (1.1)-(1.2) for the Feller property.

Source	$\mathbb{M}$	$\Pi_{ij}(x)$	V	$g_1(x,.,.)$	U	$c(x,.,.)$	$g_2(x,.,.)$
[22]	Finite	Continuous	-	$=0$	$\mathbb{R}$	Measurable	$=0$
[23]	Countable	Lipschitz	-	$=0$	$\mathbb{R}$	Measurable	$=0$
[24]	Countable	Continuous	-	$=0$	$\mathbb{R}$	Measurable	$=0$
[25], [26]	Countable	Lipschitz	$\mathbb{R}$	Locally Lipschitz	$\mathbb{R}$	Measurable	$=0$
[27]	Finite	Lipschitz	$\mathbb{R}$	Locally Lipschitz	$\mathbb{R}$	Measurable	Bounded

kernel of a regime-switching diffusion, let alone for a regime-switching jump-diffusion that involves hybrid jumps. To improve the situation, this paper studies the Feller property of the transition kernel of a regime-switching jump-diffusion process $\{X_t, \theta_t\}$, which is the solution of the pair of Itô-Skorohod SDEs of [18]:

$$dX_t = a(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t + \int_V g_1(X_{t-}, \theta_{t-}, v)q_1(dt, dv) + \int_U g_2(X_{t-}, \theta_{t-}, u)p_2(dt, du) \quad (1.1)$$

$$d\theta_t = \int_U c(X_{t-}, \theta_{t-}, u)p_2(dt, du) \quad (1.2)$$

with $\{W_t\}$ standard Brownian motion, $q_1(dt, dv)$ a martingale random measure with intensity measure $\mu_1(dv)$ on mark space V, and $p_2(dt, du)$ a Poisson random measure with intensity measure $\mu_2(du)$ on mark space U. $\{W_t\}, q_1(dt, dv)$ and $p_2(dt, du)$ are independent of each other and of (X_0, θ_0). The coefficient $c(x, \theta, u)$ in (1.2) is defined in such a way that $\{\theta_t\}$assumes values in $\mathbb{M} = \{e_1, .., e_N\}$, according to X_t-conditional transition rates $\Pi_{ij}(X_t)$:

$$P\{\theta_{t+\Delta} = e_j \,| \theta_t = e_i, X_t = x, (X_s, \theta_s; s < t)\} = \Pi_{ij}(x)\Delta + o(\Delta) \quad (1.3)$$

with $\lim_{\Delta\to 0} o(\Delta)/\,|\Delta| = 0$.

For specific versions of (1.1)-(1.2), the Feller property of the corresponding transition kernel has been shown by [22–27]; an overview is given in Table 1. Yin and Zhu [22] study the Feller property if $g_1(.) = g_2(.) = 0$, which simplifies (1.1) to the regime-switching diffusion:

$$dX_t = a(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t \quad (1.4)$$

Yin and Zhu [22, Lemma 2.14] first remind that the Feller property is satisfied, if $\Pi_{ij}(x)$ is x-invariant. Subsequently, Yin and Zhu [22, Theorem 2.18] develop a novel approach in proving that the Feller property is satisfied if $\Pi_{ij}(x)$ is continuous in x. Shao [23] and Zhang [24] extend the results of [22] to situations of countable $\mathbb{M} = \{1, 2, ...\}$. Xi et al. [25] and Kunwai and Zhu [26] extend the Feller property results of [22, 23] to regime-switching jump-diffusions of type (1.1)-(1.2) with locally Lipschitz $g_1(x, .)$. [22–26] have $g_2(.) = 0$ in common, i.e. there are no hybrid jumps.

In unpublished report [27], Feller property is proven in case of hybrid jumps and finite M under the conditions that $\Pi_{ij}(x)$ is Lipschitz and $g_2(.,.,.)$ is bounded. This proof makes use of a stochastic continuity result by Gihman and Skorohod [28] for a non-switching jump-diffusion solution of (1.1) for $|\mathbb{M}| = 0$.

The objective of this paper is to relax the Lipschitz and bound conditions that [27] imposes on $\Pi_{ij}(x)$ and $g_2(x, ., .)$ to continuity and linear growth conditions respectively. To make this feasible, section 2 introduces a partition of the mark space U, that differs from the common partition [9, 16–18, 22–27]. In addition, the following generalizations of the mark space U and the mapping $g_2(x,.,.)$ are adopted. Firstly, a multi-dimensional mark space is assumed, i.e. $U = \mathbb{R} \times \mathbb{R}^d$. Secondly, the measure μ_2 is assumed to be a product of a Lebesque measure on $\mathbb{R}$ and probability measure on $\mathbb{R}^d$. Thirdly, while only one mark component of u influences coefficient $c(x,\theta,u)$ in (1.2), all mark components of u influence coefficient $g_2(x,\theta,u)$ in (1.1).

The novel mark space partition and the generalization of U and $g_2(x,.,.)$ imply the need of a novel proof of the existence of a pathwise unique solution of (1.1–1.2), as well as a novel route in using the stochastic continuity result of [28] in the derivation of the Feller property for the corresponding transition kernel.

This paper is organized as follows. Section 2 starts presenting the novel partitioning and extension of the mark space U and definitions of $g_2(x,\theta,u)$ and $c(x,e,u)$. Subsequently, Section 2 develops relevant characterizations and an illustrative hybrid jump example. Section 3 starts with the background of pathwise uniqueness of a jump-diffusion solution of a non-switching Itô-Skorohod SDE of type (1.1) if $|\mathbb{M}| = 0$. Subsequently, Section 3 derives the existence of pathwise unique solutions of Itô-Skorohod SDEs (1.1)-(1.2) under the novel mark space partition and generalization of U and $g_2(x,.,.)$. Finally, Section 3 characterizes the corresponding Markov transition kernel. Section 4 proves that this Markov transition kernel satisfies the Feller property. Section 5 provides a discussion of results.

2. Composition and characterization of $p_2(dt,du)$ generated discontinuities

2.1. Composition

Throughout the paper all processes are defined on a stochastic basis $(\Omega,\mathcal{F},F,P)$, with $(\Omega,\mathcal{F})$ a measurable space that is equipped with a collection $F = \{\mathcal{F}_t, t \in [0,\infty)\}$ of increasing right-continuous sub σ-algebra's $\mathcal{F}_t$ of $\mathcal{F}$; P is a probability measure defined on the σ-algebra $\mathcal{F}$.

Itô-Skorohod SDEs (1.1)-(1.2) evolve on hybrid space $\mathbb{R}^n \times \mathbb{M}$, where $\mathbb{M} = \{e_1, e_2, .., e_N\}$ is a finite set of unit vectors $e_i \in \mathbb{R}^N$, $i \in \{1,..,N\}$. We assume Borel measurable mappings $a : \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^n$, $b : \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^n \times \mathbb{R}^m$, $g_1 : \mathbb{R}^n \times \mathbb{M} \times V \to \mathbb{R}^n$, $g_2 : \mathbb{R}^n \times \mathbb{M} \times U \to \mathbb{R}^n$, $c : \mathbb{R}^n \times \mathbb{M} \times U \to \mathbb{R}^N$, with $U = \mathbb{R} \times \mathbb{R}^d$; and that $\{W_t\}$, $q_1(dt,dv)$, and $p_2(dt,du)$ satisfy (A1)-(A4):

(A1) $\{W_t\}$ is an $m-$dimensional standard Wiener process;
(A2) $q_1(dt,dv)$ is a martingale random measure on a Blackwell space $\{V,\mathcal{V}\}$, associated to a homogeneous Poisson random measure $p_1(dt,dv)$ with intensity $dt \times \mu_1(dv)$, where $\mu_1(dv)$ is a positive σ-finite measure on $\{V,\mathcal{V}\}$;
(A3) $p_2(dt,du)$ is a homogeneous Poisson random measure on a Blackwell space $\{U,\mathcal{U}\}$ with intensity $dt \times \mu_2(du)$, where $\mu_2(du)$ is a positive σ-finite measure on $\{U,\mathcal{U}\}$;
(A4) Wiener process $\{W_t\}$ and Poisson random measures $p_1(dt,dv)$ and $p_2(dt,du)$ are mutually independent, and are also independent of initial condition $\xi_0 = (x_0,\theta_0)$.

The right-continuous sub σ-algebra $\mathcal{F}_t$ is assumed to be endowed with the restriction to $[0,t]$ of Wiener process $\{W_t\}$ and Poisson random measures $p_1(dt,dv)$ and $p_2(dt,du)$.

Composition of the $p_2(dt,du)$ related measure $\mu_2(du)$ and mappings $c(.,.,.)$ and $g_2(.,.,.)$ are specified in (C0) below:

(C0) Conditions imposed on $\mu_2(du)$, and on coefficients $g_2(.,.,.)$ and $c(.,.,.)$:

Measure $\mu_2(du)$ satisfies:

$$\mu_2(du_0 \times d\underline{u}) = m(du_0) \times \mu(d\underline{u}), \tag{2.1a}$$

with $m(du_0)$ Lebesque measure on $\mathbb{R}$ and $\mu(d\underline{u})$ a probability measure on $\mathbb{R}^d$. Mappings $c(.,.,.)$ and $g_2(.,.,.)$ are composed as follows:

$$c(x, e_i, (u_0, \underline{u})) = \sum_{j=1}^{N} \left[(e_j - e_i) 1_{\Delta_{ij}(x)}(u_0) \right] = \begin{cases} e_j - e_i, & \text{if } u_0 \in \Delta_{ij}(x),\ \underline{u} \in \mathbb{R}^d \\ 0, & \text{otherwise} \end{cases} \tag{2.1b}$$

$$g_2(x, e_i, (u_0, \underline{u})) = \sum_{j=1}^{N} \left[\varphi_{ij}(x, \underline{u}) 1_{\Delta_{ij}(x)}(u_0) \right] = \begin{cases} \varphi_{ij}(x, \underline{u}), & \text{if } u_0 \in \Delta_{ij}(x),\ \underline{u} \in \mathbb{R}^d \\ 0, & \text{otherwise} \end{cases} \tag{2.1c}$$

with measurable mappings $\varphi_{ij} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n, i,j = 1,2,..,N$; and where $\Delta_{ij}(x)$, $i,j = 1,2,..,N$, are the following intervals on the real line:

$$\Delta_{ij}(x) \triangleq \left[(j-1)C_\lambda, (j-1)C_\lambda + \lambda_{ij}(x) \right], \tag{2.1d}$$

with constant $C_\lambda < \infty$, and measurable mappings $\lambda_{ij} : \mathbb{R}^n \to [0, C_\lambda), i,j = 1,2,..,N$.

Remark 2.1. The $\mu_2(du)$ product composition in (2.1a) and the $g_2(.,.,.)$ composition in (2.1c) follow [16]. The composition of $c(.,.,.)$ in (2.1b) is similar to the one in [22–27]. However, the $\Delta_{ij}(x)$'s defined by (2.1d) differ from those in [16, 22–27] on the following three aspects. Firstly, in addition to the sets $\Delta_{ij}(x)$,$i \neq j$, (2.1d) also defines non-empty sets $\Delta_{ii}(x)$. Secondly, $\Delta_{ij}(x)$ in (2.1d) has an x-invariant left-hand-side boundary. Thirdly for $i \neq k$ the sets $\Delta_{ij}(x)$ and $\Delta_{kj}(x)$ in (2.1d) have overlap.

2.2. Characterizations

Condition (C0) leads to the following characterizations.

Characterization 2.2: Thanks to the $\Delta_{ij}(x)$ definition in (2.1d), for all $e_i \in \mathbb{M}$, $x, y \in \mathbb{R}^n$:

$$\Delta_{ik}(x) \cap \Delta_{ij}(y) = \emptyset, \text{ if } k \neq j. \tag{2.2}$$

The non-overlap property in (2.2) will be used in proving existence of pathwise uniqueness of solutions of (1.1)-(1.2) and subsequently in proving the Feller property of the Markov kernel.

Characterization 2.3: If eq. (1.2) admits a pathwise unique solution $\{\theta_t\}$ that assumes values in $[0, \infty) \times \mathbb{M}$ then $\{\theta_t\}$ is an $\{X_t\}$-dependent $N-$state switching process with $N \times N$ transition rate matrix $\Pi(X_{t-}) = \left[\Pi_{ij}(X_{t-}) \right]$, with $\Pi_{ii}(x) = -\sum_{j \neq i} \Pi_{ij}(x)$ and off-diagonal components satisfying:

$$\Pi_{ij}(x) = \lambda_{ij}(x),\ j \neq i. \tag{2.3}$$

Characterization 2.4: (1.1)-(1.2) includes the special case that the $\{\theta_t\}$ transition rate matrix is x-invariant, i.e. $\Pi(x) = \Pi$. Then $p_2(dt, du)$ in (1.1)-(1.2) can still generate simultaneous jumps in $\{X_t\}$ and in $\{\theta_t\}$, as a result of which the process $\{\theta_t\}$ is not independent of the process $\{X_t\}$. Because in this case the term $1_{\Delta_{ij}(x)}(u_0)$ in (2.1b-c) is x-invariant, pathwise uniqueness and Feller property of the joint solution of (1.1)-(1.2) follows from the classical reasoning under the additional conditions that $\lambda_{ii}(x)$ is x-invariant and $\varphi_{ij}(x)$ satisfies a local Lipschitz condition.

Characterization 2.5: If $p_2(dt, du)$ generates a joint random mark $(u_0, \underline{u})$ at moment τ, and if $X_{\tau-} = x$ and $\theta_{\tau-} = e_k$, then the jump in $\{X_t, \theta_t\}$ satisfies:

$$\begin{bmatrix} X_\tau - X_{\tau-} \\ \theta_\tau - \theta_{\tau-} \end{bmatrix} = \begin{bmatrix} X_\tau - x \\ \theta_\tau - e_k \end{bmatrix} = \begin{bmatrix} g_2(x, e_k, u_0, \underline{u}) \\ c(x, e_k, u_0, \underline{u}) \end{bmatrix} = \sum_{j=1}^{N} \left(1_{\Delta_{kj}(x)}(u_0) \begin{bmatrix} \varphi_{kj}(x, \underline{u}) \\ (e_j - e_k) \end{bmatrix} \right). \tag{2.4}$$

where the last equality follows from substituting eq. (2.1b,c). From eq. (2.4) can be seen when the different types of jumps in $\{X_t, \theta_t\}$ happen:

(1) Jump in $\{X_t\}$ only: if $u_0 \in \Delta_{kk}(x)$ and $\varphi_{kk}(x, \underline{u}) \neq 0$;
(2) Jump in $\{\theta_t\}$ only: if $\exists j \neq k$ such that $u_0 \in \Delta_{kj}(x)$ and $\varphi_{kj}(x, \underline{u}) = 0$;
(3) Hybrid jump: if $\exists j \neq k$ such that $u_0 \in \Delta_{kj}(x)$ and $\varphi_{kj}(x, \underline{u}) \neq 0$;
(4) No jump: if none of conditions 1) - 3) hold true.

2.3. Hybrid jump example of (1.1)-(1.2) under (C0)

In air traffic, spontaneous changes of aircraft dynamics pose safety risk. For example, the aircraft flight mode may suddenly switch from level flight to climb or to descent. Consider a discrete-valued state space consisting of three vertical flight modes: Level flight ($i = 1$), Climb ($i = 2$), and Descent ($i = 3$). The vertical flight mode process $\theta_t^\perp$ evolves according to SDE (1.2), with $c_\perp(x, \theta, u)$ satisfying (2.1b) and (2.1d), with non-zero rates $\lambda_{12}(x)$, $\lambda_{13}(x)$,$\lambda_{21}(x)$ and $\lambda_{31}(x)$. Because an aircraft undergoes a finite number of discontinuities on a finite time interval, X_t is modeled by (1.1) with $g_1(.,.) = 0$, where the X_t components in vertical direction are position $X_{y,t}$ and velocity $X_{v,t}$. The evolution in vertical position satisfies $dX_{y,t} = X_{v,t}dt$, i.e. $a_y((y, v), \theta) = v$ and $b_y((y, v), \theta) = 0$. In between two successive mode switching's, the evolution in vertical velocity satisfies $dX_{v,t} = f_\theta(v_\theta - X_{v,t}) + \beta_\theta dw_{v,t}$, i.e. $a_v((X_y, X_v), \theta) = f_\theta(v_\theta - X_v)$ and $b_v((X_y, X_v), \theta) = \beta_\theta$, with f_θ a feedback factor to keep the aircraft near the desired vertical velocity v_θ under mode θ, and β_θ the level of Brownian motion disturbance in vertical velocity under mode θ.

At moments of mode switching the random measure $p_2(dt, du)$ also generates a jump in vertical velocity. This is modeled using eq. (2.1c) with $d = 1$, $\underline{u} \in \mathbb{R}_+$, and with

$$\varphi_{v,ij}((y, v), \underline{u}) = \begin{cases} -v & \text{if } (i,j) = (2,1) \\ -v, & \text{if } (i,j) = (3,1) \\ -v + \underline{u}, & \text{if } (i,j) = (1,2) \\ -v - \underline{u}, & \text{if } (i,j) = (1,3) \\ 0, & \text{else} \end{cases} , \tag{2.5}$$

where the first two lines capture a reset of the vertical velocity to zero simultaneously with a mode switching from Climb or from Descent to Level flight. The third and fourth lines capture a jump in the vertical velocity to random value $\underline{u}$ or $-\underline{u}$ from probability measure $\mu(d\underline{u})$, simultaneously with a mode switching from Level flight to Climb or to Descent.

3. Existence of pathwise unique solutions

Characterization 2.4 explains that the classical route to pathwise unique joint solution of (1.1)-(1.2) falls short if one or more of the following three conditions apply: 1) $\Pi(x) \neq \Pi$; 2) $\lambda_{ii}(x) \neq \lambda_{ii}$, for some i; and 3) $\varphi_{ij}(x, \underline{u})$ is non-Lipschitz in x, for some i, j. To address these cases, in subsection 3.1 a non-Lipschitz route in literature for an Itô-Skorohod SDE on a Euclidean space is presented. Next, in subsection 3.2, the existence of a pathwise unique solution of (1.1)-(1.2) under (C0) is proven under weak conditions, by elaboration of the mapping of Itô-Skorohod SDEs and pathwise uniqueness conditions from Euclidean space to hybrid space.

3.1. Itô-Skorohod SDE on Euclidean space

The Itô-Skorohod SDE considered is:

$$d\xi_t = \tilde{a}(\xi_t)dt + \tilde{b}(\xi_t)dW_t + \int_V \tilde{g}_1(\xi_{t-}, v)q_1(dt, dv) + \int_U \tilde{g}_2(\xi_{t-}, u)p_2(dt, du), \tag{3.1}$$

with $\tilde{a} : \mathbb{R}^{n'} \to \mathbb{R}^{n'}$, $\tilde{b} : \mathbb{R}^{n'} \to \mathbb{R}^{n'} \times \mathbb{R}^m$, $\tilde{g}_1 : \mathbb{R}^{n'} \times V \to \mathbb{R}^{n'}$, $\tilde{g}_2 : \mathbb{R}^{n'} \times U \to \mathbb{R}^{n'}$ measurable mappings, and with $\{W_t\}$, $q_1(dt, dv)$ and $p_2(dt, du)$ satisfying (A1)-(A4). Moreover, the coefficients $\tilde{a}$, $\tilde{b}$, $\tilde{g}_1$, and $\tilde{g}_2$ satisfy conditions (L1)-(L3):

(L1) There exists a constant $c_G < \infty$ such that for all $\xi \in \mathbb{R}^{n'}$:

$$\left|\tilde{a}(\xi)\right|^2 + \sum_{k=1}^m \left|\tilde{b}_k(\xi)\right|^2 + \int_V \left|\tilde{g}_1(\xi, v)\right|^2 \mu_1(dv) + \int_U \left|\tilde{g}_2(\xi, u)\right|^2 \mu_2(du) \leq c_G(1 + |\xi|^2);$$

(L2) For each positive integer r there is constant $c_L^r < \infty$ such that for all $|\xi|, |\zeta| < r$:

$$\left|\tilde{a}(\xi) - \tilde{a}(\zeta)\right|^2 + \sum_{k=1}^m \left|\tilde{b}_k(\xi) - \tilde{b}_k(\zeta)\right|^2 + \int_V \left|\tilde{g}_1(\xi, v) - \tilde{g}_1(\zeta, v)\right|^2 \mu_1(dv) \leq c_L^r \, |\xi - \zeta|^2);$$

(L3) There is a constant $c_J < \infty$ such that $\sup_{\xi \in \mathbb{R}^{n'}} \{\int_U 1\{\tilde{g}_2(\xi, u) \neq 0\}\mu_2(du)\} \leq c_J$.

If $\tilde{g}_2(.,.) = 0$, then conditions for existence of pathwise unique cadlag solution of (3.1) are well known under (L1) and (L2), e.g. [20, 21]. If $\tilde{g}_2(.,.) \neq 0$, (L3) assures that $p_2(dt, du)$ generates a finite number of additional discontinuities in $\{\xi_t\}$ on a finite interval, as a result of which no Lipschitz condition on $\tilde{g}_2(\xi, .)$ is needed to prove existence of pathwise unique solution of (3.1) [28–30].

Proposition 3.1. *Let assumptions (A1)-(A4) hold true and let the coefficients of Itô-Skorohod SDE (3.1) satisfy conditions (L1)-(L3). Then for each $\mathcal{F}_0$-measurable square integrable $\mathbb{R}^{n'}$-valued random variable η, SDE (3.1) with initial condition $\xi_0 = \eta$ admits a pathwise unique $\mathcal{F}_t$-measurable cadlag solution $\xi_t^{0,\eta}$, $t \in [0, \infty)$, satisfying $E\left|\xi_t^{0,\eta}\right|^2 < \infty$. Moreover, for each $\xi \in \mathbb{R}^{n'}$ there exists a random measurable function $\phi(\xi, t, \omega), t \in [0, \infty)$, such that $\phi(\xi, t) = \xi_t^{0,\xi} \, P - a.s.$*

In having shown existence and pathwise uniqueness of a solution of an SDE driven by Brownian motion and Poisson random measure, the classical approach in showing stochastic continuity of this solution is to make use of Lipschitz continuity for all terms in (3.1). Because Proposition 3.1 does not assume Lipschitz continuity on the term involving $\tilde{g}_2(\xi, u)$, another proof is needed to show that the solution process of (3.1) is stochastically continuous. Such proof has been given by Gihman and Skorohod [28, pp. 249–254], the result of which is stated in Proposition 3.2 below.

Proposition 3.2. *Let, in addition to the assumptions and conditions of Proposition 3.1, $\tilde{g}_2(\xi, u)$ be continuous in ξ, for almost all u in measure $\mu_2(du)$. Then for any converging sequence $\{\eta^\kappa\}$ that is independent of $\{W_t\}$, $p_1(dt, dv)$, $p_2(dt, du)$ and ξ_0, and has limit $\lim_{\kappa\to\infty} \eta^\kappa = \eta \in \mathbb{R}^{n'}$,*

$$\lim_{\kappa\to\infty} P(\sup_{t\leq r} \left|\xi_t^{0,\eta^\kappa} - \xi_t^{0,\eta}\right| > \varepsilon) = 0, \text{ all } r > 0 \text{ and } \varepsilon > 0.$$

3.2. Itô-Skorohod SDE on a hybrid space

To prove existence of a pathwise unique solution of (1.1)-(1.2), in addition to condition (C0), we adopt the following conditions (C1)-(C3):

(C1) There exists a constant ℓ_G such that for all $e_i \in \mathbb{M}$ and all $x \in \mathbb{R}^n$:

$$|a(x, e_i)|^2 + \sum_{k=1}^{m} |b_k(x, e_i)|^2 + \int_V \left|g_1(x, e_i, v)\right|^2 \mu_1(dv) \leq \ell_G(1 + |x|^2);$$

(C2) For each positive integer r there exists a constant ℓ_L^r such that for all $e_i \in \mathbb{M}$ and all $|x|, |y| < r$:

$$\left|a(x, e_i) - a(y, e_i)\right|^2 + \sum_{k=1}^{m} \left|b_k(x, e_i) - b_k(y, e_i)\right|^2$$
$$+ \int_V \left|g_1(x, e_i, v) - g_1(y, e_i, v)\right|^2 \mu_1(dv) \leq \ell_L^r \left|x - y\right|^2;$$

(C3) There exists a constant $\ell_\varphi < \infty$ such that for all $i, j = 1, 2..., N$ and all $x \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^d} \left|\varphi_{ij}(x, \underline{u})\right|^2 \mu(d\underline{u}) \leq \ell_\varphi(1 + |x|^2).$$

To prove existence of a pathwise unique solution, the system of equations (1.1)-(1.2) is mapped on (3.1). Subsequently it is verified that conditions (C0)-(C3) imply that conditions (L1)-(L3) in Proposition 3.1 hold true. Hence, Proposition 3.1 yields existence of a pathwise unique solution of (1.1)-(1.2).

Theorem 3.3. *Let assumptions (A1)-(A4) and conditions (C0)-(C3) hold true. Then for each $\mathcal{F}_0$-measurable square integrable $\mathbb{R}^n \times \mathbb{M}$-valued random variable η, SDE (1.1)-(1.2) with initial condition $\xi_0 = (x_0, \theta_0) = \eta$, admits a pathwise unique $\mathcal{F}_t$-measurable cadlag solution $\xi_t^{0,\eta}$, $t \in [0, \infty)$, satisfying $E\left|\xi_t^{0,\eta}\right|^2 < \infty$. Moreover, for each $\xi \in \mathbb{R}^n \times \mathbb{M}$ there exists a random measurable function $\phi(\xi, t, \omega)$, $t \in [0, \infty)$, such that $\phi(\xi, t) = \xi_t^{0,\xi}$ $P - a.s.$*

Proof. See Appendix A, which also explains differences with the proof in [27]. □

A well-known consequence of Theorem 3.3 is that joint solutions of (1.1)-(1.2) define a Markov transition kernel, e.g. [19, 20].

Corollary 3.4. *Theorem 3.3's pathwise unique $\mathcal{F}_t$-measurable SDE solution process $\xi_t^{0,\eta}$, $t \in [0,\infty)$ is a time-homogeneous Markov process with respect to $F = \{\mathcal{F}_t\}$, with Markov transition kernel $Q_t(\xi;B) \triangleq P\{\phi(\xi,t) \in B\}$, $t \in [0,\infty), \xi \in \mathbb{R}^n \times \mathbb{M}, B \in \beta(\mathbb{R}^n \times \mathbb{M})$, forming a semigroup, i.e.*

$$Q_{t+\Delta}(\xi;B) = \int_{\mathbb{R}^n \times \mathbb{M}} Q_\Delta(u;B)Q_t(\xi;du), \text{ for all } t, \Delta \in [0,\infty). \tag{3.2}$$

Remark 3.5. Conditions (C0) through (C3) adopted in Theorem 3.3 mean that for the existence of a pathwise unique solution of (1.1)-(1.2) it is sufficient for $\lambda_{ij}(x)$ and $\varphi_{ij}(x,.)$ to be measurable only. In the next section it will be shown that the Feller property holds true if $\lambda_{ij}(x)$ and $\varphi_{ij}(x,\underline{u})$ are continuous in x, for each $i, j, \underline{u}$.

4. Feller property of the Markov transition kernel

Let $C_{\mathbb{R}^n \times \mathbb{M}}$ denote the space of bounded functions on $\mathbb{R}^n \times \mathbb{M}$ that are continuous on $\mathbb{R}^n$ for each value in $\mathbb{M}$. Following [19], transition kernel $Q_t(\xi;B), \xi \in \mathbb{R}^n \times \mathbb{M}, B \in \beta(\mathbb{R}^n \times \mathbb{M})$ is said to satisfy the Feller property if $\int_{\mathbb{R}^n \times \mathbb{M}} f(y)Q_t(\xi;dy) \in C_{\mathbb{R}^n \times \mathbb{M}}$, for each $f \in C_{\mathbb{R}^n \times \mathbb{M}}, t \in [0,\infty)$.

The objective is to prove that the Markov transition kernel of Corollary 3.4 satisfies the Feller property. This is accomplished by adopting additional conditions (C4)-(C5):
(C4) $\lambda_{ij}(x)$ is continuous in x for all $i, j = 1, 2..., N$.
(C5) $\varphi_{ij}(x,\underline{u})$ is continuous in x, for every $\underline{u} \in \mathbb{R}^d$, for all $i, j = 1, 2..., N$.

Under these additional assumptions, it will be proven in Theorem 4.2 that solution $\xi_t^{0,\eta}$ of (1.1)-(1.2) is stochastically continuous in $\eta \in \mathbb{R}^n \times \mathbb{M}$. To prepare, we first prove in Lemma 4.1 that $g_2(x, e_i, u)$ and $c(x, e_i, u)$ are continuous in x, for almost all u in measure $\mu_2(du)$.

Lemma 4.1. *Under (C0), (C4) and (C5), the mapping*

$$\tilde{g}_2((x,e_i),u)) = \begin{bmatrix} g_2(x,e_i,u) \\ c(x,e_i,u) \end{bmatrix} \tag{4.1}$$

is continuous in x, for almost all u in measure $\mu_2(du)$.

Proof. Substitution of (2.1b,c) in (4.1) yields:

$$\tilde{g}_2((x,e_i),(u_0,\underline{u})) = \sum_{j=1}^{N} \left(1_{\Delta_{ij}(x)}(u_0) \begin{bmatrix} \varphi_{ij}(x,\underline{u}) \\ (e_j - e_i) \end{bmatrix} \right)$$

Due to (C5), $\varphi_{ij}(x,\underline{u})$ is continuous in $x \in \mathbb{R}^n$ for every $i, j = 1, 2..., N$ and every $\underline{u} \in \mathbb{R}^d$. Hence, for the completion of the proof, it remains to be shown that the process $\{ 1_{\Delta_{ij}(x)}(u_0) \}$ is continuous in x, for almost all u_0 in measure $m(du_0)$. To prove the latter, for each sequence $\{x^\kappa \in \mathbb{R}^n\}$, that is independent of $\{W_t\}$, $p_1(dt,dv)$, $p_2(dt,du)$ and ξ_0, and which converges to

$x \in \mathbb{R}^n$, i.e. $\lim_{\kappa\to\infty} x^\kappa = x$, we evaluate:

$$\int_{\mathbb{R}} \left| \sum_{j=1}^{N} [(1_{\Delta_{ij}(x)}(u_0) - 1_{\Delta_{ij}(x^\kappa)}(u_0))] \right| m(du_0)$$
$$= \int_{\mathbb{R}} \left| \sum_{j=1}^{N} [(1_{\Delta_{ij}(x)\setminus\Delta_{ij}(x^\kappa)}(u_0) - 1_{\Delta_{ij}(x^\kappa)\setminus\Delta_{ij}(x)}(u_0))] \right| du_0.$$

Thanks to non-overlap property (2.2), there is only one non-zero term in each of the summations over j. Hence the summation can be moved outside $||$. Due to Fubini's theorem, the summation can subsequently be moved outside the integral. These two moves yield:

$$\int_{\mathbb{R}} \left| \sum_{j=1}^{N} [(1_{\Delta_{ij}(x)}(u_0) - 1_{\Delta_{ij}(x^\kappa)}(u_0))] \right| m(du_0)$$
$$= \sum_{j=1}^{N} \int_{\mathbb{R}} \left| 1_{\Delta_{ij}(x)\setminus\Delta_{ij}(x^\kappa)}(u_0) - 1_{\Delta_{ij}(x^\kappa)\setminus\Delta_{ij}(x)}(u_0) \right| du_0$$
$$= \sum_{j=1}^{N} \left| \lambda_{ij}(x) - \lambda_{ij}(x^\kappa) \right|.$$

Hence $\lim_{\kappa\to\infty} \int_{\mathbb{R}} \left| \sum_{j=1}^{N} [(1_{\Delta_{ij}(x)}(u_0) - 1_{\Delta_{ij}(x^\kappa)}(u_0))] \right| m(du_0) = \lim_{\kappa\to\infty} \sum_{j=1}^{N} \left| \lambda_{ij}(x) - \lambda_{ij}(x^\kappa) \right|$

Following (C4), $\lambda_{ij}(x)$ is continuous in x, which means $\lim_{\kappa\to\infty} \left| \lambda_{ij}(x) - \lambda_{ij}(x^\kappa) \right| = 0$. Hence

$$\lim_{\kappa\to\infty} \int_{\mathbb{R}} \left| \sum_{j=1}^{N} [(1_{\Delta_{ij}(x)}(u_0) - 1_{\Delta_{ij}(x^\kappa)}(u_0))] \right| m(du_0) = 0$$

The latter implies that the mapping $1_{\Delta_{ij}(x)}(u_0)$ is continuous in x for every $i, j = 1, 2, .., N$ and almost all u_0 in measure $m(du_0)$. □

Remark 4.2. There are two differences with [27]. The first difference is that under the common partition, non-overlap property (2.2) does not hold true. Therefore [27; Appendix C] cannot shift the summation outside $||$, which makes the proof more demanding and needs Lipschitz $\lambda_{ij}(x)$. The second difference is that instead of proving, [27; Remark 2.4] assumes that $g_2(x, e_i, u)$ is continuous in x for almost all u in measure $\mu_2(du)$.

Next, Proposition 3.2 and Lemma 4.1 are used to prove in Theorem 4.3 that solution $\xi_t^{0,\eta}$ of (1.1)-(1.2) is stochastically continuous in $\eta \in \mathbb{R}^n \times \mathbb{M}$. From this stochastic continuity result the Feller property follows in Theorem 4.4.

Theorem 4.3. *Let in addition to the assumptions and conditions of Theorem 3.3, conditions (C4) and (C5) hold true. Then solution $\xi_t^{0,\eta}$ of (1.1)-(1.2) is stochastically continuous in $\eta \in \mathbb{R}^n \times \mathbb{M}$, i.e. for any converging sequence $\{\eta^\kappa\}$, $\eta^\kappa = (x^\kappa, e_i) \in \mathbb{R}^n \times \mathbb{M}$, that is independent of*

$\{W_t\}$, $p_1(dt, dv)$, $p_2(dt, du)$ *and* ξ_0, *and that converges to* $\lim_{\kappa\to\infty} \eta^\kappa = \eta = (x, e_i) \in \mathbb{R}^n \times$ M*:*

$$\lim_{\kappa\to\infty} P\left(\sup_{t\le r}\left|\xi_t^{0,\eta^\kappa} - \xi_t^{0,\eta}\right| > \varepsilon\right) = 0, \textit{ for all } r > 0 \textit{ and } \varepsilon > 0. \tag{4.2}$$

Proof. From the proof of Theorem 3.3 we know that SDE (1.1)-(1.2) has a pathwise unique $\mathcal{F}_t$-measurable solution for each initial value in $\mathbb{R}^n \times \mathbb{M}$. From Lemma 4.1 we know that $\tilde{g}_2((x, e_i), (u_0, \underline{u}))$ is continuous in x, for almost all $(u_0, \underline{u})$ in measure $\mu_2(du)$. Then, from Proposition 3.2 follows for all $r > 0$ and $\varepsilon > 0$:

$$\lim_{\kappa\to\infty} P\left(\sup_{t\le r}\left|\xi_t^{0,\eta^\kappa} - \xi_t^{0,\eta}\right| > \varepsilon\right) = 0. \tag{4.3}$$

Hence solutions of SDE (3.1) are stochastically continuous. Because SDE (3.1) embeds SDE (1.1)-(1.2), limit (4.3) also holds true for (1.1)-(1.2). □

Theorem 4.4. *Under assumptions and conditions of Theorem 4.3, the Markov transition kernel* $Q_t(\xi; B)$ *satisfies the Feller property.*

Proof. Because solutions $\xi_t^{0,\eta}$ of (1.1)-(1.2) are stochastically continuous w.r.t. $\eta \in \mathbb{R}^n \times \mathbb{M}$ (see Theorem 4.3), for all $f \in C_{\mathbb{R}^n\times\mathbb{M}}$ the process $\{f(\xi_t^{0,\eta})\}$ also is stochastically continuous w.r.t. $\eta \in \mathbb{R}^n \times \mathbb{M}$. This means that $E\{f(\xi_t^{0,\eta})\} \in C_{\mathbb{R}^n\times\mathbb{M}}$ for all $f \in C_{\mathbb{R}^n\times\mathbb{M}}$. Because we know

$$E\{f(\xi_t^{0,\xi})\} = \int_{\mathbb{R}^n\times\mathbb{M}} f(y)P\{\xi_t^{0,\xi} \in dy\} = \int_{\mathbb{R}^n\times\mathbb{M}} f(y)P\{\phi(\xi, t) \in dy\} = \int_{\mathbb{R}^n\times\mathbb{M}} f(y)Q_t(\xi; dy),$$

this implies $\int_{\mathbb{R}^n\times\mathbb{M}} f(y)Q_t(\xi; dy) \in C_{\mathbb{R}^n\times\mathbb{M}}$, i.e. the Feller property holds true for $Q_t(\xi; B)$. □

Corollary 4.5. *Under the assumptions and conditions of Theorem 4.3, the time-homogeneous Markov process solution* $\{\xi_t^{0,\eta}\}$ *of (1.1)-(1.2) is a strong Markov process, i.e. for any stopping time* τ *with* $P\{\tau < \infty\} = 1$*, for every* $t \in [0, \infty)$ *and* $B \in \beta(\mathbb{R}^n \times \mathbb{M})$*:*

$$P\{\xi_{\tau+t}^{0,\eta} \in B \big| \mathcal{F}_\tau\} = Q_t(\xi_\tau^{0,\eta}, B) \textit{ a.s.}$$

Proof. See Friedman [19, Theorem 2.4]. □

5. Discussion of results

This paper has studied Feller property of the Markov transition kernel of regime-switching jump diffusion processes with hybrid jumps. The processes considered evolve in a hybrid state space, as solutions of SDE's that are driven by Brownian motion and Poisson random measures. The results obtained significantly enhance Feller results of [22–27] for SDE's that involve regime-switching.

These results also open directions for relevant follow-on research. One is the extension to a countable number of modes [23–26]. Another extension is to further relax growth and local Lipschitz conditions for the mode-dependent drift coefficient [22, 25, 26]. Thirdly, [11–15, 17, 18] show relevant larger classes of continuous-time hybrid state Markov processes; the study of the Feller property of their Markov transition kernels forms another direction for relevant follow-on research.

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A. Proof of Theorem 3.3

SDEs (1.1)-(1.2) on $\mathbb{R}^n \times \mathbb{M}$ are mapped on SDE (3.1) on $\mathbb{R}^{n'}$, with $n' = n + N$ and:

$$\tilde{a} : \mathbb{R}^{n+N} \to \mathbb{R}^{n+N},\ \tilde{a}((x,z)) = \left[\begin{array}{cc} \sum_{i=1}^{N} [\alpha_i(z) a(x, e_i)] & O^N \end{array} \right]^T,\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^N;$$

$$\tilde{b} : \mathbb{R}^{n+N} \to \mathbb{R}^{(n+N)} \times \mathbb{R}^m,\ \tilde{b}(x,z) = \left[\begin{array}{cc} \sum_{i=1}^{N} [\alpha_i(z) b(x, e_i)] & O^{N\times m} \end{array} \right]^T;$$

$$\tilde{g}_1 : \mathbb{R}^{n+N} \times V \to \mathbb{R}^{n+N},\ \tilde{g}_1((x,z),v) = \begin{bmatrix} \sum_{i=1}^{N} [\alpha_i(z) g_1(x,e_i,v)] & O^N \end{bmatrix}^T;$$

$$\tilde{g}_2 : \mathbb{R}^{n+N} \times U \to \mathbb{R}^{n+N},\ \tilde{g}_2((x,z),u) = \begin{bmatrix} \sum_{i=1}^{N} [\alpha_i(z) g_2(x,e_i,u)] & \sum_{i=1}^{N} [\alpha_i(z) c(x,e_i,u)] \end{bmatrix}^T,$$

where O^N and $O^{N\times m}$ denote zero N-vector and zero $N \times m$-matrix respectively, and $\alpha_i(z)$ is an elevated cosine on the set $\{z \in \mathbb{R}^N;\ |z - e_i| \le 1\}$ and zero outside this set, i.e.

$$\alpha_i(z) \triangleq \begin{cases} \frac{1}{2} + \frac{1}{2}\cos(2\pi\,|z - e_i|), & \text{if } |z - e_i| \le \frac{1}{2} \\ 0, & \text{else} \end{cases}.$$

This implies that $\alpha_i(z)$ is Lipschitz continuous and that $\alpha_i(z)\alpha_j(z) = 0$ for each $i \neq j, z \in \mathbb{R}^N$.

We have to verify that posing conditions (C0)-(C3) on coefficients $a(x,e_i)$, $b(x,e_i)$, $g_1(x,e_i,.)$, $g_2(x,e_i,.)$, and $c(x,e_i,.)$, for $(x,e_i) \in \mathbb{R}^n \times \mathbb{M}$, imply that conditions (L1)-(L3) of Proposition 3.1 hold true for $\tilde{a}(\xi)$, $\tilde{b}(\xi)$, $\tilde{g}_1(\xi,.)$, and $\tilde{g}_2(\xi,.)$ for $\xi = (x,z) \in \mathbb{R}^{n+N}$. This verification is demonstrated in subsections *A.1*, *A.2* and *A.3* for (L1), (L2), and (L3) respectively.

During this verification we point to the e_i that is at smallest distance from $z \in \mathbb{R}^N$ by using the pointer $i^z \triangleq \min\{\arg\max_i |z - e_i|\}$. This pointer is of specific use if $\exists i \neq j$ such that $|z - e_i| = |z - e_j|$; in such case $\alpha_{i^z}(z) = 0$.

Remark A: [27] also maps (1.1)-(1.2) on (3.1) in a way similar as done above, and subsequently verifies that conditions imposed on (1.1)-(1.2) imply those imposed on (3.1) for the existence of a pathwise unique solution. In doing so, there are two differences. The first difference is that [27; Appendix B] restricts the mapping on (3.1) to $(x,e_i) \in \mathbb{R}^n \times \mathbb{M}$, which means $\alpha_i(e_i) = 1$. This restricted mapping makes the verification of growth and Lipschitz conditions simpler, though also requires $\lambda_{ij}(x)$ to be Lipschitz. The second difference is that [27, Appendix A] assumes that $\tilde{g}_2(.,.)$ in (3.1) is bounded; this asks $g_2(.,.)$ also to be bounded.

A.1. Verification that (C0), (C1) and (C3) imply condition (L1) in Proposition 3.1

For $\xi \in \mathbb{R}^{n+N}$ we get:

$$\left|\tilde{a}(\xi)\right|^2 + \sum_{k=1}^{m}\left|\tilde{b}_k(\xi)\right|^2 + \int_V \left|\tilde{g}_1(\xi,v)\right|^2 \mu_1(dv) + \int_U \left|\tilde{g}_2(\xi,u)\right|^2 \mu_2(du)$$

$$= \left|\tilde{a}(x,e_i)\right|^2 + \sum_{k=1}^{m}\left|\tilde{b}_k(x,e_i)\right|^2 + \int_V \left|\tilde{g}_1(x,e_i,v)\right|^2 \mu_1(dv) + \int_U \left|\tilde{g}_2(x,e_i,u)\right|^2 \mu_2(du)$$

$$= \left|\sum_{i=1}^{N}[\alpha_i(z)a(x,e_i)]\right|^2 + \sum_{k=1}^{m}\left|\sum_{i=1}^{N}[\alpha_i(z)b_k(x,e_i)]\right|^2 + \int_V \left|\sum_{i=1}^{N}[\alpha_i(z)g_1(x,e_i,v)]\right|^2 \mu_1(dv)$$

$$+ \int_U \left|\sum_{i=1}^{N}\left[\alpha_i(z)\begin{bmatrix} g_2(x,e_i,u) \\ c(x,e_i,u) \end{bmatrix}\right]\right|^2 \mu_2(du)$$

$$= \sum_{i=1}^{N} |\alpha_i(z)a(x,e_i)|^2 + \sum_{k=1}^{m}\sum_{i=1}^{N} |\alpha_i(z)b_k(x,e_i)|^2 + \int_V \sum_{i=1}^{N} \left|\alpha_i(z)g_1(x,e_i,v)\right|^2 \mu_1(dv)$$

$$+ \int_U \sum_{i=1}^{N} \left|\alpha_i(z)\begin{bmatrix} g_2(x,e_i,u) \\ c(x,e_i,u) \end{bmatrix}\right|^2 \mu_2(du)$$

$$= |\alpha_{i^z}(z)a(x,e_{i^z})|^2 + \sum_{k=1}^{m} |\alpha_{i^z}(z)b_k(x,e_{i^z})|^2 + \int_V \left|\alpha_{i^z}(z)g_1(x,e_{i^z},v)\right|^2 \mu_1(dv)$$

$$+ \int_U \left|\alpha_{i^z}(z)\begin{bmatrix} g_2(x,e_{i^z},u) \\ c(x,e_{i^z},u) \end{bmatrix}\right|^2 \mu_2(du)$$

$$\leq |a(x,e_{i^z})|^2 + \sum_{k=1}^{m} |b_k(x,e_{i^z})|^2 + \int_V \left|g_1(x,e_{i^z},v)\right|^2 \mu_1(dv) + \int_U \left|\left[\begin{bmatrix} g_2(x,e_{i^z},u) \\ c(x,e_{i^z},u) \end{bmatrix}\right]\right|^2 \mu_2(du)$$

$$\leq \ell_G(1+|x|^2) + \int_U \left|\sum_{j=1}^{N}\left[1_{\Delta_{i^z j}(x)}(u_0)\begin{bmatrix} \varphi_{i^z j}(x,\underline{u}) \\ (e_j - e_{i^z}) \end{bmatrix}\right]\right|^2 \mu_2(du),$$

where ℓ_G is the constant in (C1), and the last term follows from (2.1b,c) of (C0). Thanks to (2.1d) of (C0) we can make use of (2.2), which implies that there is only one non-zero term in the summation over j. Hence the summation can be moved outside $||$. Due to Fubini's theorem this summation can also be moved outside the integral, i.e.

$$\int_U \left|\sum_{j=1}^{N}\left[1_{\Delta_{i^z j}(x)}(u_0)\begin{bmatrix} \varphi_{i^z j}(x,\underline{u}) \\ (e_j - e_{i^z}) \end{bmatrix}\right]\right|^2 \mu_2(du) = \sum_{j=1}^{N}\int_U \left|1_{\Delta_{i^z j}(x)}(u_0)\begin{bmatrix} \varphi_{i^z j}(x,\underline{u}) \\ (e_j - e_{i^z}) \end{bmatrix}\right|^2 \mu_2(du)$$

$$\leq C_\lambda \sum_{j=1}^{N}\int_{\mathbb{R}^d} \left|\left[\begin{matrix} \varphi_{i^z j}(x,\underline{u}) \\ (e_j - e_{i^z}) \end{matrix}\right]\right|^2 \mu(d\underline{u}) = C_\lambda \sum_{j=1}^{N}\int_{\mathbb{R}^d} [\left|\varphi_{i^z j}(x,\underline{u})\right|^2 + \left|e_j - e_{i^z}\right|^2]\mu(d\underline{u})$$

$$= C_\lambda \sum_{j=1}^{N}\int_{\mathbb{R}^d} \left|\varphi_{i^z j}(x,\underline{u})\right|^2 \mu(d\underline{u}) + 2C_\lambda(N-1) \leq C_\lambda N \ell_\varphi(1+|x|^2) + 2C_\lambda N.$$

This implies: $\left|\tilde{a}(\xi)\right|^2 + \sum_{k=1}^{m}\left|\tilde{b}_k(\xi)\right|^2 + \int_V \left|\tilde{g}_1(\xi,v)\right|^2 \mu_1(dv) + \int_U \left|\tilde{g}_2(\xi,u)\right|^2 \mu_2(du) \leq \ell_G(1+|x|^2) + C_\lambda N\ell_\varphi(1+|x|^2) + 2C_\lambda N \leq c_G(1+|x|^2)$. Hence (L2) is satisfied by setting $c_G = \max\{2C_\lambda N, (\ell_G + C_\lambda N\ell_\varphi)\}$, where ℓ_G and ℓ_φ are the constants in (C1) and (C3).

A.2. Verification that (C1) and (C2) imply condition (L2) in Proposition 3.1

For $\xi = (x,z)$, $\zeta = (y,\vartheta)$ we define:

$$L_1(x,y,z) \triangleq \left|\tilde{a}(x,z) - \tilde{a}(y,z)\right|^2 + \sum_{k=1}^{m}\left|\tilde{b}_k(x,z) - \tilde{b}_k(y,z)\right|^2$$

$$+\int_V \left|\tilde{g}_1(x,z,v)-\tilde{g}_1(y,z,v)\right|^2 \mu_1(dv),$$

$$L_2(y,z,\vartheta) \triangleq \left|\tilde{a}(y,z)-\tilde{a}(y,\vartheta)\right|^2+\sum_{k=1}^{m}\left|\tilde{b}_k(y,z)-\tilde{b}_k(y,\vartheta)\right|^2 + \int_V \left|\tilde{g}_1(y,z,v)-\tilde{g}_1(y,\vartheta,v)\right|^2 \mu_1(dv).$$

Evaluation of $L_1(x,y,z)$ for $|x|,\left|y\right| < r$ yields:

$$\begin{aligned}
L_1(x,y,z) &= \left|\tilde{a}(x,z)-\tilde{a}(y,z)\right|^2+\sum_{k=1}^{m}\left|\tilde{b}_k(x,z)-\tilde{b}_k(y,z)\right|^2 \\
&\quad + \int_V \left|\tilde{g}_1(x,z,v)-\tilde{g}_1(y,z,v)\right|^2 \mu_1(dv) \\
&= \left|\sum_{i=1}^{N}\alpha_i(z)[a(x,e_i)-a(y,e_i)]\right|^2+\sum_{k=1}^{m}\left|\sum_{i=1}^{N}\alpha_i(z)[b_k(x,e_i)-b_k(y,e_i)]\right|^2 \\
&\quad + \int_V \left|\sum_{i=1}^{N}\alpha_i(z)[g_1(x,e_i,v)-g_1(y,e_i,v)]\right|^2 \mu_1(dv) \\
&\le \left|a(x,e_{i^z})-a(y,e_{i^z})\right|^2+\sum_{k=1}^{m}\left|b_k(x,e_{i^z})-b_k(y,e_{i^z})\right|^2 \\
&\quad + \int_V \left|g_1(x,e_{i^z},v)-g_1(y,e_{i^z},v)\right|^2 \mu_1(dv) \\
&\le \ell_L^r\left|(x,e_{i^z})-(y,e_{i^z})\right|^2 \le \ell_L^r\left|x-y\right|^2, \text{ with } \ell_L^r \text{ the constant in (C2).}
\end{aligned}$$

Evaluation of $L_2(y,z,\vartheta)$ for $\left|y\right| < r$ yields:

$$\begin{aligned}
L_2(y,z,\vartheta) &= \left|\tilde{a}(y,z)-\tilde{a}(y,\vartheta)\right|^2+\sum_{k=1}^{m}\left|\tilde{b}_k(y,z)-\tilde{b}_k(y,\vartheta)\right|^2 \\
&\quad + \int_V \left|\tilde{g}_1(y,z,v)-\tilde{g}_1(y,\vartheta,v)\right|^2 \mu_1(dv) \\
&= \left|\sum_{i=1}^{N}[\alpha_i(z)-\alpha_i(\vartheta)]a(y,e_i)\right|^2+\sum_{k=1}^{m}\left|\sum_{i=1}^{N}[\alpha_i(z)-\alpha_i(\vartheta)]b_k(y,e_i)\right|^2 \\
&\quad + \int_V \left|\sum_{i=1}^{N}[\alpha_i(z)-\alpha_i(\vartheta)]g_1(y,e_i,v)\right|^2 \mu_1(dv) \\
&= \left|\sum_{i=i^z,i^\vartheta}[\alpha_i(z)-\alpha_i(\vartheta)]a(y,e_i)\right|^2+\sum_{k=1}^{m}\left|\sum_{i=i^z,i^\vartheta}[\alpha_i(z)-\alpha_i(\vartheta)]b_k(y,e_i)\right|^2
\end{aligned}$$

$$+\int_V \left| \sum_{i=i^z,i^\vartheta} [\alpha_i(z) - \alpha_i(\vartheta)] g_1(y, e_i, v) \right|^2 \mu_1(dv)$$

$$\leq \left| \sum_{i=i^z,i^\vartheta} \frac{1}{2}\pi \, |z-\vartheta| a(y,e_i) \right|^2 + \sum_{k=1}^m \left| \sum_{i=i^z,i^\vartheta} \frac{1}{2}\pi \, |z-\vartheta| b_k(y,e_i) \right|^2$$

$$+\int_V \left| \sum_{i=i^z,i^\vartheta} \frac{1}{2}\pi \, |z-\vartheta| \, g_1(y,e_i,v) \right|^2 \mu_1(dv)$$

$$\leq 1/4\,\pi^2 \, |z-\vartheta|^2 \left[\left| \sum_{i=i^z,i^\vartheta} a(y,e_i) \right|^2 + \sum_{k=1}^m \left| \sum_{i=i^z,i^\vartheta} b_k(y,e_i) \right|^2 \right.$$

$$\left. +\int_V \left| \sum_{i=i^z,i^\vartheta} g_1(y,e_i,v) \right|^2 \mu_1(dv) \right]$$

$$= 1/4\,\pi^2 \, |z-\vartheta|^2 \left[\left| a(y,e_{i^z}) + a(y,e_{i^\vartheta}) \right|^2 + \sum_{k=1}^m \left| b_k(y,e_{i^z}) + b_k(y,e_{i^\vartheta}) \right|^2 \right.$$

$$\left. +\int_V \left| g_1(y,e_{i^z},v) + g_1(y,e_{i^\vartheta},v) \right|^2 \mu_1(dv) \right]$$

$$\leq \pi^2 \, |z-\vartheta|^2 \, \ell_G (1 + |y|^2) \leq \pi^2 \, |z-\vartheta|^2 \, \ell_G (1 + |r|^2),$$

with ℓ_G the constant in (C1).

Taking the two bounds together yields for $|x|, |y| < r$:

$$L_1(x,y,z) + L_2(y,z,\vartheta) \leq \ell_L^r \, |x-y|^2 + \pi^2 \, |z-\vartheta|^2 \, \ell_G (1 + r^2).$$

Hence with $\xi = (x,z)$, $\zeta = (y,\vartheta)$ we get:

$$\left| \tilde{a}(\xi) - \tilde{a}(\zeta) \right|^2 + \sum_{k=1}^m \left| \tilde{b}_k(\xi) - \tilde{b}_k(\zeta) \right|^2 + \int_V \left| \tilde{g}_1(\xi,v) - \tilde{g}_1(\zeta,v) \right|^2 \mu_1(dv)$$

$$= \left| \tilde{a}(x,z) - \tilde{a}(y,\vartheta) \right|^2 + \sum_{k=1}^m \left| \tilde{b}_k(x,z) - \tilde{b}_k(y,\vartheta) \right|^2 + \int_V \left| \tilde{g}_1(x,z,v) - \tilde{g}_1(y,\vartheta,v) \right|^2 \mu_1(dv)$$

$$\leq 2L_1(x,y,z) + 2L_2(y,z,\vartheta) \leq 2\ell_L^r \, |x-y|^2 + 2\pi^2 \, |z-\vartheta|^2 \, \ell_G (1 + r^2)$$

$$\leq 2 \max\{\ell_L^r, \pi^2 \ell_G (1 + r^2)\} \, |\xi - \zeta|^2$$

Hence, (L2) is satisfied by setting $c_L^r = 2 \max\{\ell_L^r, \pi^2 \ell_G (1 + r^2)\}$.

A.3. Verification that (C0) implies (L3) in Proposition 3.1

Using $\xi = (x, z)$ we get:

$$1\left\{\tilde{g}_2(\xi, u) \neq 0\right\} = 1\left\{\tilde{g}_2((x,z), u) \neq 0\right\} = 1\left\{\sum_{i=1}^{N}\left[\alpha_i(z)\begin{bmatrix} c(x, e_i, u) \\ g_2(x, e_i, u)\end{bmatrix}\right] \neq 0\right\}$$

$$= 1\left\{\left[\alpha_{i^z}(z)\begin{bmatrix} c(x, e_{i^z}, u) \\ g_2(x, e_{i^z}, u)\end{bmatrix}\right] \neq 0\right\} \leq 1\left\{\begin{bmatrix} c(x, e_{i^z}, u) \\ g_2(x, e_{i^z}, u)\end{bmatrix} \neq 0\right\}.$$

Due to equations (2.1b,c) of (C0) we know: $1\left\{\begin{bmatrix} c(x, e_{i^z}, u) \\ g_2(x, e_{i^z}, u)\end{bmatrix} \neq 0\right\} \leq \sum_{j=1}^{N} 1\left\{u_0 \in \Delta_{i^z j}(x)\right\}$.

Hence:

$\int_U 1\{\tilde{g}_2(\xi, u) \neq 0\}\mu_2(du) = \int_U 1\{\tilde{g}_2((x,z), u) \neq 0\}\mu_2(du) \leq \int_U \sum_{j=1}^{N} 1\left\{u_0 \in \Delta_{i^z j}(x)\right\}\mu_2(du)$.

By using (2.1a) and (2.1d) from (C0) we get:

$$\int_U 1\{\tilde{g}_2(\xi, u) \neq 0\}\mu_2(du) = \int_U \sum_{j=1}^{N} 1\left\{u_0 \in \Delta_{i^z j}(x)\right\}\mu_2(du)$$

$$= \int_{\mathbb{R}} \sum_{j=1}^{N} 1\left\{u_0 \in \Delta_{i^z j}(x)\right\} m(du_0) = \sum_{j=1}^{N} \lambda_{i^z j}(x).$$

Due to (C0) we also know $\lambda_{ij}(x) < C_\lambda < \infty$ for all i, j, x; hence

$$\sup_{\xi \in \mathbb{R}^{n+N}} \left\{\int_U 1\{\tilde{g}_2(\xi, u) \neq 0\}\mu_2(du)\right\} \leq \sup_{(x,z) \in \mathbb{R}^{n+N}} \left\{\sum_{j=1}^{N} \lambda_{i^z j}(x)\right\} \leq \sum_{j=1}^{N} C_\lambda = NC_\lambda < \infty.$$

The latter verifies (L3) by setting $c_J = NC_\lambda$.