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Efficient multigrid based solvers for B-spline MPM

Roel Tielen[1*], Matthias Möller[1] , Kees Vuik[1]
[1]Department of Applied Mathematics, Delft University of Technology, Delft, The Netherlands

[*]E-mail: r.p.w.m.tielen@tudelft.nl

ABSTRACT

The Material Point Method (MPM) has been applied successfully to problems in engineering which involve large deformations and history-dependent material behavior. However, the classical method suffers from some shortcomings which influence the quality of the numerical solution significantly. High-order B-spline basis functions solve the problem of so-called 'grid crossing errors' completely due to their higher continuity at inter-element boundaries. Adopting a consistent mass matrix instead of its lumped counterpart, which is common practice in standard MPM, further improves the convergence properties of the MPM. However, solving a linear system of equations resulting from a B-spline discretization is considered a challenging task. In this paper, we present a solution technique using *p*-multigrid methods to efficiently solve linear systems arising in B-spline MPM.

KEY WORDS: B-spline Material Point Method; Iterative solvers; p-Multigrid;

INTRODUCTION

The Material Point Method (MPM) (Sulsky et al., 1994; Sulsky et al., 1995) has been applied to a wide range of challenging problems in engineering. For example, the modelling of sea ice dynamics (Sulsky et al., 2007) or multiphase flows (Zhang et al., 2008). However, the method suffers from some shortcomings which seriously influence the quality of the numerical solution.

The use of piecewise-linear basis functions often leads to unphysical oscillations triggered by particles crossing the discontinuity of the gradient of these basis functions at element boundaries. These so-called 'grid-crossing errors' significantly deteriorate the quality of the numerical solution and may even lead to a lack of convergence (Steffen et al., 2008). Over the years, different mitigation strategies have been proposed in the literature, for example, the Generalized Interpolation Material Point Method (GIMP) (Bardenhagen et al., 2004), the Dual Domain Material Point Method (DDMPM) (Zhang et al., 2011) and the use of reconstruction techniques (Gong et al., 2015; Wobbes et al., 2018).

As an alternative solution, high-order B-spline basis functions can be adopted in MPM, which overcome the problem of grid-crossing errors completely due to their higher inter-element continuity. Furthermore, they lead to a continuous representation of stress fields and, potentially, higher convergence rates. Recently, the use of high-order B-spline basis functions within MPM has become more popular (Tielen et al., 2017; Gan et al., 2018).
In MPM, typically, the consistent mass matrix is replaced by its lumped counterpart when solving the equation of motion. Although the use of the lumped mass matrix decreases the computational costs of solving the resulting linear system of equations, it also limits the spatial accuracy to $O(h^2)$ (Steffen et al., 2010) and causes the loss of conservation of energy and linear momentum (Love et al., 2006).

Combining high-order B-spline basis functions with a consistent mass matrix gives rise to a new challenge, the efficient solution of the equation of motion at every time step. Since for B-splines the condition number of the mass matrix scales exponentially with the order of the basis functions p, the use of (standard) iterative methods becomes computationally expensive for higher values of p. Recently, different solvers have been developed for a discretization resulting from high-order B-spline basis functions (Sangalli et al., 2016).

An alternative approach lies in the use of multigrid methods in which a hierarchy of different levels is constructed. The general idea of multigrid methods is to obtain a computational less expensive solution at the coarsest level combined with the application of a basic iterative method at all other levels.

In *h*-multigrid methods, a hierarchy is constructed based on discretizations with coarser and finer meshes. *h*-Multigrid methods have been applied successfully for B-spline discretizations of elliptic partial differential equations (Gahalaut et al., 2013; Hofreither et al., 2017). In this paper, we propose the use of *p*-multigrid methods to solve these linear systems. In *p*-multigrid methods, a hierarchy is constructed based on discretizations resulting from different approximation orders *p* of the basis functions. In contrast to *h*-multigrid methods, the coarse grid correction is obtained at level *p=1,* where B-spline basis functions coincide with piecewise-linear Lagrange basis functions. This enables us to adopt solution techniques known for standard Lagrange finite elements.

The structure of this papers is as follows. In Section 2, the *p*-multigrid method is described in detail. Numerical results obtained with *p*-multigrid as a solver are presented and compared with other solvers in Section 3. Finally, conclusions are drawn in Section 4.

P-MULTIGRID METHOD

This paper focuses on solving the equation of motion, resulting from a discretization with high-order B-spline basis functions, needed at every time step of the MPM. One-dimensional B-spline basis functions of order *p* are defined by a knot vector $\Xi = \{ \xi_1 , \xi_2 , \dots , \xi_{n+p+1} \}$, which consists of a sequence of real and non-decreasing numbers ξ_i called knots. A knot vector of length $n+p+1$ defines *n* basis functions of order *p* by starting from the constant B-spline basis functions:

$$\phi_{i,0} = \begin{cases} 1, if \xi_i \leqslant \xi < \xi_{i+1}, \\ 0, else. \end{cases} \tag{1}$$

The Cox-de Boor recursion formula (De Boor C., 2001) is then used to define higher-order B-spline basis functions:

$$\phi_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} \phi_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} \phi_{i+1,p-1}(\xi), \tag{2}$$

where ξ in $[\xi_1, \xi_{n+p+1}]$. In this paper, *open uniform* knot vectors are considered, implying that the knots are equally distributed and the first and last knot are repeated *p* times. As a consequence, basis functions of order *p* are C^{p-1} continuous, which ensures a continuous gradient (or better) for $p \geq 2$. The one-dimensional B-spline basis functions can be extended to multiple dimensions by taking the tensor product.
By considering the weak formulation of the conservation of linear momentum equation and approximating the solution by a linear combination of the basis functions $\phi_{i,p}$, the following approximation of the equation of motion is obtained:

$$M_{\mathrm{h,p}} a_{\mathrm{h,p}} = \mathrm{F}_{\mathrm{h,p}} \tag{3}$$

Here, $M_{h,p}$ denotes the mass matrix, $a_{h,p}$ the acceleration vector and $F_{h,p}$ the force vector. A single entry of the mass matrix is defined as follows:

$$\left(M_{h,p}\right)_{i,j} = \int_{\Omega} \phi_{i,p} \phi_{j,p} \rho d\Omega \tag{4}$$

where ρ denotes the density field and Ω the considered domain. The subscripts *h* and *p* denote the mesh width and approximation order of the basis functions, respectively. Equation (3) can then be solved by a *p*-multigrid method. Starting from an initial guess $a_{h,p}^{(0)}$, the steps of a two-grid correction scheme are as follows:

(1) Apply ν_1 presmoothing steps on the high-order problem:

$$a_{h,p}^{(0,m+1)} = a_{h,p}^{(0,m)} + S\left(F_{h,p} - M_{h,p}a_{h,p}^{(0,m)}\right), m = 1, \dots, \nu_{1,} \tag{5}$$

where *S* is a smoother, a basic iterative method like Jacobi or Gauss-Seidel.

(2) Transfer the residual to level *p-1* by applying the restriction operator I_p^{p-1}:

$$r_{h,p-1} = I_p^{p-1}\left(F_{h,p} - M_{h,p}a_{h,p}^{(\nu_1)}\right). \tag{6}$$

(3) Obtain the coarse grid correction $e_{h,p-1}$ at level *p-1* by solving the residual equation:

$$M_{h,p-1}e_{h,p-1} = r_{h,p-1}. \tag{7}$$

(4) Transfer the coarse grid correction to level *p* by applying the prolongation operator I_{p-1}^{p} and update the solution $a_{h,p}^{(\nu_1)}$ by adding the transferred coarse grid correction:

$$a_{h,p}^{(0,\nu_1)} := a_{h,p}^{(0,\nu_1)} + I_{p-1}^{p}\left(e_{h,p-1}\right). \tag{8}$$

(5) Apply ν_2 post smoothing steps on the high-order problem:

$$a_{h,p}^{(0,\nu_1+m+1)} = a_{h,p}^{(0,\nu_1+m)} + S\left(F_{h,p} - M_{h,p}a_{h,p}^{(0,\nu_1+m)}\right), m = 1, \dots, \nu_{2,} \tag{9}$$

The two-grid correction scheme described above is applied recursively until level *p=1* is reached, resulting in a *V*-cycle. However, different schemes can be applied, for example a *W*-cycle as shown in Figure 1. At level *p=1* the residual equation is solved by adopting a Conjugate Gradient (CG) solver.

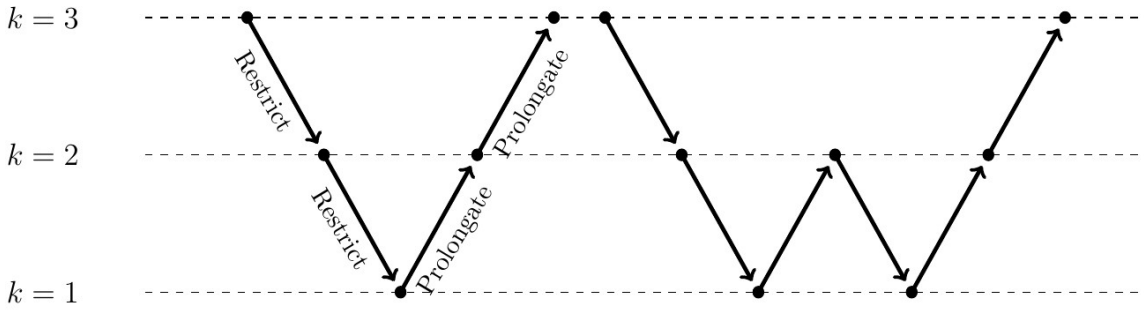

Figure 1 Description of a *V*-cycle and *W*-cycle

To restrict the residual and prolongate the coarse grid correction, transfer operators have to be defined. The prolongation operator I_{k-1}^{k} that transfers the residual from level *k-1* to level *k* is defined as follows:

$$I_{k-1}^{k} := \left(P_k^k\right)^{-1} P_{k-1}^{k}, \tag{10}$$

where P_k^{k-1} is given by

$$P_k^{k-1} = \int_\Omega \Phi_k^T \Phi_{k-1} d\Omega \tag{11}$$

Here, $\Phi_k^T = [\phi_1^k, \dots, \phi_n^k]$ denotes the vector consisting of basis functions of approximation order *k.* The restriction operator to transfer the coarse grid correction from level *k* to level *k-1* is defined in a similar way:

$$I_k^{k-1} := \left(P_{k-1}^{k-1}\right)^{-1} P_{k-1}^{k} \tag{12}$$

To circumvent the solution of a linear system, both P_k^k and P_{k-1}^{k-1} are lumped in the prolongation and restriction operator, respectively. A mathematical derivation of the transfer operations can be found in (Sampath et al., 2010).

At each level $p \geq 1$, a fixed number of smoothing steps is applied. As a smoother, an incomplete LU factorization with dual treshold strategy (Saad., 1994) is adopted. The operators $M_{h,p}$, needed at every level, are obtained by means of rediscretization.

NUMERICAL RESULTS

To illustrate the potential of *p*-multigrid solvers, we consider Equation (3) obtained from a discretization with B-spline basis functions of order p and mesh width h. The domain Ω is chosen to be the unit square. Here, we assume a constant density equal to *1* [kg/m^3]. The force vector is chosen to be constant and equal to *-9.81 [m/s^2]*, simulating a gravitational force. For different values of p and h the resulting linear system is solved with the *p*-multigrid method as described in the previous section. For all numerical experiments, the number of pre- and postsmoothing steps is chosen to be constant ($\nu_1 = \nu_2 = 1$). As a stopping criterion, a reduction of the initial residual is chosen:

$$\frac{r_{h,p}^{(i)}}{r_{h,p}^{(0)}} < \epsilon = 10^{-8} \tag{13}$$

Here, $r_{h,p}^{(i)}$ denotes the residual after iteration *i.* As an initial guess, the zero vector is adopted for all numerical experiments. At level *p=1,* the residual equation is solved by means of a CG solver, using the same stopping criterion with $\epsilon = 10^{-4}$. The number of iterations required by the *p*-multigrid to converge are presented in Table 1. Note that for higher values of *p,* the number of iterations remains constant. *p*-Multigrid exhibits furthermore, as *h*-multigrid methods, the *h*-independence property, which implies that the number of iterations necessary is independent of the mesh width *h*.

Table 1 Number of iterations needed with *p*-multigrid for different values of *h* and *p*.

	p=1	*p=2*	*p=3*	*p=4*	*p=5*
$h=2^{-4}$	1	3	3	2	2
$h=2^{-5}$	1	3	3	3	3
$h=2^{-6}$	1	3	3	3	3
$h=2^{-7}$	1	3	3	3	2

For comparison, the number of iterations needed with a CG method, adopting the same tolerance, are presented in Table 2. The number of iterations are significantly higher compared to the *p*-multigrid method, especially for higher values of *p*.

Table 2 Number of iterations needed with the Conjugate Gradient method for different values of *h* and *p*.

	p=1	*p=2*	*p=3*	*p=4*	*p=5*
$h=2^{-4}$	18	34	66	92	156
$h=2^{-5}$	18	46	93	178	242
$h=2^{-6}$	18	46	95	196	252
$h=2^{-7}$	17	44	91	187	227

CONCLUSIONS

In this paper, a *p*-multigrid method has been presented to solve the equation of motion in MPM resulting from a discretization with high-order B-spline basis functions and the use of a consistent mass matrix. Numerical results show that the use of *p*-multigrid methods leads to *h*-independence. Furthermore, the number of iterations does not depend on the approximation order p of the B-spline basis functions. *p*-Multigrid methods have the potential to solve the equation of motion efficiently. Further research remains to be done to apply *p*-multigrid methods on more advanced geotechnical problems.

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