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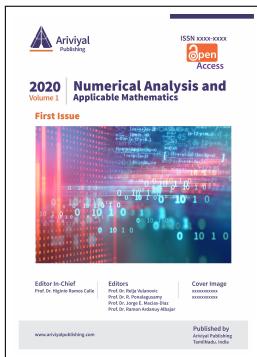
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## Study of Differential Equations with Exponential Nonlinearities via the Lower and Upper Solutions' Method

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**Abstract:** We present original results in study of the second-order differential equation with exponential nonlinearities, subjected to the Dirichlet boundary conditions. Using the proper substitution techniques, we reduce the given problem to the study of its lower and upper solutions.

**Keywords:** lower and upper solutions; nonlinear boundary-value problem; monotone method; exponential nonlinearity; Dirichlet boundary conditions

## 1. Introduction

In the classical theory of ordinary (ODEs), partial(PDEs) and fractional differential equations (FDEs), an essential attention is paid to the study of existence and/or uniqueness of solutions and their analytical constructions. Due to the applied character of some of these problems, e.g. in economics (see<sup>[14,15]</sup>), geophysics (see discussions in<sup>[5,6,16-23]</sup>) etc., it is important to develop precise tools not only to carry out the qualitative analysis, but also to represent solutions of the studied problems explicitly.

Even though there is a big variety of numerical methods that may be applied to the nonlinear equations,<sup>[7-9]</sup> they don't give a full picture about the solution. More helpful and representative may be the so-called numerical-analytic methods, widely used in study of the nonlinear boundary value problems (BVPs) for systems of ODEs (see discussions in<sup>[25-27]</sup>) and recently applied to the FDEs, subjected to periodic,<sup>[13-15]</sup> antiperiodic<sup>[21]</sup> and two-point boundary conditions.

On the other hand, the monotone iterative method, also well known as the lower and upper solutions' method (or sometimes referred to as sub- and supersolution's method), is a good tool for localization and/or approximation of solutions of semi- and nonlinear ordinary and partial differential equations.(see<sup>[5,12,18,19]</sup>)

In the current paper we present some new results on the lower and upper solutions' method in analysis of a special type second order ODE, subjected to the homogeneous Dirichlet boundary conditions. This problem occurs in the applied setting, in particular in modelling of the geophysical flow of the Antarctic Circumpolar Current (ACC, for short; see discussions in<sup>[16,17,20,22,23]</sup>).

In<sup>[16]</sup> it was shown, that the ACC flow can be modelled in terms of the semilinear elliptic equation

$$\Delta\psi + 8\omega \frac{1 - (x^2 + y^2)}{(1 + x^2 + y^2)^3} - \frac{4F(\psi)}{(1 + x^2 + y^2)^2} = 0, \quad (1)$$

defined in the domain

$$\mathcal{O} = \{(x, y) : r_- < r = \sqrt{x^2 + y^2} < r_+\} \quad (2)$$

for suitable constants  $r_1$  and  $r_2$  with  $0 < r_- < r_+ < 1$ .

Here  $\Delta = \partial_x^2 + \partial_y^2$  is the Laplace operator,  $\psi$  is the stream function,  $\omega > 0$  is the nondimensional form of the Coriolis parameter,  $F(\psi)$  is the oceanic vorticity. and

Moreover, it was proved, that the flow corresponds to a radially symmetric solution  $\psi = \psi(r)$  of the equation (1). Thus, with

$$0 < t_1 = -\ln(r_+) < t_2 = -\ln(r_-),$$

the change of variables  $r = e^{-t/2}$  and

$$\psi(r) = u(t), \quad t_1 < t < t_2,$$

transforms the PDE (1) to the second-order ODE

$$u''(t) - \frac{e^t}{(1+e^t)^2} F(u(t)) + \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} = 0, \quad (3)$$

for  $t_1 < t < t_2$ .

One of the physically relevant boundary conditions in the mathematical model of the ACC are the Dirichlet boundary conditions:

$$\begin{cases} u(t_1) = \alpha_1, \\ u(t_2) = \alpha_2, \end{cases} \quad (4)$$

expressing the fact that  $r = r_{\pm}$  are streamlines, with  $\psi = \alpha_1$  on  $r = r_-$  and  $\psi = \alpha_2$  on  $r = r_+$ .

There are recent developments in study of the ODE (3), subjected to the homogeneous and non-homogeneous Dirichlet boundary constraints in the case of constant, linear and nonlinear function  $F$  (see discussions in [17,20,23]).

In the next section we give a short overview of these results.

## 2. Overview of existing results

In [20] for a linear BVP

$$\begin{aligned} u''(t) - a(t) u(t) &= b(t), & t_1 < t < t_2, \\ u(t_1) &= \alpha_1, \quad u(t_2) = \alpha_2, \end{aligned} \quad (5)$$

where

$$a(t) := \frac{p(t) e^t}{(1+e^t)^2}, \quad b(t) := \frac{q(t) e^t}{(1+e^t)^2} - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3}, \quad t \in [t_1, t_2], \quad 0 < \omega < \infty,$$

the existence and uniqueness results were obtained. It was proved that, if the homogeneous BVP

$$\begin{cases} u''(t) - a(t) u(t) = 0, & t_1 < t < t_2, \\ u(t_1) = u(t_2) = 0, \end{cases}$$

has the unique trivial solution, then for every continuous function  $b : [t_1, t_2] \rightarrow \mathbb{R}$  the corresponding non-homogeneous BVP (5) has the unique solution.

In the case of homogeneous boundary conditions (that is, if  $\alpha_1 = \alpha_2 = 0$ ), this solution is given by

$$u(t) = \int_{t_1}^{t_2} b(s) G(t, s) ds, \quad t \in [t_1, t_2].$$

where for  $t \in [t_1, t_2]$  and  $s \in (t_1, t_2)$ , the associated Green's function is

$$G(t, s) = \begin{cases} u_1(t) u_2(s), & t_1 \leq t < s, \\ u_1(s) u_2(t), & s \leq t < t_2, \end{cases}$$

where  $u_1(t)$  and  $u_2(t)$  are linearly independent solutions of the homogeneous differential equation with  $u_1(t_1) = u_2(t_2) = 0$  and the Wronskian  $W(u_1, u_2) = 1$ .

Moreover, some explicit solutions of the problem (3), (4) were constructed. In particular, if in the differential equation (3) (i)  $F = 0$ , then the unique solution of (3) is given by

$$u(t) = c_1 + c_2 t + \omega \tanh(t/2) + q \ln(1+e^t), \quad t \in (t_1, t_2),$$

for some suitably chosen constants  $c_1$  and  $c_2$  that accommodate the two boundary conditions in (4);

(ii)  $F(u) = q$ , where  $q$  is some real constant, the unique solution of (3) is given by

$$u(t) = c_1 + c_2 t + \omega \tanh(t/2) + q \ln(1+e^t), \quad t \in (t_1, t_2),$$

for some suitably chosen constants  $c_1$  and  $c_2$  that accommodate the two boundary conditions (4);

(iii)  $F(u) = -2u$  the general solution of the differential equation in (2) is of the form

$$u(t) = c_1 \tanh(t/2) + c_2 (2 - t \tanh(t/2)).$$

In the nonlinear and more general case of the BVP (3), (4) two important results were proved.

In [17] we studied a general form of the BVP (3), (4). Assuming, that there exist constants  $m_0, M_0 > 0$  such that the continuous function  $F : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$uF(u) + m_0|u| \geq 0 \quad \text{for } |y| \geq M_0.$$

we have shown that there exists a solution  $u \in C^2[t_1, t_2]$  of (3), (4).

In the later work [23] we used a general result of [24], which ensures existence and uniqueness of the solution to (3), (4) for  $t_1 = 0, t_2 = 1$ , provided that for every  $\varepsilon \in (0, 1)$  we have:

(H1) all solutions of the initial-value problem

$$\begin{cases} u''(t) = \frac{e^t}{(1+e^t)^2} F(u) - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3}, & t \in (0, 1), \\ u(0) = \alpha_1, \\ u'(0) = u_1, \end{cases} \quad (6)$$

exist on  $[0, 1 + \varepsilon]$  for all  $\alpha_1, u_1 \in \mathbb{R}$ ;

(H2) there do not exist two solutions on  $[0, t^*]$  to the two-point boundary-value problem

$$\begin{cases} u''(t) = \frac{e^t}{(1+e^t)^2} F(u) - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3}, & t \in (0, 1), \\ u(0) = \alpha_1, \\ u(t^*) = u^*, \end{cases} \quad (7)$$

for any  $t^* \in (1 - \varepsilon, 1 + \varepsilon)$  and  $u^* \in \mathbb{R}$ .

It was proved that, if the continuous function  $F : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$M + \int_0^u F(\xi) d\xi \geq W^{-1}(F^2(u)), \quad u \in \mathbb{R},$$

for some constant  $M > 0$  and some strictly increasing function  $W : [0, \infty) \rightarrow [0, \infty)$  with  $W(0) = 0, W(s) > 0$  for  $s > 0$  and satisfying

$$\int_1^\infty \frac{du}{W(u)} = \infty,$$

and if

$$\lim_{|u| \rightarrow \infty} \int_0^u F(\xi) d\xi = \infty,$$

then all solutions of (6) are global in time. Moreover, if the continuous function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is monotone nondecreasing on  $\mathbb{R}$ , then the solution of (7) is unique. By combining of these two results we proved that the original BVP (3), (4) (with  $t_1 = 0, t_2 = 1$ ) admits a unique solution.

As already mentioned, in this paper we present the current developments in study of special type of the BVP (3), (4). In particular, we investigate the case, when function  $F$  in the right hand-side of (3) has the form of an exponential function. Such nonlinearity appears in the geophysical context and is known as the Stuart-type vorticity (see discussions in [5,6]). It is one of the few known nonlinear vorticities, for which an exact smooth analytical solution of the nonlinear second order ODE can be found. This fact and the applied background of the problem explain the high interest to it.

In our work we will use two important results.

Study of the differential equation with an exponential nonlinearity leads us to the approach, suggested by D. Crowdy. In [6] he studied a problem of finding the exact and explicit solution of the PDE

$$\nabla^2 \psi = ce^{d\psi} + g, \quad (8)$$

where  $c, d, g$  are real constants. He showed that if these constants are related in a specific way, solution of the differential equation (8) can be written in a closed form.

In [28] G. Scorza Dragoni considered the Dirichlet boundary value problem

$$u'' = f(t, u, u'), \quad u(a) = A, \quad u(b) = B. \quad (9)$$

He assumed the existence of functions  $\alpha$  and  $\beta \in C^2([a, b])$  such that

$$\alpha(t) \leq \beta(t) \quad \text{on } [a, b]$$

and

$$\alpha'' - f(t, \alpha, y) \geq 0$$

if  $t \in [a, b], y \leq \alpha'(t), \alpha(a) \leq A, \beta(b) \leq B$ ,

$$\beta'' - f(t, \beta, y) \geq 0$$

if  $t \in [a, b]$ ,  $y \geq \beta'(t)$ ,  $\beta(a) \geq A$ ,  $\beta(b) \geq B$ .

He obtained existence of a solution  $u$  of the problem (9) together with its localization  $\alpha \leq u \leq \beta$ . Moreover, assuming the regularity of function  $f$ , it was shown, that it is continuous and bounded on

$$E := \{(t, u, v) \in [a, b] \times \mathbb{R}^2 \mid \alpha(t) \leq u \leq \beta(t)\}.$$

Hence, in the Sections 3–5 we apply the approach of D. Crowdy to simplify the given differential equation and the upper and lower solutions' method, suggested by G. Scorza Dragoni, for the localization of its solution.

### 3. Problem setting

Consider a nonlinear boundary-value problem for a second-order differential equation, subjected to the homogeneous Dirichlet boundary constraints:

$$u''(t) = \frac{e^t}{(1+e^t)^2}(ae^{bu} + c) - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3}, \quad t \in (0, 1), \quad (10)$$

$$u(0) = u(1) = 0, \quad (11)$$

where  $u : [0, 1] \rightarrow \mathbb{R}$  is an unknown twice continuously differentiable on  $[0, 1]$  function,  $a, b, c$  are suitable real constants, such that  $ab > 0$ , and  $0 < \omega < \infty$  is a real-valued parameter.

Note, if we let  $\omega \rightarrow \infty$ , then (10) will be a singularly perturbed problem. In this case it will be of multiscale nature, where different methods are used (see discussions in <sup>[3,4,10,11]</sup>). We postpone this problem to our future work in a more general setting.

Let us rewrite the differential equation (10) in the form

$$u''(t) - \frac{e^t}{(1+e^t)^2} \cdot ae^{bu} - \frac{ce^t}{(1+e^t)^2} + \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} = 0 \quad (12)$$

and introduce the change of variables

$$u(t) = v(t) + A \ln \left( \frac{1+e^t}{e^{t/2}} \right), \quad (13)$$

$$e^{bu} = e^{bv} \left( \frac{1+e^t}{e^{t/2}} \right)^{Ab}, \quad (14)$$

$$u'' = v'' + \frac{Ae^t}{(1+e^t)^2}, \quad (15)$$

where the constant  $A \in \mathbb{R}$  to be defined later.

The substitutions (13)–(15) transform (12) into the differential equation

$$v'' - ae^{bv} \left( \frac{1+e^t}{e^{t/2}} \right)^{Ab-2} + \frac{(A-c)e^t}{(1+e^t)^2} + \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} = 0. \quad (16)$$

To study solutions of (16), let us first evaluate real parameters  $A, b$  and  $c$  present there.

Note that the choice of  $A = \frac{2}{b}$  leads to the equation of the form

$$v'' - ae^{bv} + \frac{(\frac{2}{b}-c)e^t}{(1+e^t)^2} + \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} = 0,$$

that might be rewritten as

$$v'' - ae^{bv} = \frac{(c - \frac{2}{b} - 2\omega)e^t}{(1+e^t)^2} + \frac{4\omega e^{2t}}{(1+e^t)^3}.$$

Let us pick the value of  $c$  equal to  $\frac{2}{b} + 2\omega$ . Taking into account the evaluated parameters  $A$  and  $c$ , as well as the substitutions (13)–(15), we come to the BVP:

$$\begin{cases} v'' - ae^{bv} = \frac{4\omega e^{2t}}{(1+e^t)^3}, \\ v(0) = -\frac{2}{b} \ln 2, \quad v(1) = -\frac{2}{b} \ln \left( \frac{1+e}{e^{1/2}} \right). \end{cases} \quad (17)$$

Even though the differential equation in (17) still contains the exponential nonlinearity, its multiplier is now a constant. This fact allows us to proceed with the qualitative analysis of the Dirichlet BVP (17) using the approach of lower and upper solutions.

## 4. Reduction of the original BVP

We are looking for a solution of the BVP (17) in the form

$$v(t) = \xi(t) + \xi_0(t), \quad (18)$$

where  $\xi_0 : [0, 1] \rightarrow \mathbb{R}$  is the classical solution of the homogeneous semilinear differential equation subjected to the non-homogeneous Dirichlet boundary conditions

$$\begin{cases} \xi_0'' - ae^{b\xi_0} = 0, \\ \xi_0(0) = -\frac{2}{b} \ln 2, \quad \xi_0(1) = -\frac{2}{b} \ln \left( \frac{1+e}{e^{1/2}} \right). \end{cases} \quad (19)$$

Calculations show, that the general solution of the differential equation in (19) is given by the function

$$\xi_0(t) = \frac{1}{b} \left( -\ln(2) + \ln \left( \frac{C_1}{a \cos \left( \frac{(C_2+t)\sqrt{C_1 b^2}}{2} \right)} \right) \right),$$

where  $C_1, C_2$  are real constants to be found from the boundary conditions in (19).

Then function  $\xi$  in (18) is the solution of the BVP with the homogeneous constraints

$$\begin{cases} -\xi'' + ae^{b\xi_0}(e^{b\xi} - 1) + \frac{4\omega e^{2t}}{(1+e^t)^3} = 0, \\ \xi(0) = \xi(1) = 0. \end{cases} \quad (20)$$

Let us show the connection between solutions of the nonlinear problem (20) and the corresponding linear BVP.

Consider the linear BVP in the form:

$$w'' = \frac{4\omega e^{2t}}{(1+e^t)^3}, \quad t \in (0, 1), \quad (21)$$

$$w(0) = w(1) = 0. \quad (22)$$

It is easy to see that the general solution of the differential equation (21) is given by formula

$$w(t) = \frac{2\omega}{e^t + 1} + 2\omega \ln(e^t + 1) + c_1 t + c_2.$$

Taking into account the boundary restrictions (22), we get the explicit solution of the BVP (21), (22):

$$w(t) = \frac{2\omega}{e^t + 1} + 2\omega \ln(e^t + 1) - 2\omega \left( \frac{1}{1+e} - \frac{1}{2} + \ln \frac{e+1}{2} \right) t - \omega(1 + 2 \ln 2). \quad (23)$$

Using the analytic form of the exact solution of (21), (22) we conclude, that function  $w(t)$  is bounded, for all  $t \in [0, 1]$  and  $0 < \omega < \infty$ , and satisfies the inequalities:

$$-0.18\omega \leq w(t) \leq 0.$$

Let us recall the definition.

**Definition 4.1.** We call functions  $\alpha \in C^2([0, 1])$  and  $\beta \in C^2([0, 1])$  correspondingly a lower and an upper solution of the BVP (20), if they satisfy the inequalities

$$-\alpha'' + ae^{b\xi_0}(e^{b\alpha} - 1) + \frac{4\omega e^{2t}}{(1+e^t)^3} \leq 0, \quad t \in (0, 1), \quad (24)$$

$$\alpha(0) \leq 0, \quad \alpha(1) \leq 0, \quad (25)$$

and

$$-\beta'' + ae^{b\xi_0}(e^{b\beta} - 1) + \frac{4\omega e^{2t}}{(1+e^t)^3} \geq 0, \quad t \in (0, 1) \quad (26)$$

$$\beta(0) \geq 0, \quad \beta(1) \geq 0 \quad (27)$$

respectively, where  $\xi_0$  is the solution of the BVP (19).

## 5. Main results

Now, using the result of D. Scorsa Dragoni mentioned in the Introduction (see also discussions in [1,5,28]), we can prove the following theorem.

**Theorem 5.1.** *Functions*

$$\alpha(t) = w(t), \beta(t) = 0,$$

where  $w(t)$  is defined by relation (23) are the lower and upper solutions of the BVP (20). Moreover  $\alpha(t) < \beta(t)$  for  $t \in (0, 1)$ .

*Proof.* Let us first prove, that function  $\alpha(t) = w(t)$  is the lower solution of (20). The substitution  $\alpha(t) = w(t)$  into the left hand side of the differential inequality (24) leads to

$$\begin{aligned} -w'' + ae^{b\xi_0}(e^{bw} - 1) + \frac{4\omega e^{2t}}{(1+e^t)^3} \\ \leq -w'' + ae^{b\xi_0} - ae^{b\xi_0} + \frac{4\omega e^{2t}}{(1+e^t)^3} = -w'' + \frac{4\omega e^{2t}}{(1+e^t)^3} = 0. \end{aligned}$$

Moreover, since function  $w(t)$  satisfies the boundary restrictions (22) and thus, inequalities (25), we conclude, that it is the lower solution of the BVP (20).

On the other hand, taking  $\beta(t) = 0$  in (26), (27) results:

$$ae^{b\xi_0}(e^{b\cdot 0} - 1) + \frac{4\omega e^{2t}}{(1+e^t)^3} = \frac{4\omega e^{2t}}{(1+e^t)^3} > 0,$$

i.e., according to the Definition 4.1, it is the upper solution of the BVP (20).

This completes the proof.  $\square$

Thus, we determined the pair of functions  $\{w(t), 0\}$ , that are the lower and the upper solution of the BVP (20). Now we can formulate the following existence result.

**Theorem 5.2.** *Let the pair of functions  $\{\alpha(t), \beta(t)\} = \{w(t), 0\}$  satisfies Theorem 5.1. Then the original BVP (10), (11) has at least one solution  $u(t)$  such that*

$$w(t) + \frac{2}{b} \ln \left( \frac{1+e^t}{e^{t/2}} \right) + \xi_0(t) \leq u(t) \leq \frac{2}{b} \ln \left( \frac{1+e^t}{e^{t/2}} \right) + \xi_0(t), t \in [0, 1], \quad (28)$$

where  $\xi_0(t)$ ,  $w(t)$  are solutions of the BVPs (19) and (21), (22) accordingly.

*Proof.* From the substitutions (13), (18) we know, that the solution  $u(t)$  of the BVP (10), (11) can be written as

$$u(t) = \xi(t) + \xi_0(t) + A \ln \left( \frac{1+e^t}{e^{t/2}} \right),$$

where  $A = \frac{2}{b}$ ,  $\xi(t)$  and  $\xi_0(t)$  are solutions of the BVPs (19) and (20) accordingly.

Moreover, in Theorem 5.1 we proved, that the pair of functions  $\{w(t), 0\}$  are the lower and the upper solution of the BVP (20). Therefore, there exists a solution  $u(t)$  of the BVP (10), (11), for which the two-sided inequality (28) holds.  $\square$

## 6. Conclusions

The paper was devoted to a special type Dirichlet boundary value problem for a second order ordinary differential equation with an exponential nonlinearity in the right hand-side. The existence of solutions of the given problem was proved via the method of lower and upper solutions. The obtained results might be broaden onto more general cases of the boundary value problems with applications. In particular, in study of the BVPs for the elliptic PDEs, hyperbolic PDEs with a pre-history in the boundary restrictions, FDEs subjected to different types of boundary conditions, etc. These problems occur in modeling of global economics of different countries, gas sorption, pipes heating by a stream of hot water, geophysics.

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The authors contributed equally to this study.

## Conflicts of Interest

The authors declare no conflict of interest.

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