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Key Points:

- The loading efficiency is 1 for soft material (sand) and much smaller for stiff material (sandstone)
- The head below a stream reacts directly to a stream stage change when the loading efficiency is 1
- Graphs are presented to determine when loading efficiency needs to be included in groundwater models

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The effect of loading efficiency on the groundwater response to water level changes in shallow lakes and streams

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Abstract The loading efficiency (sometimes called the tidal efficiency) is often neglected when simulating the head response in an aquifer to water level changes in lakes and streams. This is not appropriate when the lake or stream only partially penetrates the aquifer. In such cases, the aquifer extends below the lake or stream and is hydraulically connected through a semiconfining layer of lower permeability. The loading efficiency is the ratio between the instantaneous head response below a lake or stream and the water level change in the lake or stream. In sand and clay, whose particles are not cemented together, the instantaneous head response below a stream or lake is nearly equal to the stage change, and the loading efficiency is close to 1. New semianalytic solutions are presented for the groundwater response to water level changes in shallow lakes and streams that account for the loading efficiency of the aquifer. It is shown that the loading efficiency may have a significant effect on the head response. The effect is larger for larger values of the vertical resistance of the semiconfining layer and larger width of the stream and is much more pronounced in confined aquifers than in unconfined aquifers. The importance of the loading efficiency declines with time and with distance from the lake or stream. Graphs are presented that may be used to determine whether a certain combination of parameters gives a significant difference in the head at the lake shore or river bank when the loading efficiency is taken into account.

1. Introduction

The effect of stream stage variations on groundwater heads and flow has been studied analytically since the 1960s. An extensive overview of analytical solutions published in the twentieth century is provided in *Barlow and Moench* [1998]. Virtually all these solutions are for streams that fully penetrate the aquifer. Interesting work on fully penetrating streams continued in the 21st century [e.g., *Hantush*, 2005; *Srivastava*, 2006; *Intaraprasong and Zhan*, 2009]. *Zlotnik and Huang* [1999] were the first to publish an analytical solution for the effect of stream stage variations in a shallow stream that partially penetrates the aquifer. The stream has a finite width and is separated from the aquifer by a clogged streambed. The analytical solution is obtained in the Laplace domain and the stream stage variation may be arbitrary, as long as it can be transformed to the Laplace domain. *Zlotnik and Huang* [1999] showed that there was a significant effect of the partial penetration near the stream, but that the effect became negligible farther from the stream.

In parallel to the work on the effect of stream stage variations, there exists a considerable body of work on the effect of tidal fluctuations in seas and rivers on groundwater heads and flow, which vary sinusoidally in these studies. An overview of analytical studies is presented by *Li and Jiao* [2003]. The majority of these solutions consider a vertical coastal boundary (i.e., full penetration). Only a few solutions consider an aquifer that extends below the sea or tidal river, separated by a semiconfining or impermeable layer [e.g., *Maas and de Lange*, 1987; *Li and Jiao*, 2001; *Chuang and Yeh*, 2007; *Li et al.*, 2007, 2008; *Wang et al.*, 2012, 2014].

An important difference between the tidal solutions for partially penetrating streams and seas and the solution for an arbitrary stage change of *Zlotnik and Huang* [1999] is that the former include an additional term in the differential equation representing the tidal efficiency, which is the ratio between the instantaneous head response below the stream or sea and the water level change in the stream or sea [*Jacob*, 1940]. It is noted that common computer programs that can simulate the groundwater response to stage changes in shallow lakes and streams (e.g., MODFLOW [*Harbaugh*, 2005]; SUTRA [*Voss and Provost*, 2010]) do not take the tidal efficiency into account either [*Wang et al.*, 2014]. The tidal efficiency term applies to any change in

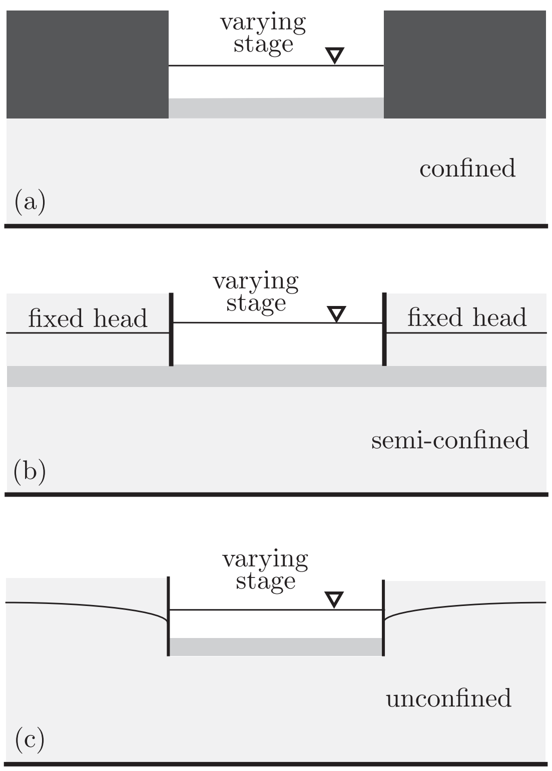

Figure 1. Cross section normal to the stream for three types of flow adjacent to the stream: (a) confined flow, (b) semiconfined flow, and (c) unconfined flow.

the loading of an aquifer, not just tides [*Domenico and Schwartz*, 1990] and should, in that respect, be called the loading efficiency.

The objective of this paper is to determine the effect of the loading efficiency on the groundwater response to water level changes in shallow lakes and streams. First, the problem description is given and the derivation of the loading efficiency is reviewed, taking into account the compressibility of the water, solids, and porous medium. It is shown that the tidal efficiency presented by *Jacob* [1940] is a special case of the general formulation given by *Verruijt* [2015]. Next, new semianalytic solutions are derived for infinitely wide lakes and streams of finite width, and the effect of the loading efficiency on the head response in the aquifer is evaluated. Finally, it is determined when the loading efficiency has a significant effect on the groundwater response to a step change in the surface water level.

2. Problem Description

Consider a long, straight, shallow stream with a semipervious layer at the bottom and a varying stage. An aquifer extends below the stream. A cross section normal to the stream is shown for three types of aquifers (Figure 1). In Figure 1a, the aquifer is covered by a thick layer of very low permeability such that the aquifer may be considered confined. A large portion of the confining layer has eroded away below the stream, where flow is semiconfined. In Figure 1b, the aquifer is semiconfined everywhere. The water level above the semiconfining layer is fixed, for example, by ditches and drains, and constant through time, while the stage in the river varies with time. In Figure 1c, the aquifer is unconfined. The bottom of the stream is formed by low permeable material such that flow below the stream is again semiconfined. The goal is to derive equations for the head h in the aquifer as a function of position and time in response to variations in the stream stage $h^*(t)$ for the case that the stream has infinite width, which represents a shallow lake or sea, and for the case that the stream has finite width. The stream stage h^* is always above the bottom of the stream and the aquifer is approximated as a porous elastic material.

3. The Storage Equation

The elastic storage in aquifers has been the topic of significant debate in the twentieth century after its first introduction in 1928 [e.g., *Meinzer*, 1928; *Jacob*, 1940; *de Wiest*, 1966; *Cooper*, 1966; *Verruijt*, 1969; *Domenico and Schwartz*, 1990; *Wang*, 2000; *Verruijt*, 2015]. *Verruijt* [2015] presented a detailed derivation of the storage equation which is valid for both soft soils such as sand and clay and stiffer porous media such as sandstone and other porous rocks. (The geological term "unconsolidated material" is not used here to avoid confusion with the process called "consolidation" in soil mechanics.) The rate of increase in storage R may be defined as [*Verruijt*, 2015, equation (1.36)]

$$R=\alpha\frac{\partial\varepsilon}{\partial t}+S_v\frac{\partial p}{\partial t} \tag{1}$$

where ε is the volume strain of the porous medium, p is the pore water pressure, t is time, and S_v is the storativity defined by *Verruijt* [2015] as

$$S_v = nC_w + (\alpha - n)C_s \tag{2}$$

where n is the porosity, C_w and C_s are the compressibility of the water and the solid particle material, respectively, and α is Biot's coefficient defined as

$$\alpha = 1 - C_s/C_m \tag{3}$$

where C_m is the compressibility of the porous medium. The water, solid particle material, and porous medium are all considered to be linear elastic materials, where the compressibilities relate a pressure or stress change $\Delta\sigma$ to a volume change

$$\frac{\Delta V_i}{V_i} = C_i \Delta\sigma \qquad i = w, s, m \tag{4}$$

Biot's coefficient α is close to 1 for uncemented materials such as sand and clay but is significantly smaller than 1 for sandstone or other porous rock [e.g., *Detournay and Cheng*, 1993].

In groundwater flow, it is common to assume that there are no horizontal deformations in the aquifer, so that the volume strain ε may be replaced with the vertical strain ε_{zz}, which in turn may be approximated as a linear function of the effective stress σ' [e.g., *Strack*, 1989; *Domenico and Schwartz*, 1990; *Verruijt*, 2015]

$$\varepsilon = \varepsilon_{zz} = -m_v \sigma'_{zz} \tag{5}$$

where m_v is the vertical compressibility of a laterally confined soil sample. The effective stress may be expressed by Terzaghi's relation using Biot's modification as [e.g., *Verruijt*, 2015]

$$\sigma'_{zz} = \sigma_{zz} - \alpha p \tag{6}$$

where σ_{zz} is the total vertical stress. Using (5) and (6), the rate of increase in storage (1) may be rewritten as

$$R = -\alpha m_v \frac{\partial \sigma_{zz}}{\partial t} + (\alpha^2 m_v + S_v) \frac{\partial p}{\partial t} \tag{7}$$

Using the definition of head, the pressure derivative is written in terms of a head derivative as

$$\frac{\partial p}{\partial t} = \gamma_w \frac{\partial h}{\partial t} \tag{8}$$

where γ_w is the specific weight of water. The change in total vertical stress σ_{zz} is expressed in terms of the change in water level h^* on top of the aquifer as

$$\frac{\partial \sigma_{zz}}{\partial t} = \gamma_w \frac{\partial h^*}{\partial t} \tag{9}$$

Combination of (7)–(9) gives

$$R = -\alpha m_v \gamma_w \frac{\partial h^*}{\partial t} + (\alpha^2 m_v + S_v)\gamma_w \frac{\partial h}{\partial t} \tag{10}$$

which may be written as

$$R = -\beta S_s \frac{\partial h^*}{\partial t} + S_s \frac{\partial h}{\partial t} \tag{11}$$

where β is the loading efficiency defined as

$$\beta = \frac{\alpha m_v}{\alpha^2 m_v + S_v} \tag{12}$$

and S_s is the specific storage coefficient defined as

$$S_s = \gamma_w (\alpha^2 m_v + S_v) \tag{13}$$

The Dupuit approximation is adopted for flow in the aquifer, which means that the resistance to vertical flow is neglected and the vertical pressure distribution in an aquifer is hydrostatic [e.g., *Strack*, 1989]. Darcy's law is applied and the specific weight γ_w and transmissivity of the aquifer T are approximated as homogeneous so that continuity of mass may be written as

$$T\nabla^2 h = DR - N \tag{14}$$

where ∇^2 is the two-dimensional Laplacian, D is the saturated thickness of the aquifer, and N is an areal source term. Substitution of (11) for R in (14) and division by T gives

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{N}{T} - \frac{\beta S}{T}\frac{\partial h^*}{\partial t} \tag{15}$$

where $S = S_s D$ is the storage coefficient of the aquifer. Below a lake or stream, the areal source term is written as

$$N = \frac{h^* - h}{c} \tag{16}$$

where c is the resistance to vertical flow of the semiconfining layer. Flow through the semiconfining layer is approximated as vertical and the storage capacity of the semiconfining layer is neglected.

Differential equation (15) is valid below the stream shown in Figure 1. In the aquifer adjacent to the stream, the load on the aquifer is constant ($\partial h^*/\partial t = 0$) so that the last term on the right-hand side of (15) is zero and the head is governed by the diffusion equation with an areal source term N, which is the common differential equation for transient flow in many textbooks [e.g., *Strack*, 1989; *Domenico and Schwartz*, 1990].

The loading efficiency β (12) is an extended version of the tidal efficiency first introduced by *Jacob* [1940]. Jacob's tidal efficiency, called β_J here, is defined as (using the notation of this paper)

$$\beta_J = \frac{m_v}{m_v + nC_w} \tag{17}$$

When the compressibility of the soil particles is neglected, $S_v = nC_w$, and when in addition Biot's coefficient α is set to 1, β reduces to β_J. Similarly, the specific storage coefficient S_s (13) is an extended form of the specific storage coefficient commonly found in groundwater textbooks [e.g., *Strack*, 1989; *Domenico and Schwartz*, 1990], written here as S_s^*

$$S_s^* = \gamma_w(m_v + nC_w) \tag{18}$$

Again, S_s reduces to S_s^* when the compressibility of the soil particles is neglected ($S_v = nC_w$) and Biot's coefficient α is set to 1. Differential equation (15) is used to describe the flow in an aquifer below the seabed, but with (17) and (18) for β and S_s, by, e.g., *Li and Jiao* [2001], *Chuang and Yeh* [2007], and *Wang et al.* [2014].

The loading efficiency is the ratio between the instantaneous head response below a stream and the change in the stream stage. In sand and clay, whose particles are not cemented together, the compressibility of the soil (m_v) is much larger than the compressibility of water (C_w), so that the loading efficiency is near 1 (compare (17)). Hence, in uncemented sands and clays, the instantaneous head response below a stream is nearly equal to the stage change, even when the stream is separated from the aquifer by a leaky bed or the aquifer is separated from the surficial aquifer by a low permeable or impermeable confining unit [e.g., *Maas and de Lange*, 1987; *Wang et al.*, 2012, 2014].

4. Solution for a Semi-infinite Lake

Consider the aquifer response to water level changes $h^*(t)$ in a semi-infinite lake (Figure 2). The origin of the x coordinate is chosen at the lake shore. Heads are measured with respect to the initial steady state head distribution. The aquifer consists of two parts (Figure 2). Flow is semiconfined below the lake (the left part), where the transmissivity is T_1, the storage coefficient is S_1, the resistance of the semiconfining layer is c_1, and the loading efficiency is β_1. Flow below the lake is governed by (15) with only one spatial coordinate, and with areal source term (16)

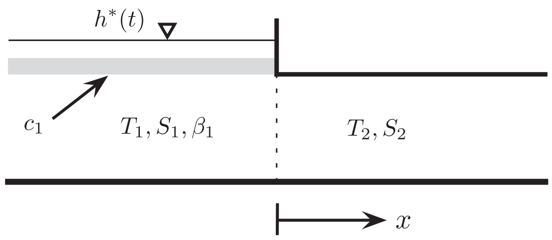

Figure 2. An infinitely wide lake ($x \leq 0$) with a varying water level $h^*(t)$.

$$\frac{\partial^2 h}{\partial x^2}=\frac{S_1}{T_1}\frac{\partial h}{\partial t}+\frac{h-h^*}{c_1 T_1}-\frac{\beta_1 S_1}{T_1}\frac{\partial h^*}{\partial t} \qquad x \leq 0 \tag{19}$$

In the right part of the aquifer, flow is semi-confined, unconfined, or confined, where the transmissivity and storage coefficient are T_2 and S_2, respectively. Variations of the saturated thickness in case of unconfined flow are neglected. S_2 represents the specific yield in case of unconfined flow and the elastic storage coefficient of the aquifer otherwise. The resistance to vertical flow of the overlying leaky layer equals c_2 while the head above the leaky layer equals zero. c_2 is set to infinity in case of unconfined or confined flow (no areal leakage). The governing equation in the right part of the aquifer is

$$\frac{\partial^2 h}{\partial x^2}=\frac{S_2}{T_2}\frac{\partial h}{\partial t}+\frac{h}{c_2 T_2} \qquad x \geq 0 \tag{20}$$

Boundary conditions are that the head and flow are continuous at $x=0$:

$$\begin{aligned} h(x=0^-)&=h(x=0^+) \\ -T_1\frac{\partial h}{\partial x}(x=0^-)&=-T_2\frac{\partial h}{\partial x}(x=0^+) \end{aligned} \tag{21}$$

where 0^- and 0^+ mean evaluation just to the left and just to the right at $x=0$, respectively.

A solution to the stated problem is obtained in the Laplace domain. Laplace transformation of equations (19) and (20) gives

$$\frac{\partial^2 \bar{h}}{\partial x^2}=\left(\frac{pS_1}{T_1}+\frac{1}{c_1 T_1}\right)\bar{h}-\left(\frac{pS_1\beta_1}{T_1}+\frac{1}{c_1 T_1}\right)\bar{h}^* \qquad x \leq 0 \tag{22}$$

$$\frac{\partial^2 \bar{h}}{\partial x^2}=\left(\frac{pS_2}{T_2}+\frac{1}{c_2 T_2}\right)\bar{h} \qquad x \geq 0 \tag{23}$$

where $\bar{h}$ is the Laplace-transformed head and p is the complex Laplace-transform parameter. Boundary conditions in the Laplace domain are given by (21) where h is replaced by $\bar{h}$. A particular solution to (22) is

$$\bar{h}_p=\bar{h}^*(pS_1\beta_1 c_1+1)/(pS_1 c_1+1) \tag{24}$$

Differential equation (23) and the homogeneous form of (22) are modified Helmholtz equations and their solution for the stated boundary conditions is

$$\bar{h}=\bar{h}_p-\frac{T_2\omega_2\bar{h}_p}{T_1\omega_1+T_2\omega_2}\exp(\omega_1 x) \qquad x \leq 0 \tag{25}$$

$$\bar{h}=\frac{T_1\omega_1\bar{h}_p}{T_1\omega_1+T_2\omega_2}\exp(-\omega_2 x) \qquad x \geq 0 \tag{26}$$

where

$$\omega_n=\sqrt{\frac{pS_n}{T_n}+\frac{1}{c_n T_n}} \qquad n=1,2 \tag{27}$$

Finally, a solution in the time domain is obtained through numerical integration of the Bromwich integral using the algorithm of *De Hoog et al.* [1982].

The variation of the lake level may be any function of time that can be transformed to the Laplace domain. In this paper, the groundwater response to a unit step in the lake level at $t=0$ is considered, so that $h^*(t)$ is the Heavyside step function $H(t)$ and the Laplace transformation of h^* is

$$\bar{h}^*=\frac{1}{p} \tag{28}$$

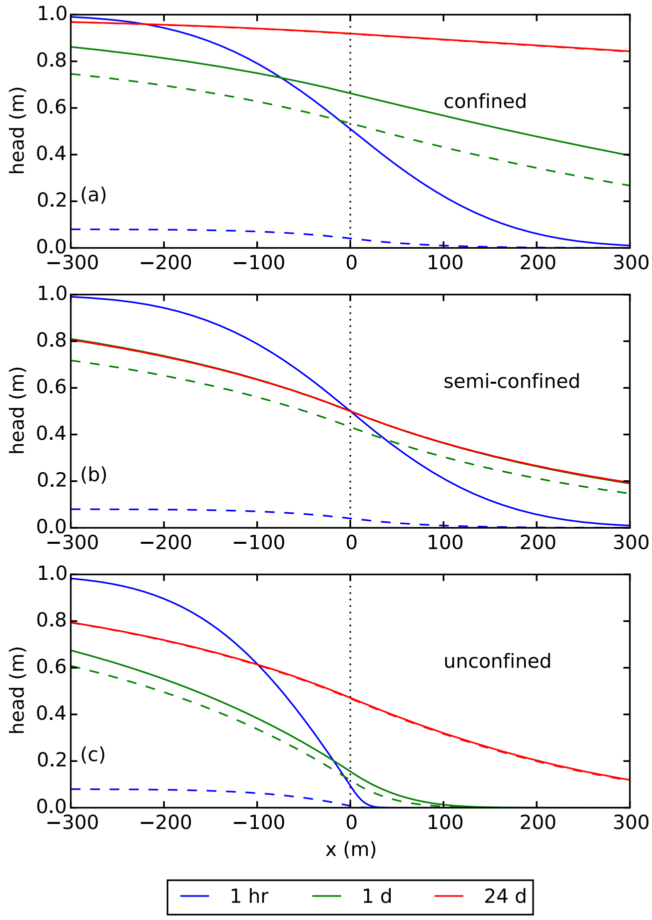

Figure 3. Example 1, variation of head with distance at 1 h, 1 day, and 24 days after a step change in the lake level. (a) Confined aquifer, (b) semiconfined aquifer, and (c) unconfined aquifer. Solution for soft soils ($\beta_1 = 1$; solid lines), and solution where loading efficiency is neglected ($\beta_1 = 0$; dashed lines).

It is noted that for the limiting case that the resistance c_1 of the semiconfining layer below the lake is infinitely large (an impermeable layer), there is still a head response in the aquifer when the loading efficiency is larger than zero. The head response in the aquifer is only zero when the loading efficiency below the lake is also zero, as for a perfectly stiff rock. This can be seen from the governing differential equations, as $h^*(t)$ (the driving force) drops out of the equations when $\beta_1 = 0$ and $c_1 = \infty$. It can also be seen from the solution in the Laplace domain (24)–(27), as the Laplace parameter p drops out of the solution when $\beta_1 = 0$ and $c_1 = \infty$.

5. Example 1: Unit Step Change in Lake Level

Consider an aquifer with $T_1 = T_2 = 200$ m^2/d and $S_1 = 0.001$. The resistance to vertical flow of the semiconfining layer below the lake is $c_1 = 500$ days. Three types of flow are considered: confined flow ($S_2 = 0.001$, $c_2 = \infty$), semiconfined flow ($S_2 = 0.001$, $c_2 = c_1$), and unconfined flow ($S_2 = 0.1$, $c_2 = \infty$). The response to a unit step in the lake level is analyzed. The head in the aquifer is computed for three times: 1 h (blue lines), 1 day (green lines), and 24 days (red lines) after the lake level change and is plotted as a function of x for all three aquifer types in Figure 3. The solid lines represent the solution for a loading efficiency $\beta_1 = 1$ (uncemented material) while the dashed lines are for the case that the loading efficiency is neglected ($\beta_1 = 0$). The red solid and dashed lines overlap for all three aquifer types while they also overlap the green solid line for semiconfined flow.

First consider the case of $\beta_1 = 1$ (solid lines in Figure 3). The head below the lake increases abruptly to $h = 1$ m in response to the unit step change in the lake level. After the initial rise, the heads below the lake start to decline so that water may leak through the semiconfining layer and flow to the part of the aquifer adjacent to the lake. The heads below the lake start to rise again after a certain time. The head rises rapidly in the confined aquifer adjacent to the lake, where the storage coefficient is small and keeps rising until it approaches the change in the lake level. The head initially rises at the same pace in the semiconfined aquifer, but the steady state position is virtually reached within a day (green line in Figure 3b). In the unconfined aquifer, the head rise is much slower, as expected, since the storage coefficient ($S_2 = 0.1$) is 2 orders of magnitude larger than the elastic storage coefficient used for confined and semiconfined flow ($S_2 = 0.001$). The solutions for $\beta_1 = 0$ (dashed lines) differ significantly from the solutions for $\beta_1 = 1$ (solid lines) for early times but are negligible for later times.

The head response over time is plotted for three observation wells and for all three aquifer types in Figure 4 (storage in the observation wells is neglected). The observation wells are located at the lake shore (Figure 4a), 100 m from the lake shore (Figure 4b), and 200 m from the lake shore (Figure 4c). Time is plotted on a log scale and $t = 1$ h and $t = 24$ days are indicated explicitly, as they correspond to the blue and red lines in

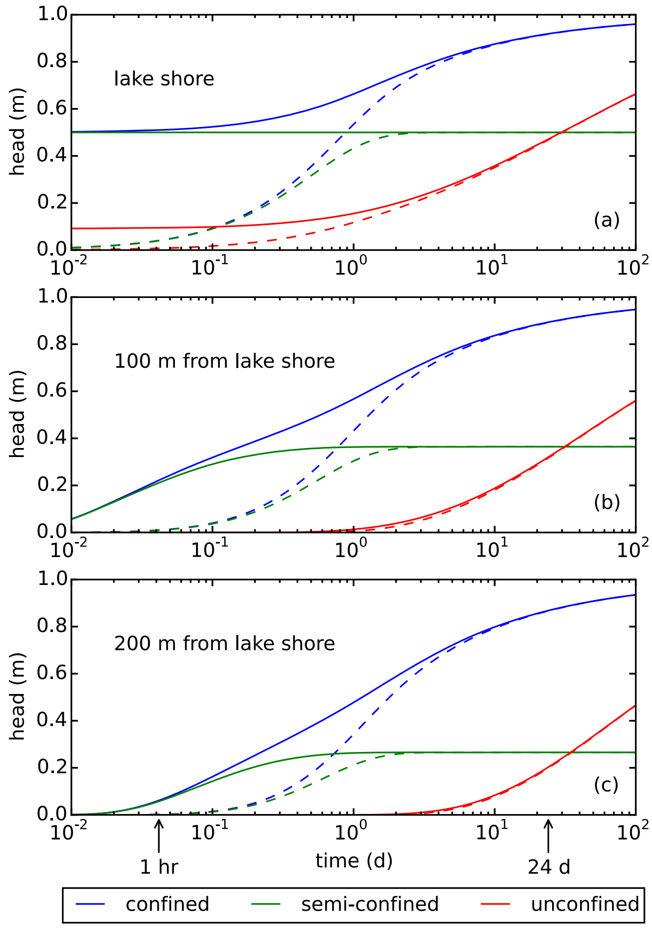

Figure 4. Example 1, variation of head with time after a step change in the lake level at (a) $x=0$ (lake shore), (b) $x=100$ m, and (c) $x=200$ m for three aquifer types. Solution for $\beta_1=1$ (solid lines) and $\beta_1=0$ (dashed lines).

Figure 3, respectively. For each observation well, the head response in a confined aquifer (blue), semiconfined aquifer (green), and unconfined aquifer (red) are shown. As stated, the head response in the confined and semiconfined aquifers is similar for early times after which they deviate. In the semiconfined aquifer, the head at $x=0$ is equal to 0.5 for the entire period as the problem is symmetric. The solid and dashed lines again represent the solutions for $\beta_1=1$ and $\beta_1=0$, respectively. The effect of the loading efficiency is much larger for the confined and semiconfined aquifers than for the unconfined aquifer (the red dashed line is very close to the red solid line, except for early times at the lake shore) but vanishes for all aquifer types after a few days for the specific parameter values of this example.

6. Solution for a Stream of Finite Width

Consider the aquifer response to stage changes $h^*(t)$ in a stream of finite width W (Figure 5). The origin of the x coordinate is at the center of the stream and heads are again measured with respect to the steady state situation. Flow is semiconfined below the stream and confined, semiconfined, or unconfined in the aquifer to the left and right of the stream. The aquifer properties to the right of the stream are the same as the properties to the left of the stream (Figure 5). Differential equation (19) is now valid for $|x| \leq W/2$ while differential equation (20) is valid for $|x| \geq W/2$. Boundary conditions (21) now apply at $x=-W/2$ and $x=+W/2$. The solution in the Laplace domain is only slightly more complicated than for the lake, and is

$$\bar{h}=\bar{h}_p+\frac{-\omega_2 T_2 \bar{h}_p \cosh(\omega_1 x)}{\omega_1 T_1 \sinh(\omega_1 W/2)+\omega_2 T_2 \cosh(\omega_1 W/2)} \qquad |x| \leq W/2 \tag{29}$$

$$\bar{h}=\frac{\omega_1 T_1 \sinh(\omega_1 W/2)\bar{h}_p \exp[-\omega_2(|x|-W/2)]}{\omega_1 T_1 \sinh(\omega_1 W/2)+\omega_2 T_2 \cosh(\omega_1 W/2)} \qquad |x| \geq W/2 \tag{30}$$

where ω_1 and ω_2 are given by (27) and $\bar{h}_p$ is given by (24). The above solution reduces to the solution of Zlotnik and Huang [1999] when the loading efficiency is set to zero in $\bar{h}_p$ (24).

7. Example 2: Unit Step Change in Stream Stage

The effect of the loading efficiency is evaluated in the aquifer at the river bank ($|x|=W/2$). The aquifer properties are $T_1=T_2=200$ m^2/d, $S_1=0.001$, and $c_1=100$ days. Three types of flow are considered: confined flow ($S_2=0.001$, $c_2=\infty$, blue), semiconfined flow ($S_2=0.001$, $c_2=c_1$, green), and unconfined flow ($S_2=0.1$, $c_2=\infty$, red). Three widths of the stream are considered $W=100$ m (Figure 6a), $W=50$ m (Figure 6b), and $W=20$ m (Figure 6c). The solid lines are for $\beta_1=1$ and the dashed lines are for $\beta_1=0$. The difference

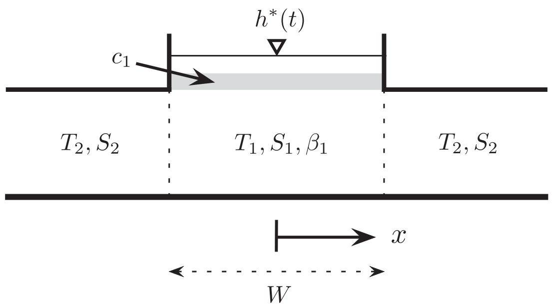

Figure 5. Stream with finite width W and varying head $h^*(t)$.

between the solutions with and without the loading efficiency decreases with the width of the river. For the parameters used in this example, the difference becomes negligible after 1–2 days for confined flow, and 2–5 h for semiconfined and unconfined flow.

The head variation for confined flow (blue solid lines in Figure 6) is interesting, as the head first decreases and then increases, which is different from the behavior at the shore of an infinitely wide lake (Figure 4a). What happens is that the head increases instantaneously below the river. When the water starts to flow laterally away from the river, flow needs to infiltrate from the river into the aquifer, which can only happen if the head below the river is lower than the stage in the lake or river. The effect of a head that, after a quick instantaneous rise, first decreases and then increases was observed next to the Deerfield River in Massachusetts by *Boutt* [2010], who modeled the behavior with a two-dimensional poroelasticity model.

8. When Does the Loading Efficiency Matter?

The remaining question is when the loading efficiency matters. The previous examples showed that the effect of loading efficiency decreases with time and with distance from the lake or stream. The effect may always be computed with the presented solutions, but here guidance is presented to determine whether the effect of loading efficiency is significant or can probably be neglected. The answer depends on, among others, the purpose of a model and the spatial and temporal scales. The time scale may be a few days for an aquifer test to multiple years for seasonal effects. Guidance is provided in terms of dimensionless numbers. The governing differential equation below the lake or stream (19) may be made dimensionless as

$$\frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} = \frac{\partial \tilde{h}}{\partial \tilde{t}} + \tilde{h} - \tilde{h}^* - \beta_1 \frac{\partial \tilde{h}^*}{\partial \tilde{t}} \tag{31}$$

while the differential equation adjacent to the lake or stream (20) becomes

$$\frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} = \frac{T_1 S_2}{T_2 S_1} \frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{c_1}{c_2} \tilde{h} \tag{32}$$

where the dimensionless variables $\tilde{h}$, $\tilde{x}$, and $\tilde{t}$ are

$$\tilde{h} = \frac{h}{\Delta h} \qquad \tilde{x} = \frac{x}{\sqrt{c_1 T_1}} \qquad \tilde{t} = \frac{t}{c_1 S_1} \tag{33}$$

where Δh is used to make the heads dimensionless. Here Δh is chosen equal to the step change in the lake or stream stage. The solution for a semi-infinite

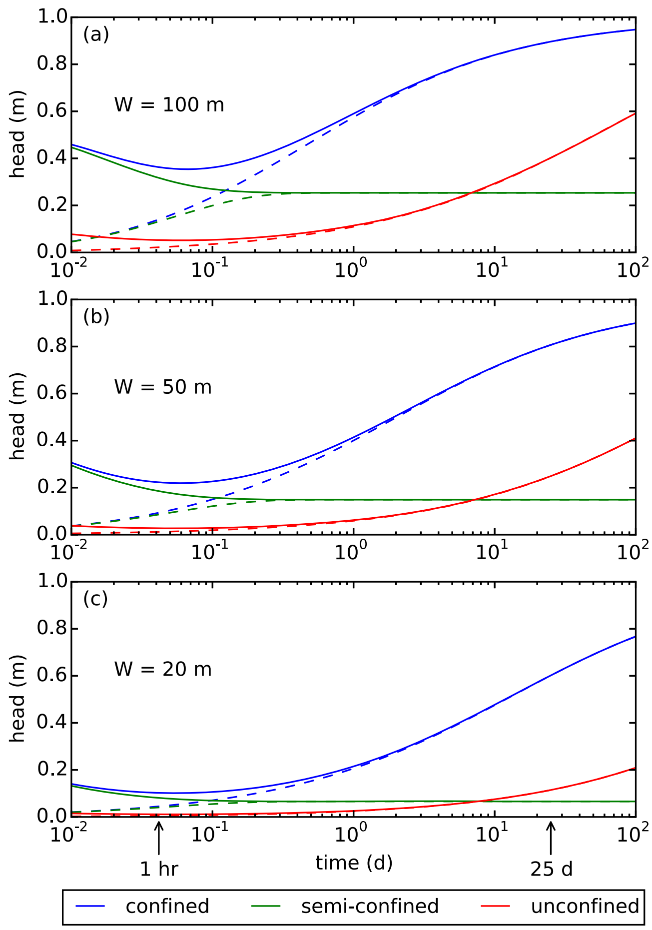

Figure 6. Example 2, variation of head with time in the aquifer at the river bank after a step change in the stream stage for three types of aquifers. The stream width is (a) $W = 100$ m, (b) $W = 50$ m, and (c) $W = 20$ m. Solution for $\beta_1 = 1$ (solid lines) and $\beta_1 = 0$ (dashed lines).

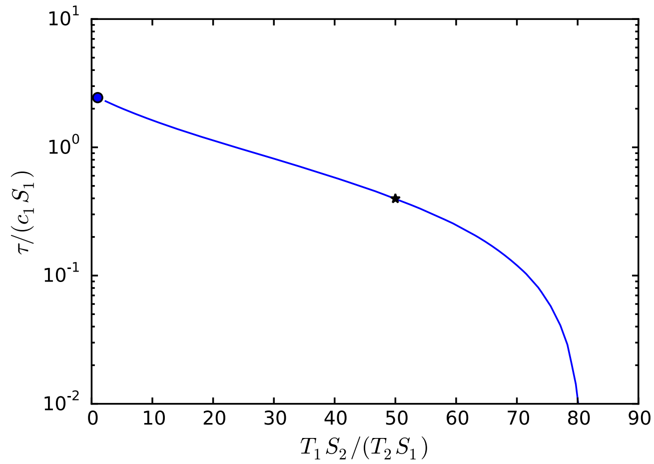

Figure 7. The effect of loading efficiency at the lake shore is more than 10% of the step in the lake level at time τ when $\tau/(c_1S_1)$ is below the curve. The dot and star represent $T_1S_2/(T_2S_1)$ equal to 1 and 50, respectively.

lake is governed by three dimensionless numbers: $T_1S_2/(T_2S_1)$, c_1/c_2, and β_1, while the solution for a stream of finite width is governed by a fourth dimensionless number $W/\sqrt{c_1T_1}$. Note that the ratio of the transmissivity over the storage coefficient is known as the aquifer diffusivity, and hence the first dimensionless number is the ratio of the aquifer diffusivities.

The difference between the solutions with loading efficiency $\beta_1=1$ and without loading efficiency $\beta_1=0$ is compared at the lake shore or stream bank. Conditions are presented for which the difference between the two solutions is at least 10% of the step in the surface water level at time $t=\tau$ after the step. Only confined flow and unconfined flow are considered, so that for all cases $c_2=\infty$ and $c_1/c_2=0$. The guidance for confined flow may be used for semiconfined flow.

First, consider a semi-infinite lake where the aquifer diffusivities below the lake and adjacent to the lake are equal $T_1S_2/(T_2S_1)=1$. The difference between the two solutions at the lake shore is at least 10% of the step in the surface water level at time $t=\tau$ when $\tau/(c_1S_1) \leq 2.44$. So for example, when $c_1=100$ days and $S_1=10^{-3}$, then the difference between the two solutions is at least 10% of the step when $t=\tau \leq 0.244$ day, or for almost 6 h after the step change.

Second, consider a semi-infinite lake where the flow adjacent to the lake is either confined or unconfined and the aquifer diffusivity below the lake T_1/S_1 differs from the aquifer diffusivity T_2/S_2 of the aquifer adjacent to the lake. The difference between the solutions with $\beta_1=1$ and $\beta_1=0$ at the lake shore at time $t=\tau$ is at least 10% of the step in the surface water level when the value of $\tau/(c_1S_1)$ is below the curve shown in Figure 7. The dot in the figure represents the case that the aquifer diffusivities are equal, which is the case considered in the previous paragraph. For example, consider the case when the aquifer adjacent to the lake is unconfined with $S_2=0.1$ and the storage coefficient below the lake is $S_1=0.002$ while the transmissivities are equal. Hence, $T_1S_2/(T_2S_1)=50$ and the loading efficiency is significant for $\tau/(c_1S_1) \leq 0.39$ (the star in Figure 7).

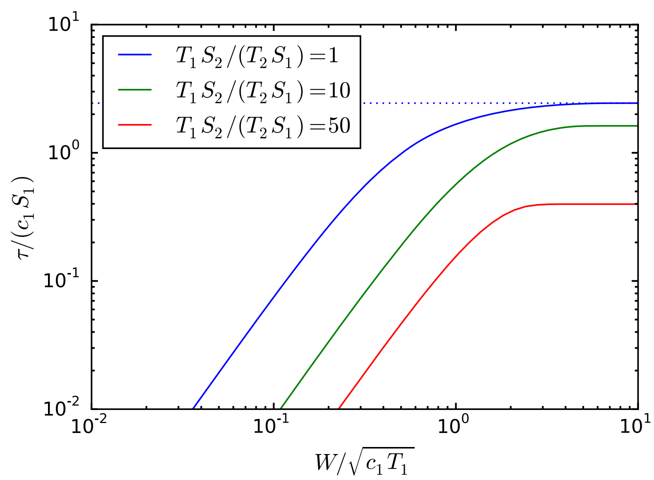

Figure 8. The effect of loading efficiency at the stream bank is more than 10% of the step in the stream stage at time τ when $\tau/(c_1S_1)$ is below the curve. Three curves represent three different ratios of aquifer diffusivity. The dotted blue line is for a semi-infinite lake with equal aquifer diffusivities (the dot in Figure 7).

Third, consider the effect of a step change in the stream stage of a stream with width W. The aquifer diffusivities below the stream and adjacent to the stream are again set equal. The difference between the solutions with $\beta_1=1$ and $\beta_1=0$ at the stream bank at time $t=\tau$ is at least 10% of the step in the stream stage when the value of $\tau/(c_1S_1)$ is below the blue curve shown in Figure 8. The curve approaches the value of $\tau/(c_1S_1)$ for a semi-infinite lake (the dotted line in the graph) for large values of $W/\sqrt{c_1T_1}$. Two additional curves are shown in Figure 8. The green line represents the case when $T_1S_2/(T_2S_1)=10$, and the red line represents the case when $T_1S_2/(T_2S_1)=50$.

9. Conclusions and Discussion

New semianalytic solutions were presented for the head response to water levels changes in shallow lakes and streams that take into account the loading efficiency of the aquifer, as derived from the storage equation presented by *Verruijt* [2015]. The lake or stream only partially penetrates the aquifer and is hydraulically connected to the aquifer by a low permeable layer, so that flow below the lake or stream is semiconfined. Existing numerical and semianalytic solutions for such cases commonly do not take the loading efficiency into account, except for tidal fluctuations. It was shown that the loading efficiency may have a significant effect on the head response, especially when the vertical resistance of the semiconfining layer is large. The difference declines with time and with distance from the lake or stream. Graphs were developed in terms of dimensionless parameters that may be used to determine whether and when a certain combination of parameters gives a significant difference in the head (10% of the step in water level in the lake or steam) at the lake shore or river bank. The effect of the loading efficiency is much more pronounced in confined aquifers and much smaller in unconfined aquifers. The effect may be significant when the vertical resistance of the semiconfining layer below the lake or stream is large and may be insignificant when the resistance of the semiconfining layer is small.

The analysis presented in this paper was based on a number of approximations, some of which deserve further attention. In the derivation of the governing differential equation from the storage equation, it was assumed that there were no horizontal deformations in the aquifer. This is common, but also a bit odd, of course, as the flow is predominantly horizontal. Other assumptions are possible that allow for horizontal deformations [e.g., *Boutt*, 2010; *Verruijt*, 2015]. For unconfined flow, the differential equation was linearized by approximating the saturated thickness as constant. A recent discussion of this effect may be found in *Sheets et al*. [2015]. The storage of the semiconfining layer below the lake was neglected, but that may be taken into account in semianalytic solutions of multilayer transient flow [e.g., *Hemker and Maas*, 1987; *Li et al*., 2008; *Bakker*, 2013]. In that respect, the solution may be extended to the solution for a multilayer system. Especially, the effect of the loading efficiency in deeper aquifers, where flow is confined, deserves further attention.

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