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Quantum breakdown of superconductivity in low-dimensional materials

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In order to understand the emergence of superconductivity it is useful to study the reverse process and identify the various pathways that lead to its destruction. One way is to increase the amount of disorder, as this leads to an increase in Coulomb repulsion that overpowers the attractive interaction responsible for Cooper pair formation. A second pathway—applicable to uniformly disordered materials—is to utilise the competition between superconductivity and Anderson localisation, as this leads to electronic granularity in which phase and amplitude fluctuations of the superconducting order parameter play a role. Finally, a third pathway is to construct an array of superconducting islands coupled by some form of proximity effect that leads from a superconducting state to a state with finite resistivity, which appears like a metallic groundstate. This Review Article summarizes recent progress in understanding of these different pathways, including experiments in low dimensional materials and application in superconducting quantum devices.

In parallel to the continuous discovery of new superconducting materials, a different research strategy can be followed that concentrates on the destruction of superconductivity in a given materials system. Instead of changing the chemical composition of the material, the approach is to change a parameter intrinsic to the material such as electron density, disorder, or dimensionality, and so drive a transition from a superconducting state to another state, which is often found to be an insulator or a metal. It is experimentally unavoidable that this transition must be monitored at finite temperature (T), but the physics is focused on the transition that is expected to occur at zero temperature reflecting a transition from one ground state to another, a quantum phase transition[1]. This quantum phase transition has—with some theoretical bias—traditionally been called a superconductor–insulator transition[2], and has been studied in numerous experimental configurations and materials, encompassing amorphous thin films, granular superconductors, nanowires, gate-tunable superconducting oxide interfaces, aluminum-based Josephson junction arrays, proximitised metals and semiconductors or two-dimensional (2D) crystalline superconductors.

In the last decade, the detailed study of the transition has revealed many surprises that have drastically changed our understanding of conventional superconductivity and has shown some links to phenomena observed in high temperature cuprate superconductors. These new developments are the focus of this Review. To remove interpretative bias in the terminology we use in throughout, we adopt a more neutral term 'quantum breakdown of superconductivity' (QBS) instead of the commonly used term 'superconductor–insulator transition' (SIT). We restrict use of the term SIT to situations involving a transition to an insulating state and use the phrase 'superconductor–metal transition' (SMT) if it transitions into a metal-like state. As we will see, even these 'insulator' and 'metal' states can also be based on ingredients related to the superconducting state, without showing zero resistance.

Main paradigms

Various means can be employed to study experimentally the evolution of superconductivity towards breakdown. Some obvious ones are the application of an external magnetic field (B) or a d.c. current, which lead to the transition to resistive superconducting states. Suitable materials can be created by either increasing disorder, changing the carrier density by field-effect gating or by using a lower effective dimensionality in thin-film, nanowire geometries or single-atom-thick layers. Understanding how the evolution of the superconducting order parameter $\Psi = \Delta e^{i\phi}$ correlates with the suppression of the zero-resistance state is the primary question that provides insight on several fundamental concepts and paradigms of condensed matter physics.

The first key paradigm concerns the complex nature of the superconducting order parameter. The suppression of superconductivity can follow two main paths[3], either a suppression of the amplitude Δ or a loss of the stiffness of the phase ϕ. The former involves the interaction between the electrons that sets the strength of the attractive interaction leading to Cooper pairing, and which may, if pushed from attractive to repulsive, restore the normal state[4,5]. The second one yields a less intuitive mechanism in which Cooper pairing remains and resistive properties emerge because the macroscopic phase loses long-range order and varies in time[6]. These two antagonistic mechanisms that both result in the destruction of superconductivity have been dubbed the fermionic and bosonic scenarios. Both yield resistive states, terminating the zero-resistance superconductivity, but are made from very different charge carriers, one formed of fermionic single electrons and the other of incoherent bosonic Cooper pairs.

Another important aspect to grasp is the effective dimensionality of the system. For single-electron coherence the relevant comparison is between the inelastic length and the thickness of a thin film or the width of a nanowire. However, for a superconductor the superconducting coherence length ξ is the scale to compare with the sample dimensions. Superconductivity is quasi-2D when one of the sample dimensions is smaller than ξ, and quasi-1D when two of them are. This reduced dimensionality has major consequences for the superconducting state[7]: it enhances fluctuations and generates topological defects—vortices in 2D or phase slips in 1D—that drastically modify the transport properties and eventually suppress the phase stiffness through a Berezinskii–Kosterlitz–Thouless transition or through 1D (quantum) phase slips.

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Last but not least, disorder is an unavoidable ingredient that has deep consequences in limiting electron transport in general and which contrasts sharply with the zero-resistance property of superconductivity. Although disorder is at first sight incompatible with superconductivity, experiments and theory show that superconductivity can develop in systems in which the single electrons are localized, leading to an insulating state in the $T = 0$ limit. If Cooper pairing can develop with localized single-electron states, then the natural question to ask is whether one may observe localization of Cooper pairs. In the past decade, a large body of work has demonstrated that this type of localization occurs in some amorphous superconductors. Probing and understanding the properties of insulating systems with localized Cooper pairs poses further challenges and intriguing questions.

These rather generic ingredients—reduced dimensions, disorder and Coulombic inter-electron interactions—can be introduced and controlled in many superconductors. Through decades of research, primarily using magnetotransport and analysis of quantum criticality, it has been found that the breakdown of superconductivity does not follow a universal path. Instead, there appears to be almost as many QBSs as systems under study. This led to a classification of various types of transition according to their structural aspects, meaning whether they take place in granular or homogeneously disordered (amorphous) systems; by the level of charge carrier density; by their effective dimensionality; by the type of weak link in Josephson junction arrays; or by the parameter that drives the quantum phase transition. In this Review, we have chosen to leave out a previously studied specific model system of Josephson tunnel junction arrays with competing Josephson and charging energies[8].

Our review will highlight the new phenomena and concepts that emerged recently and led the community to revisit several long-standing paradigms in the field of the QBS. Experimentally, much of the recent progress is due to the use of very low temperature spectroscopy with local probes that unveils the emergent electronic granularity in homogeneously disordered thin films and the existence of the pseudogap for preformed Cooper pairs. These discoveries led to the unexpected breakdown of the often-cited fermionic and bosonic dichotomy and demand a new microscopic description of superconductivity subject to strong disorder. Moreover, in a number of experimental systems the transition to the superconducting state is found to be incomplete, that is, terminated by a metal-like state down to the lowest accessible temperature[9]. This phenomenon might be called a superconductor–metal transition, although the origin of this vividly discussed[10,11] metallic state has not yet been determined unambiguously. We also discuss how new gate-tunable semiconductor and low-dimensional materials can be used to couple superconducting islands creating proximitized Josephson junction arrays, providing new insights into the QBS physics utilizing insights from mesoscopic physics. Finally, we address how strong disorder modifies the electrodynamics near the QBS and how this electrodynamics can be utilised in hybrid quantum circuits.

Amplitude- versus phase-driven transition

Thin superconducting films have become the prototypical systems to study the quantum breakdown of superconductivity. Due to their quasi-2D nature, thermal and quantum fluctuations of the order parameter $\Psi = \Delta e^{i\phi}$ play a crucial role[7,12]. Thin films are furthermore enhanced by disorder that reduces the superconducting coherence length. At the same time, the diffusive motion of electrons impeded by the reduced dimensionality tends to enhance the strength of the Coulombic inter-electron interaction[13]. These two phenomena generate two distinct pathways to QBS (Fig. 1): either a suppression of the superconducting phase stiffness by quantum phase fluctuations—the phase-driven pathway—or a Coulomb-assisted suppression of the Cooper-pair attraction—the amplitude pathway.

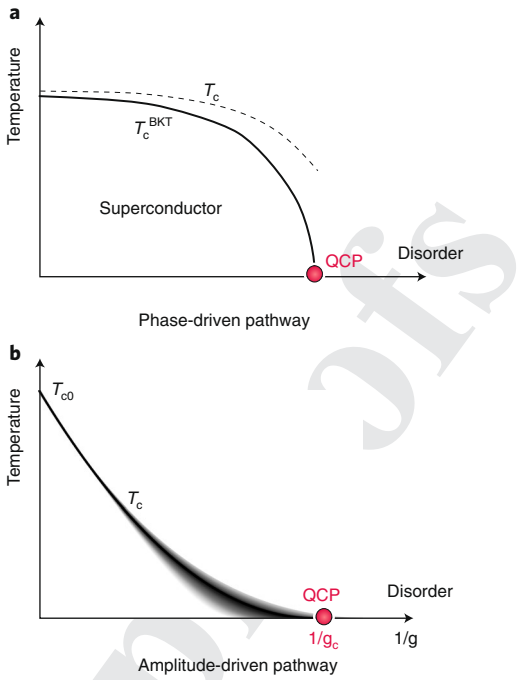

Fig. 1 | The phase diagrams of 2D superconductors. a, Temperature-disorder phase diagram of 2D superconductors[6]. The transition to the superconducting state is defined by the critical temperature T_c at which the Cooper pairs preform. The superfluid stiffness, indicating superconductivity, develops at a lower temperature, below T_c^{BKT}, the critical temperature of the Berezinski–Kosterlitz–Thouless transition[129,130]. On increasing disorder, T_c is reduced by the Finkel'stein mechanism[4,5] illustrated in **b**. Similarly, T_c^{BKT} is suppressed faster due to the disorder-enhanced phase fluctuations, till a critical disorder that defines the quantum critical point (QCP). The existence of the two critical temperatures opens up a sizeable temperature regime for phase fluctuations[12] between T_c^{BKT} and T_c, which grows with increasing disorder. At the quantum critical point the superfluid stiffness vanishes, but without the destruction of the Cooper pairs. Without stipulating the nature of the state terminating superconductivity, metal or insulator, one recognizes that this scenario defines a prototypical continuous quantum phase transition[1], driven by quantum fluctuations of the phase of the order parameter. **b**, Coulomb suppression of the critical temperature T_c as a function of disorder $1/g$, with $g = h/e^2R_\square$ according to Finkel'stein theory. The shaded grey area around the T_c line represents the fluctuations of the local critical temperature $T_c(\mathbf{r})$ that develop and grow as $\frac{\delta T_c}{T_c} \approx \frac{0.4}{g(g-g_c)}$ upon approaching the critical point $1/g_c$ (ref. [39]). Notice that the disorder dependence of T_c takes the form drawn in **a** when plotted as a function of $(1/g)$, see for instance Figs. 4 and 5 in ref. [131].

Phase-driven pathway. A seminal approach to the phase-driven QBS in thin-film superconductors was proposed by Matthew Fisher[6], who developed the quantum critical scaling theory of the dirty boson model. It is a 2D model of hard-core, interacting bosons of charge $2e$ (describing Cooper pairs) in a random potential. The fundamental concept of this work, which motivated considerable experimental activity on thin superconducting films, is the possibility that, when the magnetic field is increased at $T = 0$, the density of pinned vortices will increase and they can delocalize and undergo Bose condensation. A similar effect occurs for the vortex–anti-vortex phase when disorder is increased at $B = 0$ (Fig. 1a). In analogy, this implies that the localisation of charge-$2e$ bosons must be mandatory for the condensation of the charge-$2e$ bosons in the superfluid phase. As a result, the QBS in this model is the competition between condensation of Cooper pairs and of vortices.

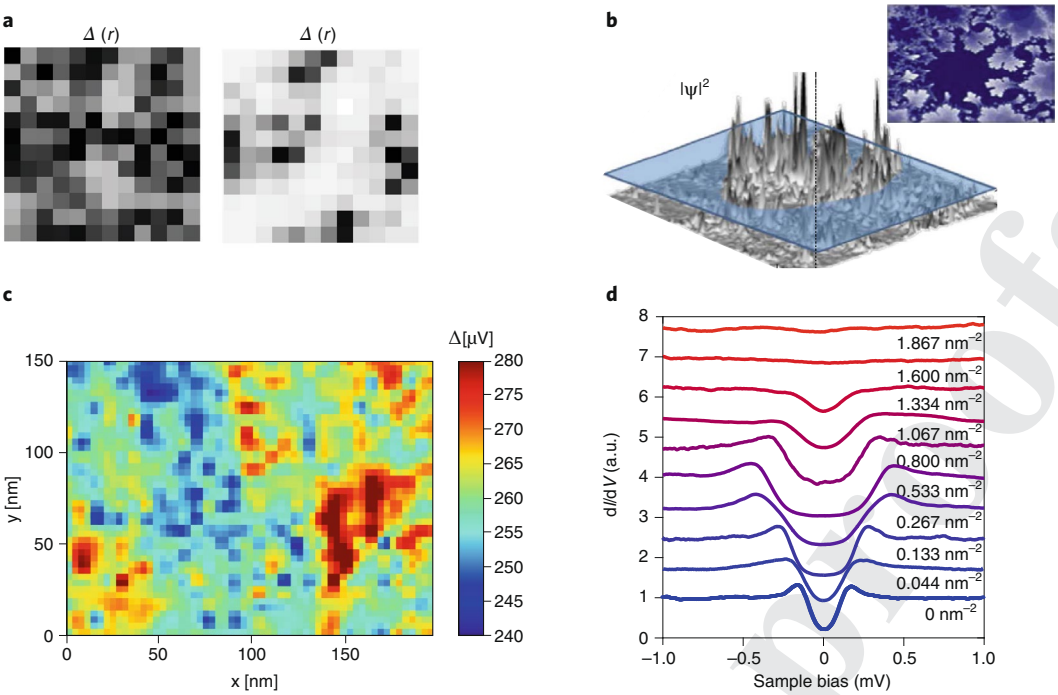

Fig. 2 | Emergent superconducting granularity. a, Spatial map of the pairing amplitude $\Delta(\mathbf{r})$ obtained by numerical solution of the 2D disordered attractive Hubbard model[36] with a disorder level equal to the nearest-neighbor hopping (left) and to twice the nearest-neighbor hopping (right). Sites with darker gray-scale indicate larger $\Delta(\mathbf{r})$. **b**, Fractal wavefunction intensity $|\Phi|^2$ at the mobility edge for the Anderson problem. The fractal nature is readily seen in the inset that shows the spatial distribution of the wavefunction at the intensity indicated by the blue plane. The wavefunction occupies only a fraction of the available volume. **c**, Spatial map of the superconducting gap measured by scanning tunnelling spectroscopy on a TiN thin film[26] near the QBS ($T_c \simeq 0.3T_{c0}$ and $R_\square = 3.5\text{k}\Omega$ reached before the superconducting transition). Measurements were performed at 0.05 K. **d**, Tunnelling conductance dI/dV measured by scanning tunnelling spectroscopy on an epitaxial monolayer of $NbSe_2$ covered by Si adatoms[34]. Each spectrum corresponds to a different surface density of adatoms, that is, different level of disorder. The superconducting gap evolves non-monotonously with the surface density of adatoms. Figure reproduced with permission from: **a**, ref. [36], APS; **b**, V. Kravtsov; **c**, ref. [26], APS; **d**, ref. [34], Springer Nature Ltd.

Two major outcomes of this theory have laid the groundwork for decades of work on this specific theory for the QBS. First, the resistance at the quantum critical point is predicted to be metallic because the vortices and charges diffuse simultaneously and, if we assume a self-duality between the charges and vortices, it should reach the quantum resistance for charge $2e$, $h/(2e)^2$. Secondly, the quantum scaling analysis led to a frequently used scaling dependence of the resistance around the quantum critical point:

$$R = \frac{h}{4e^2}\mathscr{F}(|\delta|/T^{1/z\nu}), \qquad (1)$$

which relates the behaviour of the resistance to the critical exponents ν and z of the diverging spatial and dynamical correlation lengths[1], via the scaling functional $\mathscr{F}$. The variable δ is defined as $\delta = (X - X_c)/X_c$, the distance to the critical point X_c for the tuning parameter X.

Experimentally, Hebard and Palaneen[14] were the first to uncover an intriguing crossing point in the magnetoresistance isotherms of amorphous InO (a:InO) films. The nearly T-independent resistance at the crossing point reached a value of $0.7h/4e^2$ and was identified as the quantum critical point of the magnetic-field-driven SIT. Scaling analysis of the resistance data around it, taken at various T, were shown to collapse on a single functional, in accordance with equation (1), providing direct access to the critical exponents ν and z. Another seminal work, conducted by Haviland et al.[15], modified the disorder in quench-condensed amorphous bismuth films by tuning their thickness and discovered a QBS where the critical temperature is continuously reduced upon increasing sheet resistance, reaching full suppression for a resistance of the order of $h/(2e)^2$.

These experimental findings, together with Fisher's theory, lent support to a phase-driven QBS scenario and stimulated a large body of experimental work to establish the universal character of the phase-driven QBS independently in various systems and materials. The studies carried out over the past three decades repeatedly confirmed the presence of a, sometimes approximate, crossing point in the magnetoresistance and reproduced with varying degrees of success the quantum scaling of the data. Unfortunately, the resulting collection of critical exponents extracted from scaling analysis covers a large range of values, ranging from $z\nu = 0.6$ to 2.4. In order to reconcile this dispersion, different universality classes were invoked, for example for classical and quantum percolation[16] (see for example the reviews in refs. [17,18]). Even then, one finds a lack of universality for the critical exponents, often a limited range of temperature or field used for the scaling analysis, a non-perfect crossing point that sometimes transforms into multiple crossing points in different temperature ranges that is then interpreted as multiple quantum criticality[19]. All these observations have undermined the confidence in arguments based on the data scaling analysis. In retrospect, the quantum scaling analysis, which continues to be widely used as an indicator for quantum criticality, has become a mandatory figure for a phase-driven QBS while, at the same time, it has not been able to bring fruitful insight into the QBS physics, such as the understanding of the microscopic origin of the magnetoresistance crossing point.

Likewise, the critical resistance is found to cover a range from 1 to 30kΩ, although with variation from sample to sample around

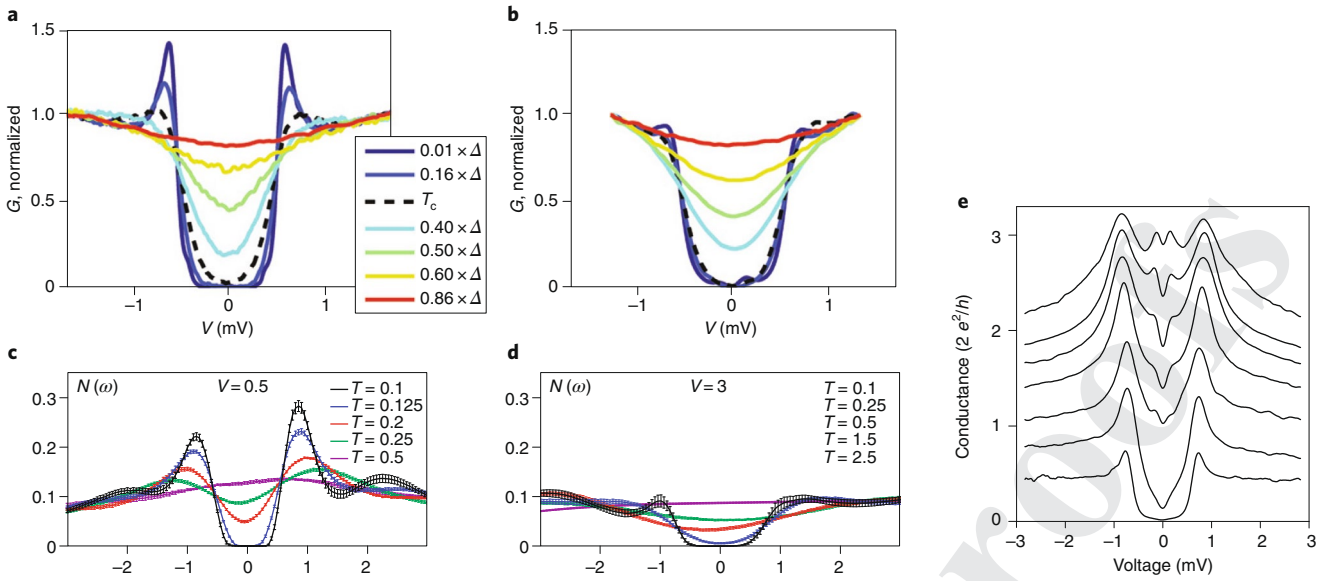

Fig. 3 | Pseudogap and collective gap of preformed Cooper pairs. a,**b**, Temperature evolution of the local tunnelling conductance G versus voltage bias V, characterized by the presence (**a**) or absence (**b**) of superconducting coherent peaks[29]. Both sets of data were measured at two different locations in superconducting samples. The tunnelling spectra are selected at temperatures equal to fractions of the low-T spectral gap. The spectral gap values are $\Delta = 560\mu$eV and $\Delta = 500\mu$eV for **a** and **b**, respectively. The black dashed lines show the spectra measured at T_c. The clear pseudogap without coherence peaks and without a state at the Fermi level ($V = 0$) is the signature of preformed Cooper pairs. Both sets of data are representative of superconducting samples with emergent granularity, for which superconducting islands show spectra with coherence peaks (**a**) and the surrounding matrix a gap without coherence peaks (**b**). The latter being a spectral signature of localized preformed Cooper pairs[29]. **c**, Numerical simulations of the disorder-averaged density of states $N(\omega)$ for the 2D disordered attractive Hubbard model. A pseudogap develops for $T \simeq T_c = 0.14$ and the coherence peaks vanish at $T \sim T_c$, in remarkable agreement with the experimental data in **a**. **d**, Similar simulations but for the insulating state at high disorder. The temperature evolution of $N(\omega)$ resembles that of the superconducting case in **c** but without coherence peaks. This set of simulations indicate that the insulator is gapped due to electron pairing. **e**, Point contact Andreev spectroscopy: evolution of the local differential conductance $G = dI/dV$ versus bias voltage measured on an a:InO sample at $T = 0.065$ K and at the same position for different values of the point-contact conductance. The conductance curves are normalized to $2e^2/h$ and have not been vertically shifted. The evolution from the tunnelling to the Andreev spectroscopy unveils a new pair of peaks inside the single-electron gap, which relate to the collective gap Δ_{col}. Figure reproduced with permission from: **a**,**b**, ref. [29], Springer Nature Ltd; **c**,**d**, ref. [41], Springer Nature Ltd; **e**, ref. [66], Springer Nature Ltd.

the resistance quantum $h/(2e)^2$ for strongly disordered systems[20]. However, sometimes it is remarkably accurate, such as in the high-T_c cuprate thin films[21] and some graphene tin-decorated hybrid devices[22]. But it can also approach h/e^2 as observed in the disorder-tuned QBS in TiN thin films[23] without any metallic separatrix between the superconducting and insulating films, contrary to what is expected for the dirty boson model. Consequently, such a diversity of critical resistance values points towards the need to consider carefully ingredients other than just long-wavelength phase fluctuations. These additional ingredients may be system specific, which calls for less emphasis on the conjectured universal behavior, and more on the microscopics of the models for specific system classes.

Amplitude-driven pathway. The second major pathway to the QBS is based on the enhancement of the effective Coulomb repulsion due to the decrease in the diffusive motion of electrons[13]. This disorder-driven enhancement of the Coulomb interaction competes with the phonon-mediated attractive part of the interaction in the Cooper channel. The resulting continuous reduction of the effective attractive interaction leads to an amplitude-driven QBS, in practice a SMT with a vanishing pairing amplitude at the critical disorder. First calculated with a perturbative diagrammatic technique[24,25], the full dependence of the critical temperature on the sheet resistance of the normal state, $T_c(R_\square)$, was obtained via the renormalization group method by Finkel'stein[4,5]. This subsequently provided a simple analytical prediction for the critical disorder: $g_c = 1/2\pi[\ln(1/T_{c0}\tau)]^2$, where $g = h/(e^2R_\square)$ is the dimensionless sheet conductance. Remarkably, g_c is completely defined by only two parameters: the unsuppressed T_{c0} and the elastic scattering time τ. The typical dependence of T_c on disorder is illustrated in Fig. 1b.

The Finkel'stein theory successfully describes the suppression of T_c in some thin films with low critical disorder[5], defined as $g_c \geq 4$. In this case, the state found after terminating superconductivity is a 'bad' metal subject to weak localization effects at experimentally accessible temperatures. Its intrinsic mechanism, which is just an extension of mean-field Bardeen–Cooper–Schrieffer (BCS) theory including disorder-enhanced interaction, is expected to be somewhat universal. However, on a quantitative level, many systems exhibit a critical disorder beyond the range of applicability of the Finkel'stein theory ($R_\square \sim h/4e^2$), where phase fluctuations are expected to contribute significantly[3]. When the sheet resistance reaches the resistance quantum, quantum phase fluctuations, localization effects and disorder-induced spatial inhomogeneities of the electronic properties disturb the standard dichotomy between phase and amplitude-driven pathways, leading to new scenarios where the two are intertwined.

Emergent granularity of superconductivity at the local scale

An early attempt to understand the diversity of experimental data on the QBS was to make a distinction between granular and homogeneous systems. The granular systems were usually thought of in analogy to Josephson tunnel-junction arrays, following a

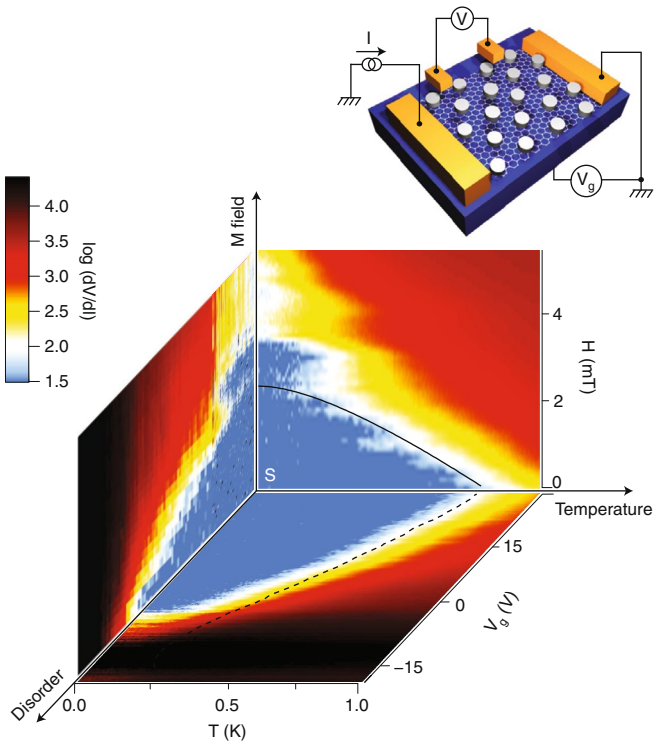

Fig. 4 | Quantum breakdown of superconductivity in a mesoscopic device. A 3D phase diagram showing the superconducting state reconstructed from measurements of the array resistance in back-gate voltage, V_g, magnetic field, H, and temperature, T, space[92]. The resistance in (V_g, H) space is measured at 0.06 K. Notice that traces of a re-entrance of superconductivity above the first critical field are visible in both the (V_g, H) and (V_g, T) planes, a clear signature of the mesoscopic superconductivity[57]. Inset: a schematic of the mesoscopic sample — a graphene Hall bar decorated with an array of superconducting discs. Figure reproduced with permission from ref. [92], Springer Nature Ltd.

phase-driven pathway to the breakdown of superconductivity. In contrast, the atomic structure of the homogeneous systems were expected to show 'homogeneous', short-range correlated disorder that should give rise to a homogeneous superconducting state, subject to Coulomb suppression of superconductivity. However, the last decade has unveiled a different, more complex and rich situation. In the strong scattering limit when the mean free path is of the order of the interatomic distance, disorder showed up as a strong disturbing agent that generates strong spatial fluctuations of the superconductivity-related spectral properties, enhanced by the proximity to the critical disorder[26–34]. Therefore, the classification between homogeneous and granular disordered materials became problematic because homogeneously disordered materials showed self-induced electronic inhomogeneities—that is, emergent granularity of the superconductivity—without an evident correlation with any structural granularity.

Emergent superconducting granularity. The concept of disorder-induced inhomogeneities of the superconducting state was proposed back in 1971 by Larkin and Ovchinnikov[35]. It was understood that relatively small spatial variations in the strength of the Cooper attraction $\lambda(\mathbf{r})$ induced by disorder could lead to strong fluctuations of the local 'transition temperature' $T_c(\mathbf{r})$. Since $T_c(\mathbf{r}) \propto \exp(-1/\lambda(\mathbf{r}))$, the variation in T_c is given by $\delta T_c(\mathbf{r})/T_c \sim \delta\lambda(\mathbf{r})/\bar{\lambda}^2$ with the spatial average $\bar{\lambda} \ll 1$. These possible fluctuations and their impact on the superconducting properties have attracted considerable theoretical attention recently[36–44].

The concept of spatial inhomogeneity of superconductivity was later analyzed within a different theoretical approach initiated by Ma and Lee[45]. Assuming that, for some reason, the Finkel'stein mechanism is ineffective, the destruction of superconductivity is considered to be the result of Anderson localization[36,38,45–49]. Earlier work[36,38] demonstrated, by numerically solving the self-consistent Bogoliubov-De Gennes equations, that in the presence of significant local disorder of the on-site electron energies, the order parameter defined as the quantum-statistical average, $\Delta_{op}(\mathbf{r}) = \lambda\langle\psi_\uparrow((r)\psi_\downarrow(\mathbf{r})\rangle$, becomes strongly inhomogeneous in space. Figure 2a shows the resulting fluctuations for the pairing amplitude calculated by Ghosal et al.[36]. Furthermore, the electron excitation spectrum of this inhomogeneous superconducting state has some unusual features: the spectral gap does not coincide with the order parameter, as it does for the standard BCS superconducting state, and the gap edge singularities (coherence peaks) in the density of states become smeared. The spatial fluctuations of the height of the coherence peak provides another indicator characteristic of the superconducting inhomogeneities.

The numerical simulations[36,38] demonstrated that sufficiently strong disorder suppresses the coherence peaks completely, while the spectral gap stays intact, thus opening an important question on the nature of the state terminating superconductivity. More recent numerical work[40,41,43], including some based on the quantum Monte Carlo method[41] that takes into account the quantum phase fluctuations between the self-induced superconducting islands, showed that the ground state is an insulator with a spectral gap caused by the attractive interaction. This type of electron pairing is similar to the so-called negative-U Hubbard model introduced by Anderson[50].

The body of numerical work reviewed above demonstrates the concept of disorder-induced 'emergent superconducting granularity' and predicts new spectral features that contrast with the weak-disorder BCS superconductivity. However, numerical approaches are limited to small system sizes and to the strong coupling limit, whereas superconductivity is in essence a 'weak effect'. Very recent results[44] indicate a way to solve this problem.

Superconductivity near the mobility edge. In order to make progress in understanding the strong-disorder regime, a combined analytical and numerical approach has been developed[48,49] to extend the theory of weak-coupling superconductivity. The crucial aspect of this new theoretical development lies in the fractal nature of the nearly localized electron wavefunctions when the Fermi level E_F is close to the Anderson mobility edge E_c (ref. [51]). These fractal electron wavefunctions $\psi_i(\mathbf{r})$ feature many unusual properties including intensities $\psi_i^2(\mathbf{r})$ that fluctuate very strongly in space (see Fig. 2b), multifractal statistics[52], and matrix elements $M_{ij} = \int d\mathbf{r}\psi_i^2(\mathbf{r})\psi_j^2(\mathbf{r})$ that grow as $(E_c/\omega)^\gamma$ when the energy difference ω between states i and j is small. In addition, right at $E_F = E_c$ the inverse participation ratio $P_i = \int d\mathbf{r}\psi_i^4(\mathbf{r})$ scales with the system size L as $P \propto L^{-d_2}$, where d_2 is the fractal dimension related to the exponent $\gamma = 1 - d_2/d$. Note that the three-dimensional Anderson mobility edge is characterized by $d_2 \approx 1.3$ and $\gamma \approx 0.6$ (ref. [52]). Importantly, fractal properties of wavefunctions are very robust at strong disorder, even half-way the mobility edge[51].

These unusual properties profoundly modify the superconductivity and have led to the concept of 'fractal superconductivity'. Using a generalized mean-field theory, Feigel'man et al.[49] showed that the power-law scaling of $M(\omega)$ leads to a new dependence of the critical temperature on the microscopic parameters:

$$T_c(\lambda) \sim E_F\lambda^{1/\gamma}. \quad (2)$$

This equation leads to the unexpected prediction that, for a constant λ, the critical temperature increases upon approaching the mobility edge compared to its weak disorder value. Later, the same enhance-

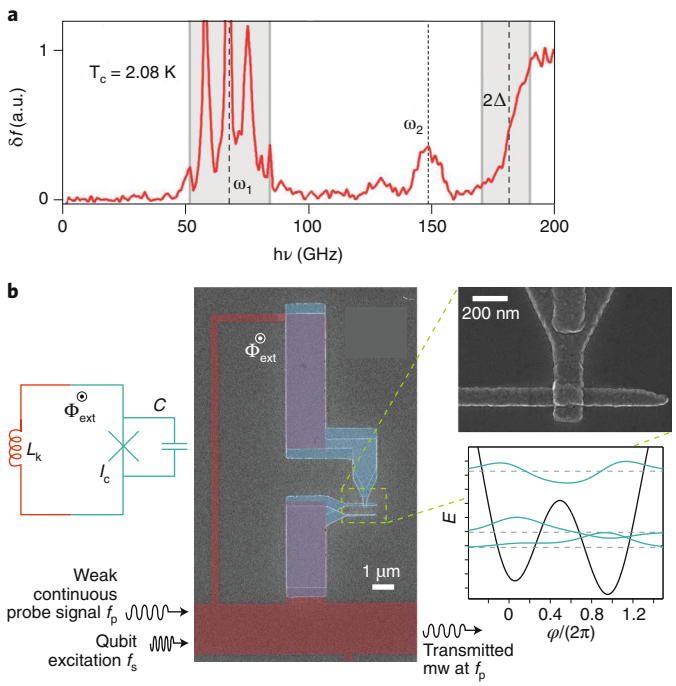

Fig. 5 | Examples of applications of high microwave kinetic inductance materials. a, Typical absorption spectrum of a superconducting resonator made from granular aluminum in use for astronomical detectors and as superinductor, showing the conventional onset of absorption at 2Δ and subgap resonances[112], identified as two distinct groups of absorption lines labeled as ω_1 and ω_2. These modes are most likely collective modes coupled to the phase differences. The signal is the shift of the resonant frequency in the 2–6 GHz range due to the absorption of the radiation in the frequency range from 10–200 GHz. **b**, Hybrid RF SQUID used as a qubit and consisting of an Al SIS superconducting tunnel junction, shunted by a high kinetic inductance loop made of TiN. Figure reproduced with permission from: **a**, ref. [112], APS; **b**, ref. [106], Springer Nature Ltd, under CC-BY-4.0.

ment of T_c due to fractality was addressed by means of the renormalization group approach[53,54] in the 2D limit. It was shown that the partial suppression of the (screened) Coulomb amplitude makes the enhancement of T_c by disorder possible in two dimensions as well.

The results of refs. [48,49,53,54] were obtained within a generalized mean-field approximation that neglects thermal phase fluctuations[12]. However, it was shown in ref. [49] that the intensity of the phase fluctuations is only moderate for superconductivity at the mobility edge, so the major conclusion about the T_c enhancement is valid. On the other hand, phase fluctuations do become crucial and destroy superconductivity when the Fermi level moves sufficiently far into the Anderson-localized band.

Another important feature of superconductivity near the mobility edge is the highly inhomogeneous superconducting order parameter. The dispersion $\sqrt{\langle\Delta^2(\mathbf{r})\rangle}$ of the order parameter $\Delta(\mathbf{r})$ is much larger[49] than its mean value $\langle\Delta(\mathbf{r})\rangle$, leading to a picture of a superconducting state splintered into superconducting islands. Such a strongly inhomogeneous superconductivity is a direct consequence of the fact that fractal wavefunctions occupy only a small fraction of the available volume (Fig. 2b). A direct consequence of the inhomogeneity of $\Delta(\mathbf{r})$ is a strong vortex pinning and measurable critical currents even extremely close to the upper critical field[55].

Emergent granularity within the amplitude pathway. The theoretical approaches reviewed above predicted the emergence of an inhomogeneous superconducting state in the strong disorder limit. However, these theories do not include the unavoidable Coulomb interaction that affects the attractive coupling via Finkel'stein's mechanism. Solving the combined effect of Anderson localization and Coulomb interaction is known to be a notoriously difficult task that has not yet been accomplished. Nevertheless, some progress[39] was made on the perturbative level by including quantum interference effects—the universal conductance fluctuations—to Finkel'stein's theory of the SMT. This extended Finkel'stein theory predicts significant spatial fluctuations of the local transition temperature on approaching the quantum critical point of the theory, g_c, due to the disorder enhancement of the mesoscopic fluctuations of the effective Coulomb amplitude. The fluctuations of T_c are given by $\frac{\delta T_c}{T_c} \approx \frac{0.4}{g(g-g_c)}$, which shows that these fluctuations necessarily become strong upon approaching the quantum critical point. This is illustrated by the shaded region around the mean T_c in Fig. 1b.

Intermediate conclusion regarding the theory. The major outcome of these recent theoretical developments is a clear breakdown of the conventional dichotomy between the amplitude- and phase-driven pathways. An initial normal state with homogeneous electronic properties can yield an inhomogeneous superconducting state whose spatial fluctuations get enhanced in the vicinity of the critical disorder. This formation of superconducting islands immersed in a non-superconducting matrix that would be metallic in the Finkel'stein SMT scenario and insulating in the strong disorder limit, intimately involves phase fluctuations between the weakly-coupled superconducting islands. A new scenario has emerged that involves a remarkably complex and subtle interplay between localization phenomena (effect of disorder and multifractality), Coulomb interaction and phase fluctuations. The breadth of consequences of these superconducting inhomogeneities on the transport and thermodynamic properties is not yet understood and will certainly produce new conceptual advances. In addition, it makes the system sensitive to external conditions in a variety of experiments.

We briefly mention a few more theoretical consequences. The spontaneous formation of superconducting islands leads to an enhancement[56] of the electron dephasing rate $1/\tau_\phi$ at low temperatures. Andreev reflections of electrons between the superconducting islands (with uncorrelated fluctuations of their phases) become the dominant mechanism of decoherence[56], far exceeding the usual Coulomb contribution $1/\tau_\phi^C \sim T/g$. This has implications for the possible existence of some anomalous metallic phases[9]. Under a perpendicular magnetic field, the self-induced inhomogeneities are predicted to induce multiple reentrant superconducting phases above the upper critical field in mesoscopic samples[57]. More generally, disorder-induced inhomogeneities lead[58] to the breakdown of the scaling theory[59] of the quantum SMT, which states that large-scale superconducting fluctuations are irrelevant due to long-range of proximity coupling via the metal matrix.

Real-space visualization of superconducting inhomogeneities. In experiments, a continuous reduction of T_c with increasing sheet resistance, consistent with Finkel'stein's theory, has long been the hallmark of structurally homogenous materials[15,60], and, consequently, assumed homogeneous superconductivity. Progress in low-temperature scanning tunnelling microscopy means that the local density of states can be measured with sub-kelvin resolution. These tunnelling experiments have shed new light on this assumed homogeneity and have revealed an emergent granularity of superconductivity on the local scale.

In 2008, Sacépé et al.[26] reported the first scanning tunnelling spectroscopy of thin superconducting films near the QBS. They studied TiN films, which exhibit a QBS[23] with a continuous reduction of T_c and a high critical disorder of the order of 30kΩ. The spectra of the local tunnelling density of states revealed significant spatial fluctuations of the superconducting gap $\Delta(\mathbf{r})$ on the scale of

tens of nanometers. Figure 2c shows a typical spatial map of $\Delta(\mathbf{r})$ measured at 0.05K. Upon approaching the critical disorder, these fluctuations were shown to increase from $\delta\Delta(\mathbf{r})/\Delta \simeq 0.15$ (Δ is here the average superconducting gap) for an intermediate disorder ($T_c \simeq 0.3T_{c0}$) up to $\delta\Delta(\mathbf{r})/\Delta \simeq 0.5$ for a nearly critical sample with $T_c \simeq 0.1T_{c0}$. Similar results were obtained in a series of studies by Raychaudhuri and co-workers on thick NbN films[30,43] and by Roditchev and co-workers[32,61,62] on ultra-thin NbN films where the initial suppression of T_c with disorder also seems to follow the amplitude pathway. Cabrillet et al.[61] combined topography and scanning tunnelling spectroscopy data to demonstrate the absence of any spatial correlation between small-scale structural grains and larger-scale fluctuations of the gap $\Delta(\mathbf{r})$. In a more recent work, Cabrillet et al.[62] also showed a clear anti-correlation between the width of the gap and the slope of the high-voltage anomaly in the tunnelling conductance, attributed to the 'soft Coulomb gap'[63]. This anti-correlation can be accounted for because regions with larger local resistance are expected to have smaller $\Delta(\mathbf{r})$ due to the disorder-enhanced Coulomb effects[4,5]. Measurements in a magnetic field add an additional perspective tothis picture. Ganguly et al.[33] showed that NbN films far from the QBSthat exhibit uniform superconducting properties at $B = 0$ develop strong spatial inhomogeneities under perpendicular magnetic field values of 4 to 7.5 T, in agreement with theory[57,64].

In general, the experiments on both TiN (ref. [26]) and NbN (refs. [32,33,43,61,62]) films demonstrate the same trend. An increase of the sheet resistance close to $R_Q = h/4e^2$ and a related suppression of T_c are systematically accompanied by an increase of the gap fluctuations $\Delta(\mathbf{r})$. Furthermore, the coherence peaks also fluctuate spatially and provide a measure of the amplitude of the local order parameter[29,43].

The disorder-induced enhancement of the superconducting gap inhomogeneities is accompanied by a significant increase of the ratio Δ/T_c in TiN films (ref. [26]), NbN (ref. [28]) and in MoGe (ref. [65]), upon approaching the critical disorder. In TiN films, it grows up to a value of 4, far above the weak-coupling value of 1.76. This indicates a serious deviation from the standard ratio of the BCS theory. The same evolution is seen in a:InO films that are much thicker than the superconducting coherence length[29,66]. In this case the ratio of the spectral gap to T_c grows from 2.5 to 5.5 when T_c is reduced from 3.5 K to 1.2 K by increasing disorder.

A remarkable consequence of the increase of Δ/T_c with disorder is the non-vanishing spectral gap in nearly critical samples in spite of the enhanced spatial fluctuations. The anomalously large and increasing Δ/T_c ratio indicates that, at the critical disorder defined by $T_c = 0$, the spectral gap remains finite and potentially persists into the insulator phase[26,31]. This behaviour would be consistent with the prediction of a gapped insulating phase (Fig. 3d) in the disordered attractive Hubbard model[36,38,41] discussed in the previous section. Furthermore, as shown in a:InO[29], locations with a vanishing local order parameter—evidenced by a lack of coherence peaks—remain fully gapped, although other materials have sub-gap states[26,30,32,43,62,67]. The origin of the sub-gap states in nearly critical films with very low T_c remains unclear but could be the result of pair breaking due to interactions[68], or more trivially limited by the energetic resolution of tunnelling spectroscopy, which is known to be notoriously sensitive to the filtering of the electromagnetic environment[69,70].

Recently, Zhao et al. used low-temperature scanning tunnelling spectroscopy to investigate an epitaxial monolayer of $NbSe_2$ on which disorder was controlled through in situ adatom deposition prior to tunnelling measurements[34]. They observed an initial increase of the superconducting gap with disorder, followed by a sharp drop, as shown in Fig. 2d. This may possibly constitute evidence for enhancement of 2D superconductivity by disorder[53], although additional checks for alternative mechanisms are needed.

These scanning tunnelling experiments provided compelling evidence for the emergent superconducting granularity in homogeneously disordered materials. The new picture of the superconducting state in the vicinity of the QBS is that of superconducting puddles embedded in a matrix with vanishing local order parameter, a matrix that can be gapped or gapless in case of a SIT or SMT, respectively. Consequently, quantum phase fluctuations in such weakly coupled puddles will definitely be crucial in the ultimate suppression of superconductivity. These superconducting inhomogeneities that emerge near criticality lead us to conclude that there is a progressive evolution from amplitude-driven to phase-driven mechanisms of QBS upon increasing disorder, at least in TiN, NbN and a:InO, the materials investigated so far.

Preformed pairs and their localization at the QBS

In strongly disordered superconductors one can not a priori assume that the attractive interaction leading to Cooper pair formation is uniformly spread through the material. One possibility is that the attractive interaction fluctuates from one point to another with a significant amplitude so that the Cooper pairs are formed locally. As this situation does not necessarily lead to long range coherence, these pairs are called 'preformed pairs', anticipating the development of a state of superconductivity by phase coherence between these localised pairs. Such a theoretical possibility, which is obviously difficult to identify based on transport measurements, has recently received experimental support from local spectroscopy data. A suppression of the quasiparticle tunnelling density of states was observed at temperatures far above the superconducting transition temperature, in strongly disordered TiN thin films[27]. A similar suppression is routinely observed in underdoped cuprate superconductors and labeled a 'pseudo-gap'[71]. Assuming that this pseudo-gap (PG) is related to superconductivity, one is led to conclude that on cooling down, first the preformed pairs of electrons appear, and then they become coherent at much lower temperatures, at the critical temperature T_c. An early theoretical analysis of the tunnel current above T_c in the regime of superconducting fluctuations was carried out by Varlamov and Dorin[72], and showed an apparent pseudo-gap formation.

The presence of preformed electron pairs in a conventional low-T_c superconductor with strong disorder was suggested by experiments of Sacépé et al.[29] in scanning tunnelling measurements on a:InO films. A nearly full-width tunnelling gap, Δ_{PG}, was visible at the temperature where zero resistance develops, defined as T_c, while a considerable suppression of the density of states was detected up to temperatures of $4T_c$. Right below T_c coherence peaks developed near the edges of the gap at a number of locations of the investigated area, while at other points no coherence peaks are found down to the lowest $T \ll T_c$. These observations, summarized in Fig. 3a,b, stimulated the theory of the QBS, to be discussed below. Similar pseudo-gap-type features were reported in subsequent experiments on a:InO, by Sherman et al.[31], on NbN, by Mondal et al.[28], Chand et al.[30], and Carbillet et al.[61,62], and on MoGe by Mandal et al.[73]. In all cases, the observations are carried out on films with a T_c strongly suppressed by disorder.

In interpreting these experiments, it is reasonable to assume that a relatively large tunnelling gap is related to electron pairing, but it cannot serve as a direct measure of the superconducting coherence. In addition, the value is too large in comparison to superconducting transition temperature T_c to be understood by the BCS-type theory or its extensions like the Eliashberg theory. Eventually, numerical work showed[38,41] that the single-particle gap survives into the range of strong disorder, without the typical indications of superconducting coherence. An alternative interpretation of the data has been suggested, where the suppression of the tunnelling density-of-states near the Fermi-energy is due to the dynamical Coulomb blockade[63,74]. However, in low-T_c disordered superconductors, it is experimentally not difficult to distinguish between dynamical Coulomb blockade and superconductivity-related pseudo-gap, due to the large difference in relevant energy scales: about 1 meV for

pseudo-gap and on the order of 0.1 meV for dynamical Coulomb blockade. An example of a joint analysis of dynamical Coulomb blockade together with the Finkel'stein mechanism of superconductivity suppression has recently been discussed by Carbillet et al.[62]. We believe that a theory based on preformed Cooper pairs is the most plausible candidate.

We proceed by addressing the formation of a collective gap for the condensation of the preformed Cooper pairs. Within BCS theory, the collective gap $\Delta_{col} = 2\Delta_{BCS}$, corresponds to the minimal excitation energy above the superconductive ground state, if single-particle excitations are forbidden. Experimentally, as proposed by Deutscher[75,76], the collective gap of cuprate superconductors can be measured by point contact spectroscopy using Andreev reflection. The voltage threshold for two-electron transfer, $2eV_{col} = 2\Delta_{col}$, which coincides with the threshold for single-electron tunnelling $eV = \Delta_{BCS}$, if the BCS relation is valid.

A similar idea was implemented recently by Dubouchet et al.[66] for a:InO close to the QBS. The point-contact differential conductance dI/dV was measured starting from the purely tunnelling regime to a highly transmissive regime with a contact resistance of a few kΩ, at which both single-electron and Andreev processes are relevant. In the highly-transmissive regime, additional peaks emerged in dI/dV (Fig. 3e) at low voltages, $eV_{col} \ll \Delta_{PG}$, and only in the superconducting state. Moreover, V_{col} was found to be weakly dependent on the tip location, but strongly temperature-dependent, similar to the $\Delta_{BCS}(T)$ dependence. Both features are in contrast with the behavior of the single-particle threshold Δ_{PG}. The magnitude of $\Delta_{col} \equiv eV_{col}$ was interpreted as a genuine collective gap that develops together with the macroscopic superconducting coherence. This type of behaviour was found only for strongly disordered a:InO films. In contrast, less disordered films (with $T_c \geq 3$K) demonstrate single-gap BCS-type behaviour without any pseudo-gap feature.

These observations were analyzed by invoking the notion of the 'parity gap', Δ_{par}, introduced in the seminal paper of Matveev and Larkin[77], who developed the theory for an ultra-small superconducting grain. For sufficiently small grains the single-electron level spacing $\delta_1 = 1/\nu\mathcal{V}$ (ν the density-of-states per unit of volume and $\mathcal{V}$ the volume of the grain) exceeds the value of the superconducting gap Δ_0 of the bulk of the material. The level spacing δ_1 does not exceed the Debye energy. The condition $\delta_1 \gg \Delta_0$ prevents the formation of many-body coherence, of the numerous single-electron states. However two electrons, residing in the same localized orbital state with opposite spins, still attract each other and gain some energy $\Delta_{par} \propto \delta_1$, with respect to the case when those electrons populate different localized orbitals.

We implement the idea of the parity gap to describe the pseudo-gap in strongly disordered bulk (or 2D) materials following Ma and Lee.[45], Ghosal et al.[38], and Feigelman et al.[48,49]. We assume, going beyond the theory described in Section IV, that the Fermi level for non-interacting electrons lies in the localized part of the band, but close to the mobility edge E_c. The single-electron eigenfunctions are localized, with relatively long localization length $L_c \sim l[(E_F - E_c)/E_c]^{-\nu}$, where the exponent $\nu \approx 1.5$ in 3D. Then the matrix elements $P_i = \int d\mathbf{r}\psi_i^4(\mathbf{r})$ are non-zero in the thermodynamic limit and scale as $P_i \propto L_c^{-d_2} \sim [(E_F - E_c)/E_c]^{\nu d_2}$, due to the fractal nature of the electron eigenfunctions. The parity gap due to the local attraction between two electrons $\Delta_{par} = \lambda/\nu P_i$ scales in the same way. This situation, similar to the one for ultra-small grains[77], is realized when the single-electron spacing $\delta_1 = (\nu L_c^d)^{-1}$ far exceeds the transition temperature T_{c0} for the critical disorder ($L_c \to \infty$), see equation (2). Then it is possible to show[49] that the value of the parity gap Δ_{par} lies between two other energy scales:

$$T_{c0} \ll \Delta_{par} \sim \delta_1 \left(\frac{T_{c0}}{\delta_1}\right)^{\gamma} \ll \delta_1 \qquad (3)$$

When such a bulk disordered system, with a tendency to electron pairing, does not develop superconducting coherence, the parity gap is seen both as a pseudo-gap in tunnelling experiments, and as an activation gap in transport.

This physically rich theoretical model was developed to reach an understanding of the experimental data on a:InO (refs.[78,79]). Large activation gaps, up to 10–15 K, were reported in electronic transport measurements. It was noted by Shahar and Ovadyahu[78] that the experimental data rules out conventional scaling of the activation gap $\propto \delta_1$. Feigelman et al.[49] argued that the same data are consistent with a modified scaling of Δ_{par} presented in equation (3). Superconductivity will coexist with a parity gap when the ratio of δ_1/T_{c0} is not too large. The very presence of a solution with nonzero T_c much smaller than both Δ_{par} and δ_1 is a new feature of the theory of ref. [49] that was not anticipated in the original approach by Ma and Lee[45]. It is due to two effects that enhance the overlap matrix elements M_{ij}: the fractal nature of the eigenfunctions, and the Mott resonances[80] between localized eigenstates with a small energy difference $\omega \ll \delta_1$. The collective gap $\Delta_{col}(T)$ at $T \ll T_c$ appears to be of the same order as T_c, and much less than Δ_{par}. An additional feature expected for the pseudo-gapped superconductors is the violation of the usual BCS rule that the full optical spectral weight is insensitive to superconducting transition[49]. Upon a further increase of disorder and of the level spacing δ_1, the transition temperature T_c and the collective gap Δ_{col} gets smaller and eventually vanishes, while Δ_{PG} stays nonzero[41,49]. The resulting ground state is an insulator with preformed electron pairs.

For a system with preformed electron pairs, close to the QBS, the values for both T_c and Δ_{col} are much smaller than the pseudogap Δ_{PG}. Therefore, this transition can be understood in terms of the Anderson pseudo-spin model[81], which describes hopping of the preformed pairs between different localized orbitals $\psi_i(\mathbf{r})$. This specific case of the QBS can be called a 'pseudo-spin' QBS. The hopping terms in the effective Hamiltonian are given by $H_{hop} = -\sum_{ij} J_{ij}(S_i^+ S_j^- + \text{h.c.})$ where $S_i^{\pm}$ are the creation and annihilation operators of a preformed pair in the ith orbital. These hopping terms compete with the random on-site potential energy given by $H_{loc} = \sum_i 2\xi_i S_i^z$. The preformed pairs are just hard-core bosons, and hence the operator set S_i^z, S_i^+, S_i^- is formally equivalent to the spin-$\frac{1}{2}$ operators $\mathbf{S}_i$. The superconducting state is then described by the non-vanishing quantum-statistical average values such as $\Delta_i = \langle S_i^- \rangle$, while in the insulating state all $\Delta_i \equiv 0$ and the preformed pairs are localized.

This pseudo-spin QBS appears to be in the same universality class as the order-disorder transition in the quantum XY spin-$\frac{1}{2}$ model with random transverse fields. The transition is controlled by the value of the effective coupling strength $J \sim ZJ_{ij}$, with Z the typical number of connections per 'spin'. The theory worked out by Mézard, Ioffe and Feigelman[82] leads to the following conclusions:

- A $T = 0$ transition between superconducting and insulating states occurs at some critical value of the coupling strength J_c.
- At $J > J_c$, the superconducting state exists below a critical temperature $T_c(J)$, whose magnitude drops very sharply to zero when $J \to J_c$.
- In a wide range of $J > J_c$, the ordered state is extremely inhomogeneous, with a very broad probability distribution $\mathcal{P}(\Delta_i)$. Subsequent numerical studies for a 2D model by Lemarié et al.[43] have confirmed this result.
- A typical value of the order parameter $\Delta_{typ} = \exp(\overline{\ln(\Delta_i)})$ demonstrates an unusual exponential scaling near the $T = 0$ critical point: $-\ln\Delta_{typ} \propto (J - J_c)^{-1}$. This type of scaling was found earlier by Carpentier and Le Doussal[83] for the disorder-driven Berezinskii–Kosterlitz–Thouless transition.

We would like to draw the attention to several unusual and important predictions of this pseudo-spin QBS. Firstly, the inhomo-

geneity of all superconducting properties close to the pseudo-spin QBS is much stronger than the emergent granularity in the bulk of the superconducting region. Moreover, the former develops at much larger spatial scale, which diverges close to the QBS. Eventually, the pseudo-spin QBS acquires superconducting features which are reminiscent of the classical percolation transition. This aspect might provide a framework to understand the unexpected size effects found near the QBS by Kowal and Ovadyahuv[84], as well the unusual behavior of the Nernst coefficient at $T \gg T_c$ in near-critical a:InO films[85,86].

The mechanism we have described for the pseudo-spin QBS has one common feature with the 'bosonic' scenario. In essense, hard-core bosons defined on a lattice are formally equivalent to spin-1/2 spins sitting on lattice sites. The crucial physical difference is that the pseudo-spin scenario does not assume any local superconducting order with many-body correlations unless global superconductivity has developed. In particular, no trace of the coherence peaks is expected in the insulating state, in contrast to the standard bosonic scenario based on the Josephson tunnel-junction model: The superconducting grain of a size large enough to have $\delta_1 \ll \Delta$ would show smeared coherence peaks in the tunnelling conductance on either side of the QBS.

Another important feature of the pseudo-spin scenario for the QBS is that disorder plays a crucial role, unlike the commonly used Coulomb-blockade, Josephson tunnel-junction model. The pseudo-spin QBS model we describe here is in that sense similar to the infinite-disorder renormalization group theory[87,88]. The strong disorder effects and the need to account for the whole probability distribution of the relevant variable[83,89] make the use of an elementary scaling analysis problematic near such phase transitions. This might be the reason for the broad range of critical exponents found in the literature.

Mesoscopic approach to inhomogeneous superconducting materials

A major research direction in the past few decades has been on networks of superconducting islands coupled by tunnel barriers through which Josephson tunnelling can occur[8]. The main emphasis was on an analysis based on Berezinskii–Kosterlitz–Thouless physics, the competition between Coulomb blockade and Josephson coupling, and the magnetic field dependence expressed as the number of flux quanta per closed loop of the network. They have been also used as model systems to understand the QBS in real materials. While these systems are useful, these tunnel-junction-based systems ignore two important ingredients that are relevant for real materials. One ingredient is the use of large superconducting islands with a well-defined macroscopic quantum phase. In practice, much smaller pockets of superconductivity with a small number of electrons with a poorly defined phase may occur. Secondly, the coupling between the islands might in practice be much more transmissive than for a tunnel barrier, either because of a few transmissive quantum channels or by a diffusive proximity effect. These arrays were first made several decades ago[90] and were recently addressed, with better lithography, in the work by Eley et al.[91]. In the latter work a series of samples was studied with different coupling strengths between the islands, due to a variation in distance between the superconducting islands. The analysis was, as in the earlier work, based on the thermal Berezinskii–Kosterlitz–Thouless phase transition, assuming a Ginzburg–Landau proximity-effect description, with the coupling strength varying exponentially with distance.

The discovery of tunable 2D materials such as graphene in combination with superconductors makes it possible to construct a Josephson arrays where the coupling strength can be tuned in situ. This has been realized in a recent experiment by Han et al.[92]. A layer of graphene was decorated with a triangular array of circular tin disks, each with a diameter of 400 nm and a mutual distance between the centers of the disks of 1 μm. The total number of disks covered an area of 5 × 10 μm^2. The sample was equipped with a gate that allowed the tuning of the carrier density in the graphene, which meant that the Josephson coupling could be tuned. The conductivity of the array could be tuned from a full superconducting state through an 'anomalous metal' phase, going over to an approximate insulating phase, with diffusive transport in the graphene. The data as a function of gate voltage for different magnetic fields and temperatures clearly resemble (Fig. 4) the phase diagram laid out by M. Fisher[6]. These results are already interesting and important as such, showing the importance of quantum phase fluctuations in Berezinskii–Kosterlitz–Thouless physics, but the experimental method allows for several other routes to be explored. In principle, one could consider smaller islands in order to get the islands with an energy spacing for the electrons larger than the superconducting energy gap, although this might require the use of superconductors with a low carrier density. A second route is to use ballistic graphene, which is in principle possible by encapsulating the graphene in exfoliated boron nitride. Thirdly, the small scale of the array, about 20 sites, may lead to a contribution from finite-size effects on the observed quantum Berezinskii-Kosterlitz-Thouless-like transition. These need to be evaluated by making arrays with more sites.

Bøttcher et al.[93] carried out a similar experiment using a rectangular array of square-shaped superconducting aluminium islands deposited on a tunable two-dimensional electron gas in the semiconductor InGaAs system, using the materials systems developed for Majorana physics. The published results are analogous to data presented in earlier work on other 2D arrays and analyzed by the conventional scaling analysis[6]. In principle this experimental system has the potential to benefit from insights in mesoscopic superconductivity. The Andreev reflection process which mediates the phase coupling between the superconducting islands has been thoroughly analyzed by Kjaergaard et al.[94]. The critical new step is to integrate the understanding of the proximity effect from the perspective of Andreev processes at the interface between the normal metal and the superconductor. This knowledge is well-developed and continues to be tested in various hybrid systems. The important ingredient of the study of Josephson-coupled arrays is the fate of the quantum phase of each superconducting island. Further research, with considerably smaller superconducting islands appears to be within reach.

The emphasis on the tunability of the Josephson coupling energy in controlling the macroscopic transport properties of the arrays highlights the need for a full understanding of the dependence on environmental noise. Martinis and co-workers[70,95] have addressed this subject in the context of macroscopic quantum tunnelling. Additionally, the thermal blackbody radiation was found to contribute significantly to the performance of superconducting quantum circuits[96]. A reminder of the importance of the sensitivity to environmental radiation for materials research was shown in recent experiments by Tamir et al.[10], which showed it in the resistive transition of a:InO, but also of a crystalline 2D superconductor, H2-$NbSe_2$. A similar dependence was found recently by Dutta et al.[11] in the onset of resistance in the vortex state of a-MoGe films. It is a reminder that experiments on 'weak' superconductors, on the verge of quantum breakdown, need to be carried out in an electromagnetically well-shielded cryogenic environment, common for mesoscopic research but not typical for materials research.

Applications to quantum circuits and qubits

One of the most interesting and fascinating potential applications of strongly disordered superconductors is the superinductor. A superinductor is a non-dissipative element of an electrical circuit with an impedance that depends on the capacitance C and inductance L as $Z = \sqrt{(L/C)}$, and is much larger than the resistance quantum $R_Q = h/(2e)^2$. The inductance of a superconducting wire consists of the geometric inductance, which stores the energy of the

electromagnetic field, and the kinetic inductance of the superconducting condensate, which stores the kinetic energy of the moving superfluid. The geometric inductance scales with the length of the wire and depends only logarithmically on the diameter of the wire. Its value is typically 1 pH μm^{-1}, which makes the geometric inductance unusable, a problem well known in for example RF electronics[97]. The alternative is to exploit the kinetic inductance of a low-dimensional material such as graphene[97], which for normal conduction leads to high losses and hence low quality factor if used in a resonator. The low loss can be provided by a superconductor of which the kinetic inductance is in general given by $L_k = \hbar R_n/\pi\Delta$, which means that a high normal state resistivity and a low superfluid density can be used to maximize the kinetic inductance. Ideally, one would like to achieve at least an inductance per unit length of 0.1 nH μm^{-1}, which implies that films with $0.5 - 2.5$nH □ are required. With such a superinductor one can construct protected qubits reviewed by Doucot and Ioffe[98], with new ideas put forward by Brooks et al.[99], Groszkowski et al.[100] and Smith et al.[101].

Motivated by these ideas about protected qubits and another proposal put forth by Mooij, Nazarov and Harmans[102,103] on quantum phase slip junctions, Astafiev and co-workers have studied the quantum coherence of superconducting nanowires for a range of materials like a:InO (ref. [104]), NbN (ref. [105]), and TiN (refs. [106,107]). They considered coherent quantum phase slip processes, rather than thermally activated phase slips. The critical quantity is the transition amplitude for quantum phase slips, expressed in energy as E_s, reaching values in the 100 GHz range. In addition, one needs an impedance of the environment larger than the quantum unit of resistance of 6.45 kΩ. The scale of the inductive energy E_L is given by ${\Phi_0}^2/2L$ with L the kinetic inductance of the wire. In order to reach $E_s \gg E_L$ one needs superconducting materials with a high normal state resistivity, such as strongly disordered superconductors, which are central in this review (Fig. 5b).

An alternative experimental strategy is the work of Kuzmin et al.[108]. Instead of a model system to study the breakdown of superconductivity they constructed a model system in which they could experimentally focus on the collective electromagnetic phase mode. They created a linear chain of 40,000 Josephson tunnel junctions and used the device to study the collective modes of microwave photons. These modes have a velocity as low as $v \approx 10^6$ m s^{-1} and a wave impedance as high as $Z \approx R_Q$, with $R_Q = h/(2e)^2 \approx 6.5$kΩ, the superconducting resistance quantum. A SIT is obtained by changing the area of the junction[108], which leads to a smaller area in which the charge can be quantised on the superconducting island, and hence to strong fluctuations of the quantum phase on the islands, visible, for example, as a breakdown of the collective modes.

In the search for suitable materials, thin films of granular aluminium (GrAl) are being explored as a possible candidate[109–111]. This subclass of materials has been known for a long time to be easily accessible, but at the same time difficult to understand. The material is made by depositing aluminium in a partial pressure of oxygen leading to an assembly of small aluminium grains, surrounded by an AlO_x layer, hence the name 'granular'. For increasing resistivity the critical temperature increases up to a large factor of 3 to 4, an observation that has so far defied a convincing explanation. A number of promising properties have been measured, at the same time some detrimental properties have also been identified, including the microwave properties[112] (Fig. 5a). The same material is also studied by Gershenson and co-workers[113,114]. The parameter details for granular aluminium are not very well known. It is possible that they can be considered as random arrays of tunnel junctions, small aluminium particles surrounded by an oxide tunnel barrier, which connects the aluminium grains. The difference from the experiment by Kuzmin et al.[108], as well as in the earlier work of Fazio et al.[8] is that the aluminium islands in granular aluminum are very small. The size of the grains may lead to a situation in which energy level spacing exceeds the pairing gap, which may be important for the understanding of the results reported by Lévy-Bertrand et al.[112].

The experimental work described in the previous paragraphs makes clear that an ideal superinductor does require thoughtful experimental work. At the same time, the research on the QBS has uncovered the various ways in which superconductors with high normal state resistivity lose their superconducting properties. These pathways take into account the nature of the coupling of the superconducting phases, as well as the possibility of localized pairs and the pairing strength. With the ultimate goal of lossless inductors the materials requirements have been discussed in recent publications by Feigel'man and Ioffe[115,116]. Many challenges towards a usable superinductor are ahead of us and will need to be compared with the use of arrays of tunnel junctions aiming for the same functionality[108].

Open problems and conclusions

The key challenge for understanding superconductivity is that a macroscopic quantum state emerges from or gets destroyed by microscopic quantum properties. In this Review we have identified several pathways for this process, all of which may occur in real materials. Since our interest is not so much in one specific model system, nor in just plausible interpretations of experiments, but in a description of real world materials, the additional challenge is to identify which pathway is most likely applicable to a specific material system, while at the same time acknowledging that perhaps there are other pathways that have not been identified or properly solved yet.

To close, we would like to highlight several experimental observations have not yet been understood:

- For homogeneously disordered superconductors, assessing an effective Coulomb interaction is conceptually difficult: The absence of grain boundaries makes it difficult to define an equivalent capacitance that determines charging effects in tunnel junctions and the ensuing phase fluctuations. Therefore, addressing quantum phase fluctuations in superconductors on the basis of Josephson junction models remains purely qualitative and should not disguise the fact that exact role of Coulomb interaction on phase fluctuations in disordered superconductors is an open question.
- In the vicinity of a SMT, a strange metal state is frequently observed[9]. Do these observations indicate the existence of an intrinsic equilibrium quantum state, which may result from the intricate interplay between Cooper pairing, Coulomb repulsion and Anderson localization? Or are these observations in need of a careful reexamination of the experimental conditions under which the data have been taken, because of a possible coupling to the non-equilibrium environment?
- The experimentally observed high sensitivity[10,11] to radio-wave interference, as known for Josephson junctions with either an insulating or normal metal weak link[70], calls for an analysis of the effect of microwaves on the study of superconducting materials, which are expected to be spatially inhomogeneous. Both the common Josephson response due to a local voltage difference that affects the local phase differences and the non-equilibrium effects that change the Josephson coupling should be considered.
- For the magnetic-field-driven QBS in moderately disordered materials, the nature of the strongly pinned vortex glass in the $T = 0$ limit is still unclear[55]. The theoretical problem is to describe a vortex glass with a macroscopic superfluid density ρ_s. In the strong disorder limit, a combined account of the localization of preformed pairs and Coulomb interaction is still to be developed.
- Can one avoid the low-energy collective modes in pseudogap superconductors? This is an issue of practical importance for the

development of various quantum circuits where a high kinetic inductance proportional to $1/\rho_s$ should be combined with an absence of dissipation at frequencies in the sub-gigahertz range.

- The giant magnetoresistance peak terminating superconductivity in some materials (a:InO, refs. [117–119]; TiN, ref. [120]; nano-patterned a-Bi, refs. [121,122]; proximitized graphene, ref. [22]) continues to defy understanding. New experimental approaches must be employed to probe this insulating state beyond transport measurements that have proven to be limited by non-equilibrium effects[123].
- On the insulating side of a SIT we may encounter a many-body localized state, with extremely weak electron–phonon coupling[124] and conductivity vanishing at some nonzero temperature[125].
- Lastly, understanding which microscopic parameters and phenomena define the metallic-like or insulating nature of the ground state terminating superconductivity for given material, that is, SMT versus SIT, remains a major question.

Future research is expected to focus on the interaction between the three themes addressed in this review: theory, experiments and applications. Experiments unavoidably will have to focus on model systems, which allow them to evaluate identified theoretical pathways. Applications have the virtue of providing large amounts of experimental data, assuming they have been taken under relevant experimental circumstances and with a good knowledge of the details of how the system interacts with its environment. From the latter point of view the increased use of strongly disordered superconducting materials for astronomical instrumentation and quantum computation holds the potential of providing data that are significant for evaluation based on the available theoretical models.

Finally, we expect that the subject of quantum breakdown of superconductivity will benefit from the increasing availability of gatable 2D superconducting systems[126–128] that constitute well-defined atomically uniform model systems in which specific pathways could be tested in much detail.

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Competing interests

The authors declare no competing interests. Q17

Additional information

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Q8:	In the sentence beginning 'Andreev reflections', please define the superscript C.
Q9:	In the sentence beginning 'However, in low', 'tenths of meV' was changed to 'on the order of 0.1 meV', is this okay?
Q10:	In the sentence beginning 'We assume, going', what does 'Section IV' refer to? If this is an earlier section of this paper, could this be replaced by 'above', or the name of the section?
Q11:	In the sentence beginning 'When such a bulk disordered system', what is meant by 'with a tendency to electron pairing'? Could this be reworded as 'with a tendency to undergo electron pairing' or similar, for clarity?
Q12:	In the sentence beginning 'The hopping terms', please define h.c. Is this the Hermitian conjugate?
Q13:	In the sentence beginning 'The total number of disks', please confirm that the units are correct; they were changed

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	from μm to μm^2.
Q14:	In the sentence beginning 'The low loss', please define the variable R_n and Δ. Are these the 'normal state resistivity' and 'superfluid density' as mentioned in the sentence? Please also confirm the units used in the sentence that follows.
Q15:	Fig. 5: Please provide definitions for \delta f, f_p, f_s, h \nu, \Phi_ext, E and \phi.
Q16:	In the sentence beginning 'This is an issue', please define \rho_s.
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