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DOI
10.1007/s11854-018-0013-3

Publication date
2018

Document Version
Accepted author manuscript

Published in
Journal d'Analyse Mathematique

CONICAL SQUARE FUNCTION ESTIMATES AND FUNCTIONAL CALCULI FOR PERTURBED HODGE-DIRAC OPERATORS IN L^p

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Abstract. Perturbed Hodge-Dirac operators and their holomorphic functional calculi, as investigated in the papers by Axelsson, Keith and the second author, provided insight into the solution of the Kato square-root problem for elliptic operators in L^2 spaces, and allowed for an extension of these estimates to other systems with applications to non-smooth boundary value problems. In this paper, we determine conditions under which such operators satisfy conical square function estimates in a range of L^p spaces, thus allowing us to apply the theory of Hardy spaces associated with an operator, to prove that they have a bounded holomorphic functional calculus in those L^p spaces. We also obtain functional calculi results for restrictions to certain subspaces, for a larger range of p. This provides a framework for obtaining L^p results on perturbed Hodge Laplacians, generalising known Riesz transform bounds for an elliptic operator L with bounded measurable coefficients, one Sobolev exponent below the Hodge exponent, and L^p bounds on the square-root of L by the gradient, two Sobolev exponents below the Hodge exponent. Our proof shows that the heart of the harmonic analysis in L^2 extends to L^p for all $p \in (1,\infty)$, while the restrictions in p come from the operator-theoretic part of the L^2 proof. In the course of our work, we obtain some results of independent interest about singular integral operators on tent spaces, and about the relationship between conical and vertical square functions.

Mathematics Subject Classification (2010): 47A60, 47F05, 42B30, 42B37

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1. Introduction

In [19], Axelsson, Keith, and the second author introduced a general framework to study various harmonic analytic problems, such as boundedness of Riesz transforms or the construction of solutions to boundary value problems, through the holomorphic functional calculus of certain first order differential operators that generalise the Hodge-Dirac operator $d + d^*$ (where d is the exterior derivative) of Riemannian geometry. By proving that such Hodge-Dirac operators have a bounded holomorphic functional calculus in L^2, they recover, in particular, the solution of Kato's square root problem obtained by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [10]. Their results also provide the harmonic analytic foundation to new approaches to problems in PDE (see e.g. [6–8]) and geometry (see e.g. [16]).

The main result in [19] is of a perturbative nature. Informally speaking, it states that the functional calculus of the standard Hodge-Dirac operator in L^2 is stable under perturbation by rough coefficients. It is natural, and important in applications, to know whether or not such a result also holds in L^p for $p \in (1, \infty)$. There are two main approaches to this question. The first one uses the extrapolation method pioneered by Blunck and Kunstmann in [21], and developed by Auscher in [4] (see also [31]) to show that the relevant L^2 bounds remain valid in certain intervals (p_-, p_+) about 2 which depend on the operator involved. This approach has been mostly developed to study second order differential operators, but has also been adapted to first order operators by Ajiev [1] and by Auscher and Stahlhut in [17,18]. The other approach to L^p estimates for the holomorphic functional calculus of Hodge-Dirac operators consists in adapting the entire machinery of [19] to L^p. This was done in the series of papers [34–36] by the second and third authors, together with Hytönen, using ideas from (UMD) Banach space valued harmonic analysis.

At the technical level, all these results are fundamentally perturbation results for square function estimates. In L^2, the heart of [19] is an estimate of the form

$$\left(\int_0^\infty \int_{\mathbb{R}^n} |t\Pi_B(I + t^2{\Pi_B}^2)^{-1}u(x)|^2 \frac{dxdt}{t}\right)^{\frac{1}{2}} \lesssim \left(\int_0^\infty \int_{\mathbb{R}^n} |t\Pi(I + t^2\Pi^2)^{-1}u(x)|^2 \frac{dxdt}{t}\right)^{\frac{1}{2}} \quad \forall u \in \mathcal{R}(\Gamma),$$

where $\Pi = \Gamma + \Gamma^*$ is a first order differential (Hodge-Dirac) operator with constant coefficients, and $\Pi_B = \Gamma + B_1\Gamma^* B_2$ is a perturbation by L^∞ coefficients B_1, B_2. See Section 2 for precise definitions. In L^p, the papers [34–36] establish analogues of the form

$$\|(\int_0^\infty |t\Pi_B(I+t^2{\Pi_B}^2)^{-1}u(.)|^2\frac{dt}{t})^{\frac{1}{2}}\|_p \lesssim \|(\int_0^\infty |t\Pi(I+t^2\Pi^2)^{-1}u(.)|^2\frac{dt}{t})^{\frac{1}{2}}\|_p \quad \forall u \in \mathcal{R}(\Gamma).$$

While these (vertical) L^p square function estimates are traditionally used to establish the boundedness of the holomorphic functional calculus (see e.g. [25]), the same result could also be obtained using the conical L^p square function estimates:

$$\|(t,x)\mapsto (t\Pi_B(I+t^2{\Pi_B}^2)^{-1})^M u(x)\|_{T^{p,2}} \lesssim \|(t,x)\mapsto (t\Pi(I+t^2\Pi^2)^{-1})^M u(x)\|_{T^{p,2}} \quad \forall u \in \mathcal{R}(\Gamma),$$

where M is a suitably large integer and $T^{p,2}$ is one of Coifman-Meyer-Stein's tent spaces (see [23] and Section 2 for precise definitions). This fact has been noticed in the development of a Hardy space theory associated with bisectorial operators (starting with [16,26,32], see also [37, Theorem 7.10]).

In this paper, we prove such conical L^p square function estimates for the Hodge-Dirac operators introduced in [19]. This allows us to strengthen the results from [34–36] (in the scalar-valued setting) by eliminating the R-boundedness assumptions. Similar results have been independently obtained in [18] as a further development of the extrapolation method. Here we aim, as in [34–36], to obtain not just L^p results, but L^p analogues of the techniques of [19]. Instead of relying on the probabilistic/dyadic methods of [35,36], we use the more flexible theory of Hardy spaces associated with operators, and recent results about integral operators on tent spaces. Our proof then exhibits an interesting phenomenon. As in [19] and other papers on functional calculus of Hodge-Dirac operators or Kato square root estimates, we consider separately the "high frequency" part of the estimate (involving $\|(t,x)\mapsto (t\Pi_B(I+t^2{\Pi_B}^2)^{-1})^M(I+t^2\Pi^2)^{-M})u(x)\|_{T^{p,2}}$), and the "low frequency" part (involving $\|(t,x)\mapsto (t\Pi_B(I+t^2{\Pi_B}^2)^{-1})^M(I-(I+t^2\Pi^2)^{-M})u(x)\|_{T^{p,2}}$). In L^2, the proof of the high frequency estimate is purely operator theoretic, while the low frequency requires the techniques from real analysis used in the solution of the Kato square root problem. In the approach to the L^p case given in [34–36], the same is true, but both the high and the low frequency estimate use an extra assumption: the R-bisectoriality of Π_B in L^p. With the approach through conical square function given here, we obtain the low frequency estimate for all $p\in(1,\infty)$ without any assumption on the L^p behaviour of the operator Π_B. Restrictions in p, and appropriate assumptions (which are necessary, as can be seen in [4]), are needed for the high frequency part. We believe that this will be helpful in future projects, as the theory moves away from the Euclidean setting (see e.g. the work of Morris [43], Bandara and the second author [20]). Dealing with a specific Hodge-Dirac operator in a geometric context, one can hope to prove sharp high frequency estimates using methods specific to the context at hand, and combine them with the harmonic analytic machinery developed here to get the full square function estimates, and hence the functional calculus result.

Another feature of the approach given here is that we obtain, from L^p assumptions, not just functional calculus results in L^p, but also functional calculus results on some subspaces of L^q for certain $q<p$. In particular, we obtain Riesz transform estimates for $q\in(p_*,2]$, and reverse Riesz transform estimates for $q\in(p_{**},2]$. Here p_* and p_{**} denote

the first and second Sobolev exponents below p. This can also be relevant in geometric settings, where one expects the results to depend not only on the geometry, but on the different levels of forms.

The paper is organised as follows. In Section 2, we give the relevant definitions and recall the main results from the theories that this paper builds upon. In Section 3, we state our main results - relevant high and low frequency square function estimates - and establish their functional calculus consequences as corollaries in Section 4. In Section 5, we prove low frequency estimates by developing L^p conical square function versions of the tools used in [19]. In Section 6, we prove high frequency estimates for $p \in (\max(1, 2_*), 2]$. In this range, the proof is straightforward, and does not require any L^p assumption. In dimensions 1 and 2 this already gives the result for all $p \in (1, \infty)$. In Section 7 we establish the relevant L^p-L^2 off-diagonal bounds for the resolvents of our Hodge-Dirac operator. In Section 8, we use them to bound a conical square function by a vertical square function related to the functional calculus of our Hodge-Dirac operator. In Section 9, we use these off-diagonal bounds to prove the high frequency estimates. This uses singular integral operator theory on tent spaces, and, in particular, Schur-type extrapolation results established in Section 10. We believe the latter results are of independent interest.

1.1. **Acknowledgments.** All three authors gratefully acknowledge support from the Australian Research Council through the Discovery Project DP120103692. This work is a key outcome of DP120103692. Frey and McIntosh also acknowledge support from ARC DP110102488. Portal is further supported by the ARC through the Future Fellowship FT130100607. The authors thank Pascal Auscher and Sylvie Monniaux for stimulating discussions, and Pascal Auscher in particular for keeping us aware of the progress of his student Sebastian Stahlhut on related questions. We are also grateful to the anonymous referee for giving us the opportunity to correct a mistake in the original manuscript, and for several other useful suggestions.

2. Preliminaries

2.1. **Notation.** Throughout the paper n and N denote two fixed positive natural numbers. We express inequalities “up to a constant" between two positive quantities a, b with the notation $a \lesssim b$. By this we mean that there exists a constant $C > 0$, independent of all relevant quantities in the statement, such that $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$.
We denote $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. For a Banach space X, we write $\mathcal{L}(X)$ for the set of all bounded linear operators on X.
For $p \in (1, \infty)$ and an unbounded linear operator A on $L^p(\mathbb{R}^n; \mathbb{C}^N)$, we denote by $\mathcal{D}_p(A), \mathcal{R}_p(A), \mathcal{N}_p(A)$ its domain, range and null space, respectively.
We use upper and lower stars to denote Sobolev exponents: For $p \in [1, \infty)$, we denote $p_* = \frac{np}{n+p}$ and $p^* = \frac{np}{n-p}$, with the convention $p^* = \infty$ for $p \geq n$.
For a ball (resp. cube) $B \subseteq \mathbb{R}^n$ with radius (resp. side length) $r > 0$ and given $\alpha > 0$, we write αB for the ball (resp. the cube) with the same centre and radius (resp. side length) αr. We define dyadic shells by $S_1(B) := 4B$ and $S_j(B) := 2^{j+1}B \setminus 2^j B$ for $j \geq 2$.

2.2. **Holomorphic functional calculus.** This paper deals with the holomorphic functional calculus of certain bisectorial first order differential operators. The fundamental results concerning this calculus have been developed in [15,25,39,41]. References for this theory include the lecture notes [2] and [40], and the book [28].

Definition 2.1. *Let* $0 \leq \omega < \mu < \frac{\pi}{2}$. *Define* closed and open sectors and double sectors *in the complex plane by*

$$\begin{aligned} S_{\omega+} &:= \{z \in \mathbb{C} : |\arg z| \leq \omega\} \cup \{0\}, \qquad S_{\omega-} := -S_{\omega+}, \\ S^o_{\mu+} &:= \{z \in \mathbb{C} : z \neq 0, |\arg z| < \mu\}, \qquad S^o_{\mu-} := -S^o_{\mu+}, \\ S_\omega &:= S_{\omega+} \cup S_{\omega-}, \qquad S^o_\mu := S^o_{\mu+} \cup S^o_{\mu-}. \end{aligned}$$

Denote by $H(S^o_\mu)$ the space of all holomorphic functions on S^o_μ. Let further

$$\begin{aligned} H^\infty(S^o_\mu) &:= \{\psi \in H(S^o_\mu) : \|\psi\|_{L^\infty(S^o_\mu)} < \infty\}, \\ \Psi^\beta_\alpha(S^o_\mu) &:= \{\psi \in H(S^o_\mu) : \exists C > 0 : |\psi(z)| \leq C|z|^\alpha(1 + |z|^{\alpha+\beta})^{-1} \ \forall z \in S^o_\mu\} \end{aligned}$$

for every $\alpha, \beta > 0$, and set $\Psi(S^o_\mu) := \bigcup_{\alpha,\beta>0} \Psi^\beta_\alpha(S^o_\mu)$. We say that $\psi \in \Psi(S^o_\mu)$ is *non-degenerate* if neither of the restrictions $\psi|_{S^o_{\mu\pm}}$ vanishes identically.

Definition 2.2. *Let* $0 \leq \omega < \frac{\pi}{2}$. *A closed operator D acting on a Banach space X is called* ω-bisectorial *if* $\sigma(D) \subset S_\omega$, *and for all* $\theta \in (\omega, \frac{\pi}{2})$ *there exists* $C_\theta > 0$ *such that*

$$\|\lambda(\lambda I - D)^{-1}\|_{\mathcal{L}(X)} \leq C_\theta \quad \forall \lambda \in \mathbb{C} \setminus S_\theta.$$

We say that D is *bisectorial* if it is ω-bisectorial for some $\omega \in [0, \frac{\pi}{2})$.
For D bisectorial with angle $\omega \in [0, \frac{\pi}{2})$ and $\psi \in \Psi(S^o_\mu)$ for $\mu \in (\omega, \frac{\pi}{2})$, we define $\psi(D)$ through the Cauchy integral

$$\psi(D) = \frac{1}{2\pi i} \int_\gamma \psi(z)(zI - D)^{-1} dz,$$

where γ denotes the boundary of S_θ for some $\theta \in (\omega, \mu)$, oriented counter-clockwise.

Definition 2.3. *Let* $0 \leq \omega < \frac{\pi}{2}$ *and* $\mu \in (\omega, \frac{\pi}{2})$. *An ω-bisectorial operator D, acting on a Banach space X, is said to have a* bounded H^∞ functional calculus with angle μ *if there exists* $C > 0$ *such that for all* $\psi \in \Psi(S^0_\mu)$

$$\|\psi(D)\|_{\mathcal{L}(X)} \leq C\|\psi\|_\infty.$$

For such an operator, the functional calculus extends to a bounded algebra homomorphism from $H^\infty(S^o_\mu)$ to $\mathcal{L}(X)$. More precisely, for all bounded functions $f : S^o_\mu \cup \{0\} \to \mathbb{C}$ which are holomorphic on S^o_μ, one can define a bounded operator $f(D)$ by

$$f(D)u = f(0)\mathbb{P}_{\mathcal{N}(D)} u + \lim_{n\to\infty} \psi_n(D)u, \qquad u \in X,$$

where $\mathbb{P}_{\mathcal{N}(D)}$ denotes the bounded projection onto $\mathcal{N}(D)$ with null space $\overline{\mathcal{R}(D)}$, and the functions $\psi_n \in \Psi(S^o_\mu)$ are uniformly bounded and tend locally uniformly to f on S^o_μ; see [2,25]. The definition is independent of the choice of the approximating sequence $(\psi_n)_{n\in\mathbb{N}}$.

2.3. **Off-diagonal bounds.** The operator theoretic property that captures the relevant aspect of the differential nature of our operators, is the following notion of off-diagonal bounds. This notion plays a central role in many current developments of singular integral operator theory. We refer to [4] for more information and references.

Definition 2.4. *Let $p \in [1,2]$. A family of operators $\{U_t\ ;\ t \in \mathbb{R}^*\} \subset \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^N))$ is said to have L^p-L^2 off-diagonal bounds of order $M > 0$ if there exists $C_M > 0$ such that for all $t \in \mathbb{R}^*$, all Borel sets $E, F \subseteq \mathbb{R}^n$ and all $u \in L^p(\mathbb{R}^n;\mathbb{C}^N)$ with $\operatorname{supp} u \subseteq F$, we have*

$$\|U_t u\|_{L^2(E)} \le C_M |t|^{-n(\frac{1}{p}-\frac{1}{2})} \left(1 + \frac{\operatorname{dist}(E,F)}{|t|}\right)^{-M} \|u\|_{L^p(F)}, \tag{2.1}$$

where $\operatorname{dist}(E,F) = \inf\{|x-y|;\ x \in E, y \in F\}$.

We also use the following variant, where, given a ball B, $\dot W^{1,p}_{\overline{B}}(\mathbb{R}^n;\mathbb{C}^N)$ denotes the space $W^{1,p}_{\overline{B}}(\mathbb{R}^n;\mathbb{C}^N) = \{u \in W^{1,p}(\mathbb{R}^n;\mathbb{C}^N); \operatorname{supp} u \in \overline{B}\}$ when taken with the norm $\|\nabla u\|_p$. These spaces interpolate in the complex method with respect to $p \in (1,\infty)$, as they can be identified with $W^{1,p}_0(B)$ through extension and restriction.
We also use the homogeneous spaces $\dot W^{1,p}(\mathbb{R}^n;\mathbb{C}^N)$ of distributions modulo constants, taken with the norm $\|\nabla u\|_p$.

Definition 2.5. *Let $p \in [1,2]$. A family of operators $\{U_t\ ;\ t \in \mathbb{R}^*\} \subset \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^N))$ is said to have $\dot W^{1,p}$-L^2 off-diagonal bounds of order $M > 0$ on balls, if there exists $C_M > 0$ such that for all $t \in \mathbb{R}^*$, all balls B of radius $|t|$, all $j \in \mathbb{N}$, and all $u \in W^{1,p}_{\overline{B}}(\mathbb{R}^n;\mathbb{C}^N)$ we have*

$$\|U_t u\|_{L^2(S_j(B))} \le C_M |t|^{-n(\frac{1}{p}-\frac{1}{2})} 2^{-jM} \|\nabla u\|_{L^p}, \tag{2.2}$$

The following properties of off-diagonal bounds with respect to composition and interpolation are essentially known (see [13]). We nonetheless include some proofs.

Lemma 2.6. *Let $p \in (1,2]$. Let $\{T_t\ ;\ t \in \mathbb{R}^*\} \subset \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^N))$ have L^2-L^2 off-diagonal bounds of every order, $\{V_t\ ;\ t \in \mathbb{R}^*\} \subset \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^N))$ have L^2-L^2 off-diagonal bounds of order M>0, $\{U_t\ ;\ t \in \mathbb{R}^*\} \subset \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^N))$ have $\dot W^{1,2}$-L^2 off-diagonal bounds of every order on balls, $\{Z_t\ ;\ t \in \mathbb{R}^*\} \subset \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^N))$ have $\dot W^{1,p}$-L^2 off-diagonal bounds of every order on balls, and $\{S_t\ ;\ t \in \mathbb{R}^*\} \subset \mathcal{L}(L^2(\mathbb{R}^n;\mathbb{C}^N))$ have L^p-L^2 off-diagonal bounds of order M, Then*

1. *If $\sup\limits_{t \in \mathbb{R}^*} \||t|^{n(\frac{1}{p}-\frac{1}{2})} T_t\|_{\mathcal{L}(L^p,L^2)} < \infty$, then for all $q \in (p,2]$, $\{T_t\ ;\ t \in \mathbb{R}^*\}$ has L^q-L^2 off-diagonal bounds of every order.*
2. *If $\sup\limits_{t \in \mathbb{R}^*} \sup\limits_{B=B(x,|t|)} \||t|^{n(\frac{1}{p}-\frac{1}{2})} U_t\|_{\mathcal{L}(\dot W^{1,p}_{\overline{B}},L^2)} < \infty$, then for all $q \in (p,2]$, $\{U_t\ ;\ t \in \mathbb{R}^*\}$ has $\dot W^{1,q}$-L^2 off-diagonal bounds of every order on balls.*
3. *If $\{T_t\ ;\ t \in \mathbb{R}^*\}$ has L^p-L^q off-diagonal bounds of every order for some $q \in [p,2]$, then $\sup\limits_{t \in \mathbb{R}^*} \|T_t\|_{\mathcal{L}(L^p)} < \infty$.*
4. *$\{V_t S_t\ ;\ t \in \mathbb{R}^*\}$ has L^p-L^2 off-diagonal bounds of order M, and $\{T_t Z_t\ ;\ t \in \mathbb{R}^*\}$ has $\dot W^{1,p}$-L^2 off-diagonal bounds of every order on balls.*

(5) For all $t \in \mathbb{R}^*$, T_t extends to an operator $T_t : L^\infty(\mathbb{R}^n;\mathbb{C}^N) \to L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{C}^N)$ with

$$\|T_t u\|_{L^2(B(x_0,t))} \lesssim |t|^{\frac{n}{2}} \|u\|_\infty \quad \forall u \in L^\infty(\mathbb{R}^n;\mathbb{C}^N),\ x_0 \in \mathbb{R}^n.$$

Proof. For (1), use Stein's interpolation [45, Theorem 1] for the analytic family of operators $\{|t|^{n(\frac{1}{p}-\frac{1}{2})z}(1+\frac{\text{dist}(E,F)}{|t|})^{M'(1-z)}\mathbb{1}_E T_t \mathbb{1}_F\ ;\ z \in S\}$, where $S = \{z \in \mathbb{C}\ ;\ Re(z) \in [0,1]\}$ and $M' \in \mathbb{N}$. For $q \in [p,2]$, this gives L^q-L^2 off-diagonal bounds of order $\max(0, M'(1-\frac{\frac{1}{q}-\frac{1}{2}}{\frac{1}{p}-\frac{1}{2}}))$, which implies the result by choosing M' large enough. A similar proof gives (2), this time using the fact that the spaces $\dot{W}^{1,p}_{\overline{B}}(\mathbb{R}^n;\mathbb{C}^N)$ interpolate in p by the complex method.

We refer to [4, Lemma 3.3] for a proof of (3).

We now turn to (4). Let $E, F \subset \mathbb{R}^n$ be two Borel sets, and $t \in \mathbb{R}^*$. Set $\delta = \text{dist}(E,F)$ and $G = \{x \in \mathbb{R}^n\,;\, \text{dist}(x,F) < \frac{\delta}{2}\}$. Then $\text{dist}(E,\overline{G}) \geq \frac{\delta}{2}$ and $\text{dist}(\mathbb{R}^n \setminus G, F) \geq \frac{\delta}{2}$. Observe that the assumptions on V_t and S_t in particular imply that $\sup_{t\in\mathbb{R}^*} \|V_t\|_{\mathcal{L}(L^2)} < \infty$ and $\sup_{t\in\mathbb{R}^*} \||t|^{n(\frac{1}{p}-\frac{1}{2})} S_t\|_{\mathcal{L}(L^p,L^2)} < \infty$ (taking $E = F = \mathbb{R}^n$ in the definition of off-diagonal bounds). We have the following for all $u \in L^p$:

$$\begin{aligned}
\|\mathbb{1}_E V_t S_t \mathbb{1}_F u\|_2 &\leq \|\mathbb{1}_E V_t \mathbb{1}_G S_t \mathbb{1}_F u\|_2 + \|\mathbb{1}_E V_t \mathbb{1}_{\mathbb{R}^n\setminus G} S_t \mathbb{1}_F u\|_2 \\
&\lesssim (1+\frac{\text{dist}(E,\overline{G})}{|t|})^{-M}\|\mathbb{1}_G S_t \mathbb{1}_F u\|_2 + \|\mathbb{1}_{\mathbb{R}^n\setminus G} S_t \mathbb{1}_F u\|_2 \\
&\lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}((1+\frac{\text{dist}(E,\overline{G})}{|t|})^{-M} + (1+\frac{\text{dist}(\mathbb{R}^n\setminus G,F)}{|t|})^{-M})\|\mathbb{1}_F u\|_p \\
&\lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}(1+\frac{\text{dist}(E,F)}{|t|})^{-M}\|\mathbb{1}_F u\|_p.
\end{aligned}$$

This proves (4) for $\{V_t S_t\ ;\ t \in \mathbb{R}^*\}$. For $\{T_t Z_t\ ;\ t \in \mathbb{R}^*\}$, a ball B of radius $|t|$, $j \in \mathbb{N}$, and $u \in \dot{W}^{1,p}_{\overline{B}}$, we have, for all $N \in \mathbb{N}$, that

$$\begin{aligned}
\|\mathbb{1}_{S_j(B)} T_t Z_t u\|_2 &\leq \sum_{k=1}^{\infty} \|\mathbb{1}_{S_j(B)} T_t \mathbb{1}_{S_k(B)} Z_t u\|_2 \lesssim \sum_{k=1}^{\infty} 2^{-|j-k|(N+1)} \|\mathbb{1}_{S_k(B)} Z_t u\|_2 \\
&\lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})} \sum_{k=1}^{\infty} 2^{-|j-k|(N+1)} 2^{-k(N+1)} \|\nabla u\|_p \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})} 2^{-jN} \|\nabla u\|_p.
\end{aligned}$$

(5) The extension $T_t : L^\infty(\mathbb{R}^n;\mathbb{C}^N) \to L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{C}^N)$ can be defined as

$$\mathbb{1}_Q(T_t u) = \lim_{\rho\to\infty} \sum_{\substack{R \in \Delta_{|t|} \\ \text{dist}(Q,R)<\rho}} \mathbb{1}_Q(T_t(\mathbb{1}_R u)),$$

where $u \in L^\infty(\mathbb{R}^n;\mathbb{C}^N)$, and $Q \in \Delta_{|t|}$ a dyadic cube in $\mathbb{R}^n$ (see the beginning of Section 5 for a definition of $\Delta_{|t|}$). It is shown in [19, Corollary 5.3] that the limit exists and the extension is well-defined. □

The next lemma was shown in [37, Lemma 7.3] (as in [16, Lemma 3.6]).

Lemma 2.7. *Suppose $M > 0$, $0 \leq \omega < \theta < \mu < \frac{\pi}{2}$. Let D be an ω-bisectorial operator in $L^2(\mathbb{R}^n;\mathbb{C}^N)$ such that $\{z(zI-D)^{-1}\,;\, z \in \mathbb{C}\setminus S_\theta\}$ has L^2-L^2 off diagonal bounds of order M in the sense that*

$$\|z(zI-D)^{-1}u\|_{L^2(E)} \leq C_M\,(1+|z|\operatorname{dist}(E,F))^{-M}\,\|u\|_2$$

for all $z \in \mathbb{C}\setminus S_\theta$, all Borel subsets $E,F \subseteq \mathbb{R}^n$, and all $u \in L^2(\mathbb{R}^n;\mathbb{C}^N)$ with $\operatorname{supp} u \subseteq F$. If $\psi \in \Psi_\beta^\alpha(S_\mu^o)$ for $\alpha > 0$, $\beta > M$, then $\{\psi(tD)\ ;\ t \in \mathbb{R}^*\}$ has L^2-L^2 off-diagonal bounds of order M.

2.4. **Tent spaces.** Let $p,q \in [1,\infty]$. Recall that the tent space $T^{p,q}(\mathbb{R}^{n+1}_+)$, first introduced by Coifman, Meyer, and Stein in [23], is the space of measurable functions $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ such that the following norm is finite:

$$\|F\|_{T^{p,q}} = \Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_{B(x,t)} |F(t,y)|^q\,\frac{dydt}{t^{n+1}}\Big)^{\frac{p}{q}}\,dx\Big)^{\frac{1}{p}},$$

for $p,q \in [1,\infty)$,

$$\|F\|_{T^{\infty,q}} = \sup_{(r,x)\in\mathbb{R}_+\times\mathbb{R}^n}\Big(r^{-n}\int_0^r\int_{B(x,r)} |F(t,y)|^q\,\frac{dydt}{t}\Big)^{\frac{1}{q}},$$

for $q \in (1,\infty)$.
For $p \in [1,\infty)$, we define $T^{p,\infty}$ as the space of measurable functions $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ such that the following norm is finite:

$$\|F\|_{T^{p,\infty}} = \|x \mapsto \operatorname{ess\,sup}\{|F(t,y)|\ ;\ (t,y) \in \mathbb{R}^{n+1}_+\ :\ |y-x| \leq t\}\|_{L^p}.$$

Note that, contrary to [23], we do not impose continuity of the functions in $T^{p,\infty}$, and thus use an essential supremum rather than a supremum. See [3] for a theory of $T^{p,\infty}$ spaces defined in this way. We use the facts that the tent spaces interpolate by the complex method, in the sense that $[T^{p_0,2},T^{p_1,2}]_\theta = T^{p_\theta,2}$ for $\theta \in [0,1]$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We also use that, for $p,q \in (1,\infty)$, the space $C_c^\infty(\mathbb{R}^{n+1}_+)$ is dense in $T^{p,q}$. We recall a basic result about tent spaces, and another about operators acting on them.

Lemma 2.8. *[5] Let $p \in [1,\infty)$, $\alpha \geq 1$ and $T^{p,2}_\alpha(\mathbb{R}^{n+1}_+)$ denote the completion of $C_c^\infty(\mathbb{R}^{n+1}_+)$ with respect to the norm*

$$\|F\|_{T^{p,2}_\alpha} = \Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_{B(x,\alpha t)} |F(t,y)|^2\,\frac{dydt}{t^{n+1}}\Big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}}.$$

Then $T^{p,2}_\alpha(\mathbb{R}^{n+1}_+) = T^{p,2}(\mathbb{R}^{n+1}_+)$ with the equivalence of norms

$$\|F\|_{T^{p,2}} \leq \|F\|_{T^{p,2}_\alpha} \lesssim \alpha^{\frac{n}{\min\{p,2\}}}\|F\|_{T^{p,2}} \quad \forall F \in T^{p,2}(\mathbb{R}^{n+1}_+).$$

Lemma 2.9. *[37, Theorem 5.2] Let $p \in (1,\infty)$. Let $\{T_t\}_{t>0}$ be a family of operators acting on $L^2(\mathbb{R}^n)$ with L^2-L^2 off-diagonal bounds of order $M > \frac{n}{min\{p,2\}}$. Then there exists $C > 0$ such that for all $F \in T^{p,2}(\mathbb{R}^{n+1}_+)$*

$$\|(t,x) \mapsto T_tF(t,\,.\,)(x)\|_{T^{p,2}} \leq C\|F\|_{T^{p,2}},$$

in the sense that the operator $\mathcal{T}$, initially defined for $F \in C_c^\infty(\mathbb{R}_+^{n+1})$ by $\mathcal{T}(F) : (t,x) \mapsto T_tF(t,\,.\,)(x)$ in $L^2_{loc}(\mathbb{R}_+^{n+1})$, extends to a bounded operator on $T^{p,2}$.

2.5. **Hardy spaces associated with bisectorial operators.** We consider Hardy spaces associated with bisectorial operators. We refer to [16,26,30,32,33,37] and the references therein for more details about such spaces, and just recall here the main definition and result.

Let $0 \leq \omega < \mu < \frac{\pi}{2}$, and D be an ω-bisectorial operator in $L^2(\mathbb{R}^n;\mathbb{C}^N)$ such that $\{(I + itD)^{-1}\ ;\ t \in \mathbb{R}\setminus\{0\}\}$ has L^2-L^2 off-diagonal bounds of order $M > \frac{n}{2}$. Assume further that D has a bounded H^∞ functional calculus with angle $\theta \in (\omega,\mu)$. Given $u \in L^2(\mathbb{R}^n;\mathbb{C}^N)$ and $\psi \in \Psi(S^o_\mu)$, write

$$\mathcal{Q}_\psi u(x,t) := \psi(tD)u(x), \qquad x \in \mathbb{R}^n,\ t > 0.$$

Definition 2.10. *Let $p \in [1,\infty)$, let $\psi \in \Psi(S^o_\mu)$ be non-degenerate. The Hardy space $H^p_{D,\psi}(\mathbb{R}^n;\mathbb{C}^N)$ associated with D and ψ is the completion of the space*

$$\{u \in \overline{\mathcal{R}_2(D)}\ :\ \mathcal{Q}_\psi u \in T^{p,2}(\mathbb{R}_+^{n+1};\mathbb{C}^N)\}$$

with respect to the norm

$$\|u\|_{H^p_{D,\psi}} := \|\mathcal{Q}_\psi u\|_{T^{p,2}}.$$

Let us also recall [37, Theorem 7.10]:

Theorem 2.11. *Let $\varepsilon > 0$. Let $p \in (1,2]$ and $\psi,\tilde{\psi} \in \Psi_\varepsilon^{\frac{n}{2}+\varepsilon}(S^o_\mu)$, or $p \in [2,\infty)$ and $\psi,\tilde{\psi} \in \Psi^\varepsilon_{\frac{n}{2}+\varepsilon}(S^o_\mu)$, where $\mu > \omega$ and both ψ and $\tilde{\psi}$ are non-degenerate. Then*

(1) $H^p_{D,\psi}(\mathbb{R}^n;\mathbb{C}^N) = H^p_{D,\tilde{\psi}}(\mathbb{R}^n;\mathbb{C}^N) =: H^p_D(\mathbb{R}^n;\mathbb{C}^N)$;

(2) For all $u \in H^p_D(\mathbb{R}^n;\mathbb{C}^N)$, and all $f \in \Psi(S^o_\mu)$, we have

$$\|(t,x) \mapsto \psi(tD)f(D)u(x)\|_{T^{p,2}} \lesssim \|f\|_\infty \|u\|_{H^p_D}.$$

In particular, D has a bounded H^∞ functional calculus on $H^p_D(\mathbb{R}^n;\mathbb{C}^N)$.

2.6. **Hodge-Dirac operators.** Throughout the paper, we work with the following class of Hodge-Dirac operators. It is a slight modification of the classes considered in [19] and [36].

Definition 2.12. *A* Hodge-Dirac operator with constant coefficients *is an operator of the form $\Pi = \Gamma + \Gamma^*$, where $\Gamma = -i\sum_{j=1}^n \hat{\Gamma}_j\partial_j$ is a Fourier multiplier with symbol defined by*

$$\hat{\Gamma} = \hat{\Gamma}(\xi) = \sum_{j=1}^n \hat{\Gamma}_j\xi_j \qquad \forall \xi \in \mathbb{R}^n,$$

with $\hat{\Gamma}_j \in \mathscr{L}(\mathbb{C}^N)$, the operator Γ is nilpotent, i.e. $\hat{\Gamma}(\xi)^2 = 0$ for all $\xi \in \mathbb{R}^n$, and there exists $\kappa > 0$ such that

$$\kappa|\xi||w| \leq |\hat{\Pi}(\xi)w| \qquad \forall w \in \mathcal{R}(\hat{\Pi}(\xi)),\ \forall \xi \in \mathbb{R}^n. \tag{Π1}$$

We list some results about these operators.

Proposition 2.13. *Suppose $p \in (1,\infty)$.*
(1) The operator identity $\Pi = \Gamma + \Gamma^*$ holds in $L^p(\mathbb{R}^n;\mathbb{C}^N)$, in the sense that $\mathcal{D}_p(\Pi) = \mathcal{D}_p(\Gamma) \cap \mathcal{D}_p(\Gamma^*)$ and $\Pi u = \Gamma u + \Gamma^* u$ for all $u \in \mathcal{D}_p(\Pi)$.
(2) There holds $\mathcal{N}_p(\Pi) = \mathcal{N}_p(\Gamma) \cap \mathcal{N}_p(\Gamma^*)$.
(3) Π Hodge decomposes $L^p(\mathbb{R}^n;\mathbb{C}^N)$ in the sense that

$$L^p(\mathbb{R}^n;\mathbb{C}^N) = \mathcal{N}_p(\Pi) \oplus \overline{\mathcal{R}_p(\Gamma)} \oplus \overline{\mathcal{R}_p(\Gamma^*)}\ ,$$

or equivalently, $L^p = \mathcal{N}_p(\Gamma) \oplus \overline{\mathcal{R}_p(\Gamma^*)}$ and $L^p = \mathcal{N}_p(\Gamma^*) \oplus \overline{\mathcal{R}_p(\Gamma)}$.
(4) $\mathcal{N}_p(\Gamma)$, $\mathcal{N}_p(\Gamma^*)$, $\overline{\mathcal{R}_p(\Gamma)}$ and $\overline{\mathcal{R}_p(\Gamma^*)}$ each form complex interpolation scales, $p \in (1,\infty)$.
(5) Hodge-Dirac operators with constant coefficients have a bounded H^∞ functional calculus in $L^p(\mathbb{R}^n;\mathbb{C}^N)$.
(6) There holds $\|\nabla \otimes u\|_p \lesssim \|\Pi u\|_p$ for all $u \in \mathcal{D}_p(\Pi) \cap \overline{\mathcal{R}_p(\Pi)}$.
(7) There exists a bounded potential map $S_\Gamma : \overline{\mathcal{R}_p(\Gamma)} \to \dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^N)$ such that $\Gamma S_\Gamma = I$ on $\overline{\mathcal{R}_p(\Gamma)}$; and there exists a bounded potential map $S_{\Gamma^*} : \overline{\mathcal{R}_p(\Gamma^*)} \to \dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^N)$ such that $\Gamma^* S_{\Gamma^*} = I$ on $\overline{\mathcal{R}_p(\Gamma^*)}$.

Proof. See [36], Lemma 5.3, Proposition 5.4. For (4), see [34]. Part (5) is proven in [36, Theorem 3.6]. Part (6) is a consequence of (Π1), as shown in [36, Proposition 5.2]. To prove part (7) for Γ, first note that, for all $u \in \overline{\mathcal{R}_p(\Gamma)}$, $u = \lim_{k\to\infty} \Gamma w_k$ where $w_k = k^2\Gamma^*(I + k^2\Pi^2)u$, so that $\|\nabla \otimes w_k\|_p \lesssim \|u\|_p$ by (6),
and further $(w_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^N)$. Define $S_\Gamma u = \lim_{k\to\infty} w_k$ in $\dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^N)$, and we obtain $\Gamma S_\Gamma u = u$, since Γ is a bounded operator from $\dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^N)$ to L^p. The same proof applies to Γ^*. □

We now consider *perturbed Hodge-Dirac operators.*

Definition 2.14. *A* perturbed Hodge-Dirac operator *is an operator of the form*

$$\Pi_B := \Gamma + \Gamma_B^* := \Gamma + B_1\Gamma^* B_2,$$

where $\Pi = \Gamma + \Gamma^*$ is a Hodge-Dirac operator with constant coefficients, and B_1, B_2 are multiplication operators by $L^\infty(\mathbb{R}^n;\mathscr{L}(\mathbb{C}^N))$ functions which satisfy

$$\Gamma^* B_2 B_1 \Gamma^* = 0 \quad \textit{in the sense that} \quad \mathcal{R}_2(B_2B_1\Gamma^*) \subset \mathcal{N}_2(\Gamma^*);$$
$$\Gamma B_1 B_2 \Gamma = 0 \quad \textit{in the sense that} \quad \mathcal{R}_2(B_1B_2\Gamma) \subset \mathcal{N}_2(\Gamma);$$

$$\mathrm{Re}(B_1\Gamma^* u, \Gamma^* u) \geq \kappa_1 \|\Gamma^* u\|_2^2, \qquad \forall u \in \mathcal{D}_2(\Gamma^*) \qquad \textit{and}$$
$$\mathrm{Re}(B_2\Gamma u, \Gamma u) \geq \kappa_2 \|\Gamma u\|_2^2, \qquad \forall u \in \mathcal{D}_2(\Gamma)$$

for some $\kappa_1, \kappa_2 > 0$. Let the angles of accretivity be

$$\omega_1 := \sup_{u\in\mathcal{R}(\Gamma^*)\setminus\{0\}} |\arg(B_1u,u)| < \frac{\pi}{2},$$
$$\omega_2 := \sup_{u\in\mathcal{R}(\Gamma)\setminus\{0\}} |\arg(B_2u,u)| < \frac{\pi}{2},$$

and set $\omega := \frac{1}{2}(\omega_1 + \omega_2)$.

Such operators satisfy the invertibility properties (denoting $\frac{1}{p'} = 1 - \frac{1}{p}$)

$$(\Pi_B(p)) \quad \|u\|_p \leq C_p \|B_1 u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma^*)} \quad \text{and} \quad \|v\|_{p'} \leq C_{p'} \|B_2{}^* v\|_{p'} \quad \forall v \in \overline{\mathcal{R}_{p'}(\Gamma)}$$

when $p = 2$.

In many cases they satisfy $(\Pi_B(p))$ for all $p \in (1, \infty)$, for example if B_1 and B_2 are invertible in L^∞, though in general all we can say is that the set of p for which $(\Pi_B(p))$ holds is open in $(1, \infty)$. This follows on applying the extrapolation result of Kalton and Mitrea ([38], Theorem 2.5) to the interpolation families $B_1 : \overline{\mathcal{R}_p(\Gamma^*)} \to L^p(\mathbb{R}^n)$ and $B_2{}^* : \overline{\mathcal{R}_{p'}(\Gamma)} \to L^{p'}(\mathbb{R}^n)$.

As noted in [36], it is a consequence of $(\Pi_B(p))$ that Γ_B^* is a closed operator in L^p with adjoint $(\Gamma_B^*)^* = B_2{}^* \Gamma B_1{}^*$ acting in $L^{p'}$, that $\overline{\mathcal{R}_p(\Gamma_B^*)} = B_1 \overline{\mathcal{R}_p(\Gamma^*)}$, and that $\overline{\mathcal{R}_{p'}(B_2{}^* \Gamma B_1{}^*)} = B_2{}^* \overline{\mathcal{R}_{p'}(\Gamma)}$. Moreover, if $(\Pi_B(p))$ holds for all p in a subinterval of $(1, \infty)$, then the spaces $\overline{\mathcal{R}_p(\Gamma_B^*)}$ interpolate for those p also.

Definition 2.15. *A perturbed Hodge-Dirac operator* Π_B Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^N)$ *for some* $p \in (1, \infty)$, *if* $(\Pi_B(p))$ *holds and there is a splitting into complemented subspaces*

$$L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(\Pi_B) \oplus \overline{\mathcal{R}_p(\Pi_B)} = \mathcal{N}_p(\Pi_B) \oplus \overline{\mathcal{R}_p(\Gamma)} \oplus \overline{\mathcal{R}_p(\Gamma_B^*)}.$$

It is proved in [19, Proposition 2.2] that Π_B Hodge decomposes $L^2(\mathbb{R}^n; \mathbb{C}^N)$.

In investigating the property of Hodge Decomposition, let $\mathbb{P}_q$ denote the bounded projection of $L^q(\mathbb{R}^n; \mathbb{C}^N)$ onto $\overline{\mathcal{R}_q(\Gamma^*)}$ with nullspace $\mathcal{N}_q(\Gamma)$, and let $\mathbb{Q}_q$ denote the bounded projection of $L^q(\mathbb{R}^n; \mathbb{C}^N)$ onto $\overline{\mathcal{R}_q(\Gamma)}$ with nullspace $\mathcal{N}_q(\Gamma^*)$ $(1 < q < \infty)$. When Π_B Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^N)$, we denote by $\mathbb{P}_{\overline{\mathcal{R}_p(\Pi_B)}}$ the projection of $L^p(\mathbb{R}^n; \mathbb{C}^N)$ onto $\overline{\mathcal{R}_p(\Pi_B)}$ with nullspace $\mathcal{N}_p(\Pi_B)$.

Proposition 2.16. *Let* Π_B *be a perturbed Hodge-Dirac operator, and let* $p \in (1, \infty)$. *Then* (i) Π_B *Hodge decomposes* $L^p(\mathbb{R}^n; \mathbb{C}^N)$ *if and only if both* (A) *and* (B) *hold, where*

$$\text{(A)} \qquad \|u\|_p \lesssim \|B_1 u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma^*)} \quad \text{and} \quad L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(\Gamma) \oplus B_1 \overline{\mathcal{R}_p(\Gamma^*)} \ ;$$

$$\text{(B)} \qquad \|v\|_{p'} \lesssim \|B_2{}^* v\|_{p'} \quad \forall v \in \overline{\mathcal{R}_{p'}(\Gamma)} \quad \text{and} \quad L^{p'}(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_{p'}(\Gamma^*) \oplus B_2{}^* \, \overline{\mathcal{R}_{p'}(\Gamma)} \ .$$

(ii) *Moreover* (A) *is equivalent to* (A'), *and* (B) *is equivalent to* (B') *where*

$$\text{(A')} \qquad \mathbb{P}_p B_1 : \overline{\mathcal{R}_p(\Gamma^*)} \to \overline{\mathcal{R}_p(\Gamma^*)} \quad \text{is an isomophism} \ ;$$

$$\text{(B')} \qquad \mathbb{Q}_{p'} B_2{}^* : \overline{\mathcal{R}_{p'}(\Gamma)} \to \overline{\mathcal{R}_{p'}(\Gamma)} \quad \text{is an isomophism} \ .$$

Proof. (i) Under the invertibility assumption $(\Pi_B(p))$, Π_B Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^N)$ if and only if both $L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(\Gamma) \oplus \overline{\mathcal{R}_p(\Gamma_B^*)}$ and $L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(\Gamma_B^*) \oplus \overline{\mathcal{R}_p(\Gamma)}$ hold, i.e. if and only if $L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(\Gamma) \oplus B_1 \overline{\mathcal{R}_p(\Gamma^*)}$ and $L^{p'}(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_{p'}(\Gamma^*) \oplus B_2{}^* \, \overline{\mathcal{R}_{p'}(\Gamma)}$ [36, Lemmas 6.1, 6.2]. This gives the proof of (i).

(ii) (A) implies (A'): Let $u \in \overline{\mathcal{R}_p(\Gamma^*)}$. Then $\mathbb{P}_p B_1 u = -(I - \mathbb{P}_p)B_1 u + B_1 u$ with, by (A), $\|\mathbb{P}_p B_1 u\|_p \approx \|(I - \mathbb{P}_p)B_1 u\|_p + \|B_1 u\|_p$, so that $\|u\|_p \lesssim \|B_1 u\|_p \lesssim \|\mathbb{P}_p B_1 u\|_p$. It remains to prove surjectivity. Let $v \in \overline{\mathcal{R}_p(\Gamma^*)}$. By (A), there exist $w \in \mathcal{N}_p(\Gamma)$ and $u \in \overline{\mathcal{R}_p(\Gamma^*)}$ such that $v = w + B_1 u$, and hence $v = \mathbb{P}_p v = \mathbb{P}_p B_1 u$ as claimed.

(A') implies (A): First we have that if $u \in \overline{\mathcal{R}_p(\Gamma^*)}$, then $\|u\|_p \lesssim \|\mathbb{P}_p B_1 u\|_p \lesssim \|B_1 u\|_p$. Next we show that $\mathcal{N}_p(\Gamma) \cap B_1\overline{\mathcal{R}_p(\Gamma^*)} = \{0\}$. Indeed if $u \in \mathcal{N}_p(\Gamma)$, and $u = B_1 v$ with $v \in \overline{\mathcal{R}_p(\Gamma^*)}$, then $\mathbb{P}_p B_1 v = \mathbb{P}_p u = 0$, so by (A'), $v = 0$ and thus $u = 0$. Now we show that every element $u \in L^p(\mathbb{R}^n; \mathbb{C}^N)$ can be decomposed as stated. Let $u \in L^p(\mathbb{R}^n; \mathbb{C}^N)$. Then

$$\begin{aligned} u &= (I - \mathbb{P}_p)u + \mathbb{P}_p u \\ &= (I - \mathbb{P}_p)u + \mathbb{P}_p B_1 v \qquad \text{for some } v \in \overline{\mathcal{R}_p(\Gamma^*)} \quad \text{(by (A'))} \\ &= (I - \mathbb{P}_p)(u - B_1 v) + B_1 v \\ &\in \mathcal{N}_p(\Gamma) \quad + \quad B_1\overline{\mathcal{R}_p(\Gamma^*)} \end{aligned}$$

with $\|B_1 v\|_p \lesssim \|v\|_p \lesssim \|\mathbb{P}_p u\|_p \lesssim \|u\|_p$. This gives the claimed direct sum decomposition.

The proof that (B) is equivalent to (B') follows the same lines, with p, Γ, B_1 replaced by $p', \Gamma^*, {B_2}^*$. □

Proposition 2.17. *The set of all p for which Π_B Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^N)$, is an open interval (p_H, p^H), where $1 \leq p_H < 2 < p^H \leq \infty$.*

Proof. By the interpolation properties of $\overline{\mathcal{R}_p(\Gamma^*)}$, the set of p for which (A') holds, is an open interval which contains 2, and the same can be said about (B'). So the set of all p for which Π_B Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^N)$ is the intersection of these two intervals, and thus is itself an open interval which we denote by (p_H, p^H), with $1 \leq p_H < 2 < p^H \leq \infty$. □

An investigation of Π_B involves the related operator $\underline{\Pi_B} = \Gamma^* + B_2 \Gamma B_1$, which is also a perturbed Hodge-Dirac operator with $(\Gamma, \Gamma^*, B_1, B_2)$ replaced by $(\Gamma^*, \Gamma, B_2, B_1)$, and for it we need the invertibility properties

$$(\underline{\Pi_B}(p)) \quad \|u\|_p \leq C_p \|B_2 u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma)} \quad \text{and} \quad \|v\|_{p'} \leq C_{p'} \|{B_1}^* v\|_{p'} \quad \forall v \in \overline{\mathcal{R}_{p'}(\Gamma^*)} \; .$$

The formulae connecting Π_B and $\underline{\Pi_B}$ are, for $\theta \in (\omega, \frac{\pi}{2})$, $f \in H^\infty(S^o_\theta)$ and $u \in \mathcal{D}_2(\Gamma^*)$,

$$\begin{aligned} f(\underline{\Pi_B})(\Gamma^* u) &= B_2 f(\Pi_B)(B_1 \Gamma^* u), && \text{when } f \text{ is odd,} \\ B_1 g(\underline{\Pi_B})(\Gamma^* u) &= g(\Pi_B)(B_1 \Gamma^* u), && \text{when } g \text{ is even.} \end{aligned} \tag{2.3}$$

Proposition 2.18. *Suppose Π_B is a perturbed Hodge-Dirac operator which Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^N)$ for all $p \in (p_H, p^H)$. Then:*

(1) ${\Pi_B}^* = \Gamma^* + {B_2}^* \Gamma {B_1}^*$ is a perturbed Hodge-Dirac operator which Hodge decomposes $L^q(\mathbb{R}^n; \mathbb{C}^N)$ for all $q \in ((p^H)', (p_H)')$, i.e. $(\Pi_B(q))$ holds and

$$L^q(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_q({\Pi_B}^*) \oplus \overline{\mathcal{R}_q(\Gamma^*)} \oplus \overline{\mathcal{R}_q({B_2}^* \Gamma B_1^*)} \; .$$

(2) The perturbed Hodge-Dirac operator $\underline{\Pi_B}$ Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^N)$ for all $p \in (p_H, p^H)$, i.e. $(\underline{\Pi_B}(p))$ holds and

$$L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(\underline{\Pi_B}) \oplus \overline{\mathcal{R}_p(\Gamma^*)} \oplus \overline{\mathcal{R}_p(B_2 \Gamma B_1)} \; .$$

(3) If, for some $p \in (p_H, p^H)$*,* Π_B *is* ω*-bisectorial in* $L^p(\mathbb{R}^n;\mathbb{C}^N)$*, then* $\underline{\Pi_B}$ *is also* ω*-bisectorial in* $L^p(\mathbb{R}^n;\mathbb{C}^N)$*.*

Proof. (1) First note that the invertibility condition $(\Pi_B(p))$ for Π_B is the same as the invertibility condition $({\Pi_B}^*(p'))$ for ${\Pi_B}^* = \Gamma^* + {B_2}^*\Gamma {B_1}^*$. Using this, it is proved in [36], Lemma 6.3 that the Hodge decomposition for ${\Pi_B}^*$ in $L^{p'}(\mathbb{R}^n;\mathbb{C}^N)$ is equivalent to the Hodge decomposition for Π_B in $L^p(\mathbb{R}^n;\mathbb{C}^N)$.
(2) On applying Proposition 2.16, $\underline{\Pi_B}$ Hodge decomposes $L^p(\mathbb{R}^n;\mathbb{C}^N)$ if and only if

(A”) $$\mathbb{Q}_p B_2 : \overline{\mathcal{R}_p(\Gamma)} \to \overline{\mathcal{R}_p(\Gamma)} \quad \text{is an isomophism and}$$

(B”) $$\mathbb{P}_{p'} {B_1}^* : \overline{\mathcal{R}_{p'}(\Gamma^*)} \to \overline{\mathcal{R}_{p'}(\Gamma^*)} \quad \text{is an isomophism .}$$

Using the Hodge decompositions for the unperturbed operators to identify the dual of $\overline{\mathcal{R}_p(\Gamma^*)}$ with $\overline{\mathcal{R}_{p'}(\Gamma^*)}$, we find by duality that (A’) is equivalent to (B”) and (B’) is equivalent to (A”). This proves (2).
(3) This is essentially proved in [36], Lemma 6.4. □

Remark 2.19. *We are not saying that* $(\Pi_B(p))$ *is equivalent to* $(\underline{\Pi_B}(p))$ *for general* p*.*

We now define the operators

$$\begin{aligned} R_t^B &:= (I + it\Pi_B)^{-1}, \quad t \in \mathbb{R},\\ P_t^B &:= (I + t^2{\Pi_B}^2)^{-1} = \frac{1}{2}(R_t^B + R_{-t}^B) = R_t^B R_{-t}^B, \quad t > 0,\\ Q_t^B &:= t\Pi_B(I + t^2{\Pi_B}^2)^{-1} = \frac{1}{2i}(-R_t^B + R_{-t}^B), \quad t > 0. \end{aligned}$$

In the unperturbed case $B_1 = B_2 = I$, we write R_t, P_t and Q_t for R_t^B, P_t^B and Q_t^B, respectively. If we replace Π_B by $\underline{\Pi_B}$, we replace R_t^B, P_t^B and Q_t^B by $\underline{R_t^B}$, $\underline{P_t^B}$ and $\underline{Q_t^B}$, respectively.

We state some basic results for the unperturbed operator Π, noting that when we apply [36], we do not make use of the probabilistic/dyadic methods developed there.

Proposition 2.20. *Let* $M \in 2\mathbb{N}$ *be such that* $M > n + 4$*.*

(1) For all $p \in (1, 2]$*, the family* $\{s^{n(\frac{1}{p}-\frac{1}{2})}(R_s)^M \mathbb{P}_{\overline{\mathcal{R}(\Pi)}} \; ; \; s \in \mathbb{R}^*\}$ *is uniformly bounded in* $\mathcal{L}(L^p, L^2)$*, and the family* $\{(Q_s)^M \; ; \; s \in \mathbb{R}^*\}$ *has* L^p*-*L^2 *off-diagonal bounds of every order.*

(2) For all $p \in (1, \infty)$*, the family* $\{P_s \; ; \; s \in \mathbb{R}^*\}$ *has* L^p*-*L^p *off-diagonal bounds of every order.*

(3) For all $p \in (1, \infty)$*,*

$$\|(s, x) \mapsto {Q_s}^M u(x)\|_{T^{p,2}} \approx \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Pi)}.$$

(4) For all $p \in (\max\{2_*, 1\}, 2]$*,*

$$\|(t, x) \mapsto Q_t u(x)\|_{T^{p,2}} \approx \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Pi)}.$$

Proof. (1) Let $s > 0$. By [36, Proposition 4.8], $(R_s)^M$ is a Fourier multiplier with bounded symbol $\xi \mapsto m(s\xi)$. We also have that $\Pi^{M-1}{R_1}^M$ is a Fourier multiplier with bounded symbol $\tilde{m} : \xi \mapsto \widehat{\Pi}(\xi)^{M-1}m(\xi)$. Since $|\xi|^{M-1}|m(\xi)w| \lesssim |\tilde{m}(\xi)w|$ for every $\xi \in \mathbb{R}^n$ and every $w \in \mathcal{R}(\widehat{\Pi}(\xi))$ (by (Π1)), we have that

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{M-1}|m(\xi)w| \lesssim \|\tilde{m}\|_\infty.$$

For $u \in \overline{\mathcal{R}(\Pi)} \cap L^1$, this implies

$$\|s^{\frac{n}{2}}{R_s}^M u\|_2 \lesssim \|s^{\frac{n}{2}} m(s\,.)\,\widehat{u}\|_2 \lesssim \|\widehat{u}\|_\infty \lesssim \|u\|_1.$$

Since $\mathbb{P}_{\overline{\mathcal{R}(\Pi)}}$ is a Fourier multiplier of weak type 1-1 by [36, Proposition 4.4], we have by interpolation that, for all $p \in (1, 2]$, the family $\{s^{n(\frac{1}{p}-\frac{1}{2})}(R_s)^M\mathbb{P}_{\overline{\mathcal{R}(\Pi)}}\ ;\ s \in \mathbb{R}^*\}$ is uniformly bounded in $\mathcal{L}(L^p, L^2)$. This implies that $\{s^{n(\frac{1}{p}-\frac{1}{2})}(Q_s)^M\ ;\ s \in \mathbb{R}^*\}$ is uniformly bounded in $\mathcal{L}(L^p, L^2)$. Using Lemma 2.6 to interpolate this uniform bound with the L^2-L^2 off-diagonal bounds for $\{(Q_s)^M\ ;\ s \in \mathbb{R}^*\}$ gives the second part of (1).
(2) By Proposition 2.13 (5), Π is bisectorial in $L^p(\mathbb{R}^n;\mathbb{C}^N)$, $p \in (1,\infty)$. Then the proof of [19, Proposition 5.2], showing off-diagonal estimates in L^2, carries over to $p \in (1,\infty)$.
(3) Let $p \in (1,\infty)$. By [36, Corollary 4.10], Q_s^M is a Fourier multiplier satisfying the assumptions of Mihlin's multiplier theorem uniformly in s. Moreover

$$\|(s,x) \mapsto {Q_s}^M u(x)\|_{T^{2,2}} \approx \|u\|_{L^2} \quad \forall u \in \overline{\mathcal{R}_2(\Pi)},$$

by [36, Theorem 5.1]. Therefore, the conditions of [29, Theorem 4.4 and Example 2] are satisfied, and the operator $\mathcal{Q}$ defined by

$$\mathcal{Q}u(t,x) = {Q_t}^M u(x),$$

extends to a bounded operator from L^p to $T^{p,2}$. The norm equivalence follows by Calderón reproducing formula and tent space duality as follows. For $u \in \overline{\mathcal{R}_p(\Pi)}$ and $v \in L^{p'}$, we have that

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} u(x).v(x)dx\Big| &\lesssim \int_{\mathbb{R}^n}\int_0^\infty |(Q_t)^M u(x)||((Q_t)^*)^M v(x)|\frac{dt}{t}dx \\
&\lesssim \|(t,x) \mapsto (Q_t)^M u(x)\|_{T^{p,2}}\|(t,x) \mapsto ((Q_t)^*)^M v(x)\|_{T^{p',2}} \\
&\lesssim \|(t,x) \mapsto (Q_t)^M u(x)\|_{T^{p,2}}\|v\|_{p'}.
\end{aligned}$$

(4) For $p \in (\max(1, 2_*), \infty)$ and $u \in L^p \cap L^2$,

$$\|(t,x) \mapsto Q_t u(x)\|_{T^{p,2}} \lesssim \|(t,x) \mapsto \int_0^\infty Q_t Q_s {Q_s}^M u(x)\,\frac{ds}{s}\|_{T^{p,2}}.$$

We estimate the right hand side using Corollary 10.2 of the Appendix. (Note that this Appendix does not rely on Sections 2-9.)

Noting that $Q_tQ_s = \begin{cases} \frac{s}{t}(I-P_t)P_s & \text{if } 0<s\leq t, \\ \frac{t}{s}(I-P_s)P_t & \text{if } 0<t\leq s, \end{cases}$ we consider the integral operator defined by

$$T_K F(t,x) = \int_0^\infty \min(\frac{t}{s},\frac{s}{t}) K(t,s) F(s,x)\frac{ds}{s} \qquad \forall t>0 \quad \forall x\in\mathbb{R}^n,$$

for $F\in C_c^\infty(\mathbb{R}^{n+1}_+)$ and $K(t,s) = \begin{cases} (I-P_t)P_s & \text{if } 0<s\leq t, \\ (I-P_s)P_t & \text{if } 0<t\leq s. \end{cases}$ For every $\varepsilon>0$, the integral operator defined by $\tilde{T}_K F(t,x) = \int_0^\infty \min(\frac{t}{s},\frac{s}{t})^\varepsilon K(t,s)F(s,x)\frac{ds}{s}$, for $F\in C_c^\infty(\mathbb{R}^{n+1}_+)$ and all $t>0$, $x\in\mathbb{R}^n$, extends to a bounded operator on $T^{2,2}$ by Schur's lemma. Using (2), the result thus follows by Corollary 10.2 and (3). □

We conclude the section by recalling the main result of Axelsson, Keith and the second author in [19]. Note that perturbed Hodge-Dirac operators satisfy the assumptions of [19] and [36]. In particular, $\|\nabla\otimes u\|_2 \lesssim \|\Pi u\|_2$ for all $u\in\mathcal{D}_2(\Pi)\cap\overline{\mathcal{R}_2(\Pi)}$ as stated in Proposition 2.13 (6).

Theorem 2.21. *Suppose Π_B is a perturbed Hodge-Dirac operator with angles of accretivity as specified in Definition 2.14. Then:*

(1) Π_B is an ω-bisectorial operator in $L^2(\mathbb{R}^n;\mathbb{C}^N)$.
(2) The family $\{R_t^B\ ;\ t\in\mathbb{R}\}$ has L^2-L^2 off-diagonal bounds of every order.
(3) Π_B satisfies the quadratic estimate

$$\|(t,x)\mapsto Q_t^B u(x)\|_{T^{2,2}} \approx \|u\|_{L^2}$$

for all $u\in\overline{\mathcal{R}_2(\Pi_B)}\subseteq L^2(\mathbb{R}^n;\mathbb{C}^N)$.
(4) For all $\mu>\omega$, Π_B has a bounded H^∞ functional calculus with angle μ in $L^2(\mathbb{R}^n;\mathbb{C}^N)$.

3. Main results

The main results in this paper are conical square function estimates for Hodge-Dirac operators on the ranges of Γ and Γ^*, and corollaries concerning the functional calculus of Π_B. The fundamental estimates required for the proofs of these square function estimates will be obtained in Sections 5–9. In this section, we state the main results, and show how their corollaries can be deduced.

Our main theorem gives the equivalence of the Hardy space $H^p_{\Pi_B}(\mathbb{R}^n;\mathbb{C}^N)$ with the L^p closure of $\mathcal{R}_p(\Pi_B)$ whenever $p\in(p_H,p^H)$, and corresponding results restricted to the ranges of Γ and Γ_B^* for p below p_H. We recall that $(\Pi_B(p))$ always holds for $p\in(p_H,p^H)$.

Theorem 3.1. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose that $\mu\in(\omega,\frac{\pi}{2})$, and that $\psi\in\Psi_\alpha^\beta(S_\mu^o)$ is non-degenerate with*

$$\textit{either } p\in(1,2],\ \textit{and } \alpha>0,\beta>\tfrac{n}{2};\quad \textit{or } p\in[2,\infty),\ \textit{and } \alpha>\tfrac{n}{2},\beta>0.$$

(1) Let $p \in (\max\{1,(p_H)_*\},p^H)$. Then

$$\|(t,x)\mapsto \psi(t\Pi_B)u(x)\|_{T^{p,2}} \approx \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma)} \quad \text{and}$$
$$\|(t,x)\mapsto \psi(t\underline{\Pi_B})u(x)\|_{T^{p,2}} \approx \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma^*)}.$$

In particular, $\overline{\mathcal{R}_p(\Gamma)} \subseteq H^p_{\Pi_B}(\mathbb{R}^n;\mathbb{C}^N)$ and $\overline{\mathcal{R}_p(\Gamma^*)} \subseteq H^p_{\underline{\Pi_B}}(\mathbb{R}^n;\mathbb{C}^N)$.

(2) Let $p \in (\max\{1,(p_H)_*\},p^H)$ and suppose that $(\Pi_B(p))$ holds. Then

$$\|(t,x)\mapsto \psi(t\Pi_B)u(x)\|_{T^{p,2}} \approx \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma^*_B)}.$$

In particular, $\overline{\mathcal{R}_p(\Gamma^*_B)} \subseteq H^p_{\Pi_B}(\mathbb{R}^n;\mathbb{C}^N)$.

(3) Let $p \in (p_H,p^H)$. Then

$$\|(t,x)\mapsto \psi(t\Pi_B)u(x)\|_{T^{p,2}} \approx \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Pi_B)}.$$

In particular, $H^p_{\Pi_B}(\mathbb{R}^n;\mathbb{C}^N) = \overline{\mathcal{R}_p(\Pi_B)}$.

Remark 3.2. *In Theorem 3.1 (2), one has in fact $H^p_{\Pi_B}(\mathbb{R}^n;\mathbb{C}^N) = \overline{\mathcal{R}_p(\Gamma)} \oplus \overline{\mathcal{R}_p(\Gamma^*_B)}$, for $p \in (\max\{1,(p_H)_*\},p^H)$. This follows from the fact that the Hodge projections preserve Hardy spaces, as can be seen by considering their actions on $H^1_{\Pi_B}$ molecules (as defined in [14]).*

Remark 3.3. *An inspection of our proof shows that we are actually proving that*

$$\|u\|_{H^p_{\Pi_B}} \approx \|u\|_{H^p_{\Pi}} \quad \forall u \in \overline{R_2(\Gamma)} \cap H^p_{\Pi}\ .$$

When $p>1$, we then use that $\|u\|_{H^p_\Pi} \approx \|u\|_{L^p}$. The proof still works if $(p_H)_* < 1$ and $p=1$. In this case we get that

$$\|u\|_{H^1_{\Pi_B}} \approx \|u\|_{H^1_{\Pi}} \quad \forall u \in \overline{R_2(\Gamma)} \cap H^1_{\Pi}\ .$$

As Π is a Fourier multiplier one can then relate the H^1_Π norm to the classical H^1 norm:

$$\|u\|_{H^1_\Pi} \approx \|u\|_{H^1(\mathbb{R}^n,\mathbb{C}^N)} \quad \forall u \in H^1_\Pi\ .$$

This can be done, for instance, by using the molecular theory presented in [14].

As a consequence of Theorem 3.1, we obtain functional calculus results for Π_B.

Theorem 3.4. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $\mu \in (\omega,\frac{\pi}{2})$.*

(1) Let $p \in (\max\{1,(p_H)_*\},p^H)$. Then for all $f \in \Psi(S^o_\mu)$,

$$\|f(\Pi_B)u\|_p \le C_p\|f\|_\infty\|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma)}.$$

(2) Let $p \in (\max\{1,(p_H)_*\},p^H)$ and suppose $(\Pi_B(p))$ holds. Then for all $f \in \Psi(S^o_\mu)$,

$$\|f(\Pi_B)u\|_p \le C_p\|f\|_\infty\|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma^*_B)}.$$

(3) Let $p \in (p_H,p^H)$. Then Π_B is ω-bisectorial, and has a bounded H^∞ functional calculus with angle μ in $L^p(\mathbb{R}^n;\mathbb{C}^N)$.

The derivation of Theorem 3.4 from Theorem 3.1 will be given later in this section.
The proof of Theorem 3.1 relies on two kinds (reverse and direct) of conical square function estimates. We first prove the reverse ones, before discussing the key direct estimates.

Proposition 3.5. *Suppose Π_B is a perturbed Hodge-Dirac operator. For all $p \in [2,\infty]$ and all $M \in \mathbb{N}$, we have*

$$\|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \quad \forall u \in L^2(\mathbb{R}^n;\mathbb{C}^N) \cap L^p(\mathbb{R}^n;\mathbb{C}^N). \tag{3.1}$$

Consequently, for all $p \in (1,2]$ and all $M \in \mathbb{N}$, we have

$$\|u\|_p \leq C_p \|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \quad \forall u \in \overline{\mathcal{R}_2(\Pi_B)} \cap L^p(\mathbb{R}^n;\mathbb{C}^N).$$

Proof. The result for $p = 2$ holds by Theorem 2.21. We show that $u \mapsto (Q_t^B)^M u$ maps L^∞ to $T^{\infty,2}$. The claim for $p \in (2,\infty)$ then follows by interpolation. The argument goes back to Fefferman and Stein [27], and was used in a similar context in e.g. [11, Section 3.2]. Fix a cube Q in $\mathbb{R}^n$ and split $u \in L^\infty(\mathbb{R}^n;\mathbb{C}^N)$ into $u = u\mathbb{1}_{4Q} + u\mathbb{1}_{(4Q)^c}$. Recall $S_j(Q) = 2^{j+1}Q \backslash 2^j Q$ for all $j \geq 2$. Theorem 2.21 gives

$$(\frac{1}{|Q|}\int_0^{l(Q)}\int_Q |(Q_t^B)^M \mathbb{1}_{4Q}u(x)|^2 \frac{dxdt}{t})^{\frac{1}{2}} \lesssim |Q|^{-\frac{1}{2}} \|\mathbb{1}_{4Q}u\|_2 \lesssim \|u\|_\infty.$$

On the other hand, L^2-L^2 off-diagonal bounds for $(Q_t^B)^M$ of order $N' > \frac{n}{2}$ yield

$$\begin{aligned}
&(\frac{1}{|Q|}\int_0^{l(Q)}\int_Q |(Q_t^B)^M \mathbb{1}_{(4Q)^c}u(x)|^2 \frac{dxdt}{t})^{\frac{1}{2}} \\
&\lesssim \sum_{j=2}^\infty (\frac{1}{|Q|}\int_0^{l(Q)}\int_Q |(Q_t^B)^M \mathbb{1}_{S_j(Q)}u(x)|^2 \frac{dxdt}{t})^{\frac{1}{2}} \\
&\lesssim \sum_{j=2}^\infty 2^{-jN'}(\frac{1}{|Q|}\int_0^{l(Q)} (\frac{t}{l(Q)})^{2N'} \|\mathbb{1}_{2^{j+1}Q}u\|_2^2 \frac{dt}{t})^{\frac{1}{2}} \lesssim \|u\|_\infty.
\end{aligned}$$

Consider now $p \in (1,2)$. Let $u \in \overline{\mathcal{R}_2(\Pi_B)} \cap L^p(\mathbb{R}^n;\mathbb{C}^N)$ and $v \in L^{p'}(\mathbb{R}^n;\mathbb{C}^N) \cap L^2(\mathbb{R}^n;\mathbb{C}^N)$. We apply the above result to Π_B^* in $L^{p'}(\mathbb{R}^n;\mathbb{C}^N)$, noting that Π_B^* Hodge decomposes L^q for all $q \in ((p^H)', (p_H)')$ by Proposition 2.18. By Calderón reproducing formula, tent space duality and the argument above, we have that

$$\begin{aligned}
|\int_{\mathbb{R}^n} u(x).v(x)dx| &\lesssim \int_{\mathbb{R}^n}\int_0^\infty |(Q_t^B)^M u(x)||((Q_t^B)^*)^M v(x)| \frac{dt}{t}dx \\
&\lesssim \|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \|(t,x) \mapsto ((Q_t^B)^*)^M v(x)\|_{T^{p',2}} \\
&\lesssim \|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \|v\|_{p'}.
\end{aligned}$$

This gives the assertion. □

Remark 3.6. *Note that for the proof of (3.1), we only use that $((Q_t^B)^M)_{t>0}$ satisfies L^2-L^2 off-diagonal bounds of order $N' > \frac{n}{2}$, and defines a bounded mapping from L^2 to $T^{2,2}$. In particular, we do not use any assumptions on Π_B in L^p for $p \neq 2$. The proof gives a way to define a bounded extension from L^p to $T^{p,2}$ of this mapping. In the case $p = \infty$, the above result shows that for every $u \in L^\infty(\mathbb{R}^n;\mathbb{C}^N)$, $|(Q_t^B)^M u(x)|^2 \frac{dxdt}{t}$ is a Carleson measure. We will make use of this fact in Proposition 5.5. Moreover, we can replace $(Q_t^B)^M$ by another operator of the form $\psi(t\Pi_B)$ for $\psi \in \Psi(S_\mu)$ and $\mu \in (\omega, \frac{\pi}{2})$, as long as $(\psi(t\Pi_B))_{t>0}$ satisfies L^2-L^2 off-diagonal bounds of order $N' > \frac{n}{2}$.*

We now state our main estimates.

Proposition 3.7. *Suppose Π_B is a perturbed Hodge-Dirac operator. Given $p \in (\max\{1,(p_H)_*\},\infty)$, we have*

$$\|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma)} \quad \text{and}$$
$$\|(t,x) \mapsto (\underline{Q_t^B})^M u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma^*)}\,,$$

where $M \in \mathbb{N}$ if $p \geq 2$, and $M \in \mathbb{N}$ with $M > \frac{n}{2}$ if $p < 2$.

Let us consider the first estimate in Proposition 3.7. Its proof splits into low and high frequency conical square function estimates via the operators $P_t^{\tilde{N}}$ and $(I - P_t^{\tilde{N}})$ where $\tilde{N}$ is a large natural number, say $\tilde{N} = 10n$. The low frequency estimate is an L^p version of the low frequency estimate in the main result of [19], Theorem 2.7 (and hence captures the harmonic analytic part of the proof of the Kato square root problem).

Proposition 3.8. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $M \in \mathbb{N}$ and $p \in (1,\infty)$. Then*

$$\|(t,x) \mapsto (Q_t^B)^M P_t^{\tilde{N}} u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Pi)}.$$

This result is proven in Section 5.

The high frequency conical square function estimate is an L^p version of the high frequency estimate [19, Proposition 4.8, part (i)]. Note that this operator theoretic part of the proof in the case $p = 2$, is the part that does not necessarily hold for all $p \in (1,\infty)$.

Proposition 3.9. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $M = 10n$ and $p \in (\max\{1,(p_H)_*\},2]$. Then*

$$\|(t,x) \mapsto (Q_t^B)^M (I - P_t^{\tilde{N}}) u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma)}.$$

This result is proven in Sections 6, 7 and 9.

We now show how to prove Proposition 3.7 from Propositions 3.8 and 3.9. Notice that the large (and somewhat arbitrary) value of M appearing in Proposition 3.9 is appropriately reduced as part of this proof.

Proof of Proposition 3.7 from Propositions 3.5, 3.8 and 3.9. For $p \in (2,\infty)$, the claim has been shown in Proposition 3.5. From now on, suppose $p \in (\max\{1,(p_H)_*\},2]$. Without loss of generality, we can assume that $M = 10n$. Indeed, the result for $M > \frac{n}{2}$ will then follow by Theorem 2.11. Combining Proposition 3.8 and Proposition 3.9, we have that

$$\|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \lesssim \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma)}.$$

Applying the same results to $\underline{\Pi_B}$ gives

$$\|(t,x) \mapsto (\underline{Q_t^B})^M u(x)\|_{T^{p,2}} \lesssim \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Gamma^*)}.$$

□

We next show that Theorems 3.1 and 3.4 follow from Proposition 3.7 and Proposition 3.5.

Proof of Theorem 3.1 from Propositions 3.7 and 3.5. By Theorem 2.11, it suffices to show

$$\|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \approx \|u\|_p,$$

and the corresponding equivalence for $(\underline{Q_t^B})^M$ in case (1), for $M = 10n$ and u as given in (1), (2) or (3). First suppose $p \in (\max\{1, (p_H)_*\}, 2]$. Combining Proposition 3.7 and Proposition 3.5 gives the equivalence for $(Q_t^B)^M$ and all $u \in \overline{\mathcal{R}_2(\Gamma)} \cap L^p(\mathbb{R}^n; \mathbb{C}^N)$. The same reasoning applies to $(\underline{Q_t^B})^M$ and $u \in \overline{\mathcal{R}_2(\Gamma^*)} \cap L^p(\mathbb{R}^n; \mathbb{C}^N)$. Now note that $\mathcal{N}_p(\Gamma^*) \cap L^2(\mathbb{R}^n; \mathbb{C}^N) \subseteq \mathcal{N}_2(\Gamma^*)$, and the same holds with p and 2 interchanged. Using the Hodge decomposition for the unperturbed operator Π, we therefore have

$$L^p(\mathbb{R}^n; \mathbb{C}^N) \cap L^2(\mathbb{R}^n; \mathbb{C}^N) = [\mathcal{N}_p(\Gamma^*) \cap \mathcal{N}_2(\Gamma^*)] \oplus [\overline{\mathcal{R}_p(\Gamma)} \cap \overline{\mathcal{R}_2(\Gamma)}].$$

Since this space is dense in $L^p(\mathbb{R}^n; \mathbb{C}^N)$, the space $\overline{\mathcal{R}_2(\Gamma)} \cap L^p(\mathbb{R}^n; \mathbb{C}^N)$ is dense in $\overline{\mathcal{R}_p(\Gamma)}$. This gives the above equivalence on $\overline{\mathcal{R}_p(\Gamma)}$, and, similarly, for $(\underline{Q_t^B})^M$ on $\overline{\mathcal{R}_p(\Gamma^*)}$. As stated before Definition 2.15, we have $\overline{\mathcal{R}_p(\Gamma_B^*)} = B_1 \overline{\mathcal{R}_p(\Gamma^*)}$ under $(\Pi_B(p))$. Using that M is even, the identity (2.3), $\|B_1\|_\infty < \infty$, and that $(\Pi_B(p))$ holds by assumption, we therefore deduce from the above that, for $u = B_1 \Gamma^* B_2 v \in \overline{\mathcal{R}_p(\Gamma_B^*)}$,

$$\begin{aligned}
\|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} &= \|(t,x) \mapsto (Q_t^B)^M B_1 \Gamma^* B_2 v(x)\|_{T^{p,2}} \\
&= \|(t,x) \mapsto B_1 (\underline{Q_t^B})^M \Gamma^* B_2 v(x)\|_{T^{p,2}} \lesssim \|(t,x) \mapsto (\underline{Q_t^B})^M \Gamma^* B_2 v(x)\|_{T^{p,2}} \\
&\lesssim \|\Gamma^* B_2 v\|_p \lesssim \|u\|_p.
\end{aligned}$$

This gives (2). In the case $p \in (p_H, 2]$, Π_B Hodge decomposes L^p. This yields the result on $\overline{\mathcal{R}_p(\Pi_B)}$. The case $p \in [2, p^H)$ follows by duality, cf. the proof of Theorem 3.4. ☐

Proof of Theorem 3.4 from Theorem 3.1. First suppose $p \in (\max\{1, (p_H)_*\}, 2]$. Let $M = 10n$, and $\mu \in (\omega, \frac{\pi}{2})$. Let $f \in \Psi(S_\mu^o)$ and $u \in \overline{\mathcal{R}_p(\Gamma)}$. Using Theorem 3.1 and Theorem 2.11, we have that

$$\begin{aligned}
\|f(\Pi_B)u\|_p &\lesssim \|(t,x) \mapsto (Q_t^B)^M f(\Pi_B) u(x)\|_{T^{p,2}} \\
&\lesssim \|f\|_\infty \|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{p,2}} \lesssim \|f\|_\infty \|u\|_p.
\end{aligned}$$

The same reasoning applies to $u \in \overline{\mathcal{R}_p(\Gamma_B^*)}$, assuming $(\Pi_B(p))$. Now let $p \in (p_H, 2]$. Since Π_B Hodge decomposes L^p, we have that, for all $f \in \Psi(S_\mu^o)$,

$$\|f(\Pi_B)u\|_p \lesssim \|f\|_\infty \|u\|_p \quad \forall u \in L^p(\mathbb{R}^n; \mathbb{C}^N).$$

This implies that Π_B is ω-bisectorial and has a bounded H^∞ functional calculus in L^p. Finally, we consider the case $p \in [2, p^H)$. We apply the above result to Π_B^*, which Hodge decomposes L^q for all $q \in ((p^H)', (p_H)')$ by Proposition 2.18. Hence, Π_B^* has a bounded H^∞ functional calculus in L^q for all $q \in ((p^H)', 2]$. By duality, Π_B has a bounded H^∞ functional calculus in L^p for all $p \in [2, p^H)$. ☐

We conclude this section by showing that in certain situations the results can be improved when restricted to subspaces of the form $L^p(\mathbb{R}^n; W)$, where W is a subspace of $\mathbb{C}^N$. The proof given depends on Corollary 9.3, as well as the preceding material.

Theorem 3.10. *Suppose Π_B is a perturbed Hodge-Dirac operator. Let W be a subspace of $\mathbb{C}^N$ that is stable under $\widehat{\Gamma^*}(\xi)\widehat{\Gamma}(\xi)$ and $\widehat{\Gamma}(\xi)\widehat{\Gamma^*}(\xi)$ for all $\xi \in \mathbb{R}^n$.*
(1) Let $p \in (\max\{1, r_*\}, 2]$ for some $r \in (1,2)$, and $M \in 2\mathbb{N}$ with $M \geq 10n$. Suppose further that $(\Pi_B(r))$ holds, $\{t\Gamma_B^* P_t^B \mathbb{P}_W \; ; \; t \in \mathbb{R}^*\}$ and $\{t\Gamma P_t^B \mathbb{P}_W \; ; \; t \in \mathbb{R}^*\}$ are uniformly bounded in $\mathcal{L}(L^r)$, and that $\{(R_t^B)^{\frac{M}{2}-2}\Gamma\mathbb{P}_W \; ; \; t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-2}B_1\Gamma^*\mathbb{P}_W \; ; \; t \in \mathbb{R}^*\}$ have $\dot W^{1,r}$-L^2 off-diagonal bounds of every order on balls, where $\mathbb{P}_W$ denotes the projection from $L^r(\mathbb{R}^n;\mathbb{C}^N)$ onto $L^r(\mathbb{R}^n;W)$ induced by the orthogonal projection of $\mathbb{C}^N$ onto W. Then, for $\mu \in (\omega, \frac{\pi}{2})$, we have

$$\text{(i)} \qquad \|f(\Pi_B)\Gamma u\|_p \lesssim \|f\|_\infty \|\Gamma u\|_p \quad \forall f \in H^\infty(S_\mu^o) \quad \forall u \in \mathcal{D}_p(\Gamma) \cap L^2(\mathbb{R}^n;W).$$

Moreover, $\overline{\mathcal{R}_p(\Gamma|_{L^p(\mathbb{R}^n;W)})} \subseteq H^p_{\Pi_B}(\mathbb{R}^n;\mathbb{C}^N)$ with

$$\text{(ii)} \qquad \|v\|_{H^p_{\Pi_B}} \approx \|v\|_p \qquad \forall v \in \overline{\mathcal{R}_p(\Gamma|_{L^p(\mathbb{R}^n;W)})}.$$

(2) If $L^2(\mathbb{R}^n;W) \subset \overline{\mathcal{R}_2(\Gamma_B^*)}$, $(p_H)_* > 1$, and $(\Pi_B(r))$ holds for all $r \in ((p_H)_*, 2]$, then the hypotheses and conclusions of (1) hold for all $p \in (\max\{1, (p_H)_{**}\}, 2]$. In particular,

$$\text{(iii)} \qquad \|(\Pi_B{}^2)^{1/2} u\|_p \lesssim \|\Gamma u\|_p \quad \forall u \in \mathcal{D}_p(\Gamma) \cap L^2(\mathbb{R}^n;W).$$

Proof. (1) (i) The hypotheses of Corollary 9.3 are satisfied by assumption. We therefore obtain, for all $p \in (\max\{1, r_*\}, 2]$,

$$\|(t,x) \mapsto (Q_t^B)^M (I - P_t^{\tilde N})\Gamma v(x)\|_{T^{p,2}} \lesssim \|\Gamma v\|_p \qquad \forall v \in \mathcal{D}_p(\Gamma) \cap L^p(\mathbb{R}^n;W).$$

Combined with Proposition 3.8, this gives

$$\|(t,x) \mapsto (Q_t^B)^M \Gamma v(x)\|_{T^{p,2}} \lesssim \|\Gamma v\|_p \qquad \forall v \in \mathcal{D}_p(\Gamma) \cap L^p(\mathbb{R}^n;W). \tag{3.2}$$

Therefore we have, for all $f \in \Psi(S_\mu^o)$, $v \in \mathcal{D}_p(\Gamma) \cap L^p(\mathbb{R}^n;W)$, that (i) holds:

$$\begin{aligned} \|f(\Pi_B)\Gamma v\|_p &\lesssim \|(t,x) \mapsto (Q_t^B)^M f(\Pi_B)\Gamma v(x)\|_{T^{p,2}} \approx \|f(\Pi_B)\Gamma v\|_{H^p_{\Pi_B}} \\ &\lesssim \|\Gamma v\|_{H^p_{\Pi_B}} \approx \|(t,x) \mapsto (Q_t^B)^M \Gamma v(x)\|_{T^{p,2}} \lesssim \|\Gamma v\|_p, \end{aligned}$$

where we have used Proposition 3.5, Theorem 2.11, and (3.2). The estimate holds for all $f \in H^\infty(S_\mu^o)$ on taking limits as usual.
(ii) This follows from (3.2) and the reverse inequality shown in Proposition 3.5.
(2) Let $q > p_H$ with $q_* > 1$ and $r \in (q_*, q]$. By Lemma 7.1, and the fact that $L^2(\mathbb{R}^n;W) \subset \overline{\mathcal{R}_2(\Gamma_B^*)}$ by assumption, we have that

$$\|t^{n(\frac{1}{q_*}-\frac{1}{2})} R_t^B u\|_q \lesssim \|u\|_{L^{q_*}(\mathbb{R}^n;W)} \qquad \forall t > 0 \quad \forall u \in L^2(\mathbb{R}^n;W) \cap L^{q_*}(\mathbb{R}^n;W).$$

Iterating, and interpolating with L^2-L^2 off-diagonal bounds (see Lemma 2.6), we get that $\{(R_t^B)^{\frac{M}{2}-2} \; ; \; t > 0\}$ has $L^r(\mathbb{R}^n;W)$-L^2 off-diagonal bounds of every order and that $\{R_t^B \; ; \; t > 0\}$ has $L^r(\mathbb{R}^n;W)$-$L^{\tilde r}$ off-diagonal bounds of every order for some $\tilde r \in (r, 2]$. The former implies that both $\{(R_t^B)^{\frac{M}{2}-2}\Gamma\mathbb{P}_W \; ; \; t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-2}B_1\Gamma^*\mathbb{P}_W \; ; \; t \in \mathbb{R}^*\}$ have $\dot W^{1,p}$-L^2 off-diagonal bounds of every order on balls, while the latter implies that $\{t\Gamma P_t^B \mathbb{P}_W \; ; \; t \in \mathbb{R}^*\} = \{Q_t^B \mathbb{P}_W \; ; \; t \in \mathbb{R}^*\} = \{\frac{1}{2i}(R_{-t}^B - R_t^B)\mathbb{P}_W \; ; \; t \in \mathbb{R}^*\}$ is uniformly bounded in $\mathcal{L}(L^r)$, by Lemma 2.6. This yields the hypotheses and hence the conclusions of (1). To obtain (iii), apply (i) with $f(z) = \operatorname{sgn}(z)$:

$$\|({\Pi_B}^2)^{1/2}u\|_p = \|\operatorname{sgn}(\Pi_B)\Pi_B u\|_p = \|\operatorname{sgn}(\Pi_B)\Gamma u\|_p \leq C_p\|\Gamma u\|_p \ \forall u \in \mathcal{D}_p(\Gamma) \cap L^2(\mathbb{R}^n; W),$$

noting that $u \in \overline{\mathcal{R}_2(\Gamma_B^*)}$ by assumption. □

4. Consequences

4.1. **Differential forms.** The motivating example for our formalism is perturbed differential forms, where $\mathbb{C}^N = \Lambda = \oplus_{k=0}^n \Lambda^k = \wedge_{\mathbb{C}}\mathbb{R}^n$, the complex exterior algebra over $\mathbb{R}^n$, and $\Gamma = d$, the exterior derivative, acting in $L^p(\mathbb{R}^n; \Lambda) = \oplus_{k=0}^n L^p(\mathbb{R}^n; \Lambda^k)$. If the multiplication operators B_1, B_2 satisfy the conditions of Definition 2.14, then $\Pi_B = d + B_1 d^* B_2$ is a perturbed Hodge-Dirac operator, and it is from here that it gets its name. The L^p results stated in Section 3 all apply to this operator.

Typically, but not necessarily, the operators $B_j, j = 1, 2$ split as $B_j = B_j^0 \oplus \cdots \oplus B_j^n$, where $B_j^k \in L^\infty(\mathbb{R}^n; \mathcal{L}(\Lambda^k))$, in which case Theorem 3.4 has a converse in the following sense (cf. [16, Theorem 5.14] for an analogous result for Hodge-Dirac operators on Riemannian manifolds).

Proposition 4.1. *Suppose $\Pi_B = d + B_1 d^* B_2$ is a perturbed Hodge-Dirac operator as above, with $B_j, j = 1, 2$ splitting as $B_j = B_j^0 \oplus \cdots \oplus B_j^n$, where $B_j^k \in L^\infty(\mathbb{R}^n; \mathcal{L}(\Lambda^k))$, $k = 0, \dots, n$. Suppose that for some $p \in (1, \infty)$, $(\Pi_B(p))$ holds and Π_B is an ω-bisectorial operator in $L^p(\mathbb{R}^n; \Lambda)$ with a bounded H^∞ functional calculus in $L^p(\mathbb{R}^n; \Lambda)$. Then $p \in (p_H, p^H)$.*

We do not know if this converse holds for all perturbed Hodge-Dirac operators. It does, however, hold for all examples given in this section.

Proof. We need to show that Π_B Hodge decomposes $L^p(\mathbb{R}^n; \Lambda)$, i.e. $L^p(\mathbb{R}^n; \Lambda) = \mathcal{N}_p(\Pi_B) \oplus \overline{\mathcal{R}_p(\Gamma)} \oplus \overline{\mathcal{R}_p(\Gamma_B^*)}$, where $\Gamma = d$ and $\Gamma_B^* = B_1 d^* B_2$. Since Π_B is bisectorial in $L^p(\mathbb{R}^n; \Lambda)$, we know that $L^p(\mathbb{R}^n; \Lambda) = \mathcal{N}_p(\Pi_B) \oplus \overline{\mathcal{R}_p(\Pi_B)}$. Therefore, it suffices to show that

$$\|\Gamma u\|_p + \|\Gamma_B^* u\|_p \approx \|\Pi_B u\|_p \quad \forall u \in \mathcal{D}_p(\Pi_B) = \mathcal{D}_p(\Gamma) \cap \mathcal{D}_p(\Gamma_B^*).$$

For $k = 0, \dots, n$ and $u \in L^p(\mathbb{R}^n; \Lambda)$, denote by $u^{(k)} \in L^p(\mathbb{R}^n; \Lambda^k)$ the k-th component of u. Note that $\Gamma : L^p(\mathbb{R}^n; \Lambda^k) \to L^p(\mathbb{R}^n; \Lambda^{k+1})$, $\Gamma_B^* : L^p(\mathbb{R}^n; \Lambda^{k+1}) \to L^p(\mathbb{R}^n; \Lambda^k)$, $k = 0, \dots, n-1$, and ${\Pi_B}^2 : L^p(\mathbb{R}^n; \Lambda^k) \to L^p(\mathbb{R}^n; \Lambda^k)$, $k = 0, \dots, n$. Using that $\operatorname{sgn}(\Pi_B)$, where

$$\operatorname{sgn}(z) = \begin{cases} 1, & \text{if } \operatorname{Re} z > 0, \\ -1, & \text{if } \operatorname{Re} z < 0, \end{cases} \quad \forall z \in S_\mu \setminus \{0\} \quad \text{and} \quad \operatorname{sgn}(0) = 0,$$

is bounded in $L^p(\mathbb{R}^n;\Lambda)$ since Π_B has a bounded H^∞ calculus, we therefore get for $u \in \mathcal{D}_p(\Pi_B)$:

$$\begin{aligned}\|\Gamma u\|_p + \|\Gamma_B^* u\|_p &\approx \sum_{j=0}^{n} \|(\Gamma u)^{(j)}\|_p + \sum_j \|(\Gamma_B^* u)^{(j)}\|_p \\ &\approx \sum_{k=0}^{n} (\|\Gamma u^{(k)}\|_p + \|\Gamma_B^* u^{(k)}\|_p) \approx \sum_{k=0}^{n} \|\Pi_B u^{(k)}\|_p \approx \sum_{k=0}^{n} \|({\Pi_B}^2)^{1/2} u^{(k)}\|_p \\ &\approx \sum_{k=0}^{n} \|(({\Pi_B}^2)^{1/2} u)^{(k)}\|_p \approx \|({\Pi_B}^2)^{1/2} u\|_p \approx \|\Pi_B u\|_p.\end{aligned}$$

□

4.2. **Second order elliptic operators.** Let L denote the uniformly elliptic second order operator defined by

$$Lf = -a \operatorname{div} A\nabla f = -a \sum_{j,k=1}^{n} \partial_j (A_{j,k} \partial_k f)$$

where $a \in L^\infty(\mathbb{R}^n)$ with $\mathrm{Re}(a(x)) \geq \kappa_1 > 0$ a.e. and $A \in L^\infty(\mathbb{R}^n;\mathcal{L}(\mathbb{C}^n))$ with $\mathrm{Re}(A(x)) \geq \kappa_2 I > 0$ a.e. Associated with L is the Hodge-Dirac operator

$$\Pi_B = \Gamma + \Gamma_B^* = \Gamma + B_1 \Gamma^* B_2 = \begin{bmatrix} 0 & -a \operatorname{div} A \\ \nabla & 0 \end{bmatrix} \quad \text{acting in} \quad L^2(\mathbb{R}^n;(\mathbb{C}^{1+n})) = \begin{matrix} L^2(\mathbb{R}^n) \\ \oplus \\ L^2(\mathbb{R}^n;\mathbb{C}^n) \end{matrix}$$

where

$$\Gamma = \begin{bmatrix} 0 & 0 \\ \nabla & 0 \end{bmatrix}, \; \Gamma^* = \begin{bmatrix} 0 & -\operatorname{div} \\ 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix},$$

so that

$${\Pi_B}^2 = \begin{bmatrix} L & 0 \\ 0 & \tilde{L} \end{bmatrix} \qquad (\text{where} \quad \tilde{L} = -\nabla a \operatorname{div} A)\ .$$

As shown in [19] (and recalled in Theorem 2.21), Π_B is an ω-bisectorial operator with an H^∞ functional calculus in L^2, so that in particular $\mathrm{sgn}(\Pi_B)$ is a bounded operator on $L^2(\mathbb{R}^n;\mathbb{C}^{1+n})$.

Using the expression

$$\mathrm{sgn}(\Pi_B) = ({\Pi_B}^2)^{-1/2}\Pi_B = \begin{bmatrix} 0 & -L^{-1/2}\, a \operatorname{div} A \\ \nabla L^{-1/2} & 0 \end{bmatrix},$$

on $\mathcal{D}(\Pi_B)$, and the fact that $(\mathrm{sgn}(\Pi_B))^2 u = u$ for all $u \in \overline{\mathcal{R}_2(\Pi_B)} = L^2(\mathbb{R}^n) \oplus \overline{\mathcal{R}_2(\nabla)}$, we find that $\|\nabla L^{-1/2} g\|_2 \approx \|g\|_2$ for all $g \in \mathcal{R}(L^{1/2})$, i.e. $\|\nabla f\|_2 \approx \|L^{1/2} f\|_2$ for all $f \in \mathcal{D}(L^{1/2}) = W^{1,2}(\mathbb{R}^n)$, this being the Kato conjecture, previously solved in [10] (when $a = 1$).

Turning now to L^p, we see that by our hypotheses, $(\Pi_B(p))$ holds for all $p \in (1, \infty)$, and that

$$\mathcal{N}_p(\Pi_B) = \begin{array}{c}\{0\}\\ \oplus \\ \mathcal{N}_p(\operatorname{div} A)\end{array}, \quad \overline{\mathcal{R}_p(\Gamma)} = \begin{array}{c}\{0\}\\ \oplus \\ \overline{\mathcal{R}_p(\nabla)}\end{array}, \quad \overline{\mathcal{R}_p(\Gamma_B^*)} = \begin{array}{c}L^p(\mathbb{R}^n)\\ \oplus \\ \{0\}\end{array}.$$

So $p \in (p_H, p^H)$, i.e. the Hodge decomposition $L^p(\mathbb{R}^n; (\mathbb{C}^{1+n})) = \mathcal{N}_p(\Pi_B) \oplus \overline{\mathcal{R}_p(\Gamma)} \oplus \overline{\mathcal{R}_p(\Gamma_B^*)}$ holds, if and only if $L^p(\mathbb{R}^n; \mathbb{C}^n) = \mathcal{N}_p(\operatorname{div} A) \oplus \overline{\mathcal{R}_p(\nabla)}$.

Turning briefly to Hardy space theory, we have

$$H^2_{\Pi_B} = \overline{\mathcal{R}_2(\Pi_B)} = \begin{array}{c}L^2(\mathbb{R}^n)\\ \oplus \\ \overline{\mathcal{R}_2(\nabla)}\end{array} = \begin{array}{c}H^2_L\\ \oplus \\ H^2_{\tilde{L}}\end{array} \quad \text{and} \quad H^p_{\Pi_B} = H^p_{{\Pi_B}^2} = \begin{array}{c}H^p_L\\ \oplus \\ H^p_{\tilde{L}}\end{array}$$

for all $p \in (1, \infty)$. We remark that L has a bounded H^∞ functional calculus in H^p_L, and that $\operatorname{sgn}(\Pi_B)$ is an isomorphism interchanging H^p_L and $H^p_{\tilde{L}}$.

We now state how the results of this section apply to Π_B, and have as consequences for L and its Riesz transform, results which are known, at least when $a = 1$ (see [4] and [33, Section 5]).

Corollary 4.2. *Let $L = -a \operatorname{div} A\nabla$ be a uniformly elliptic operator as above. Then the following hold:*

1. *If $p_H < p < p^H$, then Π_B is an ω-bisectorial operator in $L^p(\mathbb{R}^n; \mathbb{C}^{1+n})$ with a bounded H^∞ functional calculus.*
2. *If $\max\{1, (p_H)_*\} < p < p^H$, then $H^p_{\Pi_B} = \overline{\mathcal{R}_p(\Pi_B)}$ and Π_B is an ω-bisectorial operator in $\overline{\mathcal{R}_p(\Pi_B)}$ with a bounded H^∞ functional calculus, so that L has a bounded H^∞ functional calculus in $L^p(\mathbb{R}^n)$, and $\mathcal{D}_p(L^{1/2}) = W^{1,p}(\mathbb{R}^n)$ with $\|L^{1/2}f\|_p \approx \|\nabla f\|_p$.*
3. *If $r \in (1, 2]$, $\max\{1, \min\{r_*, (p_H)_{**}\}\} < p < p^H$, $M \in \mathbb{N}$ with $M \geq 10n$, and $\{(I + t^2L)^{-(\frac{M}{2}-1)} ; t > 0\}$ has $L^r(\mathbb{R}^n)$-$L^2(\mathbb{R}^n)$ off-diagonal bounds of every order, then $H^p_{\tilde{L}} = \overline{\mathcal{R}_p(\nabla)}$ and $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$ for all $f \in W^{1,p}(\mathbb{R}^n)$. Also $g \in H^p_L$ if and only if $\nabla L^{-1/2}g \in L^p(\mathbb{R}^n; \mathbb{C}^n)$, with $\|g\|_{H^p_L} \approx \|\nabla L^{-1/2}g\|_p$.*

We remark that the hypotheses of (3) can also be stated in terms of off-diagonal bounds for the semigroup $(e^{-tL})_{t>0}$.

Proof. As described above, Π_B is a perturbed Hodge-Dirac operator. (1) follows from Theorem 3.4 (3). (2) follows from Theorem 3.1 (1) and (2), noting that in our situation, the decomposition $\overline{\mathcal{R}_p(\Pi_B)} = \overline{\mathcal{R}_p(\Gamma)} \oplus \overline{\mathcal{R}_p(\Gamma_B^*)}$ holds for all $p \in (1, \infty)$. (3) Set $W = \mathbb{C}$. As stated before, $L^2(\mathbb{R}^n; W) \subseteq \overline{\mathcal{R}_2(\Gamma_B^*)} = L^2(\mathbb{R}^n) \oplus \{0\}$. For $w \in W$ and $\xi \in \mathbb{C}^n$, we have that $\widehat{\Gamma^*}(\xi)\widehat{\Gamma}(\xi)w = (\sum_{j=1}^{n} |\xi_j|^2)w$ and $\widehat{\Gamma}(\xi)\widehat{\Gamma^*}(\xi)w = 0$, so W is stable under $\widehat{\Gamma^*}(\xi)\widehat{\Gamma}(\xi)$ and $\widehat{\Gamma}(\xi)\widehat{\Gamma^*}(\xi)$. If $(p_H)_* > 1$, we can therefore apply Theorem 3.10, which gives (3). If $(p_H)_* \leq 1$, (3) follows from (2). □

Remark 4.3. *If $A = I$, one has $(p_H, p^H) = (1, \infty)$, so the estimates in Corollary 4.2 hold for all $p \in (1, \infty)$, in agreement with the results of [42] concerning $L = -a\Delta$.*

4.3. **First order systems of the form DA.** Results for operators of the form DA or AD, used in studying boundary value problems as in [8], can be obtained in a similar way to those in this paper, building on the L^2 theory in [9]. However they can also be obtained as consequences of the results for Π_B, as was shown in Section 3 of [19] when $p = 2$. Let us briefly summarise this in the L^p case.

Let D be a first order system which is self-adjoint in $L^2(\mathbb{R}^n; \mathbb{C}^N)$, and $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$ with $\mathrm{Re}(ADu, Du) \geq \kappa \|Du\|_2{}^2$ for all $u \in \mathcal{D}_2(D)$. Set

$$\Pi_B = \Gamma + \Gamma_B^* = \Gamma + B_1\Gamma^* B_2 = \begin{bmatrix} 0 & ADA \\ D & 0 \end{bmatrix} \quad \text{acting in} \quad L^2(\mathbb{R}^n; \mathbb{C}^{2N}) = \begin{matrix} L^2(\mathbb{R}^n; \mathbb{C}^N) \\ \oplus \\ L^2(\mathbb{R}^n; \mathbb{C}^N) \end{matrix}$$

where

$$\Gamma = \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}, \ \Gamma^* = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}.$$

Then Π_B is a Hodge-Dirac operator, and so, by [19], has a bounded H^∞ functional calculus in $L^2(\mathbb{R}^n; \mathbb{C}^{2N})$.

Turning to $p \in (1, \infty)$, we find that $(\Pi_B(p))$ holds if and only if

$$\|u\|_p \lesssim \|Au\|_p \quad \forall u \in \overline{\mathcal{R}_p(D)} \qquad \text{and}$$
$$\|v\|_{p'} \lesssim \|A^*v\|_{p'} \quad \forall v \in \overline{\mathcal{R}_{p'}(D)}\ ,$$

and that $(\underline{\Pi_B}(p))$ is the same. Assuming this (in particular if A is invertible in L^∞), we find that

$$\mathcal{N}_p(\Pi_B) = \begin{matrix} \mathcal{N}_p(D) \\ \oplus \\ \mathcal{N}_p(DA) \end{matrix}, \quad \overline{\mathcal{R}_p(\Gamma)} = \begin{matrix} \{0\} \\ \oplus \\ \overline{\mathcal{R}_p(D)} \end{matrix}, \quad \overline{\mathcal{R}_p(\Gamma_B^*)} = \begin{matrix} \overline{\mathcal{R}_p(AD)} \\ \oplus \\ \{0\} \end{matrix}.$$

and hence that Π_B Hodge decomposes $L^p(\mathbb{R}^n; \mathbb{C}^{2N})$, i.e. $p \in (p_H, p^H)$, if and only if

$$L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(DA) \oplus \overline{\mathcal{R}_p(D)}. \tag{4.1}$$

This can be seen following the arguments in Proposition 2.16: Under $(\Pi_B(p))$, (4.1) holds if and only if $L^{p'}(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_{p'}(D) \oplus \overline{\mathcal{R}_{p'}(A^*D)}$, i.e. if and only if $L^p(\mathbb{R}^n; \mathbb{C}^N) = \mathcal{N}_p(D) \oplus \overline{\mathcal{R}_p(AD)}$.

As in [19, Proof of Theorem 3.1] and [36, Corollary 8.17], we compute that, when defined,

$$f(DA)u = \begin{bmatrix} 0 & I \end{bmatrix} f(\Pi_B) \begin{bmatrix} A \\ I \end{bmatrix} u,$$

so that results concerning DA having a bounded H^∞ functional calculus in $L^p(\mathbb{R}^n; \mathbb{C}^N)$ can be obtained from our results for Π_B in $L^p(\mathbb{R}^n; \mathbb{C}^{2N})$. Moreover results concerning bounds on $f(DA)u$ when $u \in \overline{\mathcal{R}_p(D)}$ can be obtained from our results on $f(\Pi_B)v$ when

$v \in \overline{\mathcal{R}_p(\Gamma)}$ and on $f(\Pi_B)w$ when $w \in \overline{\mathcal{R}_p(\Gamma_B^*)}$.

We leave further details to the reader, as well as consideration of AD.

5. Low frequency estimates: The Carleson measure argument

In this section, we prove the low frequency estimate, Proposition 3.8. By Lemma 2.9 and Theorem 2.21, we have

$$\|(t,x) \mapsto (Q_t^B)^M P_t^{\tilde{N}} u(x)\|_{T^{p,2}} \lesssim \|(t,x) \mapsto Q_t^B P_t^{\tilde{N}} u(x)\|_{T^{p,2}}$$

so it suffices to prove

$$\|(t,x) \mapsto Q_t^B P_t^{\tilde{N}} u(x)\|_{T^{p,2}} \lesssim \|u\|_p \quad \forall u \in \overline{\mathcal{R}_p(\Pi)}. \tag{5.1}$$

According to Theorem 2.21 and Lemma 2.6, the operator Q_t^B extends to an operator $Q_t^B : L^\infty(\mathbb{R}^n;\mathbb{C}^N) \to L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{C}^N)$ with

$$\|Q_t^B u\|_{L^2(B(x_0,t))} \lesssim t^{\frac{n}{2}} \|u\|_\infty \quad \forall u \in L^\infty(\mathbb{R}^n;\mathbb{C}^N),\ x_0 \in \mathbb{R}^n,\ t > 0. \tag{5.2}$$

We can therefore define

$$\gamma_t(x)w := (Q_t^B w)(x) \qquad \forall w \in \mathbb{C}^N,\ x \in \mathbb{R}^n, \tag{5.3}$$

where, on the right-hand side, w is considered as the constant function defined by $w(x) = w$ for all $x \in \mathbb{R}^n$. Note that the definition of γ_t is different from the one in [19, Definition 5.1].

In order to prove (5.1), we use the splitting

$$Q_t^B P_t^{\tilde{N}} u = [Q_t^B P_t^{\tilde{N}} u - \gamma_t A_t P_t^{\tilde{N}} u] + \gamma_t A_t P_t^{\tilde{N}} u,$$

and refer to $\gamma_t A_t P_t^{\tilde{N}} u$ as the principal part, and $[Q_t^B P_t^{\tilde{N}} u - \gamma_t A_t P_t^{\tilde{N}} u]$ as the principal part approximation.

We use the following *dyadic decomposition* of $\mathbb{R}^n$. Let $\Delta = \bigcup_{j=-\infty}^{\infty} \Delta_{2^j}$, where $\Delta_{2^j} := \{2^j(k + (0,1]^n) : k \in \mathbb{Z}^n\}$. For a dyadic cube $Q \in \Delta_{2^j}$, denote by $l(Q) = 2^j$ its sidelength, by $|Q| = 2^{jn}$ its volume. We set $\Delta_t = \Delta_{2^j}$, if $2^{j-1} < t \leq 2^j$. The dyadic averaging operator $A_t : L^2(\mathbb{R}^n;\mathbb{C}^N) \to L^2(\mathbb{R}^n;\mathbb{C}^N)$ is defined by

$$A_t u(x) := \frac{1}{|Q_{x,t}|} \int_{Q_{x,t}} u(y)\, dy =: \langle u \rangle_{Q_{x,t}} \qquad \forall u \in L^2(\mathbb{R}^n;\mathbb{C}^N),\ x \in \mathbb{R}^n,\ t > 0,$$

where $Q_{x,t}$ is the unique dyadic cube in Δ_t that contains x.

Let us make the following simple observation: for all $\varepsilon > 0$, there exists a constant $C > 0$ such that for all $t > 0$

$$\sup_{Q \in \Delta_t} \sum_{R \in \Delta_t} \left(1 + \frac{\text{dist}(Q,R)}{t}\right)^{-(n+\varepsilon)} \leq C.$$

We first consider the principal part approximation, similar to [19, Proposition 5.5].

Proposition 5.1. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $p \in (1,\infty)$. Then*

$$\|(t,x) \mapsto Q_t^B P_t^{\tilde{N}} u(x) - \gamma_t(x) A_t P_t^{\tilde{N}} u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \qquad \forall u \in \overline{\mathcal{R}_p(\Pi)}.$$

Proof. Fix $x \in \mathbb{R}^n$. For $t > 0$, we cover the ball $B(x,t)$ by a finite number of cubes $Q \in \Delta_t$. According to Theorem 2.21, Q_t^B has L^2-L^2 off-diagonal bounds of every order $N' > 0$. This, together with the Cauchy-Schwarz inequality and the Poincaré inequality (see [19, Lemma 5.4]), yields the following for $Q \in \Delta_t$:

$$\begin{aligned}
&(\int_0^\infty \int_Q |Q_t^B P_t^{\tilde{N}} u(y) - \gamma_t(y) A_t P_t^{\tilde{N}} u(y)|^2 \frac{dydt}{t^{n+1}})^{\frac12} \\
&= (\int_0^\infty \int_Q |Q_t^B (P_t^{\tilde{N}} u - \langle P_t^{\tilde{N}} u\rangle_Q)(y)|^2 \frac{dydt}{t^{n+1}})^{\frac12} \\
&\leq (\int_0^\infty (\sum_{R\in\Delta_t} \|Q_t^B \mathbb{1}_R (P_t^{\tilde{N}} u - \langle P_t^{\tilde{N}} u\rangle_Q)\|_{L^2(Q)})^2 \frac{dt}{t^{n+1}})^{\frac12} \\
&\lesssim (\int_0^\infty (\sum_{R\in\Delta_t} (1+\frac{\operatorname{dist}(Q,R)}{t})^{-N'} \|P_t^{\tilde{N}} u - \langle P_t^{\tilde{N}} u\rangle_Q\|_{L^2(R)})^2 \frac{dt}{t^{n+1}})^{\frac12} \\
&\lesssim (\int_0^\infty \sum_{R\in\Delta_t} (1+\frac{\operatorname{dist}(Q,R)}{t})^{-N'} \|P_t^{\tilde{N}} u - \langle P_t^{\tilde{N}} u\rangle_Q\|^2_{L^2(R)} \frac{dt}{t^{n+1}})^{\frac12} \\
&\lesssim (\int_0^\infty \int_{\mathbb{R}^n} (1+\frac{\operatorname{dist}(Q,y)}{t})^{-N'+2n} |t\nabla P_t^{\tilde{N}} u(y)|^2 \frac{dydt}{t^{n+1}})^{\frac12} \\
&\lesssim \sum_{j=0}^\infty (\int_0^\infty \int_{S_j(Q)} 2^{-j(N'-2n)} |t\nabla P_t^{\tilde{N}} u(y)|^2 \frac{dydt}{t^{n+1}})^{\frac12}.
\end{aligned}$$

By change of angle in tent spaces, see Lemma 2.8, we thus get

$$\begin{aligned}
\|(t,x)\mapsto Q_t^B P_t^{\tilde{N}} u(x) - \gamma_t(x) A_t P_t^{\tilde{N}} u(x)\|_{T^{p,2}} &\lesssim \sum_{j=0}^\infty 2^{-\frac{j}{2}(N'-2n)} \|(t,x)\mapsto t\nabla P_t^{\tilde{N}} u(x)\|_{T^{p,2}_{2^j}} \\
\lesssim \sum_{j=0}^\infty 2^{-\frac{j}{2}(N'-2n)} 2^{j\frac{n}{\min\{p,2\}}} \|(t,x)\mapsto t\nabla P_t^{\tilde{N}} u(x)\|_{T^{p,2}} &\lesssim \|(t,x)\mapsto t\nabla P_t^{\tilde{N}} u(x)\|_{T^{p,2}},
\end{aligned}$$

choosing $N' > 2n + \frac{2n}{\min\{p,2\}}$. Since $P_t^{\tilde{N}}$ is a Fourier multiplier, we have that, for $u = \Pi v$ with $v \in \mathcal{D}_2(\Pi)$, and all $j = 1,...,n$:

$$t\partial_{x_j} P_t^{\tilde{N}} u = \tilde{Q}_t(\partial_{x_j} v)$$

with $\tilde{Q}_t = t\Pi P_t^{\tilde{N}}$. Therefore, using Proposition 2.20 and Theorem 2.11 (for $p \leq 2$), or Proposition 3.5 (for $p \geq 2$) and Remark 3.6, along with Proposition 2.13 (6), we have that

$$\|(t,x)\mapsto t\nabla P_t^{\tilde{N}} u(x)\|_{T^{p,2}} \lesssim \max_{j=1,\dots,n} \|(t,x)\mapsto \tilde{Q}_t(\partial_{x_j} v)(x)\|_{T^{p,2}} \lesssim \max_{j=1,\dots,n} \|\partial_{x_j} v\|_p \lesssim \|u\|_p,$$

which concludes the proof. □

Turning now to the estimate for the principal part, we first show that $\{\gamma_t A_t\}_{t>0}$ defines a bounded operator on $T^{p,2}$ for all $p \in (1,\infty)$. This is an analogue of [19, Proposition 5.7].

Lemma 5.2. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $p \in (1,\infty)$. Then*

$$\|(t,x)\mapsto \gamma_t(x) A_t F(t,\,.\,)(x)\|_{T^{p,2}} \leq C_p \|F\|_{T^{p,2}} \qquad \forall F \in T^{p,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^N).$$

Proof. First observe that, given $x \in \mathbb{R}^n$ and $t > 0$,

$$\|A_t F(t,\,.\,)\|_{L^\infty(B(x,t))} = \sup_{y\in B(x,t)} |A_t F(t,y)| = \sup_{\substack{Q\in\Delta_t \\ B(x,t)\cap Q\neq\emptyset}} |Q|^{-1}|\int_Q F(t,z)\,dz|$$
$$\leq \sup_{\substack{Q\in\Delta_t \\ B(x,t)\cap Q\neq\emptyset}} |Q|^{-\frac{1}{2}}\|F(t,\,.\,)\|_{L^2(B(x,Ct))} \lesssim t^{-\frac{n}{2}}\|F(t,\,.\,)\|_{L^2(B(x,Ct))},$$

for some constant $C > 0$ depending only on the dimension n. According to (5.2), we have on the other hand $\|\gamma_t\|_{L^2(B(x,t))} \lesssim t^{\frac{n}{2}}$, and consequently

$$(\int_0^\infty \int_{B(x,t)} |\gamma_t(y)\cdot A_t F(t,y)|^2 \,\frac{dydt}{t^{n+1}})^{\frac{1}{2}} \lesssim (\int_0^\infty \int_{B(x,Ct)} |F(t,y)|^2 \,\frac{dydt}{t^{n+1}})^{\frac{1}{2}}.$$

Taking the L^p norm with respect to $x \in \mathbb{R}^n$ then yields the assertion. □

The corresponding estimate for the principal part $\gamma_t(x)A_tP_t^{\tilde{N}}$ relies on the following factorisation result for tent spaces:

Theorem 5.3 ([22], Theorem 1.1)**.** *Let $p, q \in (1,\infty)$. If $F \in T^{p,\infty}(\mathbb{R}^{n+1}_+;\mathbb{C}^N)$ and $G \in T^{\infty,q}(\mathbb{R}^{n+1}_+;\mathbb{C}^N)$, then $FG \in T^{p,q}(\mathbb{R}^{n+1}_+;\mathbb{C}^N)$ and*

$$\|F\cdot G\|_{T^{p,q}} \leq C\|F\|_{T^{p,\infty}}\|G\|_{T^{\infty,q}},$$

with a constant C which is independent of F and G.

This plays the role of the L^p vertical square function version of Carleson's inequality proven in [35, Lemma 8.1]. Note that this conical version is substantially simpler than its vertical counterpart. Note also that the fact that we use an essential supremum in our definition of $T^{p,\infty}$ does not affect the proof of the above result.

We also use the following conical maximal function estimate for operators with L^q-L^q off-diagonal bounds.

Lemma 5.4. *Let $q \in [1,2]$ and $p \in (1,\infty)$ with $q < p$. Let $\{T_t\}_{t>0}$ be a family of operators in $\mathcal{L}(L^q(R^n;\mathbb{C}^N))$, such that for all $t > 0$, T_t has L^q-L^q off-diagonal bounds of order $N' > \frac{n}{q}$. Then*

$$\|(t,x)\mapsto A_tT_tu(x)\|_{T^{p,\infty}} \leq C_p\|u\|_p \quad \forall u \in L^q(\mathbb{R}^n;\mathbb{C}^N)\cap L^p(\mathbb{R}^n;\mathbb{C}^N).$$

Proof. Let $u \in L^q(\mathbb{R}^n;\mathbb{C}^N)\cap L^p(\mathbb{R}^n;\mathbb{C}^N)$. Using Hölder's inequality and L^q-L^q off-diagonal bounds for T_t, we obtain, given $x \in \mathbb{R}^n$, the pointwise estimate

$$\begin{aligned}
\operatorname*{ess\,sup}_{(y,t)\in\Gamma(x)} |A_tT_tu(y)| &\lesssim \sup_{(y,t)\in\Gamma(x)} (t^{-n}\int_{Q_{y,t}} |T_tu(z)|\,dz) \\
&\leq \sup_{(y,t)\in\Gamma(x)} \sum_{j=0}^{\infty} (t^{-n}\int_{Q_{y,t}} |T_t \mathbb{1}_{S_j(Q_{y,t})}u(z)|^q\,dz)^{1/q} \\
&\lesssim \sup_{(y,t)\in\Gamma(x)} \sum_{j=0}^{\infty} t^{-\frac{n}{q}} 2^{-jN'} \|u\|_{L^q(2^jQ_{y,t})} \\
&\lesssim \sup_{(y,t)\in\Gamma(x)} \sum_{j=0}^{\infty} 2^{-j(N'-\frac{n}{q})} (2^jt)^{-\frac{n}{q}} \|u\|_{L^q(2^jQ_{y,t})} \\
&\lesssim \sup_{R>0} (\frac{1}{|B(x,R)|}\int_{B(x,R)} |u(z)|^q dz)^{\frac{1}{q}} =: \mathcal{M}_q u(x).
\end{aligned}$$

Since $q < p$, the boundedness of the Hardy-Littlewood maximal operator in $L^{\frac{p}{q}}$ implies that the maximal operator $\mathcal{M}_q$ is bounded in $L^p(\mathbb{R}^n;\mathbb{C}^N)$. Thus,

$$\|(t,x) \mapsto A_tT_tu(x)\|_{T^{p,\infty}} \lesssim \|\mathcal{M}_q u\|_p \lesssim \|u\|_p.$$

□

The estimate for the principal part is a direct consequence of the results above, together with the Carleson measure estimate for $|\gamma_t(x)|^2 \frac{dtdx}{t}$.

Proposition 5.5. *Suppose* Π_B *is a perturbed Hodge-Dirac operator. Let* $(t,x) \mapsto \gamma_t(x)$ *be defined as in* (5.3)*. Suppose* $p \in (1,\infty)$*. Then*

$$\|(t,x) \mapsto \gamma_t(x)A_tP_t^{\tilde{N}}u(x)\|_{T^{p,2}} \leq C_p\|u\|_p \qquad \forall u \in \overline{\mathcal{R}_p(\Pi)}.$$

Proof. Since $A_t^2 = A_t$, Theorem 5.3 yields

$$\|(t,x) \mapsto \gamma_t(x)A_tP_t^{\tilde{N}}u(x)\|_{T^{p,2}} \lesssim \|(t,x) \mapsto A_tP_t^{\tilde{N}}u(x)\|_{T^{p,\infty}} \cdot \|(t,x) \mapsto \gamma_t(x)\|_{T^{\infty,2}}.$$

The boundedness of the last factor is shown in Proposition 3.5 and noted in Remark 3.6, as a consequence of the L^2 theory for Π_B established in [19], cf. Theorem 2.21. The first factor is bounded by a constant times $\|u\|_p$ as an application of Lemma 5.4: take $T_t := P_t^{\tilde{N}}$ and notice that, for all $t > 0$, and every $q \in (1,2]$, $P_t^{\tilde{N}}$ satisfies L^q-L^q off-diagonal bounds of every order by Proposition 2.20. This completes the proof of (5.1) and hence of Proposition 3.8. □

6. High frequency estimates for $p \in (2_*,2]$

In this section, we give a proof of Proposition 3.9 for the case $2_* < p_H < 2$. In particular, this gives a proof for $n \in \{1,2\}$, a case we have to exclude in Section 7 below for technical reasons. The proof is similar to the corresponding proof in L^2 in [19], and is less technically involved than the case $p_H \leq 2_*$ considered in the next sections.

Proposition 6.1. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $M \in \mathbb{N}$ and $p \in (2_*, 2]$. Then*

$$\|(t,x) \mapsto (Q_t^B)^M (I - P_t^{\tilde{N}}) u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \quad \forall u \in \overline{\mathcal{R}_2(\Gamma)} \cap L^p(\mathbb{R}^n; \mathbb{C}^N).$$

Proof of Proposition 6.1. Let $p \in (2_*, 2]$, $M \in \mathbb{N}$, and $u \in \mathcal{R}_2(\Gamma) \cap L^p(\mathbb{R}^n; \mathbb{C}^N)$. Lemma 2.9 and Lemma 6.2 below yield

$$\begin{aligned}\|(t,x) \mapsto (Q_t^B)^M (I - P_t^{\tilde{N}}) u(x)\|_{T^{p,2}} &= \|(t,x) \mapsto Q_t^B t\Gamma(\sum_{k=0}^{\tilde{N}-1} P_t^{k}) Q_t u(x)\|_{T^{p,2}} \\ &\lesssim \|(t,x) \mapsto Q_t u(x)\|_{T^{p,2}}.\end{aligned}$$

The assertion then follows from Proposition 2.20. □

We use the following lemma in the proof of Proposition 6.1 above. The result and its proof are a slight modification of [19, Proposition 5.2].

Lemma 6.2. *The families $\{t\Gamma_B^* Q_t^B \ ; \ t \in \mathbb{R}\}$ and $\{t\Gamma Q_t^B \ ; \ t \in \mathbb{R}\}$ have L^2-L^2 off-diagonal bounds of every order.*

Proof. We prove the result for $\{t\Gamma_B^* Q_t^B \ ; \ t \in \mathbb{R}\}$. The result for $\{t\Gamma Q_t^B \ ; \ t \in \mathbb{R}\}$ then follows, given that for all $t \in \mathbb{R}$,

$$t\Gamma Q_t^B = (I - P_t^B) - t\Gamma_B^* Q_t^B,$$

and $\{P_t^B \ ; \ t \in \mathbb{R}\}$ has L^2-L^2 off-diagonal bounds of every order by Theorem 2.21. By Theorem 2.21, we also have that the family $\{t\Gamma_B^* Q_t^B \ ; \ t \in \mathbb{R}\} = \{P_{\overline{R(\Gamma_B^*)}}(I - P_t^B) \ ; \ t \in \mathbb{R}\}$ is uniformly bounded in L^2. Let $E, F \subset \mathbb{R}^n$ be two Borel sets, $u \in L^2(\mathbb{R}^n; \mathbb{C}^N)$, and $t \in \mathbb{R}$. As in [19, Proposition 5.2], let η be a Lipschitz function supported in $\tilde{E} = \{x \in \mathbb{R}^n \ ; \ \operatorname{dist}(x, E) < \frac{1}{2} \operatorname{dist}(x, F)\}$, constantly equal to 1 on E, and such that $\|\nabla \eta\|_\infty \leq \frac{4}{\operatorname{dist}(E,F)}$. We have the following:

$$\|t\Gamma_B^* Q_t^B u\|_{L^2(E)} \leq \|\eta t\Gamma_B^* Q_t^B u\|_2 \leq \|[\eta I, t\Gamma_B^*] Q_t^B u\|_2 + \|t\Gamma_B^* \eta Q_t^B u\|_2.$$

To estimate the first term, we use that $[\eta I, t\Gamma_B^*] = tB_1[\eta I, \Gamma^*]B_2$ is a multiplication operator with norm bounded by $t\|\nabla \eta\|_\infty$, together with the off-diagonal bounds for Q_t^B. For the second term, observe that, since Π_B Hodge decomposes L^2 according to Proposition 2.17, we have that

$$\|t\Gamma_B^* \eta Q_t^B u\|_2 \lesssim \|t\Pi_B \eta Q_t^B u\|_2 \leq \|[\eta I, t\Pi_B] Q_t^B u\|_2 + \|\eta t\Pi_B Q_t^B u\|_2.$$

Here, we use that the commutator in the first part of the sum is again a multiplication operator. For the second part, we use that $t\Pi_B Q_t^B = I - P_t^B$, which satisfies L^2-L^2 off-diagonal bounds. □

7. L^p-L^2 OFF-DIAGONAL BOUNDS

We assume $n \geq 3$ throughout this section.

In this section, we show how to deduce L^{p_*}-L^p bounds from L^p bisectoriality via a Sobolev inequality. We then use this result to establish appropriate $\dot{W}^{1,p} - L^2$ off-diagonal bounds on balls, when $p_* > p_H$.

Lemma 7.1. *Suppose $n \geq 3$. Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $p \in (1,2]$ with $p_* > 1$, and assume that $p > p_H$ and that Π_B is bisectorial in $L^p(\mathbb{R}^n;\mathbb{C}^N)$. Then*

$$\sup_{t\in\mathbb{R}} \|tR_t^B u\|_p \lesssim \|u\|_{p_*} \quad \forall u \in \overline{\mathcal{R}_{p_*}(\Gamma)}. \tag{7.1}$$

Moreover, if $(\Pi_B(p_*))$ holds, then

$$\sup_{t\in\mathbb{R}} \|tR_t^B u\|_p \lesssim \|u\|_{p_*} \quad \forall u \in \overline{\mathcal{R}_{p_*}(\Gamma_B^*)}. \tag{7.2}$$

Proof. To prove (7.1), we use the potential map $S_\Gamma : \overline{\mathcal{R}_{p_*}(\Gamma)} \to \dot{W}^{1,p_*}(\mathbb{R}^n;\mathbb{C}^N) \hookrightarrow L^p(\mathbb{R}^n;\mathbb{C}^N)$ defined in Proposition 2.13(7). Then, for all $u \in \overline{\mathcal{R}_{p_*}(\Gamma)}$,

$$\begin{aligned}\|tR_t^B u\|_p &= \|t(P_t^B - iQ_t^B)\Gamma S_\Gamma u\|_p \leq \|t\Gamma P_t^B S_\Gamma u\|_p + \|t\Gamma_B^* Q_t^B S_\Gamma u\|_p \\ &\lesssim \|Q_t^B S_\Gamma u\|_p + \|(I-P_t^B)S_\Gamma u\|_p \quad (\text{since } p > p_H) \\ &\lesssim \|S_\Gamma u\|_p \quad (\text{using bisectoriality}) \\ &\lesssim \|S_\Gamma u\|_{\dot{W}^{1,p_*}} \lesssim \|\Gamma S_\Gamma u\|_{p_*} = \|u\|_{p_*}\end{aligned}$$

as claimed.
By $(\Pi_B(p_*))$, (7.2) follows from

$$\sup_{t\in\mathbb{R}} \|tR_t^B B_1 v\|_p \lesssim \|v\|_{p_*} \quad \forall v \in \overline{\mathcal{R}_{p_*}(\Gamma^*)}.$$

This is proven in the same way as (7.1), using $\underline{\Pi_B}$ instead of Π_B, and remarking that

$$tR_t^B B_1 v = t(P_t^B - iQ_t^B)B_1\Gamma^* S_{\Gamma^*} v = tB_1 \underline{P_t^B}\Gamma^* S_{\Gamma^*} v - itB_2 \underline{Q_t^B}\Gamma^* S_{\Gamma^*} v \text{ (by (2.3))}.$$

□

We use the following induction argument in which $\begin{cases} p^{*(k)} = (p^{*(k-1)})^* & \forall k \in \mathbb{N}, \\ p^{*(0)} = p, \end{cases}$ and $M_s(p)$ is the smallest natural number such that $p^{*(M_s(p))} \geq 2$. A simple induction argument gives $p^{*(M)} = \frac{np}{n-pM}$ for all $M \in \mathbb{N}$, so that $M_s(p) \geq n(\frac{1}{p} - \frac{1}{2})$.

Proposition 7.2. *Suppose $n \geq 3$. Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $p \in (p_H, 2]$ with $p_* > 1$. Assume that Π_B is bisectorial in $L^p(\mathbb{R}^n;\mathbb{C}^N)$. Assume further that for all $M \in \mathbb{N}$ such that $M \geq M_s(p)$ and all $r \in (p,2]$ (with $r = 2$ if $p = 2$), $\{|t|^{n(\frac{1}{r}-\frac{1}{2})}(R_t^B)^M \mathbb{P}_{\overline{\mathcal{R}_r(\Pi_B)}} \; ; \; t \in \mathbb{R}^*\}$ is bounded from $L^r(\mathbb{R}^n;\mathbb{C}^N)$ to $L^2(\mathbb{R}^n;\mathbb{C}^N)$ uniformly in t.*

(1) Given $q \in (p_*, 2]$ and $M \in \mathbb{N}$ such that $M \geq M_s(p)$, we have

$$\sup_{t\in\mathbb{R}} \||t|^{n(\frac{1}{q}-\frac{1}{2})}(R_t^B)^M u\|_2 \lesssim \|u\|_q \quad \forall u \in \overline{\mathcal{R}_q(\Gamma)}.$$

Moreover, assuming $(\Pi_B(p_*))$ holds when $q < p_H$, we have

$$\sup_{t\in\mathbb{R}} \||t|^{n(\frac{1}{q}-\frac{1}{2})}(R_t^B)^M u\|_2 \lesssim \|u\|_q \quad \forall u \in \overline{\mathcal{R}_2(\Gamma_B^*)} \cap L^q(\mathbb{R}^n;\mathbb{C}^N).$$

(2) Given $q \in (\max\{p_H, p_*\}, 2]$ and $M \in \mathbb{N}$ such that $M \geq M_s(p)$, we have

$$\sup_{t\in\mathbb{R}} \||t|^{n(\frac{1}{q}-\frac{1}{2})}(R_t^B)^M \mathbb{P}_{\overline{\mathcal{R}_q(\Pi_B)}} u\|_2 \lesssim \|u\|_q \quad \forall u \in L^q(\mathbb{R}^n;\mathbb{C}^N).$$

Moreover $\{(R_t^B)^M\Gamma\ ;\ t \in \mathbb{R}^*\}$ and $\{(R_t^B)^M B_1\Gamma^*\ ;\ t \in \mathbb{R}^*\}$ have $\dot{W}^{1,q}$-L^2 off-diagonal bounds of every order on balls.

Proof. (1) By assumption, we have for all $r \in (p, 2]$ and all $M \in \mathbb{N}$ such that $M \geq M_s(p)$

$$\sup_{t\in\mathbb{R}} \||t|^{n(\frac{1}{r}-\frac{1}{2})}(R_t^B)^M \mathbb{P}_{\overline{\mathcal{R}_r(\Pi_B)}} u\|_2 \lesssim \|u\|_r \quad \forall u \in L^r(\mathbb{R}^n;\mathbb{C}^N).$$

Combining this with Lemma 7.1 gives the assertion for $q = r_*$.
(2) For $q > p_H$, Π_B Hodge decomposes $L^q(\mathbb{R}^n;\mathbb{C}^N)$ by assumption. We therefore get the first estimate in (2) as a direct consequence of (1).
Since $\|\Gamma w\|_q + \|B_1\Gamma^* w\|_q \lesssim \|\nabla w\|_q$ for all balls B and all $w \in \dot{W}^{1,q}_{\overline{B}}$, with constants independent of the ball B, we have that $\sup\limits_{t\in\mathbb{R}^*} \sup\limits_{B=B(x,|t|)} \||t|^{n(\frac{1}{q}-\frac{1}{2})}(R_t^B)^M\Gamma\|_{\mathcal{L}(\dot{W}^{1,q}_{\overline{B}},L^2)} < \infty$ and that $\sup\limits_{t\in\mathbb{R}^*} \sup\limits_{B=B(x,|t|)} \||t|^{n(\frac{1}{q}-\frac{1}{2})}(R_t^B)^M B_1\Gamma^*\|_{\mathcal{L}(\dot{W}^{1,q}_{\overline{B}},L^2)} < \infty$. Moreover $\{(R_t^B)^M\Gamma\ ;\ t \in \mathbb{R}^*\}$ and $\{(R_t^B)^M B_1\Gamma^*\ ;\ t \in \mathbb{R}^*\}$ have $\dot{W}^{1,2}$-L^2 off-diagonal bounds of every order on balls, since $\{(R_t^B)^M\ ;\ t \in \mathbb{R}^*\}$ has L^2-L^2 off-diagonal bounds of every order. The result then follows from Lemma 2.6(2).

□

8. Estimating conical square functions by vertical square functions

While vertical and conical square functions look similar, the conical square functions are applied quite differently here compared with the way the vertical square functions are used in [35]. Nevertheless, as is the case classically (see e.g. [46]), there are relationships between conical and vertical square functions, as Auscher, Hofmann, and Martell have already pointed out in [11]. Here we prove a new comparison theorem that exploits $\dot{W}^{1,p}$-L^2 off-diagonal bounds on balls. The proof is based on some unpublished work of Auscher, Duong and the second author, where a similar result was obtained for operators with pointwise Gaussian bounds.

Theorem 8.1. *Let $p \in (1, 2]$, and $M \in 2\mathbb{N}$ with $M \geq 10n$. Let Π_B be a perturbed Hodge Dirac operator, and assume that $(\Pi_B(p))$ holds. Assume that $\{(R_t^B)^{\frac{M}{2}-2}\Gamma\ ;\ t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-2}B_1\Gamma^*\ ;\ t \in \mathbb{R}^*\}$ have $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls. Then*

$$\|(t,x) \mapsto (Q_t^B)^M G(t,.)(x)\|_{T^{p,2}} \leq C_p\|(\int_0^\infty |G(t,.)|^2 \frac{dt}{t})^{\frac{1}{2}}\|_p, \tag{8.1}$$

for all $G \in L^p(\mathbb{R}^n; L^2(\mathbb{R}_+, \frac{dt}{t};\mathbb{C}^N))$ such that either $G(t,.) \in \overline{\mathcal{R}_p(\Gamma)}$ or $G(t,.) \in \overline{\mathcal{R}_p(\Gamma_B^*)}$ for almost every $t > 0$. If, in addition, $p > p_H$, then the result holds for all $G \in L^p(\mathbb{R}^n; L^2(\mathbb{R}_+, \frac{dt}{t};\mathbb{C}^N))$.

Proof. We use a variant of the Blunck-Kunstmann extrapolation method established by Auscher in [4, Theorem 1.1], combined with Auscher's Calderón-Zygmund decomposition for Sobolev spaces [4, Lemma 4.12]. As pointed out in [4] both results hold in Hilbert space valued L^p spaces. Let us therefore consider $H_1 = L^2(\mathbb{R}_+, \frac{dt}{t}; \mathbb{C}^N)$, $H_2 = L^2(\mathbb{R}_+ \times \mathbb{R}^n, \frac{dtdx}{t^{n+1}}; \mathbb{C}^N)$, and an operator T defined by

$$T(G) : x \mapsto [(t,y) \mapsto \mathbb{1}_{B(y,t)}(x)(Q_t^B)^M G(t,.)(y)],$$

for $G \in L^2(\mathbb{R}^n; H_1)$. Since Π_B is bisectorial in L^2, T is a bounded operator from $L^2(\mathbb{R}^n; H_1)$ to $L^2(\mathbb{R}^n; H_2)$. Indeed $\|T(G)\|_{L^2(H_2)} \sim (\int_0^\infty \|((Q_t^B)^M G(t,.)\|_2^2 \frac{dt}{t})^{\frac{1}{2}} \lesssim \|G\|_{L^2(H_1)}$.
We are going to show the weak type estimate: For all $\alpha > 0$,

$$|\{x \in \mathbb{R}^n : \|TG(x)\|_{H_2} > \alpha\}| \lesssim \alpha^{-p} \|G\|_p^p. \tag{8.2}$$

The strong type estimate will then follow by interpolation for every $q \in (p, 2]$.
In proving bounds on $L^p(\mathbb{R}^n; H_1)$, we use the fact that the lifting $S \mapsto \tilde{S}$ from $\mathcal{L}(L^p(\mathbb{R}^n; \mathbb{C}^N))$ to $\mathcal{L}(L^p(\mathbb{R}^n; H_1))$ is bounded, where $(\tilde{S}G)(t,.) = S(G(t,.))$ for almost every $t > 0$. This is a classical consequence of Khintchine-Kahane's inequalities (see [40, Section 2]) for discrete square functions, that extends to continuous square functions using a decomposition in an orthonormal basis of H_1. When $p > p_H$, we can apply this to the Hodge projections, and obtain the Hodge decomposition in $L^p(\mathbb{R}^n; H_1)$:

$$G = \tilde{\mathbb{P}}_{N_p(\Pi_B)} G + \tilde{\mathbb{P}}_{\overline{\mathcal{R}_p(\Gamma)}} G + \tilde{\mathbb{P}}_{\overline{\mathcal{R}_p(\Gamma_B^*)}} G =: G_0 + G_1 + G_2$$

with $\|G\|_p \approx \|G_0\|_p + \|G_1\|_p + \|G_2\|_p$.
Since $T(G_0) = 0$ for all G_0 such that $G_0(t,.) \in N_p(\Pi_B)$ for almost every $t > 0$, we only have to prove the result for $G_1(t,.) \in \overline{\mathcal{R}_p(\Gamma)}$ or $G_2(t,.) \in \overline{\mathcal{R}_p(\Gamma_B^*)}$ for almost every $t > 0$. In the first case, one can use the lifted potential map $\tilde{S}_\Gamma : \overline{\mathcal{R}_p(\Gamma)} \to W^{1,p}(\mathbb{R}^n; H_1)$ to write $G_1 = \Gamma \tilde{S}_\Gamma G_1 = \Gamma f$ where $f = \tilde{S}_\Gamma G_1 \in W^{1,p}(\mathbb{R}^n; H_1)$ with $\|\nabla \otimes f\|_p \approx \|G_1\|_p \leq \|G\|_p$.
We thus only have to prove the following weak type estimate: For all $\alpha > 0$,

$$|\{x \in \mathbb{R}^n : \|TG_1(x)\|_{H_2} > \alpha\}| \lesssim \alpha^{-p} \|\nabla \otimes f\|_p^p. \tag{8.3}$$

for $G_1 = \Gamma f$ and $f \in \dot{W}^{1,p}(\mathbb{R}^n; H_1)$.
According to [4, Lemma 4.12], given $\alpha > 0$, there exists a collection of cubes (Q_j) and functions g, b_j such that $f = g + \sum_j b_j$ with $\operatorname{supp} b_j \subseteq Q_j$ and

$$G_1 = \Gamma f = \Gamma g + \sum_j \Gamma b_j, \tag{8.4}$$

satisfying

$$g \in \dot{W}^{1,2}(H_1) \cap \dot{W}^{1,p}(H_1), \qquad \|\nabla g\|_2 \lesssim \alpha^{2-p} \|\nabla f\|_p^p; \tag{8.5}$$

$$b_j \in W_0^{1,p}(Q_j, H_1), \qquad \|\nabla b_j\|_p \lesssim \alpha |Q_j|^{1/p}; \tag{8.6}$$

$$\sum_j |Q_j| \lesssim \alpha^{-p} \|\nabla f\|_p^p; \tag{8.7}$$

$$\sum_j \mathbb{1}_{Q_j} \leq N. \tag{8.8}$$

Applying this decomposition, we have

$$|\{x \in \mathbb{R}^n : \|TG_1(x)\|_{H_2} > \alpha\}| \lesssim |\{x \in \mathbb{R}^n : \|T\Gamma g(x)\|_{H_2} > \alpha/3\}| + \\ |\{x \in \mathbb{R}^n : \|\sum_j T(I - A_{r_j})\Gamma b_j(x)\|_{H_2} > \frac{\alpha}{3}\}| + |\{x \in \mathbb{R}^n : \|\sum_j T(A_{r_j})\Gamma b_j(x)\|_{H_2} > \frac{\alpha}{3}\}|.$$

By (8.5) and the L^2 estimate, we have

$$|\{x \in \mathbb{R}^n : \|T\Gamma g(x)\|_{H_2} > \alpha/3\}| \lesssim \alpha^{-2}\|\Gamma g\|_2^2 \lesssim \alpha^{-p}\|\nabla f\|_p^p$$

so we turn our attention to the functions b_j.
Define

$$\phi_t(\Pi_B) = I - (I - (P_t^B)^M)^M \qquad \forall t > 0$$

and set $A_{r_j} = \phi_{r_j}(\Pi_B)$ where r_j is the diameter of Q_j.
We will establish, in Lemma 8.2 below, the following two off-diagonal bound conditions, similar to those of [4, Theorem 1.1]. Let $B \subset \mathbb{R}^n$ be a ball of radius $r > 0$, $k \in \mathbb{N}$, and $f \in \dot{W}^{1,p}_{\overline{B}}(\mathbb{R}^n; H_1)$. The two conditions are, in our setting,

$$(\frac{1}{|2^{k+1}B|}\int_{S_k(B)} \|\phi_r(\Pi_B)\Gamma f(.,y)\|_{H_1}^2 dy)^{\frac{1}{2}} \lesssim g(j)(|B|^{-1}\int_B \|\nabla f(.,y)\|_{H_1}^p dy)^{\frac{1}{p}}, \tag{8.9}$$

for $k \geq 1$, and

$$(\frac{1}{|2^{k+1}B|}\int_{S_k(B)} \|T(I - \phi_r(\Pi_B))\Gamma f(.,y)\|_{H_2}^2 dy)^{\frac{1}{2}} \lesssim g(k)(|B|^{-1}\int_B \|\nabla f(.,y)\|_{H_1}^p dy)^{\frac{1}{p}} \tag{8.10}$$

for $k \geq 2$, with $g(k)$ satisfying $\sum_{j\in\mathbb{N}} g(k)2^{nk} < \infty$.

Let us assume these for the moment. Then

$$|\{x \in \mathbb{R}^n : \|\sum_j T(I - A_{r_j})\Gamma b_j(x)\|_{H_2} > \alpha/3\}| \\ \lesssim \sum_j |Q_j| + \alpha^{-2}\|\sum_j \mathbb{1}_{(4Q_j)^c} T(I - A_{r_j})\Gamma b_j\|_2^2,$$

and the first term is bounded by $\alpha^{-p}\|\nabla f\|_p^p$ according to (8.7). To bound the second term, let $u \in L^2(H_2)$ with $\|u\|_2 = 1$. Use Hölder's inequality in the first step, (8.10) in the second step, $\sum_k g(k)2^{kn} < \infty$ and (8.6) in the third step to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\langle u(x), \sum_j \mathbb{1}_{(4Q_j)^c}(x) T(I - A_{r_j})\Gamma b_j(x)\rangle| \, dx \\
&\leq \sum_j \sum_{k=2}^{\infty} \left(\int_{S_k(Q_j)} \|u(x)\|_{H_2}^2 \, dx \right)^{1/2} \left(\int_{S_k(Q_j)} \|T(I - A_{r_j})\Gamma b_j(x)\|_{H_2}^2 \, dx \right)^{1/2} \\
&\lesssim \sum_j \sum_{k=2}^{\infty} |2^k Q_j|^{1/2} \inf_{y \in Q_j} \mathcal{M}_2(\|u\|_{H_2})(y) g(k) |2^k Q_j|^{1/2} |Q_j|^{-1/p} \|\Gamma b_j\|_p \\
&\lesssim \alpha \sum_j \int_{Q_j} \mathcal{M}_2(\|u\|_{H_2})(y) \, dy \lesssim \alpha \int_{\bigcup_j Q_j} \mathcal{M}_2(\|u\|_{H_2})(y) \, dy,
\end{aligned}$$

where $\mathcal{M}_2$ is the maximal operator defined in the proof of Lemma 5.4. By Kolmogorov's lemma and (8.7), the last line is bounded by

$$\alpha |\bigcup_j Q_j|^{1/2} \|u\|_2 \lesssim \alpha (\sum_j |Q_j|)^{1/2} \lesssim \alpha^{1-p/2} \|\nabla f\|_p^{p/2},$$

which gives the desired estimate.
It remains to consider the last part

$$|\{x \in \mathbb{R}^n : \|\sum_j T A_{r_j} \Gamma b_j(x)\|_{H_2} > \alpha/3\}| \lesssim \alpha^{-2} \|\sum_j T A_{r_j} \Gamma b_j\|_{L^2(H_2)}^2.$$

We again dualise with $u \in L^2(H_2)$, $\|u\|_2 = 1$, now using L^p-L^2 off-diagonal bounds for A_{r_j} on $R_p(\Gamma)$. With similar arguments as before, but now including the on-diagonal term, using (8.9) and using that $T \in \mathcal{L}(L^2(H_1), L^2(H_2))$ by assumption, one obtains

$$\begin{aligned}
|\langle u, \sum_j T A_{r_j} \Gamma b_j \rangle| &\leq \sum_j \sum_{k=0}^{\infty} \left(\int_{S_k(Q_j)} \|Tu(x)\|_{H_1}^2 \, dx \right)^{1/2} \left(\int_{S_k(Q_j)} \|A_{r_j} \Gamma b_j(x)\|_{H_1}^2 \, dx \right)^{1/2} \\
&\lesssim \alpha \mathcal{M}_2(\|Tu\|_{H_1})(y) \, dy \lesssim \alpha \|Tu\|_2 (\sum_j |Q_j|)^{1/2} \lesssim \alpha^{1-p/2} \|\nabla f\|_p^{p/2}.
\end{aligned}$$

This establishes (8.3) for $G_1 = \Gamma f$.
In the case where $G(t,.) \in \overline{\mathcal{R}_p(\Gamma_B^*)}$ for almost every $t > 0$, we replace $(Q_t^B)^M u$ for $u \in \overline{\mathcal{R}_p(\Gamma_B^*)}$, by $(\underline{Q_t^B})^M v$ for $v \in \overline{\mathcal{R}_p(\Gamma^*)}$, as in the proof of Theorem 3.1, and then proceed as above. □

Lemma 8.2. *Suppose $p \in (p_H, 2]$ and $M \in 2\mathbb{N}$ with $M > n + 4$.*
Assume that $\{(R_t^B)^{\frac{M}{2}-1}\Gamma \; ; \; t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-1} B_1 \Gamma^* \; ; \; t \in \mathbb{R}^*\}$ have $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls. Then (8.9) and (8.10) hold for all balls $B \subset \mathbb{R}^n$ of radius $r > 0$, and all $f \in \dot{W}_{\overline{B}}^{1,p}(\mathbb{R}^n; H_1)$.

Proof. We write $\phi(z) = \tilde{\phi}(z)(1 + z^2)^{-1}$ for $z \in S_\theta$ and $\theta \in (0, \frac{\pi}{2})$, and notice that $\phi \in H^\infty(S_\theta)$. Moreover ϕ is a sum and product of functions of the form $z \mapsto (1 \pm iz)^{-1}$. Therefore $\{\phi(r\Pi_B) \; ; \; r > 0\}$ has L^2-L^2 off-diagonal bounds of every order. Combining this with the $\dot{W}^{1,p}$-L^2 off-diagonal bounds assumption and using Lemma 2.6, we have that

$\{\phi(r\Pi_B)\Gamma\ ;\ r>0\}$ has $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls. This gives the following:

$$\Big(\frac{1}{|2^{j+1}B|}\int\limits_{S_j(B)} \|\phi_r(\Pi_B)\Gamma f(.,y)\|_{H_1}^2 dy\Big)^{\frac12} \approx 2^{-j\frac{n}{2}} r^{-\frac{n}{2}}\Big(\int\limits_0^\infty \int\limits_{S_j(B)} |\phi_r(\Pi_B)\Gamma f(t,y)|^2 \frac{dydt}{t}\Big)^{\frac12}$$
$$\lesssim 2^{-j(M+\frac{n}{2})} r^{-\frac{n}{p}}\Big(\int\limits_0^\infty \|\nabla f(t,y)\|_p^2 \frac{dt}{t}\Big)^{\frac12} \lesssim 2^{-j(M+\frac{n}{2})}|B|^{-\frac1p}\|\int\limits_B |\nabla f(t,y)|^p dy\|_{\frac2p}^{\frac1p}$$
$$\lesssim 2^{-j(M+\frac{n}{2})}\Big(|B|^{-1}\int\limits_B \Big(\int\limits_0^\infty |\nabla f(t,y)|^2\frac{dt}{t}\Big)^{\frac{p}{2}} dy\Big)^{\frac1p} = 2^{-j(M+\frac{n}{2})}\Big(|B|^{-1}\int\limits_B \|\nabla f(.,y)\|_{H_1}^p dy\Big)^{\frac1p}.$$

This establishes (8.9).
The proof will thus be complete once we have established (8.10). To do so, we first use the straightforward integration lemma [23, Lemma 1] (an application of Fubini's theorem), and obtain that for $j\geq 2$

$$\Big(\frac{1}{|2^{j+1}B|}\int\limits_{S_j(B)} \|T(I-\phi_r(\Pi_B))\Gamma f(.,y)\|_{H_2}^2 dy\Big)^{\frac12}$$
$$\lesssim \Big(\frac{1}{|2^{j+1}B|}\int\limits_0^\infty \int\limits_{\mathcal{R}(S_j(B))} |(Q_t^B)^M(I-\phi_r(\Pi_B))\Gamma f(t,y)|^2\frac{dydt}{t}\Big)^{\frac12}$$
$$\leq \Big(\frac{1}{|2^{j+1}B|}\int\limits_0^\infty \int\limits_{(2^{j-1}B)^c} |(Q_t^B)^M(I-\phi_r(\Pi_B))\Gamma f(t,y)|^2\frac{dydt}{t}\Big)^{\frac12}$$
$$+\Big(\frac{1}{|2^{j+1}B|}\int\limits_{2^{j-2}r}^\infty \int\limits_{2^{j-1}B} |(Q_t^B)^M(I-\phi_r(\Pi_B))\Gamma f(t,y)|^2\frac{dydt}{t}\Big)^{\frac12} =: I_1+I_2,$$

where $\mathcal{R}(S_j(B)) = \bigcup\limits_{x\in S_j(B)}\{(t,y)\in\mathbb{R}_+\times\mathbb{R}^n\ ;\ |y-x|\leq t\}$. Let us first estimate I_2. Notice that

$$(Q_t^B)^M(I-\phi_r(\Pi_B)) = \Big(\frac{r}{t}\Big)^M (t^2\Pi_B^2)^M (I+t^2\Pi_B^2)^{-M}\tilde{\phi}(r\Pi_B) \qquad \forall t,r>0,$$

for $\tilde{\phi}(z) = \Big(\frac{(1+z^2)^M-1}{z(1+z^2)^M}\Big)^M$, and that $\{(t^2\Pi_B^2)^M(I+t^2\Pi_B^2)^{-M}\ ;\ t>0\}$ has L^2-L^2 off-diagonal bounds. In order to show that $\{\tilde{\phi}(r\Pi_B)\ ;\ r>0\}$ has $\dot{W}^{1,p}$-L^2 off-diagonal bounds of order $N'>\frac{n}{2}$ on balls, write $\tilde{\phi}(z) = (1+z^2)^{\frac{M}{2}-1}\tilde{\phi}(z)(1+z^2)^{-(\frac{M}{2}-1)}$. By Lemma 2.7, $\{(1+r^2\Pi_B^2)^{\frac{M}{2}-1}\tilde{\phi}(r\Pi_B)\ ;\ r>0\}$ has L^2-L^2 off-diagonal estimates of every order. On the other hand, $\{(I+r^2\Pi_B^2)^{-(\frac{M}{2}-1)}\Gamma\ ;\ r\in\mathbb{R}^*\}$ has $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls. Combining the two families of operators gives the statement (see Lemma 2.6). Thus $\{(\frac{r}{t})^{n(\frac1p-\frac12)}(t^2\Pi_B^2)^M(I+t^2\Pi_B^2)^{-M}\tilde{\phi}(r\Pi_B)\Gamma\ ;\ t>0\}$ has $\dot{W}^{1,p}$-L^2 off-diagonal bounds of order $N' > \frac{n}{2}$ on balls, again by Lemma 2.6, whenever $0 < r < t$. In particular, $\|\psi_t(\Pi_B)(I-\phi_r(\Pi_B))\|_{\mathcal{L}(L^p,L^2)} \lesssim (\frac{r}{t})^{\tilde{M}} t^{-n(\frac{1}{p}-\frac{1}{2})}$, for $\tilde{M} := M - n(\frac{1}{p}-\frac{1}{2})$. This gives

$$\begin{aligned} I_2 &= (\frac{1}{|2^{j+1}B|}\int_{2^{j-2}r}^{\infty}\int_{2^{j-1}B} |\psi_t(\Pi_B)(I-\phi_r(\Pi_B))\Gamma f(t,y)|^2 \frac{dydt}{t})^{\frac{1}{2}} \\ &\lesssim (\frac{1}{|2^{j+1}B|}\int_{2^{j-2}r}^{\infty} ((\frac{r}{t})^{\tilde{M}} t^{-n(\frac{1}{p}-\frac{1}{2})}\|\nabla f(t,.)\|_p)^2 \frac{dt}{t})^{\frac{1}{2}} \\ &\lesssim (\int_{2^{j-2}r}^{\infty} ((2^j r)^{-\frac{n}{2}}(\frac{r}{t})^{\tilde{M}} t^{-n(\frac{1}{p}-\frac{1}{2})}\|\nabla f(t,.)\|_p)^2 \frac{dt}{t})^{\frac{1}{2}}. \end{aligned}$$

We can estimate the above by

$$\begin{aligned} 2^{-j(\tilde{M}+\frac{n}{p})} r^{-\frac{n}{p}} (\int_{2^j r}^{\infty} \|\nabla f(t,.)\|_p^2 \frac{dt}{t})^{\frac{1}{2}} &\lesssim 2^{-j(\tilde{M}+\frac{n}{p})} r^{-\frac{n}{p}} \| \int_B |\nabla f(.,y)|^p dy\|_{\frac{2}{p}}^{\frac{1}{p}} \\ &\lesssim 2^{-j(\tilde{M}+\frac{n}{p})} (\frac{1}{|B|}\int_B \|\nabla f(.,y)\|_{H_1}^p dy)^{\frac{1}{p}}. \end{aligned}$$

We now estimate

$$\begin{aligned} I_1 \leq J_1 + J_2 &:= (\frac{1}{|2^{j+1}B|}\int_0^r \int_{(2^{j-1}B)^c} |(Q_t^B)^M (I-\phi_r(\Pi_B))\Gamma f(t,y)|^2 \frac{dydt}{t})^{\frac{1}{2}} \\ &+ (\frac{1}{|2^{j+1}B|}\int_r^{\infty} \int_{(2^{j-1}B)^c} |(Q_t^B)^M (I-\phi_r(\Pi_B))\Gamma f(t,y)|^2 \frac{dydt}{t})^{\frac{1}{2}}. \end{aligned}$$

For J_1, we use that for $0 < t < r$,
$\{(Q_t^B)^M \Gamma f \ ; \ t > 0\}$ and $\{(Q_t^B)^M \phi_r(\Pi_B)\Gamma \ ; \ r > 0\}$ have $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls (since $(Q_t^B)^M = (I-P_t^B)^{\frac{M}{2}}(P_t^B)^{\frac{M}{2}}$ and $\{P_t^B \ ; \ t > 0\}$ has L^2-L^2

off-diagonal bounds of every order). Thus,

$$
\begin{aligned}
&(\frac{1}{|2^{j+1}B|}\int_0^r \int_{(2^{j-1}B)^c} |(Q_t^B)^M(I-\phi_r(\Pi_B))\Gamma f(t,y)|^2 \frac{dydt}{t})^{\frac{1}{2}} \\
&\leq (\frac{1}{|2^{j+1}B|}\int_0^r \int_{(2^{j-1}B)^c} |(Q_t^B)^M\Gamma f(t,y)|^2 \frac{dydt}{t})^{\frac{1}{2}} + (\frac{1}{|2^{j+1}B|}\int_0^r \int_{(2^{j-1}B)^c} |(Q_t^B)^M\phi_r(\Pi_B)\Gamma f(t,y)|^2 \frac{dydt}{t})^{\frac{1}{2}} \\
&\lesssim (\frac{1}{|2^{j+1}B|}\int_0^r (t^{-\frac{n}{p}+\frac{n}{2}}(1+\frac{2^j r}{t})^{-N'}\|\nabla f(t,.)\|_p)^2\frac{dt}{t})^{\frac{1}{2}} + (\frac{1}{|2^{j+1}B|}\int_0^r (r^{-\frac{n}{p}+\frac{n}{2}}2^{-jN'}\|\nabla f(t,.)\|_p)^2\frac{dt}{t})^{\frac{1}{2}} \\
&\lesssim 2^{-j(\frac{n}{2}+N')}r^{-\frac{n}{p}}(\int_0^r (\frac{t}{r})^{N'-\frac{n}{p}+\frac{n}{2}}\|\nabla f(t,.)\|_p)^2\frac{dt}{t})^{\frac{1}{2}} + 2^{-j(\frac{n}{2}+N')}r^{-\frac{n}{p}}(\int_0^r \|\nabla f(t,.)\|_p)^2\frac{dt}{t})^{\frac{1}{2}} \\
&\lesssim 2^{-j(\frac{n}{2}+N')}(\frac{1}{|B|}\int_B \|\nabla f(.,y)\|_{H_1}^p dy)^{\frac{1}{p}}.
\end{aligned}
$$

Turning to J_2, we now use that $\{(\frac{r}{t})^{n(\frac{1}{p}-\frac{1}{2})}(t^2\Pi_B^2)^M(I+t^2\Pi_B^2)^{-M}\tilde{\phi}(r\Pi_B)\Gamma\ ;\ t>0\}$ has $\dot{W}^{1,p}$-L^2 off-diagonal bounds of order $N'>\frac{n}{2}$, which gives

$$
\begin{aligned}
&(\frac{1}{|2^{j+1}B|}\int_r^\infty \int_{(2^{j-1}B)^c} |(Q_t^B)^M(I-\phi_r(\Pi_B))\Gamma f(t,y)|^2 \frac{dydt}{t})^{\frac{1}{2}} \\
&\lesssim (\int_r^{2^j r} ((2^j r)^{-\frac{n}{2}}(\frac{r}{t})^{\tilde{M}}t^{-n(\frac{1}{p}-\frac{1}{2})}(\frac{2^j r}{t})^{-N'}\|\nabla f(t,.)\|_p)^2\frac{dt}{t})^{\frac{1}{2}} \\
&\quad + (\int_{2^j r}^\infty ((2^j r)^{-\frac{n}{2}}(\frac{r}{t})^{\tilde{M}}t^{-n(\frac{1}{p}-\frac{1}{2})}\|\nabla f(t,.)\|_p)^2\frac{dt}{t})^{\frac{1}{2}} \\
&\lesssim 2^{-j\frac{n}{p}}r^{-\frac{n}{p}}(\int_r^{2^j r} ((\frac{r}{t})^{\tilde{M}}(\frac{2^j r}{t})^{-(N'-n(\frac{1}{p}-\frac{1}{2}))}\|\nabla f(t,.)\|_p)^2\frac{dt}{t})^{\frac{1}{2}} + 2^{-j(\tilde{M}+\frac{n}{p})}(\frac{1}{|B|}\int_B \|\nabla f(.,y)\|_{H_1}^p dy)^{\frac{1}{p}} \\
&\lesssim (2^{-j(N'+\frac{n}{2})}+2^{-j(\tilde{M}+\frac{n}{p})})(\frac{1}{|B|}\int_B \|\nabla f(.,y)\|_{H_1}^p dy)^{\frac{1}{p}}.
\end{aligned}
$$

Combining all the estimates gives

$$
(\frac{1}{|2^{j+1}B|}\int_{S_j(B)} \|T(I-\phi_r(\Pi_B))\Gamma f(.,y)\|_{H_2}^2 dy)^{\frac{1}{2}} \lesssim j2^{-j\min\{M+\frac{n}{2},N'+\frac{n}{2}\}}(|B|^{-1}\int_B \|\nabla f(.,y)\|_{H_1}^p dy)^{\frac{1}{p}},
$$

which shows (8.10), given that $M>\frac{n}{2}$ and $N'>\frac{n}{2}$. □

9. High frequency estimates for $p \in (\max\{1,(p_H)_*\},2]$

In this section, we finally prove Proposition 3.9 in the case $p \in (\max\{1,(p_H)_*\},2]$.

The idea of the proof is to show, using Corollary 10.2, that the integral operator defined by

$$T_K(F)(t,\,.\,) := \int_0^\infty K(t,s)F(s,\,.\,)\,\frac{ds}{s}$$

with $K(t,s) := (Q_t^B)^M(I - P_t^{\tilde N}){Q_s}^{2\tilde N}$ and M sufficiently large, extends to a bounded operator on tent spaces. The square function estimate of Proposition 3.9 is then reduced to the square function estimate for the unperturbed operator shown in Proposition 2.20.

Theorem 9.1. *Let M be even and such that $M \geq 10n$, and let $p \in (1,2]$. Suppose Π_B is a perturbed Hodge-Dirac operator, such that $(\Pi_B(p))$ holds. Assume that $\{t\Gamma P_t^B \;;\; t \in \mathbb{R}^*\}$ and $\{t\Gamma_B^* P_t^B \;;\; t \in \mathbb{R}^*\}$ are uniformly bounded in $\mathcal{L}(L^p)$ and that $\{(R_t^B)^{\frac{M}{2}-2}\Gamma \;;\; t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-2}B_1\Gamma^* \;;\; t \in \mathbb{R}^*\}$ have $\dot W^{1,p}$-L^2 off-diagonal bounds of every order on balls. Then, for all $q \in (\max\{1,p_*\},2]$,*

$$\|(t,x) \mapsto (Q_t^B)^M(I - P_t^{\tilde N})u(x)\|_{T^{q,2}} \leq C_q\|u\|_q \quad \forall u \in \overline{\mathcal{R}_q(\Gamma)}.$$

Proof. Let $p \in (1,2]$. Let $u \in \mathcal{R}_p(\Pi_B)$. By Theorem 2.21, Lemma 2.9, and using the reproducing formula $u = C\int_0^\infty {Q_s}^{2\tilde N}u\frac{ds}{s}$ for some constant C, we have that:

$$\begin{aligned}\|(t,x) \mapsto \int_t^\infty (Q_t^B)^M(I - P_t^{\tilde N}){Q_s}^{2\tilde N}u(x)\frac{ds}{s}\|_{T^{p,2}} &\lesssim \|(t,x) \mapsto \int_t^\infty (I - P_t^{\tilde N}){Q_s}^{2\tilde N}u(x)\frac{ds}{s}\|_{T^{p,2}}\\ &= \|(t,x) \mapsto \psi(t\Pi)u(x)\|_{T^{p,2}},\end{aligned}$$

for $\psi(z) = (1-(1+z^2)^{-\tilde N})\int_1^\infty (\frac{zs}{1+(zs)^2})^{2\tilde N}\frac{ds}{s}$. Therefore

$$\|(t,x) \mapsto \int_t^\infty (Q_t^B)^M(I - P_t^{\tilde N}){Q_s}^{2\tilde N}u(x)\frac{ds}{s}\|_{T^{p,2}} \lesssim \|u\|_p,$$

by Theorem 2.11 and Proposition 2.20, since $\psi \in \Psi_2^{\tilde N}$. Now let $q \in (\max\{1,p_*\},2]$. For $u \in \mathcal{R}_2(\Gamma) \cap L^q(\mathbb{R}^n;\mathbb{C}^N)$, we have that

$$\begin{aligned}&\|(t,x) \mapsto \int_0^t (Q_t^B)^M(I - P_t^{\tilde N}){Q_s}^{2\tilde N}u(x)\frac{ds}{s}\|_{T^{q,2}}\\ &= \|(t,x) \mapsto \int_0^t (\frac{s}{t})t\Gamma(Q_t^B)^M(I - P_t^{\tilde N})P_s{Q_s}^{\tilde N-1}{Q_s}^{\tilde N}u(x)\frac{ds}{s}\|_{T^{q,2}}\\ &\lesssim \|(t,x) \mapsto \int_0^t (\frac{s}{t})(Q_t^B)^{M-2}(I - P_t^{\tilde N})P_s{Q_s}^{\tilde N-1}{Q_s}^{\tilde N}u(x)\frac{ds}{s}\|_{T^{q,2}},\end{aligned}$$

where in the last step we have used Lemma 6.2 and Lemma 2.9. We now consider the integral operator T_K with kernel

$$K(t,s) = \mathbb{1}_{(0,\infty)}(t-s)(Q_t^B)^{M-2}(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}.$$

Using the results of Section 10, we aim to show that $T_{K_1^+}$ extends to a bounded operator on $T^{q,2}$. The result then follows from Proposition 2.20.
From our assumption, we have that $\{(P_t^B)^{\frac{M}{2}-2}\Gamma \ ; \ t \in \mathbb{R}^*\}$ and $\{(P_t^B)^{\frac{M}{2}-2}B_1\Gamma^* \ ; \ t \in \mathbb{R}^*\}$ have $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls. Since $(Q_t^B)^{M-4} = (I - P_t^B)^{\frac{M}{2}-2}.(P_t^B)^{\frac{M}{2}-2}$ and $\{P_t^B \ ; \ t \in \mathbb{R}\}$ has L^2-L^2 off-diagonal bounds of every order, we have that $\{(Q_t^B)^{M-4}\Gamma \ ; \ t \in \mathbb{R}^*\}$ and $\{(Q_t^B)^{M-4}B_1\Gamma^* \ ; \ t \in \mathbb{R}^*\}$ have $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls. This gives, for $u \in L^2$,

$$\begin{aligned}
&\|(Q_t^B)^{M-2}(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}u\|_2 \\
&\leq \|(Q_t^B)^{M-3}\Gamma S_\Gamma t\Gamma P_t^B(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}u\|_2 + \|(Q_t^B)^{M-3}B_1\Gamma^* S_{\Gamma^*} t\Gamma^* B_2 P_t^B(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}u\|_2 \\
&\lesssim \|t\Gamma P_t^B(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}u\|_p + \|t\Gamma^* B_2 P_t^B(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}u\|_p \\
&\lesssim \|u\|_p + \|tB_1\Gamma^* B_2 P_t^B(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}u\|_p \lesssim \|u\|_p,
\end{aligned}$$

where we have used the bisectoriality of the unperturbed operator in L^p (see Proposition 2.20), the assumption that $\{t\Gamma_B^* P_t^B \ ; \ t \in \mathbb{R}^*\}$ and $\{t\Gamma P_t^B \ ; \ t \in \mathbb{R}^*\}$ are uniformly bounded in $\mathcal{L}(L^p)$, and the properties of the potential maps (see Proposition 2.13).
Using Lemma 2.6, we thus get that K satisfies (10.1) with $\max\{t,s\} = t$ for all $r \in (p,2]$. To conclude the proof using Corollary 10.2, we thus only have to show that

$$\sup_{\gamma \in \mathbb{R}} \|T_{K_{\varepsilon+i\gamma}^+}\|_{\mathcal{L}(T^{r,2})} < \infty \quad \forall \varepsilon > 0 \quad \forall r \in (p,2].$$

To do so we use Lemma 2.9, Lemma 2.7, and Theorem 8.1, and obtain the following, for $\varepsilon > 0$, $r \in (p,2]$, $F \in T^{r,2}$, and $\gamma \in \mathbb{R}$ (with implicit constants independent of F and γ):

$$\begin{aligned}
\|T_{K_{\varepsilon+i\gamma}^+}F\|_{T^{r,2}} &= \|(t,x) \mapsto \int_0^t (\frac{s}{t})^{\varepsilon+i\gamma}(Q_t^B)^{M-2}(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}F(s,x)\frac{ds}{s}\|_{T^{r,2}} \\
&\lesssim \|(t,x) \mapsto \int_0^t (\frac{s}{t})^{\varepsilon+i\gamma}(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}F(s,x)\frac{ds}{s}\|_{L^r(\mathbb{R}^n;L^2((0,\infty),\frac{dt}{t}))} \\
&= \|T_{\widetilde{K}_{\varepsilon+i\gamma}^+}F\|_{L^r(\mathbb{R}^n;L^2((0,\infty),\frac{dt}{t}))},
\end{aligned}$$

where $\widetilde{K}(t,s) = (I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}$. Since the unperturbed operator Π has a bounded H^∞ functional calculus in L^r, the family $\{\widetilde{K}(t,s) \ ; \ t,s > 0\}$ is R-bounded in L^r by [39, Theorem 5.3]. Indeed, a product of finitely many R-bounded families is R-bounded. Moreover P_t and Q_s are given by the H^∞ functional calculus of Π. Therefore, one can use a theorem of Kalton and Weis that states, that, in L^r (or more generally in a Banach space with Pisier's property (α), see e.g. [39, Section 3] and the references therein), any such family is automatically R-bounded. This is proven in [39, Theorem 5.3] (1).

Therefore, Lemma 10.3 gives

$$\|T_{K^+_{\varepsilon+i\gamma}} F\|_{T^{r,2}} \lesssim \|F\|_{L^r(\mathbb{R}^n;L^2((0,\infty),\frac{dt}{t}))}.$$

We conclude the proof using [11, Proposition 2.1] to get that $\|F\|_{L^r(\mathbb{R}^n;L^2((0,\infty),\frac{dt}{t}))} \lesssim \|F\|_{T^{r,2}}$. □

Corollary 9.2. *Suppose Π_B is a perturbed Hodge-Dirac operator. Suppose $p \in (\max\{1,(p_H)_*\},2]$ and $M \in \mathbb{N}$ with $M \geq 10n$. Then*

$$\|(t,x) \mapsto (Q_t^B)^M (I - P_t^{\tilde{N}})u(x)\|_{T^{p,2}} \leq C_p \|u\|_p \quad \forall u \in \overline{\mathcal{R}_2(\Gamma)} \cap L^p(\mathbb{R}^n;\mathbb{C}^N).$$

Proof. For $p=2$, Π_B is bisectorial in $L^2(\mathbb{R}^n;\mathbb{C}^N)$ by Theorem 2.21, and $\{(R_t^B)^{\frac{M}{2}-2}\Gamma \ ; \ t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-2}B_1\Gamma^* \ ; \ t \in \mathbb{R}^*\}$ have $\dot{W}^{1,2}$-L^2 off-diagonal bounds of every order on balls, since $\{R_t^B \ ; \ t \in \mathbb{R}^*\}$ has L^2-L^2 off-diagonal bounds of every order. Therefore for all $q \in (2_*,2]$,

$$\|(t,x) \mapsto (Q_t^B)^M (I - P_t^{\tilde{N}})u(x)\|_{T^{q,2}} \leq C_q \|u\|_q \quad \forall u \in \overline{\mathcal{R}_q(\Gamma)},$$

by Theorem 9.1. Combined with Proposition 3.8, this gives

$$\|(t,x) \mapsto (Q_t^B)^M u(x)\|_{T^{q,2}} \leq C_q \|u\|_q \quad \forall u \in \overline{\mathcal{R}_q(\Gamma)}.$$

Since the same holds for $\underline{\Pi_B}$ instead of Π_B, we have, as in the proof of Theorem 3.1, that $H^r_{\Pi_B} = \overline{\mathcal{R}_r(\Pi_B)}$ for all $r \in (\max(p_H,2_*),2]$. In particular, Π_B is bisectorial in L^r. Moreover, applying Proposition 7.2, we have that $\{(R_t^B)^{\frac{M}{2}-2}\Gamma \ ; \ t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-2}B_1\Gamma^* \ ; \ t \in \mathbb{R}^*\}$ have $\dot{W}^{r,2}$-L^2 off-diagonal bounds of every order on balls. The assumptions of Theorem 9.1 and Proposition 7.2 are now satisfied in L^r. Note that $(\Pi_B(r))$ holds as long as $r > p_H$ by Proposition 2.16. We can repeat the argument finitely many times until we reach a value of p such that $p_* < p_H$. □

If we restrict the off-diagonal bound assumptions to certain subspaces, the following restricted version of the theorem remains valid.

Corollary 9.3. *Let $p \in (1,2)$, $M \in \mathbb{N}$ be even and such that $M \geq 10n$, and let Π_B be a perturbed Hodge-Dirac operator such that $(\Pi_B(p))$ holds. Let W be a subspace of $\mathbb{C}^N$ that is stable under $\widehat{\Gamma^*}(\xi)\widehat{\Gamma}(\xi)$ and $\widehat{\Gamma}(\xi)\widehat{\Gamma^*}(\xi)$ for all $\xi \in \mathbb{R}^n$. Assume that $\{t\Gamma P_t^B \mathbb{P}_W \ ; \ t \in \mathbb{R}^*\}$ $\{t\Gamma_B^* P_t^B \mathbb{P}_W \ ; \ t \in \mathbb{R}^*\}$ are uniformly bounded in $\mathcal{L}(L^p)$, and that $\{(R_t^B)^{\frac{M}{2}-2}\Gamma\mathbb{P}_W \ ; \ t \in \mathbb{R}^*\}$ and $\{(R_t^B)^{\frac{M}{2}-2}B_1\Gamma^*\mathbb{P}_W \ ; \ t \in \mathbb{R}^*\}$ have $\dot{W}^{1,p}$-L^2 off-diagonal bounds of every order on balls, where $\mathbb{P}_W$ denotes the projection from $L^p(\mathbb{R}^n;\mathbb{C}^N)$ onto $L^p(\mathbb{R}^n;W)$.*
Then, for all $q \in (\max\{1,p_*\},2]$,

$$\|(t,x) \mapsto (Q_t^B)^M (I - P_t^{\tilde{N}})\Gamma v(x)\|_{T^{q,2}} \leq C_q \|\Gamma v\|_q \quad \forall v \in \mathcal{D}_q(\Gamma) \cap L^q(\mathbb{R}^n;W).$$

Proof. Let $p \in (1,2)$, $v \in \mathcal{S}(\mathbb{R}^n;W)$, and $s > 0$. Notice that, for all $\xi \in \mathbb{R}^n$,

$$s\widehat{\Pi}(\xi)(I + s^2\widehat{\Pi^2}(\xi))^{-1}\widehat{\Gamma}(\xi)\widehat{v}(\xi) = s\widehat{\Gamma^*}(\xi)\widehat{\Gamma}(\xi)(I + s^2\widehat{\Gamma^*}(\xi)\widehat{\Gamma}(\xi))^{-1}\widehat{v}(\xi) \in W,$$

since $\widehat{\Gamma}(\xi)$ is nilpotent, and W is stable under $\widehat{\Gamma^*}(\xi)\widehat{\Gamma}(\xi)$. We thus have that $Q_s\Gamma v$ belongs to $L^p(\mathbb{R}^n;W)$. The same reasoning also gives that $(I - P_t^{\tilde{N}})P_sQ_s^{2\tilde{N}-1}\Gamma v \in L^p(\mathbb{R}^n;W)$ for all $t, s > 0$. Therefore we have that, for all $t > 0$ and $x \in \mathbb{R}^n$,

$$\int_0^t (\frac{s}{t})(Q_t^B)^{M-2}(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1} Q_s^{\tilde{N}} \Gamma v(x) \frac{ds}{s}$$

$$= \int_0^t (\frac{s}{t})(Q_t^B)^{M-2} \mathbb{P}_W (I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1} Q_s^{\tilde{N}} \Gamma v(x) \frac{ds}{s}.$$

This allows us to use the proof of Theorem 9.1, replacing the kernel K by $\widetilde{K}(t,s) = \mathbb{1}_{(0,\infty)}(t-s)(Q_t^B)^{M-2}\mathbb{P}_W(I - P_t^{\tilde{N}})P_s Q_s^{\tilde{N}-1}$, for all $t, s > 0$. □

10. Appendix: Schur estimates in tent spaces

In this section, we establish boundedness results for singular integral operators with operator-valued kernels acting on tent spaces. These are general results of independent interest that do not rely on the specific theory of Dirac operators developed in the rest of the paper.

We consider integral operators of the form

$$T_K F(t,x) = \int_0^\infty K(t,s)F(s,.)(x)\frac{ds}{s} \qquad \forall F \in T^{p,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^N),$$

where $\{K(t,s)\ ;\ t,s > 0\}$ is a uniformly bounded family of bounded linear operators acting on $L^2(\mathbb{R}^n;\mathbb{C}^N)$, and $(t,s) \mapsto K(t,s)$ is strongly measurable. We initially define these operators for $F \in C_c^\infty(\mathbb{R}^{n+1}_+)$, and consider $T_K F(t,.)$ as a Bochner integral in $L^2(\mathbb{R}^n)$ for all $t > 0$. We then write $T_K \in \mathcal{L}(T^{p,2})$ to mean that the operator extends to a bounded operator on $T^{p,2}$.

We are interested here in Schur type estimates, i.e. estimates for integral operators with kernels satisfying size conditions of the form $\|K(t,s)\| \lesssim \min(\frac{t}{s}, \frac{s}{t})^\alpha$ for some $\alpha > 0$. The proofs are similar to those developed in [12] to treat singular integral operators with kernels satisfying size conditions of the form $\|K(t,s)\| \lesssim |t-s|^{-1}$. The appropriate off-diagonal bound assumptions are as follows.

Let $p \in [1,2]$. Let $\{K(t,s),\ s,t > 0\}$ be a uniformly bounded family of bounded linear operators acting on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ that satisfies L^p-L^2 off-diagonal bounds of the following form: there exists $C > 0$, $N' > 0$, such that for all Borel sets $E, F \subseteq \mathbb{R}^n$ and all $s, t > 0$

$$\|\mathbb{1}_E K(t,s)\mathbb{1}_F\|_{L^p \to L^2} \leq C \max\{t,s\}^{-n(\frac{1}{p}-\frac{1}{2})} \left(1 + \frac{\operatorname{dist}(E,F)}{\max\{t,s\}}\right)^{-N'}. \tag{10.1}$$

Given a kernel K, we also consider

$$K_z^+(t,s) = \mathbb{1}_{(0,\infty)}(t-s)(\frac{s}{t})^z K(t,s),$$

$$K_z^-(t,s) = \mathbb{1}_{(0,\infty)}(s-t)(\frac{t}{s})^z K(t,s), \qquad \forall t,s \in (0,\infty), \quad \forall z \in \mathbb{C}.$$

We then obtain the following result on $T^{1,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^N)$, which is a refined version of the arguments in [16, Theorem 4.9].

Proposition 10.1. *Suppose K satisfies (10.1) for some $N' > \frac{n}{2}$ and $p \in [1,2]$. Then the following holds.*
(1) Given $\alpha \in (0,\infty)$, we have $T_{K^-_\alpha} \in \mathcal{L}(T^{1,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^N))$.
(2) Given $\beta \in (\frac{n}{p'},\infty)$ and $\gamma \in \mathbb{R}$, we have $T_{K^+_{\beta+i\gamma}} \in \mathcal{L}(T^{1,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^N))$ with

$$\sup_{\gamma\in\mathbb{R}} \|T_{K^+_{\beta+i\gamma}}\|_{\mathcal{L}(T^{1,2})} < \infty.$$

Corollary 10.2. *Suppose K satisfies (10.1) for some $N' > \frac{n}{2}$ and $p \in [1,2]$. Suppose $q \in [1,p]$, $\alpha \in (0,\infty)$, and $\beta \in (n(\frac{1}{q}-\frac{1}{p}),\infty)$. If $\sup\limits_{\gamma\in\mathbb{R}} \|T_{K^+_{i\gamma}}\|_{\mathcal{L}(T^{p,2})} < \infty$, then $T_{K^-_\alpha + K^+_\beta} \in \mathcal{L}(T^{q,2}(\mathbb{R}^{n+1}_+;\mathbb{C}^N))$.*

Proof of Corollary 10.2. This follows from Proposition 10.1 by applying Stein's interpolation [45, Theorem 1] to the analytic family of operators $\{T_{K^-_{\frac{n}{p'}z}}\ ;\ Re(z) \in [0,1]\}$. We choose the spaces $T^{p,2}$ and $T^{1,2}$ as endpoints, and set $\frac{1}{q} = \frac{1-\theta}{p} + \theta$ for $\theta \in [0,1]$. This gives $\theta = p'(\frac{1}{q}-\frac{1}{p})$ and thus the condition $\beta > \frac{n}{p'}\theta = n(\frac{1}{q}-\frac{1}{p})$. □

We now turn to the proof of Proposition 10.1, which follows the one of [16, Theorem 4.9].

Proof of Proposition 10.1. Let $\alpha > 0$, $\beta > \frac{n}{p'}$. It suffices to show that

$$\|T_{K^-_\alpha + K^+_{\beta+i\gamma}} F\|_{T^{1,2}} \leq C$$

uniformly for all atoms F in $T^{1,2}$ and all $\gamma \in \mathbb{R}$.
Let F be a $T^{1,2}$ atom associated with a ball $B \subseteq \mathbb{R}^n$ of radius $r > 0$. Then

$$\iint_{T(B)} |F(s,x)|^2 \, \frac{dxds}{s} \leq |B|^{-1},$$

where $T(B) = (0,r) \times B$. Set $\tilde{K} = K^-_\alpha + K^+_{\beta+i\gamma}$, $\tilde{F} := T_{\tilde{K}}(F)$, and $\tilde{F}_1 := \tilde{F}\mathbb{1}_{T(4B)}$, $\tilde{F}_k := \tilde{F}\mathbb{1}_{T(2^{k+1}B)\setminus T(2^kB)}$, $k \geq 2$. We show that there exists $\delta > 0$, independent of γ, such that

$$\iint |\tilde{F}_k(t,x)|^2 \, \frac{dxdt}{t} \lesssim 2^{-k\delta} |2^{k+1}B|^{-1}.$$

Let $k=1$. Observe that for every $\varepsilon > 0$, $\int_0^\infty \min\left(\frac{s}{t},\frac{t}{s}\right)^\varepsilon \frac{ds}{s} \leq C$, uniformly in $t > 0$. Using Minkowski's inequality and the assumption on K, we obtain

$$\begin{aligned}
\iint_{T(4B)} |\tilde{F}(t,x)|^2 \, \frac{dxdt}{t} &\leq \int_0^{l(4B)} \int_{4B} \Big(\int_0^\infty \min(\frac{s}{t},\frac{t}{s})^{\min(\alpha,\beta)} |K(t,s)F(s,\,.\,)(x)| \, \frac{ds}{s}\Big)^2 \, \frac{dxdt}{t} \\
&\leq \int_0^{l(4B)} \Big(\int_0^\infty \min(\frac{s}{t},\frac{t}{s})^{\min(\alpha,\beta)} \|K(t,s)F(s,\,.\,)\|_{L^2(4B)} \, \frac{ds}{s}\Big)^2 \, \frac{dt}{t} \\
&\lesssim \int_0^\infty \Big(\int_0^\infty \min(\frac{s}{t},\frac{t}{s})^{\min(\alpha,\beta)} \|F(s,\,.\,)\|_{L^2(B)} \, \frac{ds}{s}\Big)^2 \, \frac{dt}{t} \\
&\lesssim \int_0^\infty \|F(s,\,.\,)\|^2_{L^2(B)} \, \frac{ds}{s} \leq |B|^{-1}.
\end{aligned}$$

For $k \geq 2$, we cover $T(2^{k+1}B) \setminus T(2^k B)$ with the two parts $(0, 2^{k-1}r) \times 2^{k+1}B \setminus 2^{k-1}B$ and $(2^{k-1}r, 2^{k+1}r) \times 2^{k+1}B$. Via Minkowski's inequality, we have

$$
\begin{aligned}
&\Big(\iint_{T(2^{k+1}B)\setminus T(2^k B)} |\tilde{F}_k(t,x)|^2 \,\frac{dxdt}{t}\Big)^{\frac{1}{2}} \\
&\leq \int_0^r \Big(\int_0^{2^{k-1}r} \min\big((\frac{t}{s})^{\alpha}, (\frac{s}{t})^{\beta}\big) \|K(t,s)F(s,\,.\,)\|^2_{L^2(2^{k+1}B\setminus 2^{k-1}B)} \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&\quad + \int_0^r \Big(\int_{2^{k-1}r}^{2^{k+1}r} (\frac{s}{t})^{\beta} \|K(t,s)F(s,\,.\,)\|^2_{L^2(2^{k+1}B)} \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&=: I_1 + I_2.
\end{aligned}
$$

For I_2, the fact that $s < t$ and the assumed L^p-L^2 boundedness of $K(t,s)$ yield

$$
\begin{aligned}
I_2 &\leq \int_0^r \Big(\int_{2^{k-1}r}^{2^{k+1}r} (\frac{s}{t})^{2\beta} \|K(t,s)F(s,\,.\,)\|^2_{L^2(\mathbb{R}^n)} \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&\lesssim \int_0^r \Big(\int_{2^{k-1}r}^{2^{k+1}r} \big((\frac{s}{t})^{\beta} t^{-n(\frac{1}{p}-\frac{1}{2})} \|F(s,\,.\,)\|_{L^p(B)}\big)^2 \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&\lesssim \int_0^r (\frac{s}{2^k r})^{\beta} (2^k r)^{-n(\frac{1}{p}-\frac{1}{2})} r^{n(\frac{1}{p}-\frac{1}{2})} \|F(s,\,.\,)\|_{L^2(B)} \,\frac{ds}{s} \\
&\lesssim 2^{-k(\beta+n(\frac{1}{p}-\frac{1}{2}))} \Big(\int_0^r (\frac{s}{r})^{2\beta} \,\frac{ds}{s}\Big)^{\frac{1}{2}} \Big(\int_0^r \|F(s,\,.\,)\|^2_{L^2(B)} \,\frac{ds}{s}\Big)^{\frac{1}{2}} \\
&\lesssim 2^{-k(\beta+n(\frac{1}{p}-\frac{1}{2})-\frac{n}{2})} |2^k B|^{-\frac{1}{2}}.
\end{aligned}
$$

Since, by assumption, $\beta > n(1-\frac{1}{p})$, this yields the desired estimate for I_2.
We split the term I_1 into the two parts

$$
\begin{aligned}
I_1 &\leq \int_0^r \Big(\int_0^s (\frac{t}{s})^{2\alpha} \|K(t,s)F(s,\,.\,)\|^2_{L^2(2^{k+1}B\setminus 2^{k-1}B)} \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&\quad + \int_0^r \Big(\int_s^{2^{k-1}r} (\frac{s}{t})^{2\beta} \|K(t,s)F(s,\,.\,)\|^2_{L^2(2^{k+1}B\setminus 2^{k-1}B)} \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&=: I_{1,1} + I_{1,2}.
\end{aligned}
$$

For $I_{1,1}$, we have $t < s$. The assumed L^p-L^2 off-diagonal bounds and the fact that $N' > n(\frac{1}{p}-\frac{1}{2})$ yield

$$
\begin{aligned}
I_{1,1} &\lesssim \int_0^r \Big(\int_0^s \big((\frac{t}{s})^{\alpha} s^{-n(\frac{1}{p}-\frac{1}{2})} (1 + \frac{\operatorname{dist}(B, 2^{k-1}B)}{s})^{-N'} \|F(s,\,.\,)\|_{L^p(B)}\big)^2 \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&\lesssim \int_0^r \|F(s,\,.\,)\|_{L^2(B)} (\frac{s}{r})^{-n(\frac{1}{p}-\frac{1}{2})} (\frac{s}{2^k r})^{N'} \Big(\int_0^s (\frac{t}{s})^{2\alpha} \,\frac{dt}{t}\Big)^{\frac{1}{2}} \,\frac{ds}{s} \\
&\lesssim 2^{-kN'} \Big(\int_0^r \|F(s,\,.\,)\|^2_{L^2(B)} \,\frac{ds}{s}\Big)^{\frac{1}{2}} \Big(\int_0^r (\frac{s}{r})^{2N'-2n(\frac{1}{p}-\frac{1}{2})} \,\frac{ds}{s}\Big)^{\frac{1}{2}} \\
&\lesssim 2^{-kN'} |B|^{-\frac{1}{2}} \lesssim 2^{-k(N'-\frac{n}{2})} |2^k B|^{-\frac{1}{2}}.
\end{aligned}
$$

Since $N' > \frac{n}{2}$, this yields the assertion for $I_{1,1}$.
For $I_{1,2}$, we have $s < t$. According to our assumptions, there exist $\beta' \in (0, \min(\beta, N' -$

$n(\frac{1}{p}-\frac{1}{2})))$ and $\tilde{N}>0$ with $\frac{n}{2}<\tilde{N}<\min(N',\beta+n(\frac{1}{p}-\frac{1}{2}))$. Using the L^p-L^2 off-diagonal bounds, we get

$$\begin{aligned}
I_{1,2} &\lesssim \int_0^r \Big(\int_s^{2^{k-1}r} \big((\frac{s}{t})^{\beta} t^{-n(\frac{1}{p}-\frac{1}{2})}(1+\frac{2^k r}{t})^{-N'}\|F(s,\,.\,)\|_{L^p(B)}\big)^2 \frac{dt}{t}\Big)^{\frac12} \frac{ds}{s}\\
&\lesssim 2^{-kN'}\int_0^r \|F(s,\,.\,)\|_{L^2(B)}\Big(\int_s^r (\frac{s}{t})^{2\beta'}(\frac{t}{r})^{2N'-2n(\frac{1}{p}-\frac{1}{2})}\frac{dt}{t}\Big)^{\frac12}\frac{ds}{s}\\
&\quad + 2^{-k\tilde{N}}\int_0^r \|F(s,\,.\,)\|_{L^2(B)}\Big(\int_r^{2^{k-1}r} (\frac{s}{t})^{2\beta}(\frac{t}{r})^{2\tilde{N}-2n(\frac{1}{p}-\frac{1}{2})}\frac{dt}{t}\Big)^{\frac12}\frac{ds}{s}\\
&\lesssim 2^{-k\tilde{N}}\int_0^r (\frac{s}{r})^{\beta'}\|F(s,\,.\,)\|_{L^2(B)}\frac{ds}{s}\\
&\lesssim 2^{-k\tilde{N}}\Big(\int_0^r (\frac{s}{r})^{2\beta'}\frac{ds}{s}\Big)^{\frac12}\Big(\int_0^r \|F(s,\,.\,)\|^2_{L^2(B)}\frac{ds}{s}\Big)^{\frac12} \lesssim 2^{-k(\tilde{N}-\frac{n}{2})}|2^kB|^{-\frac12},
\end{aligned}$$

which completes the proof. □

We conclude this section by pointing out that such estimates are much simpler in the context of vertical, rather than conical, square functions. In particular we have the following lemma (see [44, Section 5] for the relevant information regarding R-boundedness).

Lemma 10.3. *Suppose $p\in(1,\infty)$ and $\varepsilon>0$. If $\{K(t,s)\ ;\ t,s>0\}$ is a R-bounded family of bounded operators on $L^p(\mathbb{R}^n)$, then $T_{K_\varepsilon^+ + K_\varepsilon^-}\in\mathcal{L}(L^p(\mathbb{R}^n;L^2(\mathbb{R}_+;\frac{dt}{t})))$.*

Proof. Let $F\in L^p(\mathbb{R}^n;L^2(\mathbb{R}_+;\frac{dt}{t}))$. By Kalton-Weis' γ-multiplier theorem (see [44, Theorem 5.2]), we have the following:

$$\begin{aligned}
\|T_{K_\varepsilon^+}F\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+;\frac{dt}{t}))} &\le \int_0^1 s^{\varepsilon}\|(x,t)\mapsto K(t,ts)F(x,ts)\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+;\frac{dt}{t}))}\frac{ds}{s}\\
&\lesssim \int_0^1 s^{\varepsilon}\|(x,t)\mapsto F(x,ts)\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+;\frac{dt}{t}))}\frac{ds}{s}\\
&= \int_0^1 s^{\varepsilon}\|(x,t)\mapsto F(x,t)\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+;\frac{dt}{t}))}\frac{ds}{s} \lesssim \|F\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+;\frac{dt}{t}))}.
\end{aligned}$$

The same reasoning applies to $T_{K_\varepsilon^-}$. □

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