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THE ℓ^s -BOUNDEDNESS OF A FAMILY OF INTEGRAL OPERATORS ON UMD BANACH FUNCTION SPACES

EMIEL LORIST

Dedicated to Ben de Pagter on the occasion of his 65th birthday.

ABSTRACT. We prove the ℓ^s -boundedness of a family of integral operators with an operator-valued kernel on UMD Banach function spaces. This generalizes and simplifies the earlier work by Gallarati, Veraar and the author [12], where the ℓ^s -boundedness of this family of integral operators was shown on Lebesgue spaces. The proof is based on a characterization of ℓ^s -boundedness as weighted boundedness by Rubio de Francia.

1. INTRODUCTION

Over the past decades there has been a lot of interest in the L^p -maximal regularity of PDEs. Maximal L^p -regularity of the abstract Cauchy problem

$$(1.1) \quad \begin{cases} u'(t) + Au(t) = f(t), & t \in (0, T] \\ u(0) = x, \end{cases}$$

where A is a closed operator on a Banach space X , means that for all $f \in L^p((0, T]; X)$ the solution u has “maximal regularity”, i.e. both u' and Au are in $L^p((0, T]; X)$. Maximal L^p -regularity can for example be used to solve quasi-linear and fully nonlinear PDEs by linearization techniques combined with the contraction mapping principle, see e.g. [1, 8, 30, 36].

In the breakthrough work of Weis [40, 41], an operator theoretic characterization of maximal L^p -regularity on UMD Banach spaces was found in terms of the \mathcal{R} -boundedness of the resolvents of A on a sector. \mathcal{R} -boundedness is a random boundedness condition on a family of operators which is a strengthening of uniform boundedness. We refer to [7, 21] for more information on \mathcal{R} -boundedness.

In [13, 14] Gallarati and Veraar developed a new approach to maximal L^p -regularity for the case where the operator A in (1.1) is time-dependent and $t \mapsto A(t)$ is merely assumed to be measurable. In this new approach \mathcal{R} -boundedness is once again one of the main tools. For their approach the \mathcal{R} -boundedness of the family of integral operators $\{I_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R}; X)$

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is required. Here I_k is defined for $f \in L^p(\mathbb{R}; X)$ as

$$I_k f(t) := \int_{-\infty}^t k(t-s)T(t,r)f(r) dr, \quad t \in \mathbb{R},$$

where $T(t,s)$ is the two-parameter evolution family associated to $A(t)$ and \mathcal{K} contains all kernels $k \in L^1(\mathbb{R})$ such that $|k| * |g| \leq Mg$ for all simple $g : \mathbb{R} \rightarrow \mathbb{C}$.

In the literature there are many \mathcal{R} -boundedness results for integral operators, see [21, Chapter 8] for an overview. However none of these are applicable to the operator family of $\{I_k : k \in \mathcal{K}\}$. Therefore in [12] Gallarati, Veraar and the author show a sufficient condition for the \mathcal{R} -boundedness of $\{I_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R}; X)$ in the special case where $X = L^q$. This is done through the notion of ℓ^s -boundedness, which states that for all finite sequences $(I_{k_j})_{j=1}^n$ in $\{I_k : k \in \mathcal{K}\}$ and $(x_j)_{j=1}^n$ in X we have

$$\left\| \left(\sum_{j=1}^n |I_{k_j} x_j|^s \right)^{1/s} \right\|_X \lesssim \left\| \left(\sum_{j=1}^n |x_j|^s \right)^{1/s} \right\|_X.$$

For $s = 2$ this notion coincides with \mathcal{R} -boundedness as a consequence of the Kahane-Khintchine inequalities.

Our main contribution is the generalization of the main result in [12] to the setting of UMD Banach function spaces X . For the proof we will follow the general scheme of [12] with some simplifications. As in case $X = L^q$, for any UMD Banach function space the notions of ℓ^2 -boundedness and \mathcal{R} -boundedness coincide, so the following theorem in particular implies the \mathcal{R} -boundedness of $\{I_k : k \in \mathcal{K}\}$.

Theorem 1.1. *Let X be a UMD Banach function space and $p \in (1, \infty)$. Let $T : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}(X)$ be such that the family of operators*

$$\{T(t,r) : t, r \in \mathbb{R}\}$$

is ℓ^s -bounded for all $s \in (1, \infty)$. Then $\{I_k : k \in \mathcal{K}\}$ is ℓ^s -bounded on $L^p(\mathbb{R}; X)$ for all $s \in (1, \infty)$.

We will prove Theorem 1.1 in a more general setting in Section 3. In particular we allow weights in time, which in applications for example allow rather rough initial values (see e.g. [23, 26, 31, 37]).

For certain UMD Banach function spaces the ℓ^s -boundedness assumption in Theorem 1.1 can be checked by weighted extrapolation techniques, see Corollary 3.5 and Remark 3.6.

Notation. For a measure space (S, μ) we denote the space of all measurable functions by $L^0(S)$. We denote the Lebesgue measure of a Borel set $E \in \mathcal{B}(\mathbb{R}^d)$ by $|E|$. For Banach spaces X and Y we denote the vector space of bounded linear operators from X to Y by $\mathcal{L}(X, Y)$ and we set $\mathcal{L}(X) := \mathcal{L}(X, X)$. For an operator family $\Gamma \subset \mathcal{L}(X, Y)$ we set $\Gamma^* := \{T^* : T \in \Gamma\}$. For $p \in [1, \infty]$ we let $p' \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Throughout the paper we write $C_{a,b,\dots}$ and $\phi_{a,b,\dots}$ to denote a constant and a nondecreasing function on $[1, \infty)$ respectively, which only depend on the parameters a, b, \dots and the dimension d and which may change from line to line.

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2. PRELIMINARIES

2.1. Banach function spaces. Let (S, μ) be a σ -finite measure space. An order ideal X of $L^0(S)$ equipped with a norm $\|\cdot\|_X$ is called a *Banach function space* if it has the following properties:

- (i) *Compatibility:* If $\xi, \eta \in L^0(S)$ with $|\xi| \leq |\eta|$, then $\|\xi\|_X \leq \|\eta\|_X$
- (ii) *Weak order unit:* There is a $\xi \in X$ with $\xi > 0$.
- (iii) *Fatou property:* If $0 \leq \xi_n \uparrow \xi$ for $(\xi_n)_{n=1}^\infty$ in X , $\xi \in L^0(S)$ and $\sup_{n \in \mathbb{N}} \|\xi_n\|_X < \infty$, then $\xi \in X$ and $\|\xi\|_X = \sup_{n \in \mathbb{N}} \|\xi_n\|_X$.

A Banach function space is called *order continuous* if for any sequence $0 \leq \xi_n \uparrow \xi \in X$ we have $\|\xi_n - \xi\|_X \rightarrow 0$. Every reflexive Banach function space is order continuous. Order continuity ensures that the dual of X is also a Banach function space. For a thorough introduction to Banach function spaces we refer to [28, section 1.b] or [3, Chapter 1].

A Banach function space X is said to be *p -convex* for $p \in [1, \infty]$ if

$$\left\| \left(\sum_{j=1}^n |\xi_j|^p \right)^{1/p} \right\|_X \leq \left(\sum_{j=1}^n \|\xi_j\|_X^p \right)^{1/p}$$

for all $\xi_1, \dots, \xi_n \in X$ with the sums replaced by suprema if $p = \infty$. The defining inequality for p -convexity often includes a constant, but X can always be renormed such that this constant equals 1. If a Banach function space is p -convex for some $p \in [1, \infty]$, then X is also q -convex for all $q \in [1, p]$.

For a p -convex Banach function space X we can define another Banach function space by

$$X^p := \{ |\xi|^p \operatorname{sgn} \xi : \xi \in X \} = \{ \xi \in L^0(S) : |\xi|^{1/p} \in X \}$$

equipped with the norm $\|\xi\|_{X^p} := \left\| |\xi|^{1/p} \right\|_X^p$. We refer the interested reader to [28, section 1.d] for an introduction to p -convexity.

2.2. ℓ^s -boundedness. Let X and Y be Banach function spaces and let $\Gamma \subseteq \mathcal{L}(X, Y)$ be a family of operators. We say that Γ is *ℓ^s -bounded* if for all finite sequences $(T_j)_{j=1}^n$ in Γ and $(x_j)_{j=1}^n$ in X we have

$$\left\| \left(\sum_{j=1}^n |T_j x_j|^s \right)^{1/s} \right\|_Y \leq C \left\| \left(\sum_{j=1}^n |x_j|^s \right)^{1/s} \right\|_X.$$

with the sums replaced by suprema if $s = \infty$. The least admissible constant C will be denoted by $[\Gamma]_{\ell^s}$.

Implicitly ℓ^s -boundedness is a classical tool in harmonic analysis for operators on L^p -spaces (see e.g. [16, Chapter V] and [17, 18]). For Banach function spaces the notion was introduced in [40] under the name \mathcal{R}_s -boundedness, underlining its connection to the more well-known notion of \mathcal{R} -boundedness. An extensive study of ℓ^s -boundedness can be found in [24] and for a comparison between ℓ^2 -boundedness and \mathcal{R} -boundedness we refer to [25].

Lemma 2.1. *Let X and Y be Banach function spaces and let $\Gamma \subseteq \mathcal{L}(X, Y)$.*

- (i) Let $1 \leq s_0 < s_1 \leq \infty$ and assume that X and Y are order continuous. If Γ is ℓ^{s_0} - and ℓ^{s_1} -bounded, then Γ is ℓ^s -bounded for all $s \in [s_0, s_1]$ with $[\Gamma]_{\ell^s} \leq \max\{[\Gamma]_{\ell^{s_0}}, [\Gamma]_{\ell^{s_1}}\}$
- (ii) Let $s \in [1, \infty]$ and assume that Γ is ℓ^s -bounded. Then the adjoint family Γ^* is $\ell^{s'}$ -bounded with $[\Gamma^*]_{\ell^{s'}} = [\Gamma]_{\ell^s}$

Proof. Lemma 2.1(i) follows from Calderón's theory of complex interpolation of vector-valued function spaces, see [6] or [24, Proposition 2.14]. Lemma 2.1(ii) is direct from the identification $X(\ell_n^s)^* = X^*(\ell_n^{s'})$, see [28, Section 1.d] or [24, Proposition 2.17] \square

The following characterization of ℓ^s -boundedness for $s \in [1, \infty)$ will be one of the key ingredients of our main result. This characterization relating ℓ^s -boundedness to a certain weighted boundedness comes from the work of Rubio de Francia [16, 38, 39].

Proposition 2.2. *Let $s \in [1, \infty)$ and let X and Y be s -convex order continuous Banach function spaces over (S_X, μ_X) and (S_Y, μ_Y) respectively. Let $\Gamma \subseteq \mathcal{L}(X)$ and take $C > 0$. Then the following are equivalent:*

- (i) Γ is ℓ^s -bounded with $[\Gamma]_{\ell^s} \leq C$.
- (ii) For all nonnegative $u \in (Y^s)^*$, there exists a nonnegative $v \in (X^s)^*$ with $\|v\|_{(Y^s)^*} \leq \|u\|_{(X^s)^*}$ and

$$\left(\int_{S_Y} |T(\xi)|^s u \, d\mu_Y \right)^{1/s} \leq C \left(\int_{S_X} |\xi|^s v \, d\mu_X \right)^{1/s}$$

for all $\xi \in X$ and $T \in \Gamma$.

Proof. The statement is a combination of [39, Lemma 1, p. 217] and [16, Theorem VI.5.3], which for $X = Y$ is proven [2, Lemma 3.4]. The statement for $X \neq Y$ can be extracted from the proof of [2, Lemma 3.4] and can in full detail be found in [29, Proposition 6.1.3] \square

2.3. Muckenhoupt weights. A locally integrable function $w : \mathbb{R}^d \rightarrow (0, \infty)$ is called a *weight*. For $p \in (1, \infty)$ and a weight w we let $L^p(w)$ be the space of all $f \in L^0(\mathbb{R}^d)$ such that

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^d} |f|^p w \right)^{1/p} < \infty.$$

We will say that a weight w lies in the *Muckenhoupt class* A_p and write $w \in A_p$ if it satisfies

$$[w]_{A_p} := \sup_Q \frac{1}{|Q|} \int_Q w \cdot \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes.

Lemma 2.3. *Let $p \in (1, \infty)$ and $w \in A_p$.*

- (i) $w \in A_q$ for all $q \in (p, \infty)$ with $[w]_{A_q} \leq [w]_{A_p}$.
- (ii) $w^{1-p'} \in A_{p'}$ with $[w]_{A_p}^{1/p} = [w^{1-p'}]_{A_{p'}}^{1/p'}$.
- (iii) $w \in A_{p-\varepsilon}$ for $\varepsilon = \frac{1}{\phi_p([w]_{A_p})}$ with $[w]_{A_{p-\varepsilon}} \leq \phi_p([w]_{A_p})$.

The first two properties of Lemma 2.3 follow directly from the definition. The third is for example proven in [18, Exercise 9.2.4]. For a more thorough introduction to Muckenhoupt weights we refer to [18, Chapter 9].

2.4. The UMD property. A Banach space X is said to have the UMD property if the martingale difference sequence of any finite martingale in $L^p(\Omega; X)$ is unconditional for some (equivalently all) $p \in (1, \infty)$. We will work with UMD Banach function spaces, of which standard examples include reflexive Lebesgue, Lorentz and Orlicz spaces. In this Festschrift it is shown that reflexive Musielak-Orlicz spaces, so in particular reflexive variable Lebesgue spaces, have the UMD property, see [27]. The UMD property implies reflexivity, so in particular L^1 and L^∞ do not have the UMD property. For a thorough introduction to the theory of UMD Banach spaces we refer to [5, 20].

For an order continuous Banach function space X over (S, μ) there is also a characterization of the UMD property in terms of the *lattice Hardy–Littlewood maximal operator*, which for simple functions $f: \mathbb{R}^d \rightarrow X$ is given by

$$\widetilde{M}f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^d$$

where the supremum is taken pointwise in S and over all cubes $Q \subseteq \mathbb{R}^d$ with sides parallel to the coordinate axes (see [15] or [19, Lemma 5.1] for a detailed definition of \widetilde{M}). It is a deep result by Bourgain [4] and Rubio de Francia [39] that X has the UMD property if and only if \widetilde{M} is bounded on $L^p(\mathbb{R}^d; X)$ and $L^p(\mathbb{R}^d; X^*)$ for some (equivalently all) $p \in (1, \infty)$. For weighted L^p -spaces we have the following proposition, which was proven in [15]. The increasing dependence on $[w]_{A_p}$ is shown in [19, Corollary 5.3].

Proposition 2.4. *Let X be a UMD Banach function space, $p \in (1, \infty)$ and $w \in A_p$. Then for all $f \in L^p(w; X)$ we have*

$$\|\widetilde{M}f\|_{L^p(w; X)} \leq \phi_{X,p}([w]_{A_p}) \|f\|_{L^p(w; X)}.$$

The UMD property of a Banach function space X also implies that X^q has the UMD property for a $q > 1$, which is a deep result by Rubio de Francia [39, Theorem 4].

Proposition 2.5. *Let X be a UMD Banach function space. Then there is a $p > 1$ such that X is p -convex and X^q is a UMD Banach function space for all $q \in [1, p]$.*

3. INTEGRAL OPERATORS WITH AN OPERATOR-VALUED KERNEL

Before turning to our main result on the ℓ^s -boundedness of a family of integral operators on $L^p(w; X)$ with operator-valued kernels, we will first study the ℓ^s -boundedness of a family of convolution operators on $L^p(w; X)$ with scalar-valued kernels. For this define

$$\mathcal{K} := \{k \in L^1(\mathbb{R}^d) : |k| * |f| \leq Mf \text{ a.e. for all simple } f: \mathbb{R}^d \rightarrow \mathbb{C}\}.$$

As an example any radially decreasing $k \in L^1(\mathbb{R}^d)$ with $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$ is an element of \mathcal{K} . For more examples see [17, Chapter 2] and [34, Proposition 4.6].

Let X be a Banach function space. For a kernel $k \in \mathcal{K}$ and a simple function $f: \mathbb{R}^d \rightarrow X$ we define

$$T_k f := k * f = \int_{\mathbb{R}^d} k(x-y)f(y) dy.$$

As

$$\|T_k f\|_X \leq |k| * \|f\|_X \leq M(\|f\|_X),$$

and since the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$ for all $p \in (1, \infty)$ and $w \in A_p$, T_k extends to a bounded linear operator on $L^p(w; X)$ by density. This argument also shows that the family of convolution operators given by $\Gamma := \{T_k : k \in \mathcal{K}\}$ is uniformly bounded on $L^p(w; X)$.

If X is a UMD Banach function space we can say more. The following lemma was first developed by van Neerven, Veraar and Weis in [33, 34] in connection to stochastic maximal regularity. As in [33, 34], the endpoint case $s = 1$ will play a major role in the proof of our main theorem in the next section.

Proposition 3.1. *Let X be a UMD Banach function space, $s \in [1, \infty]$, $p \in (1, \infty)$ and $w \in A_p$. Then $\Gamma = \{T_k : k \in \mathcal{K}\}$ is ℓ^s -bounded on $L^p(w; X)$ with*

$$[\Gamma]_{\ell^s} \leq \phi_{X,p}([w]_{A_p}).$$

The proof is a weighted variant of [34, Theorem 4.7], which for the special case where X is an iterated Lebesgue space is presented in [12, Proposition 3.6]. For convenience of the reader we sketch the proof in the general case.

Proof. As X is reflexive and therefore order-continuous, \widetilde{M} is well-defined on $L^p(w; X)$ and we have $T_k f \leq \widetilde{M}f$ pointwise a.e. for all simple $f: \mathbb{R}^d \rightarrow X$.

If $s = \infty$ take simple functions $f_1, \dots, f_n \in L^p(w; X)$ and $k_1, \dots, k_n \in \mathcal{K}$. Using Proposition 2.4 we have

$$\begin{aligned} \left\| \sup_{1 \leq j \leq n} |T_{k_j} f_j| \right\|_{L^p(w; X)} &\leq \left\| \sup_{1 \leq j \leq n} \widetilde{M} f_j(x) \right\|_{L^p(w; X)} \\ &\leq \left\| \widetilde{M} \left(\sup_{1 \leq j \leq n} |f_j| \right)(x) \right\|_{L^p(w; X)} \\ &\leq \phi_{X,p}([w]_{A_p}) \left\| \sup_{1 \leq j \leq n} |f_j| \right\|_{L^p(w; X)}. \end{aligned}$$

The result now follows by the density of simple functions in $L^p(w; X)$.

If $s = 1$ we use duality. Note that since X is reflexive we have $L^p(w; X)^* = L^{p'}(w'; X^*)^*$ with $w' = w^{1-p'}$ under the duality pairing

$$(3.1) \quad \langle f, g \rangle_{L^p(w; X), L^{p'}(w'; X^*)} = \int_{\mathbb{R}^d} \langle f(x), g(x) \rangle_{X, X^*} dx$$

by Lemma 2.3(ii) and [20, Corollary 1.3.22]. One can routinely check that $T_k^* = T_{\tilde{k}}$ with $\tilde{k}(x) = k(-x)$ and that $k \in \mathcal{K}$ if and only if $\tilde{k} \in \mathcal{K}$. Since X^* is also a UMD Banach function space (see [20, Proposition 4.2.17]) we know from the case $s = \infty$ that the adjoint family Γ^* is ℓ^∞ -bounded on $L^{p'}(\mathbb{R}^d, w'; X^*)$, so the result follows by Lemma 2.1(ii). Finally if $s \in (1, \infty)$ the result follows by Lemma 2.1(i). \square

With these preparations done we can now introduce the family of integral operators with operator-valued kernel that we will consider. Let X and Y be a Banach function space and let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X, Y)$ such that $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$. The integral operators that we will consider are for simple $f : \mathbb{R}^d \rightarrow X$ given by

$$I_{k,T}f(x) = \int_{\mathbb{R}^d} k(x-y)T(x,y)f(y) dy$$

with $k \in \mathcal{K}$ and $T \in \mathcal{T}$. If $\|T(x,y)\|_{\mathcal{L}(X,Y)} \leq C$ for all $T \in \mathcal{T}$ and $x, y \in \mathbb{R}^d$, we have

$$\|I_{k,T}f\|_X \leq C |k| * \|f\|_X \leq C M(\|f\|_X).$$

So as before $I_{k,T}$ extends to a bounded linear operator from $L^p(w; X)$ to $L^p(w; Y)$ for all $p \in (1, \infty)$ and $w \in A_p$, and

$$\mathcal{I}_{\mathcal{T}} := \{I_{k,T} : k \in \mathcal{K}, T \in \mathcal{T}\}$$

is uniformly bounded. For the details see [12, Lemma 3.9].

If X and Y are Hilbert spaces, this implies that $\mathcal{I}_{\mathcal{T}}$ is also ℓ^2 -bounded from $L^2(\mathbb{R}^d; X)$ to $L^2(\mathbb{R}^d; Y)$, as these notions coincide on Hilbert spaces. However if X and Y are not Hilbert spaces, but a UMD Banach function space or if we move to weighted L^p -spaces, the ℓ^2 -boundedness of $\mathcal{I}_{\mathcal{T}}$ is a lot more delicate.

Our main theorem is a quantitative and more general version of Theorem 1.1 in the introduction:

Theorem 3.2. *Let X and Y be a UMD Banach function spaces and let $p, s \in (1, \infty)$. Let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X, Y)$ such that*

- (i) $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.
- (ii) The family of operators $\tilde{\mathcal{T}} := \{T(x, y) : T \in \mathcal{T}, x, y \in \mathbb{R}^d\}$ is ℓ^σ -bounded for all $\sigma \in (1, \infty)$.

Then $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for all $w \in A_p$ with

$$\begin{aligned} [\mathcal{I}_{\mathcal{T}}]_{\ell^s} &\leq \phi_{X,Y,p}([w]_{A_p}) \max\{[\tilde{\mathcal{T}}]_{\ell^\sigma}, [\tilde{\mathcal{T}}]_{\ell^{\sigma'}}\}, & \sigma &= 1 + \frac{1}{\phi_{p,s}[w]_{A_p}} \\ &\leq \phi_{X,Y,\mathcal{T},p,s}([w]_{A_p}). \end{aligned}$$

We will first prove a result assuming the ℓ^s -boundedness of $\tilde{\mathcal{T}}$ for a fixed $s \in [1, \infty)$.

Proposition 3.3. *Fix $1 \leq s \leq r < p < \infty$ and let X and Y be s -convex Banach function spaces such that X^s has the UMD property. Let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X, Y)$ such that*

- (i) $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.
- (ii) The family of operators $\tilde{\mathcal{T}} := \{T(x, y) : T \in \mathcal{T}, x, y \in \mathbb{R}^d\}$ is ℓ^s -bounded.

Then $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for all $w \in A_{p/s}$ with

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,p,r}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s}.$$

Proof. Let (S_X, μ_X) and (S_Y, μ_Y) be the measure spaces associated to X and Y respectively. For $j = 1, \dots, n$ take $I_j \in \mathcal{I}_{\mathcal{T}}$ and let $k_j \in \mathcal{K}$ and $T_j \in \mathcal{T}$ be such that $I_j = I_{k_j, T_j}$. Fix simple functions $f_1, \dots, f_n \in L^p(w; X)$ and note that

$$(3.2) \quad \left\| \left(\sum_{j=1}^n |I_j f_j|^s \right)^{1/s} \right\|_{L^p(w; Y)} = \left\| \sum_{j=1}^n |I_j f_j|^s \right\|_{L^{p/s}(w; Y^s)}^{1/s}.$$

Fix $x \in \mathbb{R}^d$, then by Hahn-Banach we can find a nonnegative $u_x \in (Y^s)^*$ with $\|u_x\|_{(X^s)^*} = 1$ such that

$$(3.3) \quad \left\| \sum_{j=1}^n |I_j f_j(x)|^s \right\|_{Y^s} = \sum_{j=1}^n \int_{S_Y} |I_j f_j(x)|^s u_x \, d\mu_Y.$$

With Proposition 2.2 we can then find a nonnegative $v_x \in (X^s)^*$ with $\|v_x\|_{(X^s)^*} \leq 1$ such that

$$(3.4) \quad \int_{S_Y} |T_j(x, y) \xi|^s v_x \, d\mu_Y \leq [\tilde{\mathcal{T}}]_{\ell^s} \int_{S_X} |\xi|^s v_x \, d\mu_X$$

for $j = 1, \dots, n$, $y \in \mathbb{R}^d$ and $\xi \in X$. Since $\|k_j\|_{L^1(\mathbb{R}^d)} \leq 1$ by [34, Lemma 4.3], Holder's inequality yields

$$(3.5) \quad |I_j f_j(x)|^s \leq \int_{\mathbb{R}^d} |k_j(x - y)| |T_j(x, y) f_j(y)|^s \, dy.$$

Applying (3.5) and (3.4) successively we get

$$\begin{aligned} \sum_{j=1}^n \int_{S_Y} |I_j f_j(x)|^s u_x \, d\mu_Y &\leq \sum_{j=1}^n \int_{S_Y} \int_{\mathbb{R}^d} |k_j(x - y)| |T_j(x, y) f_j(y)|^s \, dy u_x \, d\mu_Y \\ &= \sum_{j=1}^n \int_{\mathbb{R}^d} |k_j(x - y)| \int_{S_Y} |T_j(x, y) f_j(y)|^s u_x \, d\mu_Y \, dy \\ &\leq [\tilde{\mathcal{T}}]_{\ell^s} \sum_{j=1}^n \int_{S_X} \int_{\mathbb{R}^d} |k_j(x - y)| |f_j(y)|^s \, dy v_x \, d\mu_X \\ &\leq [\tilde{\mathcal{T}}]_{\ell^s} \left\| \sum_{j=1}^n (|k_j| * |f_j|^s)(x) \right\|_{X^s}, \end{aligned}$$

using duality and $\|v_x\|_{(X^s)^*} \leq 1$ in the last step. We can now use the ℓ^1 -boundedness result of Proposition 3.1, since $(X^s)^*$ has the UMD property by [21, Proposition 4.2.17]. Combined with (3.2) and (3.3) we obtain

$$\begin{aligned} \left\| \left(\sum_{j=1}^n |I_j f_j|^s \right)^{1/s} \right\|_{L^p(w; Y)} &\leq [\tilde{\mathcal{T}}]_{\ell^s} \left\| \sum_{j=1}^n |k_j| * |f_j|^s \right\|_{L^{p/s}(w; X^s)}^{1/s} \\ &\leq \phi_{X, p/s}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s} \left\| \sum_{j=1}^n |f_j|^s \right\|_{L^{p/s}(w; X^s)}^{1/s} \\ &\leq \phi_{X, p, r}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s} \left\| \left(\sum_{j=1}^n |f_j|^s \right)^{1/s} \right\|_{L^p(w; X)}, \end{aligned}$$

where we can pick the increasing function ϕ in the last step independent of s , since the increasing function in Proposition 3.1 depends continuously on p . This can for example be seen by writing out the exact dependence on p in Theorem 2.4 using [19, Theorem 1.3] and [32, Theorem 3.1]. \square

Using this preparatory proposition, we will now prove Theorem 3.2.

Proof of Theorem 3.2. Let $w \in A_p$. We shall prove the theorem in three steps.

Step 1. First we shall prove the theorem very small $s > 1$. By Proposition 2.5 we know that there exists a $\sigma_{X,Y} \in (1,p)$ such that X and Y are s -convex and X^s has the UMD property for all $s \in [1, \sigma_X]$. By Lemma 2.3(iii) we can then find a $\sigma_{p,w} \in (1, \sigma_{X,Y}]$ such that for all $s \in [1, \sigma_{p,w}]$

$$[w]_{A_{p/s}} \leq [w]_{A_{p/\sigma_{p,w}}} \leq \phi_p([w]_{A_p})$$

Let $\sigma_1 = \min\{\sigma_{X,Y}, \sigma_{p,w}\}$, then by Proposition 3.3 we know that $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for $s \in (1, \sigma_1]$ with

$$(3.6) \quad [\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,p,\sigma_{X,Y}}([w]_{A_{p/s}}) [\tilde{\mathcal{T}}]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p}) [\tilde{\mathcal{T}}]_{\ell^s}.$$

Step 2. Now we use a duality argument to prove the theorem for large $s < \infty$. As noted in the proof of Proposition 3.1, we have $L^p(w; X)^* = L^{p'}(w'; X^*)$ with $w' = w^{1-p'}$ under the duality pairing as in (3.1) and similarly for Y . Furthermore X^* and Y^* have the UMD property.

It is routine to check that under this duality $I_{k,T}^* = I_{\tilde{k},\tilde{T}}$ with $\tilde{k}(x) = k(-x)$ and $\tilde{T}(x,y) = T^*(y,x)$ for any $I_{k,T} \in \mathcal{I}_{\mathcal{T}}$. Trivially $\tilde{k} \in \mathcal{K}$ if and only if $k \in \mathcal{K}$ and by Proposition 3.1(ii) the adjoint family $\tilde{\mathcal{T}}^*$ is $\ell^{\sigma'}$ -bounded with

$$[\tilde{\mathcal{T}}^*]_{\ell^{\sigma'}} = [\tilde{\mathcal{T}}]_{\ell^{\sigma}}$$

for all $\sigma \in (1, \infty)$. Therefore, it follows from step 1 that there is a $\sigma_2 > 1$ such that $\mathcal{I}_{\mathcal{T}}^*$ is ℓ^s -bounded from $L^{p'}(w'; Y^*)$ to $L^{p'}(w'; X^*)$ for all $s \in (1, \sigma_2]$. Using Proposition 3.1(ii) again, we deduce that $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ for all $s \in [\sigma_2', \infty)$ with

$$(3.7) \quad [\mathcal{I}_{\mathcal{T}}]_{\ell^s} = [\mathcal{I}_{\mathcal{T}}^*]_{\ell^{s'}} \leq \phi_{X,Y,p}([w]_{A_p}) [\tilde{\mathcal{T}}]_{\ell^s}.$$

Step 3. We can finish the prove by an interpolation argument for $s \in (\sigma_1, \sigma_2')$. By Proposition 2.2(i) we get for $s \in (\sigma_1, \sigma_2')$ that $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded from $L^p(w; X)$ to $L^p(w; Y)$ with

$$(3.8) \quad [\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p}) \max\left\{[\tilde{\mathcal{T}}]_{\ell^{\sigma_1}}, [\tilde{\mathcal{T}}]_{\ell^{\sigma_2'}}\right\}.$$

Now note that by Lemma 2.3 there is a $\sigma \in (1, \infty)$ such that $\sigma < \sigma_1, \sigma_2$ and $\sigma < s < \sigma'$ and

$$\sigma = 1 + \frac{1}{\phi_{p,s}([w]_{A_p})}.$$

Thus combining (3.6), (3.7) and (3.8) we obtain

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X,Y,p}([w]_{A_p}) \max\left\{[\tilde{\mathcal{T}}]_{\ell^{\sigma}}, [\tilde{\mathcal{T}}]_{\ell^{\sigma'}}\right\} \leq \phi_{X,Y,T,p,s}([w]_{A_p}),$$

using the fact that $t \mapsto \max\{[\tilde{\mathcal{T}}]_{\ell^t}, [\tilde{\mathcal{T}}]_{\ell^{t'}}\}$ is increasing for $t \rightarrow 1$ by Proposition 2.2(i). This proves the theorem. \square

Remark 3.4.

- From Theorem 3.2 one can also conclude that $\mathcal{I}_{\mathcal{T}}$ is \mathcal{R} -bounded, since \mathcal{R} - and ℓ^2 -boundedness coincide if X and Y have the UMD property, see e.g. [21, Theorem 8.1.3].
- The UMD assumptions in Theorem 3.2 are necessary. Indeed already if $X = Y$, $w = 1$ and if $\tilde{\mathcal{T}}$ only contains the identity operator, it is shown in [22] that the ℓ^2 -boundedness of $\mathcal{I}_{\mathcal{T}}$ implies the UMD property of X .
- The main result of [12] is Theorem 3.2 for the special case $X = Y = L^q(S)$. In applications to systems of PDEs one needs Theorem 3.2 on $L^q(S; \mathbb{C}^n)$ with $s = 2$, see e.g. [13]. This could be deduced from the proof of [12, Theorem 3.10], by replacing absolute values by norms in \mathbb{C}^n . In our more general statement the case $L^q(S; \mathbb{C}^n)$ is included, since $L^q(S; \mathbb{C}^n)$ is a UMD Banach function space over $S \times \{1, \dots, n\}$

If $X = Y$ is a rearrangement invariant Banach function space on \mathbb{R}^e , we can check the ℓ^σ -boundedness of $\tilde{\mathcal{T}}$ for all $\sigma \in (1, \infty)$ by weighted extrapolation. Examples of such Banach function spaces are Lebesgue, Lorentz and Orlicz spaces. See [28, Section 2.a] for an introduction to rearrangement invariant Banach function spaces.

Corollary 3.5. *Let X be a rearrangement invariant UMD Banach function space on \mathbb{R}^e and let $p, s \in (1, \infty)$. Let \mathcal{T} be a family of operators $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{L}(X)$ such that*

- (i) $(x, y) \mapsto T(x, y)\xi$ is measurable for all $T \in \mathcal{T}$ and $\xi \in X$.
- (ii) For some $q \in (1, \infty)$ and all $v \in A_q$ we have

$$\sup_{T \in \mathcal{T}, x, y \in \mathbb{R}^d} \|T(x, y)\|_{\mathcal{L}(L^q(v))} \leq \phi_{\mathcal{T}, q}([v]_{A_q})$$

Then $\mathcal{I}_{\mathcal{T}}$ is ℓ^s -bounded on $L^p(w; X)$ for all $w \in A_p$ with

$$[\mathcal{I}_{\mathcal{T}}]_{\ell^s} \leq \phi_{X, Y, \mathcal{T}, p, q, s}([w]_{A_p}).$$

Note that in Corollary 3.5 we need that $T(x, y)$ is well-defined on $L^q(v)$ for all $T \in \mathcal{T}$ and $x, y \in \mathbb{R}^d$. This is indeed the case, since $X \cap L^q(v)$ is dense in $L^q(v)$.

Proof. Let Y be the linear span of

$$\{\mathbf{1}_K \xi : K \subseteq \mathbb{R}^e \text{ compact}, \xi \in X \cap L^\infty(\mathbb{R}^e)\}.$$

Then $Y \subseteq L^q(v)$ for all $v \in A_p$ and Y is dense in X by order continuity. Define

$$\mathcal{F} := \{(|T(x, y)\xi|, |\xi|) : T \in \mathcal{T}, x, y \in \mathbb{R}^d, \xi \in Y\}.$$

Note that X has upper Boyd index $q_X < \infty$ by the UMD property (see [21, Proposition 7.4.12] and [28, Section 2.a]). So we can use the extrapolation result for Banach function spaces in [11, Theorem 2.1] to conclude that for $\sigma \in (1, \infty)$

$$\left\| \left(\sum_{j=1}^n |T_j(x_j, y_j)\xi_j|^\sigma \right)^{1/\sigma} \right\|_X \leq C_{\mathcal{T}, q} \left\| \left(\sum_{j=1}^n |\xi_j|^\sigma \right)^{1/\sigma} \right\|_X$$

for any $T_j \in \mathcal{T}$, $x_j, y_j \in \mathbb{R}^d$ and $\xi_j \in Y$ for $j = 1, \dots, n$. By the density this extends to $\xi_j \in X$, so

$$\{T(x, y) : x, y \in \mathbb{R}^d, T \in \mathcal{T}\}$$

is ℓ^σ -bounded for all $\sigma \in (1, \infty)$. Therefore the corollary follows from Theorem 3.2. \square

Remark 3.6.

- A sufficient condition for the weighted boundedness assumption in Corollary 3.5 is that $T(x, y)\xi \leq CM\xi$ for all $T \in \mathcal{T}$, $x, y \in \mathbb{R}^d$ and $\xi \in L^q(\mathbb{R}^e)$, which follows directly from [18, Theorem 9.1.9].
- Corollary 3.5 holds more generally for UMD Banach function spaces X such that the Hardy-Littlewood maximal operator is bounded on both X and X^* (see [10, Theorem 4.6]). For example the variable Lebesgue spaces $L^{p(\cdot)}$ satisfy this assumption if $p_+, p_- \in (1, \infty)$ and $p(\cdot)$ satisfies a certain continuity condition, see [9, 35].
- The conclusion of Corollary 3.5 also holds for $X(v)$ for all $v \in A_{p_X}$ where p_X is the lower Boyd index of X and $X(v)$ is a weighted version of X , see [11, Theorem 2.1].

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