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Fractional periodic boundary value and Cauchy problems with Hilfer–Prabhakar operator

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Abstract
We introduce a successive approximations method to study one fractional periodic boundary value problem of the Hilfer-Prabhakar type. The problem is associated to the corresponding Cauchy problem, whose solution depends on an unknown initial value. To find this value we numerically solve the so-called '*determining system*' of algebraic or transcendental equations. As a result, we determine an approximate solution of the studied problem, written in a closed form. Finally, we evaluate efficiency of our method on a nonlinear numerical example.

Keywords Hilfer–Prabhakar fractional derivative · Periodic boundary conditions · Successive approximations · Determining system · Cauchy problem

1 Introduction

Differential equations of fractional order have recently proved to be valuable tools in modeling of complex phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, porous media, electromagnetic and stochastic processes, finance, inverse problems etc. Diethelm and Freed (1999); Gaul et al. (1991); Glockle and Nonnenmacher (1995); Hilfer (2000); Mainardi (1997); Metzler et al. (1995); Garra et al. (2014); Al-Abedeen (1976); Javed and Malik (2023). But these applications would not have been possible without breakthrough contributions to fundamental

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study of fractional differential equations (fDEs), published in monographs of Miller and Ross (1993); Oldham and Spanier (1974); Podlubny (1999); Kilbas et al. (2006); Samko et al. (1993); Sandev and Tomovski (2019); Mainardi (2022); Zhou (2014); Gorenflo et al. (2014) and the references therein. In these works authors obtained new results in analysis of existence of solutions to nonlinear fDEs using techniques of nonlinear analysis, such as the fixed-point theorems, Leray-Schauder theory, the upper and lower solution method, Adomian decomposition method, etc. (see also discussions in Delbosco and Rodino (1996); Li et al. (2010, ?)). However, these results are still not complete, and there is a great deal of work which needs to be done.

A particular interest of experts in fractional dynamical systems is paid to study of the *fractional order initial value problems* (IVPs), which in contrary to more traditional integer order systems depend on the type of a fractional operator we choose. Most of the known results in this direction involve solvability analysis of fDEs with the *Riemann–Liouvile operator* $(D^{\mu}_{a+}y)(x)$ on some finite interval $[a, b]$, written as

$$(D^{\mu}_{a+}y)(x) = f[x, y(x)] \tag{1}$$

and subject to initial conditions of the form:

$$(D^{\mu-k}_{a+}y)(a+) = b_k,\ b_k \in \mathbb{C},\ k = 1, 2, 3, .., n, \tag{2}$$

where $n = \Re(\mu) + 1,\ \mu \notin \mathbb{N}$ and $\mu = n$, if $\mu \in \mathbb{N}$. Note, that when $0 < \Re(\mu) < 1$ conditions (2) reduce to

$$(I^{1-\mu}_{a+}y)(a+) = b.$$

Such Cauchy type problems (1), (2) were studied by Pitcher and Sewell (1938), (Al-Bassam (1965), Theorems 2, 4, 5, 6), Al-Abedeen (1976); Al-Abedeen and Arora (1978); Arora and Alshamani (1980); Tazali (1982); Tazali and Karim (1994); El-Sayed (1988, 1992, 1993, 1996); El-Sayed and Ibrahim (1995); Hadid (1995); Lakshmikantham and Vatsala (2008a, 2007, 2008b) etc.. To our best knowledge most results, that were obtained up to now, concern analysis not of the IVPs directly, but of the corresponding Volterra integral equations. In some papers authors considered only particular cases of IVPs that underwent their robust qualitative analysis. Moreover, lately also more generalized types of fractional derivatives were introduced, among which we would like to name the *Hilfer* and *Prabhakar operators* and their regularized sub-types (see discussions in Prabhakar (1971); Hilfer (2000, 2008)). One of the known results for such kinds of dynamical systems was obtained by Tomovski in Tomovski (2012). He considered an IVP for nonlinear fDEs with Hilfer differential operator and proved existence and uniqueness of solution to this problem in the space $L[a, b]$ of Lebesgue integrable functions. In a different paper, by Furati et al. (2012), authors showed existence and uniqueness of global solutions of an IVP for a class of nonlinear fDEs involving Hilfer fractional derivative in the space of weighted continuous functions.

Another contribution to the theory of fDEs is analysis of and approximation theory for nonlinear *fractional boundary value problems* (fBVPs). Here we would like to mention papers by Fečkan, Marynets, Pantova and Wang (see Fečkan and Marynets (2023, 2018); Fečkan et al. (2019); Marynets and Pantova (2023, 2022)). Authors associate the studied problems with the corresponding Cauchy problems and introduce successive iterations techniques for approximation of their solutions. Moreover, they demonstrate efficiency of the developed methods on numerical examples that generalize mathematical models of the Antarctic Circumpolar Current, preditor-prey models with prey refuge and development of GDPs of multiple economies.

Motivated by the research above, we present our novel existence and uniqueness results in analysis of an even more complex fundamental problem – periodic BVP with regularized *Hilfer-Prabhakar (HP) differential operator*. We give two significant results: one of them is based on the Picard theorem (Theorem 2), and another one contains remarks about nonlinear higher order fractional IVP of the HP type with n-initial conditions (Proposition 7). Finally, we present a numerical example illustrating efficiency of the suggested iteration technique by applying it to a nonlinear periodic BVP. Note, that recent results of Javed and Malik in Javed and Malik (2023) show applicability of these types of operators in analysis of inverse problems, arising in image processing.

2 Preliminaries

In this section we give definitions of the main fractional differential and integral operators that are used throughout the paper.

Definition 1 Prabhakar (1971) Let $\rho, \mu, \gamma \in \mathbb{C}$, $Re(\rho)$, $Re(\mu) > 0$. The ***three-parameter*** Mittag-Leffer (ML) function is a function, defined by a power series of the form

$$E^{\gamma}_{\rho,\mu}(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} \frac{x^k}{k!}, \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 Prabhakar (1971) Let $f \in L^1[0, T]$, $0 < t < T \leq \infty$. The ***Prabhakar integral*** is an operator, defined by a relation

$$\mathbb{E}^{\gamma}_{\rho,\mu,\omega,0+} f(t) = \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] f(s)ds = (e^{\gamma}_{\rho,\mu,\omega} * f)(t), \tag{4}$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$, $Re(\rho)$, $Re(\mu) > 0$ and

$$e^{\gamma}_{\rho,\mu,\omega}(t) = t^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega t^{\rho}).$$

Definition 3 Garra et al. (2014) Let $f \in L^1[0, T]$, $0 < t < T \leq \infty$ and $f * e^{-\gamma}_{\rho,\mu,\omega}(\cdot) \in W^{m,1}[0, T]$. The ***Prabhakar derivative*** is an integral operator, defined as

$$({}^P D^{\gamma,\mu}_{\rho,\omega,0+} f)(t) = \frac{d^m}{dt^m} \left(\mathbb{E}^{-\gamma}_{\rho,(m-\mu),\omega,0+} f \right)(t). \tag{5}$$

For $n \in \mathbb{N}$, we denote by $AC^n[a, b]$ a space of all real-valued functions with $(n-1)$ continuous derivatives on $[a, b]$, such that $f^{(n-1)}(t) \in AC[a, b]$, where $AC[a, b]$ is a space of real-valued functions $f(x)$ which are absolutely continuous on $[a, b]$.

Definition 4 Garra et al. (2014); Tomovski et al. (2020) Let $0 < \nu \leq 1$, $n-1 < \mu \leq n$, $n \in \mathbb{N}$, ω, $\gamma \in \mathbb{C}$, $\rho > 0$ and let $f \in L^1[0, b]$, $f * e^{-\gamma}_{\rho,\mu,\omega}(\cdot) \in AC[0, b]$. The ***generalized HP derivative*** is defined by

$$(\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+} f)(t) = \left(\mathbb{E}^{-\gamma\nu}_{\rho,\nu(n-\mu),\omega,0+} \frac{d^n}{dt^n} \left(\mathbb{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0+} f \right) \right)(t), \tag{6}$$

where $(\mathbb{E}^{0}_{\rho,0,\omega,0+} f)(t) = f$. In particular, for $0 < \nu \leq 1$, $0 < \mu \leq 1$,

$$(\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+} f)(t) = \left(\mathbb{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0+} \frac{d}{dt} \left(\mathbb{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+} f \right) \right)(t). \tag{7}$$

A special case for $n = 1$ of this definition, was introduced and considered by Garra et al. (2014). Note, that (6) reduces to the generalized Hilfer derivative for $\gamma = 0$, defined by Hilfer in Hilfer (2008).

Definition 5 Garra et al. (2014) For $0 < \nu \leq 1$, $n - 1 < \mu \leq n$ and $n \in \mathbb{N}$, the ***regularized HP derivative*** of a function $f \in AC^n[0, b]$ is given by a relation:

$$\begin{aligned}({}^C D^{\gamma,\mu}_{\rho,\omega,0+} f)(t) &= \left(\mathbb{E}^{-\gamma\nu}_{\rho,\nu(n-\mu),\omega,0+} \mathbb{E}^{-\gamma(1-\nu)}_{\rho,\nu(n-\mu),\omega,0+} \frac{d^n}{dt^n} f\right)(t) \\ &= \left(\mathbb{E}^{-\gamma}_{\rho,\nu(n-\mu),\omega,0+} \frac{d^n}{dt^n} f\right)(t).\end{aligned} \tag{8}$$

In particular,

$$({}^C D^{\gamma,\mu}_{\rho,\omega,0+} f)(t) = \left(\mathbb{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0+} \mathbb{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+} \frac{\boldsymbol{d}}{\boldsymbol{dt}} f\right)(t) = \left(\mathbb{E}^{-\gamma}_{\rho,1-\mu,\omega,0+} \frac{\boldsymbol{d}}{\boldsymbol{dt}} f\right)(t), \tag{9}$$

that for $\mu \in (0, 1)$ can be written as follows:

$${}^C D^{\gamma,\mu}_{\rho,\omega,0+} f(t) = \mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+} \left(f(t) - f(0^+)\right). \tag{10}$$

In addition, by Definition 7 in Polito and Tomovski (2016) we get a relation between the HP derivative and its regularized version that reads:

$$({}^P D^{\gamma}_{\rho,\mu,\omega,0+} f)(t) = ({}^C D^{\gamma}_{\rho,\mu,\omega,0+} f)(t) + \sum_{k=0}^{n-1} t^{k-\mu} E^{-\gamma}_{\rho,k-\mu+1}(\omega t^\rho) f^{(k)}(0^+).$$

Consider a Cauchy problem

$${}^C D^{\gamma,\mu}_{\rho,\omega,0+} x(t) = g(t), \quad t \in [0, T], \quad \mu \in (0, 1), \tag{11}$$

$$x(0) = x_0, \tag{12}$$

where ${}^C D^{\gamma,\mu}_{\rho,\omega,0+}$ is the regularized HP derivative (8), $x * e^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega}(\cdot) \in AC^1[0, T]$ and $g(t)$ is a continuous on $[0, T]$ function.

Then the following proposition holds.

Proposition 1 *The exact solution to the Cauchy problem* (11), (12) *is given by*

$$x(t) = (\mathcal{T}x)(t) := x_0 + (\mathbb{E}^{\gamma}_{\rho,\mu,\omega,0+} g)(t), \tag{13}$$

where $\mathcal{T}$ is an inverse operator to the regularized HP fractional differential operator (8).

Proof By a semigroup property for the Prabhakar integral (4) and using equality (9), we get:

$$({}^C D^{\gamma,\mu}_{\rho,\omega,0+} x)(t) = \left(\mathbb{E}^{-\gamma}_{\rho,1-\mu,\omega,0+} \frac{d}{dt} x\right)(t) = g(t).$$

Applying this relation to the left and right hand-sides of equation (11) we derive that

$$\left(\mathbb{E}^{\gamma}_{\rho,\mu-1,\omega,0+} \left(\mathbb{E}^{-\gamma}_{\rho,1-\mu,\omega,0+} \frac{d}{dt} x\right)\right)(t) = (\mathbb{E}^{\gamma}_{\rho,\mu-1,\omega,0+} g)(t),$$

$$\frac{dx}{dt} = (\mathbb{E}^{\gamma}_{\rho,\mu-1,\omega,0+} g)(t) = \int_0^t (t-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}[\omega(t-s)^\rho] g(s) ds,$$

$$x(t) = x(0) + \int_0^t \int_0^\tau (\tau - s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}[\omega(\tau - s)^\rho] g(s) ds d\tau.$$

By writing the repeated integral in the last equality as a single integral we obtain that

$$x(t) = x(0) + \int_0^t (t - s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t - s)^\rho] g(s) ds. \tag{14}$$

This finishes the proof. □

Definition 6 For any non-negative vector $\beta \in \mathbb{R}^n$ of the form

$$\beta := \frac{T^\mu}{2^{2\mu-1}} E^{\gamma}_{\rho,\mu+1}[\omega T^\rho] M, \tag{15}$$

under a componentwise $\beta-$neighbourhood of a point $z_0 \in \mathbb{R}^n$ we understand a collection of points defined as

$$B(z, \beta) := \left\{z_0 \in \mathbb{R}^n : |z_0 - z| \le \beta\right\}, \tag{16}$$

where $M \in \mathbb{R}^n$ is a given constant vector with non-negative entries.

Definition 7 For a given bounded connected set $D_0 \subset \mathbb{R}^n$, we introduce its componentwise $\beta-$neighborhood by

$$D := B(D_0, \beta). \tag{17}$$

Definition 8 For a set $D \subset \mathbb{R}^n$, closed interval $[a, b] \subset \mathbb{R}$, Caratheodory function $f : [a, b] \times D \to \mathbb{R}^n$ and an n–dimensional square matrix K with non-negative entries, we write

$$f \in Lip(K, D) \tag{18}$$

if the inequality

$$|f(t, u) - f(t, v)| \le K\,|u - v| \tag{19}$$

holds, for all $\{u, v\} \subset D$ and a.e. $t \in [a, b]$.

3 Problem setting

In this paper we study a periodic BVP for a system of fDEs

$$^C D^{\gamma,\mu}_{\rho,\omega,0+} x(t) = f(t, x(t)),\ \ t \in [0, T], x, f \in \mathbb{R}^n,\ \ \mu \in (0, 1), \tag{20}$$

$$x(0) = x(T), \tag{21}$$

where $^C D^{\gamma,\mu}_{\rho,\omega,0+}$ is the regularised HP differential operator, defined by (9), (10), $x * e^{-\gamma}_{\rho,\mu,\omega}(\cdot) \in AC^1[0, T]$ and $x : [0, T] \to D$, with $D \subset \mathbb{R}^n$ being a closed and bounded domain.

Let us perturb differential equation (20) by a constant vector Δ:

$$^C D^{\gamma,\mu}_{\rho,\omega,0+} x(t) = f(t, x(t)) + \Delta, \tag{22}$$

and consider it together with initial condition (12).

From Proposition 1 we know that the Cauchy problem (22), (12) can be written in an equivalent integral form as follows:

$$x(t) = x_0 + \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] f(s, x(s)) ds + \Delta t^{\mu} E^{\gamma}_{\rho,\mu+1}[\omega t^{\rho}]. \quad (23)$$

In order to find the perturbation term Δ we enforce $x(t)$ in (23) to also satisfy periodic constraints (21). Simple calculations show that

$$x(0) = x_0,$$
$$x(T) = x_0 + \int_0^T (T-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}] f(s, x(s)) ds + \Delta T^{\mu} E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}],$$

and thus,

$$\int_0^T (T-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}] f(s, x(s)) ds + \Delta T^{\mu} E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}] = 0.$$

Using the last equality as an equation with respect to the unknown Δ, we can find its explicit form that reads

$$\begin{aligned} \Delta &= -\frac{\int_0^T (T-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}] f(s, x(s)) ds}{T^{\mu} E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]} \\ &= -\frac{1}{T^{\mu} E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]} \left(f * e^{-\gamma}_{\rho,\mu,\omega} \right)(T). \end{aligned}$$

Thus, an exact solution of the perturbed differential equation (22) under initial and periodic boundary conditions (12), (21) is given by

$$\begin{aligned} x(t) = x_0 &+ \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] f(s, x(s)) ds \\ &- \theta(t) \int_0^T (T-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}] f(s, x(s)) ds, \end{aligned} \quad (24)$$

where

$$\theta(t) = \frac{\int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] ds}{\int_0^T (T-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}] ds} = \left(\frac{t}{T}\right)^{\mu} \frac{E^{\gamma}_{\rho,\mu+1}[\omega t^{\rho}]}{E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]} \leq 1. \quad (25)$$

Remark 1 There are two questions that arise:

(i) How to find the exact solution (24), if function f also depends on x?
(ii) What is the relation between the original BVP (20), (21) and the perturbed Cauchy problem (22), (12).

In the following sections we will address both of these questions simultaneously.

4 Numerical-analytic approximations

Let us connect with the periodic BVP (20), (21) a parametrized sequence of functions $\{x_m(t, x_0)\}$ defined by a recursive relation:

$$x_m(t, x_0) = x_0 + \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] f(s, x_{m-1}(t, x_0)) ds$$

$$-\theta(t)\int_0^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]f(s,x_{m-1}(t,x_0))ds, \quad m\in\mathbb{N}, \tag{26}$$

where $t\in[0,T]$, $\theta(t)$ is given by (25) and

$$x_0(t,x_0)=x_0 \tag{27}$$

is taken as a zeroth approximation, with x_0 being an unknown parameter.

Note, that sequence (26), (27) is constructed in such a way that it satisfies periodic boundary conditions (21) beforehand, and it is of the form (24).

4.1 Convergence result

For the sequence of functions (26), (27) the following convergence result holds.

Theorem 2 *Assume that*

(i) there exists a non-negative vector β, satisfying inequality (16);
(ii) function $f: G_f\to\mathbb{R}^n$ satisfies Caratheodory and Lipschitz conditions $f\in Lip(K,D)$ in the domain D of the form (17) with matrix K;
(iii) for a spectral radius of matrix

$$Q=\frac{T^{\mu}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]}{2^{2\mu-1}}K \tag{28}$$

an estimate

$$r(Q)<1 \tag{29}$$

holds.

Then, for all fixed $x_0\in D_0$:

1. *Functions of the sequence (26) are absolutely continuous for $t\in[0,T]$, have values in the domain D and satisfy periodic boundary conditions*

$$x_m(0,x_0)=x_m(T,x_0).$$

2. *Sequence of functions (26) converges uniformly for $t\in[0,T]$ as $m\to\infty$ to the limit function*

$$x_{\infty}(t,x_0)=\lim_{m\to\infty}x_m(t,x_0). \tag{30}$$

3. *The limit function satisfies initial condition*

$$x_{\infty}(0,x_0)=x_0 \tag{31}$$

and periodic boundary conditions

$$x_{\infty}(0,x_0)=x_{\infty}(T,x_0).$$

4. *Function $x_{\infty}(\cdot,x_0)$ is a unique absolutely continuous solution of the integral equation*

$$x(t)=x_0+\int_0^t (t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]f(s,x(s))ds$$
$$-\theta(t)\int_0^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]f(s,x(s))ds, \tag{32}$$

where $\theta(t)$ is given by (25). In other words, $x_\infty(\cdot, x_0)$ satisfies the Cauchy problem for a modified system of fDEs:

$$^C D^{\gamma,\mu}_{\rho,\omega,0^+} x(t) = f(t, x(t)) + t^\mu E^\gamma_{\rho,\mu+1}[\omega t^\rho]\Delta(x_0), \tag{33}$$

$$x(0) = x_0, \tag{34}$$

where $\Delta : D_0 \times \Lambda \to \mathbb{R}^n$ is a mapping given by formula

$$\begin{aligned} \Delta(x_0) &:= -\frac{1}{T^\mu E^\gamma_{\rho,\mu+1}[\omega T^\rho]} \int_0^T (T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] f(s, x(s))ds \\ &= -\frac{1}{T^\mu E^\gamma_{\rho,\mu+1}[\omega T^\rho]} \left(f * e^{-\gamma}_{\rho,\mu,\omega}\right)(T). \end{aligned} \tag{35}$$

5. The following error estimate holds:

$$|x_\infty(t, x_0) - x_m(t, x_0)| \le \frac{T^\mu}{2^{2\mu-1}} E^\gamma_{\rho,\mu+1}[\omega T^\rho] Q^m (I_n - Q)^{-1} M, \tag{36}$$

where Q and $E^\gamma_{\rho,\mu+1}[\cdot]$ are given by (28) and (3) respectively, and M is such that $|f(t,x)| \le M$, for all $(t,x) \in G_f$.

In order to prove Theorem 2 we first need to show some auxiliary results.

Lemma 3 *Let $f * e^{-\gamma}_{\rho,\mu,\omega}(\cdot) \in AC^1[0,T]$. Then for all $t \in [0,T]$ the following estimate is true:*

$$\begin{aligned} &\left| \int_0^t (t-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(t-s)^\rho] f(s)ds - \theta(t) \int_0^T (T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] f(s)ds \right| \\ &\qquad \le \alpha_1(t) \max_{t\in[a,b]} |f(t)|, \end{aligned} \tag{37}$$

where $\theta(t)$ is defined by (25) and

$$\alpha_1(t) = 2\theta(t)(T-t)^\mu E^\gamma_{\rho,\mu+1}[\omega(T-t)^\rho]. \tag{38}$$

Proof It is obvious that

$$\begin{aligned} &\left| \int_0^t (t-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(t-s)^\rho] f(s)ds - \theta(t) \int_0^T (T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] f(s)ds \right| \\ &= \left| \int_0^t (t-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(t-s)^\rho] f(s)ds - \theta(t) \int_0^t (T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] f(s)ds \right. \\ &\quad \left. - \theta(t) \int_t^T (T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] f(s)ds \right| \\ &\le \int_0^t \left| (t-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(t-s)^\rho] - \theta(t)(T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] \right| |f(s)|ds \\ &\quad + \theta(t) \int_t^T (T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] |f(s)|ds \\ &\le 2\theta(t)(T-t)^\mu E^\gamma_{\rho,\mu+1}[\omega(T-t)^\rho] \max_{t\in[0,T]} |f(t)| = \alpha_1(t) \max_{t\in[0,T]} |f(t)|. \end{aligned}$$

This finishes the proof. □

Lemma 4 *Let* $\{\alpha_m(t)\}_{m\in\mathbb{N}}$ *be a sequence of continuous functions at the interval* $[0, T]$*, given by*

$$\alpha_{m+1}(t) := \int_0^t \left\{(t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] - \theta(t)(T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\right\}\alpha_m(s)ds \\ + \theta(t)\int_t^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\alpha_m(s)ds,\ m\in\mathbb{N}_0, \tag{39}$$

where $\alpha_0(t) = 1$ *and* $\alpha_1(t)$ *is defined by formula* (38)*. Then the following estimate holds:*

$$\alpha_{m+1}(t) \le \frac{T^{(m+1)\mu}}{2^{(m+1)(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^{m+1}, \tag{40}$$

for $m \in \mathbb{N}_0$ *and* $\rho > 0$.

Proof By setting in (39) $m = 0$ and using the fact that

$$\begin{aligned}\alpha_1(t) &= 2\theta(t)(T-t)^{\mu}E^{\gamma}_{\rho,\mu+1}[\omega(T-t)^{\rho}] \\ &= 2t^{\mu}\left(1-\frac{t}{T}\right)^{\mu}\frac{E^{\gamma}_{\rho,\mu+1}[\omega t^{\rho}]}{E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]}E^{\gamma}_{\rho,\mu+1}[\omega(T-t)^{\rho}],\end{aligned}$$

we get the following estimate:

$$\begin{aligned}\alpha_1(t) \quad &= 2t^{\mu}\left(1-\tfrac{t}{T}\right)^{\mu}\frac{E^{\gamma}_{\rho,\mu+1}[\omega(t)^{\rho}]}{E^{\gamma}_{\rho,\mu+1}[\omega(T)^{\rho}]}E^{\gamma}_{\rho,\mu+1}[\omega(T-t)^{\rho}] \\ &= 2T^{\mu}\left(\tfrac{t}{T}\right)^{\mu}\left(1-\tfrac{t}{T}\right)^{\mu}\frac{E^{\gamma}_{\rho,\mu+1}[\omega(t)^{\rho}]}{E^{\gamma}_{\rho,\mu+1}[\omega(T)^{\rho}]}E^{\gamma}_{\rho,\mu+1}[\omega(T-t)^{\rho}] \\ &\le \tfrac{T^{\mu}}{2^{2\mu-1}}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}],\end{aligned} \tag{41}$$

where we used an elementary inequality:

$$ab \le \frac{(a+b)^2}{4}.$$

Next for $m = 1$ from the sequence (39) and taking into account inequality (41) we obtain:

$$\begin{aligned}\alpha_2(t) &= \int_0^t \left\{(t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] - \theta(t)(T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\right\}\alpha_1(s)ds \\ &\quad + \theta(t)\int_t^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\alpha_1(s)ds \\ &\le 2\theta(t)(T-t)^{\mu}E^{\gamma}_{\rho,\mu+1}[\omega(T-t)^{\rho}]\frac{T^{\mu}}{2^{2\mu-1}}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}] \\ &\le \frac{T^{2\mu}}{2^{2(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^2.\end{aligned} \tag{42}$$

Now we assume that for $m = (n-1)$ inequality (40) holds and takes the form:

$$\alpha_n(t) = \frac{T^{n\mu}}{2^{n(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^n. \tag{43}$$

And finally we prove (40) for $m = n$:

$$\alpha_{n+1}(t) = \int_0^t \left\{(t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] - \theta(t)(T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\right\}\alpha_n(s)ds$$

$$
\begin{aligned}
&+\theta(t)\int_t^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\alpha_n(s)ds\\
&\le \left(\int_0^t \left\{(t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]-\theta(t)(T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\right\}ds\right.\\
&\left.+\theta(t)\int_t^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]ds\right)\frac{T^{n\mu}}{2^{n(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^n\\
&=\alpha_1(t)\frac{T^{n\mu}}{2^{n(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^n \le \frac{T^{(n+1)\mu}}{2^{(n+1)(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^{n+1}. \qquad (44)
\end{aligned}
$$

The last inequality coincides with (40) and thus, the proof is completed. □

Proof of Theorem 2. Statement 1 of the theorem follows by direct substitution of the sequence (26) into the periodic boundary conditions (21).

Next we show that, independently from the number of iterations, all functions x_m of the sequence (26) will remain in the domain D of their definition. To prove this we use the mathematical induction method.

Indeed, for $m=1$ we get an inequality:

$$
\begin{aligned}
&|x_1(t,x_0)-x_0(t,x_0)|\\
&=\left|\int_0^t (t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]f(s,x_0)ds-\theta(t)\int_0^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]f(s,x_0)ds\right|\\
&\le \alpha_1(t)|f(t,x_0)|\le \alpha_1(t)M\le \frac{T^{\mu}}{2^{2\mu-1}}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]M:=\beta, \qquad (45)
\end{aligned}
$$

where $M:=\max_{(t,x)\in G_f}|f(t,x)|$.

Next, for $m=2$ the following estimate holds:

$$
\begin{aligned}
&|x_2(t,x_0)-x_0(t,x_0)|\\
&=\left|\int_0^t (t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]f(s,x_1(s,x_0))ds\right.\\
&\left.-\theta(t)\int_0^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]f(s,x_1(s,x_0))ds\right|\\
&\le \alpha_1(t)M\le \frac{T^{\mu}}{2^{2\mu-1}}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]M=\beta.
\end{aligned}
$$

Finally, assume that for $(m-1)$:

$$
|x_{m-1}(t,x_0)-x_0(t,x_0)|\le \alpha_1(t)M\le \frac{T^{\mu}}{2^{2\mu-1}}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]M,
$$

and let us prove it for a general m:

$$
\begin{aligned}
&|x_m(t,x_0)-x_0(t,x_0)|\\
&=\left|\int_0^t (t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]f(s,x_{m-1}(s,x_0))ds\right.\\
&\left.-\theta(t)\int_0^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]f(s,x_{m-1}(s,x_0))ds\right|\\
&\le \frac{T^{\mu}}{2^{2\mu-1}}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]M=\beta.
\end{aligned}
$$

Indeed every function $x_m(t, x_0)$ of the sequence (26) remains in the domain D for all $m \in \mathbb{N}$.

Let us now estimate differences of the form $|x_{m+1}(\cdot, x_0) - x_m(\cdot, x_0)|$. For $m = 0$ we have already obtained the inequality

$$|x_1(t, x_0) - x_0(t, x_0)| \leq \alpha_1(t)M \leq \frac{T^{\mu}}{2^{2\mu-1}} E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]M.$$

Then for $m = 1$ it is easy to derive that

$$\begin{aligned}|x_2(t, x_0) - x_1(t, x_0)| = \Bigg| &\int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]\{f(s, x_1(s, x_0)) - f(s, x_0)\}\,ds \\ &- \theta(t)\int_0^T (T-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\{f(s, x_1(s, x_0)) - f(s, x_0)\}\,ds\Bigg| \\ &\leq K\alpha_2(t)M \leq \frac{T^{2\mu}}{2^{2(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^2 KM.\end{aligned}$$

Under assumption that for $(m-1)$ an estimate

$$|x_m(t, x_0) - x_{m-1}(t, x_0)| \leq \frac{T^{m\mu}}{2^{m(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^m K^{m-1}M$$

holds, we prove it for m. So we obtain the following result:

$$\begin{aligned}&|x_{m+1}(t, x_0) - x_m(t, x_0)| \\ &= \Bigg|\int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]\{f(s, x_m(s, x_0)) - f(s, x_{m-1}(s, x_0))\}ds \\ &\quad - \theta(t)\int_0^T (T-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\{f(s, x_m(s, x_0)) - f(s, x_{m-1}(s, x_0))\}ds\Bigg| \\ &\leq K^m\alpha_{m+1}(t)M \leq \frac{T^{(m+1)\mu}}{2^{(m+1)(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^{m+1}K^mM \\ &= \frac{T^{\mu}}{2^{2\mu-1}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])Q^mM,\end{aligned}$$

where matrix Q is given by (28).

Summarizing, in view of (26), we get the following inequality

$$\begin{aligned}|x_{m+j}(t, x_0) - x_m(t, x_0)| &\leq \sum_{k=1}^{j} |x_{m+k}(t, x_0) - x_{m+k-1}(t, x_0)| \\ &\leq \sum_{k=1}^{j} K^{m+k-1}\alpha_{m+k}(t)M \leq \sum_{k=1}^{j} \frac{T^{(m+k)\mu}}{2^{(m+k)(2\mu-1)}}(E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}])^{m+k}K^{m+k-1}M \\ &\leq \frac{T^{\mu}}{2^{2\mu-1}}E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]Q^m\sum_{k=0}^{j-1} Q^kM.\end{aligned} \tag{46}$$

Since the maximum eigenvalue of matrix Q of the form (28) is less than 1, we get the following relations:

$$\sum_{k=0}^{j-1} Q^k \leq (I_n - Q)^{-1}, \quad \lim_{m\to\infty} Q^m = O_n,$$

where O_n is the n–dimensional matrix of zeros. Letting $j \to \infty$ in (46), we derive estimate (36). According to the Cauchy criteria, sequence of functions $\{x_m\}$, defined by (40), uniformly converges in the domain $[0, T] \times D_0$ to the limit function $x_\infty(\cdot, x_0)$.

Since all functions of the sequence (26) satisfy periodic conditions (21), limit function (30) satisfies them as well. Passing in (26) to the limit as $m \to \infty$, we get that function $x_\infty(\cdot, x_0)$ satisfies integral equation (32).

In order to show that (32) has a unique continuous solution, suppose that $x_1(t)$ and $x_2(t)$ be two distinct solutions of (32). Then by evaluating their difference we get:

$$\begin{aligned}
&|x_1(t) - x_2(t)| \\
&\leq K\Bigg[\int_0^t \Big((t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] - \theta(t)(T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]\Big)ds \\
&\quad + \theta(t)\int_t^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]ds\Bigg] \max_{t\in[0,T]} |x_1(t) - x_2(t)| \\
&= K\alpha_1(t) \max_{t\in[0,T]} |x_1(t) - x_2(t)| \\
&\leq \frac{T^{\mu}}{2^{2\mu-1}} E^{\gamma}_{\rho,\mu+1}[\omega T^{\rho}]K \max_{t\in[0,T]} |x_1(t) - x_2(t)| = Q \max_{t\in[a,b]} |x_1(t) - x_2(t)|,
\end{aligned}$$

for all $t \in [0, T]$. Hence

$$\max_{t\in[0,T]} |x_1(t) - x_2(t)| \leq Q \max_{t\in[0,T]} |x_1(t) - x_2(t)|,$$

which by (29) gives $\max_{t\in[0,T]} |x_1(t) - x_2(t)| = 0$, so $x_1(t) = x_2(t)$ for all $t \in [0, T]$. Furthermore, the IVP (33), (34) is equivalent to the integral equation

$$\begin{aligned}
x(t) &= x_0 + \int_0^t (t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}](f(s, x(s))ds + \Delta(x_0)t^{\mu}E^{\gamma}_{\rho,\mu+1}[\omega t^{\rho}] \\
&= x_0 + \int_0^t (t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}]f(s, x(t))ds \\
&\quad - \theta(t)\int_0^T (T-s)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(T-s)^{\rho}]f(s, x(t))ds,
\end{aligned} \tag{47}$$

where the perturbation term $\Delta(x_0)$ is given by (35) and $\theta(t)$ is defined by formula (25).

By comparing (47) with (32), and recalling that $x_\infty(t, x_0)$ is the unique continuous solution of (47), we see that $x(t) = x_\infty(t, x_0)$ in (47), i.e., $x_\infty(t, x_0)$ is the unique solution of (33), (34). This completes the proof. □

4.2 Relation of the limit function $x_\infty(t, x_0)$ to solution of the fBVP (20), (21)

Let us consider a Cauchy problem for a differential equation with a constant perturbation:

$$\begin{aligned}
{}^{C}D^{\gamma,\mu}_{\rho,\omega,0+}x(t) &= f(t, x(t)) + t^{\mu}E^{\gamma}_{\rho,\mu+1}[\omega t^{\rho}]\mu, \\
x(0) &= x_0,
\end{aligned} \tag{48}$$

where $t \in [0, T]$ and $\mu \in \mathbb{R}^n$ being a parameter.

The following result holds.

Theorem 5 *Let $x_0 \in D_0$, $\mu \in \mathbb{R}^n$ be some given vectors, and suppose that all conditions of Theorem 2 hold for the system of fDEs (20).*

Then a solution $x = x(\cdot, x_0, \mu)$ *of the IVP* (48) *satisfies also boundary conditions* (21) *iff*

$$\mu := \Delta(x_0), \tag{49}$$

where $\Delta(x_0)$ *is given by* (35). *In that case*

$$x(t, x_0, \mu) = x_\infty (t, x_0) \,. \tag{50}$$

We will skip the proof of this theorem since it is very similar to the analogous results in Fečkan and Marynets (2023); Marynets and Pantova (2022).

Theorem 6 *Let conditions of Theorem 2 hold. Then* $x_\infty(\cdot, x_0^*)$ *is a solution of the fBVP* (20),(21) *iff parameter* x_0^* *is a solution of the determining system:*

$$\Delta(x_0) = 0, \tag{51}$$

where $\Delta(x_0)$ *is defined by formula* (35).

Proof The result follows directly from Theorem 5 by observing that the perturbed fDS (33) coincides with (20) if and only if the vector-parameter x_0^* satisfies system of determining equations (51). □

Remark 2 Some practical issues that might hinder us from calculating the exact solution $x(t)$ to the original periodic fBVP (20), (21) are hidden behind finding the limit function (31) and the exact roots x_0^* of the determining system (51). Due to the error estimate (36) that allows us to approximate the exact solution with high precision, one can re-consider the determining system in its approximate form, i.e.,

$$\Delta_m(x_0) = 0, \tag{52}$$

where $\Delta_m : D_0 \to \mathbb{R}^n$ is the m–th determining function defined by formula

$$\Delta_m(x_0) := -\frac{1}{T^\mu E^\gamma_{\rho,\mu+1}[(\omega T)^\rho]} \int_0^T (T-s)^{\mu-1} E^\gamma_{\rho,\mu}[\omega(T-s)^\rho] f(s, x_m(s, x_0)) ds,$$

and $x_m(\cdot, x_0)$ is the sequence given by (26). On each iteration step m we treat solutions $x_{0,m}$ of the approximate system (52) as the m-th approximation to their exact counterpart x_0^*. Substituting values $x_{0,m}$ into (26) we get the m-th approximation to the exact solution of the fBVP (20), (21) in the form $X_m(t) = x_m\left(t, x_{0,m}\right)$.

5 Numerical example

Example 1 Consider a periodic BVP for a nonlinear fractional differential equation of the HP type of the order $\mu = 1/2$:

$${}^C D^{\gamma,\mu}_{\rho,\omega,0+} y(t) = -\left(\frac{y(t)}{2\sqrt{2}\pi}\right)^2 - \sin(t) + t\cos^2(t) \;(:= f(t, y)), \tag{53}$$

$$y(0) = y(2\pi), \tag{54}$$

where $t \in (0, 2\pi)$, $y,\ f \in \mathbb{R}$, for numerical values of parameters γ, ρ and ω to be

$$\gamma = 1, \quad \rho = 1, \quad \omega = 0.$$

We aim to construct an approximate solution of (53), (54) *in a symmetric domain* D, *which is given by a closed interval* $[-17, 17]$, i.e.

$$D = \{y \in \mathbb{R} : |y| \leq 17\}.$$

It is easy to see that function $f(t, y)$ in the right hand-side of equation (53) is continuous in the domain $G = [0, 2\pi] \times [-17, 17]$. Moreover, direct computations show that f is bounded for all $(t, y) \in G$ and Lipschitz continuous in y, i.e. the following inequalities hold:

$$|f(t, y)| \leq 2\pi, \quad (t, y) \in G;$$
$$f(t, y) \in Lip(K, D) \text{ with } K = \frac{1}{4\pi^2}.$$

Additionally, based on the Mittag-Leffler function $E^{\gamma}_{\rho,\mu+1}[\omega(2\pi)^{\rho}]$ which was constructed using formula (3) up to the order 10, we find that constant Q in (28) is given by

$$Q = 0.0716449 < 1,$$

and that the neighborhood $B(y_0, \beta)$ in (16) of the initial value $y(0) = y_0$ of solution of the BVP (53), (54) is non-empty for

$$\beta = 17.77154.$$

This means that we can apply the numerical-analytic scheme (26), (27), described in Section 4 of the paper, to approximate solutions of the periodic BVP (53), (54). In this particular case it will be of the form:

$$\begin{aligned} y_m(t, y_0) = y_0 + \int_0^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-s)^{\rho}] f(s, y_{m-1}(t, y_0)) ds \\ -\theta(t) \int_0^{2\pi} (2\pi - s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(2\pi - s)^{\rho}] f(s, y_{m-1}(t, x_0)) ds, \quad m \in \mathbb{N}, \end{aligned} \tag{55}$$

where $t \in [0, 2\pi]$, $\theta(t) = \sqrt{\frac{t}{2\pi}}$ and

$$y_0(t, y_0) = y_0. \tag{56}$$

Moreover, Theorem 2 guarantees the uniform convergence of the sequence (55) to the exact solution of the studied problem, where the initial value y_0 will be chosen to satisfy the approximate determining equation:

$$\begin{aligned} \Delta_m(y_0) \; = -\frac{1}{(2\pi)^{\mu} E^{\gamma}_{\rho,\mu+1}[\omega(2\pi)^{\rho}]} \int_0^{2\pi} (2\pi - s)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(2\pi - s)^{\rho}] \\ f(s, y_m(s, y_0)) ds = 0. \end{aligned} \tag{57}$$

On the zeroth iteration step

$$y_0(t, y_0) = y_0$$

we obtain a quadratic determining equation

$$-2.833745 + 0.01266515 y_0^2 = 0$$

that has two real roots:

$$y_{0,0}^{-} = -14.95806, \quad y_{0,0}^{+} = 14.95806.$$

Note, that both values are in the domain D and thus, depending on the choice we make, we obtain the zeroth approximation to a positive or to a negative solution of the studied problem (53), (54).

Continuing our computations up to the order $m = 5$ we obtain the following pairs of initial values $y_{0,m}^{-}$ and $y_{0,m}^{+}$:

$$\begin{aligned} y_{0,1}^{-} &= -11.43829, \quad y_{0,1}^{+} = 16.36336; \\ y_{0,2}^{-} &= -11.15320, \quad y_{0,2}^{+} = 16.09520; \\ y_{0,3}^{-} &= -11.15880, \quad y_{0,3}^{+} = 16.07233; \\ y_{0,4}^{-} &= -11.18028, \quad y_{0,4}^{+} = 16.09158; \\ y_{0,5}^{-} &= -11.18925, \quad y_{0,5}^{+} = 16.08594. \end{aligned}$$

Substitution of each of those values into the approximate solution (55) on every iteration step $m = \overline{0, 5}$ leads to six approximations to the positive and negative solution of the periodic BVP (53), (54). We depict these approximations on Fig. 1.

Note, that both solutions co-exist in the domain D, never intersect the t axis and thus, do not have any intersection points. Hence Statement 4 of Theorem 2 about uniqueness of solution of the associated IVPs is not violated, which means that the periodic BVP (53), (54) has two solutions: one negative and one positive.

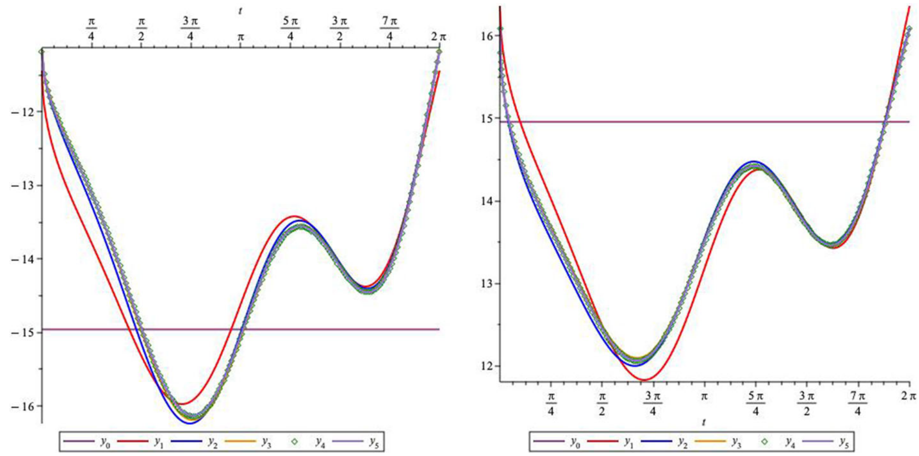

Fig. 1 Six approximations to the negative (on the left) and to the positive (on the right) solution of the periodic BVP (53), (54)

6 Cauchy type problem with *n*-initial conditions

The successive approximations approach of the previous sections can be also generalized to higher order fDEs of the HP type. We have already shown how one can relate a BVP to the corresponding IVP. In this section we would like to highlight some details about simplification and approximation of solutions of the Cauchy problems with n initial conditions.

In the space $AC^n[0, T]$ we consider a nonlinear Cauchy fractional model:

$$\left(\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}y\right)(t) = f(t, y(t)), \ \ t \in [0, T], \ f \in \mathbb{R}^n, \ 0 < \nu < 1, \ n-1 < \mu \le n, \tag{58}$$

$$y^{(k)}(0+) = b_k, \ k = 0, 1, 2, ..., n-1, \tag{59}$$

where $f \in Lip(K, D)$, with D being an open domain which contains a point $(0, y_0)$.

Since

$$\begin{aligned}
&(\mathbb{E}^{\gamma}_{\rho,\mu,\omega,0+}\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}y)(t) = \mathbb{E}^{\gamma}_{\rho,\mu,\omega,0+}\left(\mathbb{E}^{-\gamma\nu}_{\rho,\nu(n-\mu),\omega,0^+}\frac{d^n}{dt^n}\left(\mathbb{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0^+}y\right)\right)(t)\\
&= \left(\mathbb{E}^{\gamma-\gamma\nu}_{\rho,\mu+\nu(n-\mu),\omega,0^+}\frac{d^n}{dt^n}\left(\mathbb{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0^+}y\right)\right)(t)\\
&= \left(\mathbb{E}^{\gamma-\gamma\nu}_{\rho,\mu+(1-\nu)(n-\mu),\omega,0^+}({}^PD^{\gamma-\gamma\nu}_{\rho,n-(1-\nu)(n-\mu),\omega,0^+}y)\right)(t)\\
&= \left(\mathbb{E}^{\gamma-\gamma\nu}_{\rho,\mu+(1-\nu)(n-\mu),\omega,0+}\ {}^CD^{\gamma-\gamma\nu}_{\rho,\mu+(1-\nu)(n-\mu),\omega,0+}y\right)(t)\\
&\quad + \sum_{k=0}^{n-1}\mathbb{E}^{\gamma-\gamma\nu}_{\rho,\mu+\nu(n-\mu),\omega,0+}\left\{t^{k-\mu-(1-\nu)(n-\mu)}E^{-\gamma+\gamma\nu}_{\rho,k-\mu-\nu(n-\mu)+1}(\omega t^{\rho})\right\}y^{(k)}(0^+)\\
&= \left(\mathbb{E}^{\gamma-\gamma\nu}_{\rho,\mu+(1-\nu)(n-\mu),\omega,0^+}\mathbb{E}^{-\gamma+\gamma\nu}_{\rho,1-\mu-(1-\nu)(n-\mu),\omega,0^+}\frac{d^ny}{dt^n}\right)(t)\\
&\quad + \sum_{k=0}^{n-1}\mathbb{E}^{\gamma-\gamma\nu}_{\rho,\mu+\nu(n-\mu),\omega,0+}\left\{e^{-\gamma+\gamma\nu}_{\rho,k-\mu-\nu(n-\mu)+1,\omega}(t)\right\}y^{(k)}(0^+)\\
&= \frac{d^ny}{dt^n} + \sum_{k=0}^{n-1}\frac{t^k}{k!}b_k,
\end{aligned} \tag{60}$$

in view of homogeneous initial conditions (59), relation (60) simplifies to

$$\frac{d^ny}{dt^n} = (\mathbb{E}^{\gamma}_{\rho,\mu,\omega,0^+}f)(t, y(t)) - \sum_{k=0}^{n-1}\frac{t^k}{k!}b_k, \ \ t \in [0, T], \tag{61}$$

where $y^{(k)}(0+) = b_k, k = 0, 1, 2, .., n-1$. Then the following proposition holds.

Proposition 7 *The nonlinear Cauchy fractional model* (58) *with initial conditions* (59) *reduces to the ordinary integro-differential equation* (61) *under the same initial conditions.*

Example 2 We consider a function $f(t, y(t)) = \lambda y(t)$, where $\lambda \neq 0$. Then under initial conditions (59), we study a Cauchy type problem

$$\left(\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}y\right)(t) - \lambda y(t) = 0,$$

i.e.

$$\frac{d^n y}{dt^n} - \lambda(\mathbb{E}^{\gamma}_{\rho,\mu,\omega,0+}y)(t) + \sum_{k=0}^{n-1} \frac{t^k}{k!} b_k = 0.$$

Applying operator I^n_{0+} to the left hand side of the last equation, we get

$$y(t) = \lambda(\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}y)(t) - \sum_{k=0}^{n-1} \frac{t^{n+k}}{(n+k+1)!} b_k.$$

Let $y_0(t) = -\sum_{k=0}^{n-1} \frac{t^{n+k}}{(n+k+1)!} b_k$ and consider a sequence $\{y_m(t)\}_{m=1}^{\infty}$ defined by

$$y_m(t) = \lambda(\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}y_{m-1})(t) + \ y_0(t), \ m = 1, 2, \ldots.$$

Then,

$$\begin{aligned}
y_1(t) &= \lambda(\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}y_0)(t) + y_0(t), \\
y_2(t) &= \lambda(\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}y_1)(t) + y_0(t) \\
&= \lambda^2(\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}y_0)(t) + \lambda(\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}y_0)(t) + y_0(t) \\
&= \lambda^2(\mathbb{E}^{2\gamma}_{\rho,2(\mu+n),\omega,0+}y_0)(t) + \lambda(\mathbb{E}^{\gamma}_{\rho,\mu+n,\omega,0+}y_0)(t) + y_0(t).
\end{aligned}$$

By mathematical induction,

$$y_m(t) = y_0(t) + \sum_{j=1}^{m} \lambda^j (\mathbb{E}^{j\gamma}_{\rho,j(\mu+n),\omega,0+}y_0)(t),$$

where (see Theorem 4 in Kilbas et al. (2002))

$$(\mathbb{E}^{j\gamma}_{\rho,j(\mu+n),\omega,0+}y_0)(t) = \sum_{k=0}^{n-1} b_k t^{j(\mu+n)+n+k} E^{j\gamma}_{\rho,j(\mu+n)+n-k+1,\omega,0+}(\omega t^{\rho}), \ j = 1, 2, .., m.$$

Hence,

$$y_m(t) = y_0(t) + \sum_{k=0}^{n-1} b_k t^{n+k} \sum_{j=1}^{m} \lambda^j t^{(\mu+n)j} E^{j\gamma}_{\rho,j(\mu+n)+n-k+1,\omega,0+}(\omega t^{\rho}).$$

Passing in the last relation to the limit as $m \to \infty$, we obtain the following representation for solution $y(t)$:

$$y(t) = y_0(t) + \sum_{k=0}^{n-1} b_k t^{n+k} \sum_{j=1}^{\infty} \lambda^j t^{(\mu+n)j} E^{j\gamma}_{\rho,j(\mu+n)+n-k+1,\omega,0+}(\omega t^{\rho}).$$

The proof of convergence of the last series is presented in Sandev et al. (2011).

7 Final remarks

We want to stress, that the HP differential operator, used in this paper, generalizes the Riemann-Liouville and Hilfer operators for particular parameter values, and its regularized version contains the Caputo derivative, that is the most frequently used in modeling (see discussions in Diethelm and Freed (1999); Gaul et al. (1991); Glockle and Nonnenmacher (1995); Hilfer (2000); Mainardi (1997); Metzler et al. (1995); Garra et al. (2014)). An extensive literature overview does not show any evidence that the Caputo derivative is the only possible tool for description of complex phenomena in applied sciences. Thus, we believe that our results will not only contribute to the fundamental theory of fractional boundary value and Cauchy problems (which was our main aim in this work), but can also be used for validation and possible improvement of the existing mathematical models. One could think of comparison of a model with different types of fractional derivatives using data measurements, and defining which of them would most realistically reflect observations.

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