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On Vanishing Gains in Robust Adaptation of Switched Systems: A New Leakage-based Result for a Class of Euler Lagrange Dynamics

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Abstract

In the presence of unmodelled dynamics and uncertainties with no a priori constant bounds, conventional robust adaptation strategies for switched systems cannot allow the control gains of inactive subsystems to remain constant during inactive intervals: vanishing gains are typically required in order to prove bounded stability. As a consequence, these strategies, known in literature as leakage-based adaptive methods, might introduce poor transients at each switching instant. Leakage-based adaptive control becomes even more problematic in the switched nonlinear case, where non-conservative state-dependent upper bounds for uncertainties and unmodelled dynamics are required. This work shows that, for a class of switched Euler-Lagrange systems, such difficulties can be mitigated: a novel leakage-based stable mechanism is introduced which allows the gains of inactive subsystems to remain constant. At the same time, unmodelled dynamics and uncertainties with no a priori bounds can be handled by a quadratic state-dependent upper bound structure that reduces conservativeness as compared to state-of-the-art structures. The proposed design is validated analytically and its performance is studied in simulation with a pick-and-place robotic manipulator example.

Keywords: Robust adaptive control, Euler-Lagrange systems, Switched systems, Vanishing inactive gains

1. Introduction

Switched systems represent an important class of hybrid systems consisting of subsystems with continuous dynamics and a switching law to regulate the switching among subsystems. The switching can be state-dependent or time-driven, being dwell-time (DT) or average dwell-time (ADT) the most studied classes of time-driven switching [1, 2]. Over the last decade, several works have been reported for control

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of linear [3–6] and nonlinear [7–13] switched systems (see also references therein). Here, we focus specifically on *adaptive control of uncertain switched systems*, i.e. control of switched systems with possibly large parametric uncertainties. Recent advances in the field include [14–17] for switched linear systems and [18–24] for classes of switched nonlinear systems.

1.1. The issue of robust adaptation and inactive gains

In the presence of unmodelled dynamics and uncertainties *with no a priori constant bounds*, it is well known that leakage-based adaptive control is the only robust adaptive mechanism able to prove bounded stability [25, Chap. 8], since projection, switching σ-modification, dead-zone and dynamic normalization all require knowledge of the bounds of the unmodelled dynamics/uncertainties. Efforts have been made recently to design leakage-based adaptive methods for uncertain switched systems. However, it was recently demonstrated that switched leakage-based strategies face serious drawbacks as compared to their non-switched counterpart [26–28]. Most notably, [26] showed that the control gains of the inactive subsystems should decrease exponentially as a consequence of leakage, otherwise bounded stability cannot be proven. This will create poor transients whenever a subsystem that remained inactive for sufficiently long time is activated again. One would desire a situation in which the inactive gains are kept fixed during inactive intervals. Unfortunately, this was shown to be possible only in restrictive cases, such as the class of globally Lipschitz nonlinear dynamics in [28].

1.2. The issue of upper bounding uncertainty

Leakage-based adaptive control becomes even more challenging for switched nonlinear systems, where the presence of unmodelled dynamics and uncertainties with no a priori constant bounds requires suitable (possibly non-conservative) state-dependent upper bound structures. It is worth mentioning that conservative upper bound structures typically require high inputs, e.g. achieved by monotonically increasing control gains [19, 20, 28]. This work focuses on how conservative structures arise for the class of switched Euler-Lagrange (EL) systems, relevant in many application domains and recurring motif in adaptive switched literature. For example, the switched linear uncertain systems considered in [14, 15, 17, 26] (aircraft, electromechanical systems etc.) are linearized switched dynamics that should be more appropriately described as switched EL dynamics. Even the state-space linear-in-the-parameter (LIP) dynamics in [18, 22–24] can cover only a small class of EL dynamics, since the state-space EL form is in general nonlinear-in-the-parameter (NLIP) due to the inversion of the mass matrix. Even the NLIP structures in [19, 20] might be conservative for EL systems: while being extremely useful to attain strong stability results, the EL examples in [19, 20] reveal that such structures, relying on the parameter separation method pioneered in [29], require detailed structural knowledge of the system dynamics and result in a state-dependent quartic polynomial upper bound to the uncertainties. But it is known that, under mild assumptions [30], uncertainties in EL dynamics can be upper bounded by a less conservative state-dependent quadratic polynomial.

1.3. Main contributions

In light of the above discussions, leakage-based adaptive switched control presents unsolved challenges. This work proposes a new adaptation method in this direction with the following contributions:

- A novel leakage-based adaptive mechanism is proposed which avoids the undesirable phenomenon of vanishing control gains. This is achieved by introducing auxiliary gains specifically for leakage purpose, which allow the control gains of inactive subsystems to be kept at the same value they had at switch-out instant.

- Such leakage-based strategy is embedded in an adaptation framework for switched EL systems where uncertainties are upper bounded by a less conservative state-dependent quadratic polynomial structure, requiring less structural knowledge than LIP or parameter separation-based structures proposed in literature.

This work studies the same class of switched dynamics studied by some of the same authors in [31]. In addition to proposing a new leakage-based adaptation law, this work also manages to remove some structural constraints present in [31]. More specifically, as compared to [31], the switching law and leakage terms proposed in this work are independent of system dynamics terms, thus freely tunable. The rest of the paper is organized as follows: Section 2 describes the uncertain switched EL dynamics; Section 3 details the proposed control framework, with stability analysis carried out in Section 4; a simulation study is provided in Section 5, while Section 6 presents concluding remarks.

The following notations are used throughout the paper: $\lambda_{\min}(\bullet)$, $\lambda_{\max}(\bullet)$ and $||\bullet||$ represent minimum eigenvalue, maximum eigenvalue and Euclidean norm of $(\bullet)$ respectively; $\mathbf{I}$ denotes identity matrix with appropriate dimension; $\mathbb{R}^+, \mathbb{N}^+$ denote the set of positive real numbers and set of positive integers, respectively; $\Omega = [1, 2, \cdots, N]$ denotes the set subsystems and $\mathscr{N}(p)$ denotes the set of inactive subsystem corresponding to an active subsystem $p \in \Omega$.

2. System Dynamics and Problem Formulation

Consider the following class of switched Euler-Lagrange (EL) systems

$$\mathbf{M}_\sigma(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}_\sigma(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}_\sigma(\mathbf{q}) + \mathbf{F}_\sigma(\dot{\mathbf{q}}) + \mathbf{d}_\sigma = \tau_\sigma, \tag{1}$$

where $\mathbf{q} \in \mathbb{R}^n$ is the system state and $\sigma(t) : [0\ \infty) \mapsto \Omega$ is a piecewise constant function of time, called the switching signal, taking values in $\Omega = [1, 2, \cdots, N]$; for each subsystem σ, $\mathbf{M}_\sigma(\mathbf{q}) \in \mathbb{R}^{n\times n}$ is the mass/inertia matrix; $\mathbf{C}_\sigma(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n\times n}$ are Coriolis/centripetal terms; $\mathbf{G}_\sigma(\mathbf{q}) \in \mathbb{R}^n$ denotes the gravity vector; $\mathbf{F}_\sigma(\dot{\mathbf{q}}) \in \mathbb{R}^n$ represents the vector of damping and friction forces; $\mathbf{d}_\sigma(t) \in \mathbb{R}^n$ denotes bounded external disturbance and $\tau_\sigma \in \mathbb{R}^n$ is the generalized control input. The switching signal σ make the system terms $\mathbf{M}_\sigma$, $\mathbf{C}_\sigma$, $\mathbf{G}_\sigma$, $\mathbf{F}_\sigma$ and the signals $\mathbf{d}_\sigma$, τ_σ possibly jump: however, notice that the variables $\mathbf{q}, \dot{\mathbf{q}}$ are continuous (i.e. do not jump) at the switching instants.

Assumption 1. *Each subsystem in (1) obeys the following two properties, which hold for many EL systems of practical interest [30]:*
Property 1: $\exists \overline{c}_\sigma, \overline{g}_\sigma, \overline{f}_\sigma, \overline{d}_\sigma \in \mathbb{R}^+$ *such that* $||\mathbf{C}_\sigma(\mathbf{q}, \dot{\mathbf{q}})|| \leq \overline{c}_\sigma ||\dot{\mathbf{q}}||$, $||\mathbf{G}_\sigma(\mathbf{q})|| \leq \overline{g}_\sigma$, $||\mathbf{F}_\sigma(\dot{\mathbf{q}})|| \leq \overline{f}_\sigma ||\dot{\mathbf{q}}||$ *and* $||\mathbf{d}_\sigma(t)|| \leq \overline{d}_\sigma$.
Property 2: *The matrix* $\mathbf{M}_\sigma(\mathbf{q})$ *is symmetric and uniformly positive definite for all* $\mathbf{q}$. *This implies that* $\exists \underline{m}_\sigma, \overline{m}_\sigma \in \mathbb{R}^+$ *such that*

$$0 < \underline{m}_\sigma \mathbf{I} \leq \mathbf{M}_\sigma(\mathbf{q}) \leq \overline{m}_\sigma \mathbf{I}. \tag{2}$$

Further, let $\mathbf{M}_\sigma$ be decomposed as $\mathbf{M}_\sigma \triangleq \hat{\mathbf{M}}_\sigma + \Delta\mathbf{M}_\sigma$, where $\hat{\mathbf{M}}_\sigma$ and $\Delta\mathbf{M}_\sigma$ represent the nominal and perturbation terms of $\mathbf{M}_\sigma$, respectively. The nominal mass matrix is given, while the perturbation term (or better, its upper bound) is calculated accounting for the uncertainty in the physical parameters. The control design challenge in terms of available knowledge of EL system (1) stems from the fact that *only the knowledge of* $\hat{\mathbf{M}}_\sigma$ *and an upper bound for* $\Delta\mathbf{M}_\sigma$ *are available; the terms* $\mathbf{C}_\sigma, \mathbf{F}_\sigma, \mathbf{G}_\sigma$ *and* $\mathbf{d}_\sigma$ *(and their upper bounds* $\overline{c}_\sigma, \overline{f}_\sigma, \overline{g}_\sigma$ *and* $\overline{d}_\sigma$*) are completely unknown.* The following assumption defines the allowed uncertainty around $\hat{\mathbf{M}}_\sigma$:

Assumption 2. *There exist known scalars $\bar{E}_\sigma$ such that for $\mathbf{E}_\sigma \triangleq \left(\mathbf{M}_\sigma^{-1}\hat{\mathbf{M}}_\sigma - \mathbf{I}\right)$ the following holds*

$$||\mathbf{E}_\sigma|| \leq \bar{E}_\sigma < 1, \ \forall \sigma \in \Omega. \tag{3}$$

Remark 1. *Assumption 2 is not proposed here, but extensively used in literature dealing with EL systems such as inverse dynamics (cf. [30, §11]) and adaptive sliding mode [32, 33] designs. Such literature shows that the nominal mass matrix $\hat{\mathbf{M}}_\sigma$ can be selected such that (3) is satisfied by making use of Property 2 (cf. [30, §11]). Essentially, $\bar{E}_\sigma$ depends on the size of uncertainty around the nominal value of the mass matrix. The larger the uncertainty, the larger $\bar{E}_\sigma$ (subject to the fact that $\bar{E}_\sigma$ should be below 1 for stability analysis).*

For ease of control design, system (1) is represented as

$$\ddot{\mathbf{q}} = \mathbf{f}_\sigma(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{M}_\sigma^{-1}\tau_\sigma, \ \ \sigma(t) \in \Omega \tag{4}$$

where $\mathbf{f}_\sigma \triangleq -\mathbf{M}_\sigma^{-1}\left(\mathbf{C}_\sigma + \mathbf{F}_\sigma + \mathbf{G}_\sigma + \mathbf{d}_\sigma\right)$.

Let us define $\mathbf{x} \triangleq [\mathbf{q}^T \ \ \dot{\mathbf{q}}^T]^T$ which we assume to be available as feedback. Using Properties 1 and 2, the system dynamics term $\mathbf{f}_\sigma(\mathbf{x})$ can be upper bounded as:

$$||\mathbf{f}_\sigma(\mathbf{x})|| \leq \theta_{0\sigma} + \theta_{1\sigma}||\mathbf{x}|| + \theta_{2\sigma}||\mathbf{x}||^2 \triangleq \mathbf{Y}_\sigma^T(||\mathbf{x}||)\Theta_\sigma, \tag{5}$$

where $\mathbf{Y}_\sigma(||\mathbf{x}||) = [\mathbf{1} \ ||\mathbf{x}|| \ ||\mathbf{x}||^2]^T$, $\Theta_\sigma = [\theta_{0\sigma} \ \theta_{1\sigma} \ \theta_{2\sigma}]^T$ and $\theta_{i\sigma} \in \mathbb{R}^+ \ i = 0,1,2$ are finite but *unknown scalars*, according to the available knowledge of system (1).

Remark 2. *In the presence of unmodelled dynamics and uncertainties with no a priori bounds, the quadratic upper bound (5) finds its rationale in reduction of conservativeness when minimal structural knowledge is available. In fact, the quadratic structure (5) is general, i.e. it holds for many EL systems irrespective of their specific structure [30, 32, 33]. On the other hand, let us consider an alternative upper bound structure proposed in [19, 20]*

$$||\mathbf{f}_\sigma(\mathbf{x})|| \leq \varphi_\sigma(\mathbf{x})\phi_\sigma(\theta_\sigma) \tag{6}$$

where $\varphi_\sigma(\mathbf{x}) \geq 1$, $\phi_\sigma(\theta_\sigma) \geq 1$ are two scalar functions and θ_σ denotes the set of unknown system parameters. The structure (6) is extremely useful to attain strong (asymptotic) stability results, but there is no general procedure for deriving appropriate scalar functions in (6). In particular, for EL dynamics (4), deep structural knowledge of the system is required to derive such scalar functions (cf. the examples in [19, 20]).

We use the notation $\{(\sigma(t_l), t_l) \mid l \in \mathbb{N}^+ \cup \{0\}\}$ to denote the set of (subsystem, switching instant) pairs. The sequence of switch-in and switch-out instants of subsystem p, $p \in \Omega$ is given as $\{t_{p_1}, t_{p_2}, \cdots, t_{p_l}, \cdots \mid l \in \mathbb{N}^+\}$ and $\{t_{p_1+1}, t_{p_2+1}, \cdots, t_{p_l+1}, \cdots \mid l \in \mathbb{N}^+\}$, respectively. The following class of switching signals is considered:

Definition 1. *Average Dwell Time (ADT) [2]: For a switching signal $\sigma(t)$ and each $t_2 \geq t_1 \geq 0$, let $N_\sigma(t_1, t_2)$ denote the number of discontinuities in the interval $[t_1, t_2)$. Then $\sigma(t)$ has an average dwell time ϑ if for a given scalar $N_0 > 0$*

$$N_\sigma(t_1, t_2) \leq N_0 + (t_2 - t_1)/\vartheta, \ \ \forall t_2 \geq t_1 \geq 0$$

where N_0 is termed as chatter bound.

3. Controller Design

Let us consider the tracking problem for a desired trajectory $\mathbf{q}^d(t)$ satisfying the following assumption.

Assumption 3. *The desired trajectories are smooth enough, in particular* $\mathbf{q}^d, \dot{\mathbf{q}}^d, \ddot{\mathbf{q}}^d \in \mathscr{L}_\infty$.

Let $\mathbf{e}(t) \triangleq \mathbf{q}(t) - \mathbf{q}^d(t)$ be the tracking error and $\xi(t) \triangleq [\mathbf{e}(t),\ \dot{\mathbf{e}}(t)]$. We define a filtered tracking error variable $\mathbf{r}_\sigma$ as

$$\mathbf{r}_\sigma \triangleq \mathbf{B}^T \mathbf{P}_\sigma \xi, \qquad \sigma \in \Omega \tag{7}$$

where $\mathbf{P}_\sigma > \mathbf{0}$ is the solution to the Lyapunov equation $\mathbf{A}_\sigma^T \mathbf{P}_\sigma + \mathbf{P}_\sigma \mathbf{A}_\sigma = -\mathbf{Q}_\sigma$ for some $\mathbf{Q}_\sigma > \mathbf{0}$, $\mathbf{A}_\sigma \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_{1\sigma} & -\mathbf{K}_{2\sigma} \end{bmatrix}$ and $\mathbf{B} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}^T$. Here, $\mathbf{K}_{1\sigma}$ and $\mathbf{K}_{2\sigma}$ are two user-defined positive definite gain matrices and their positive definiteness guarantees $\mathbf{A}_\sigma$ to be Hurwitz.

The switched control law is designed as

$$\tau_\sigma = \hat{\mathbf{M}}_\sigma(-\Lambda_\sigma \xi - \Delta\tau_\sigma + \ddot{\mathbf{q}}^d), \tag{8a}$$

$$\Delta\tau_\sigma = \omega \rho_\sigma \frac{\mathbf{r}_\sigma}{\sqrt{||\mathbf{r}_\sigma||^2 + \varepsilon}}, \tag{8b}$$

where $\Lambda_\sigma \triangleq [\mathbf{K}_{1\sigma}\ \mathbf{K}_{2\sigma}]$; $\Delta\tau_\sigma$ tackles the uncertainties utilizing the gain ρ_σ; $\varepsilon > 0$ is a small scalar to avoid control chatter and $\omega > 1$ is a user-defined scalar. The design of ρ_σ will be discussed later. Let $\eta_\sigma \triangleq (-\Lambda_\sigma \xi - \Delta\tau_\sigma + \ddot{\mathbf{q}}^d)$. Now, substituting (8a) in (4) yields

$$\begin{aligned} \ddot{\mathbf{e}} &= \ddot{\mathbf{q}} - \ddot{\mathbf{q}}^d \\ &= \mathbf{f}_\sigma + \mathbf{M}_\sigma^{-1} \tau_\sigma - \ddot{\mathbf{q}}^d \\ &= \mathbf{f}_\sigma + \left(\mathbf{M}_\sigma^{-1} \hat{\mathbf{M}}_\sigma - \mathbf{I}\right) \eta_\sigma + \eta_\sigma - \ddot{\mathbf{q}}^d \\ &= -\Lambda_\sigma \xi - \Delta\tau_\sigma - \mathbf{E}_\sigma \Delta\tau_\sigma + \Psi_\sigma, \end{aligned} \tag{9}$$

where $\Psi_\sigma \triangleq \mathbf{f}_\sigma + \mathbf{E}_\sigma(\ddot{\mathbf{q}}^d - \Lambda_\sigma \xi)$ is treated as the overall uncertainty. Hence, using Assumptions 1 and 2, one can verify the existence of $\theta_{i\sigma}^* \in \mathbb{R}^+$ $i = 0, 1, 2$ such that for all $\sigma \in \Omega$

$$||\Psi_\sigma|| \leq \theta_{0\sigma}^* + \theta_{1\sigma}^* ||\xi|| + \theta_{2\sigma}^* ||\xi||^2 \triangleq \mathbf{Y}_\sigma^T(||\xi||) \Theta_\sigma^*, \tag{10}$$

where $\theta_{i\sigma}^*$'s are unknown finite scalars and $\Theta_\sigma^* = [\theta_{0\sigma}^*\ \theta_{1\sigma}^*\ \theta_{2\sigma}^*]^T$. After defining the structures of the upper bound of $||\Psi_\sigma||$ in (10), the gain ρ_σ in (8b) is proposed as

$$\rho_\sigma = \frac{1}{1 - \bar{E}_\sigma} \{(\hat{\theta}_{0\sigma} + \gamma_{0\sigma}) + (\hat{\theta}_{1\sigma} + \gamma_{1\sigma})||\xi|| + (\hat{\theta}_{2\sigma} + \gamma_{2\sigma})||\xi||^2\} \triangleq \frac{1}{1 - \bar{E}_\sigma} \mathbf{Y}_\sigma^T(||\xi||)(\hat{\Theta}_\sigma + \Gamma_\sigma), \tag{11}$$

where $\hat{\Theta}_\sigma \triangleq [\hat{\theta}_{0\sigma}\ \hat{\theta}_{1\sigma}\ \hat{\theta}_{2\sigma}]^T$ is the estimate of Θ_σ^*; $\Gamma_\sigma \triangleq [\gamma_{0\sigma}\ \gamma_{1\sigma}\ \gamma_{2\sigma}]^T$ is an auxiliary gain which has a crucial role in closed-loop system stabilization and it will be detailed later.

Let p denote the index of the subsystem active for $t \in [t_l\ t_{l+1})$ and $\mathscr{N}(p)$ denote the set of inactive subsystems. The gains $\hat{\theta}_{ip}, \gamma_{ip}$ are evaluated using the following laws:

$$\dot{\hat{\theta}}_{ip} = ||\mathbf{r}_p|| ||\xi||^i - \alpha_{ip} \hat{\theta}_{ip}, \quad \dot{\gamma}_{ip} = 0 \tag{12a}$$

$$\dot{\hat{\theta}}_{i\overline{p}} = 0, \ \ \dot{\gamma}_{i\overline{p}} = -\left(\beta_{i\overline{p}} + \overline{v}_{i\overline{p}} \hat{\theta}_{i\overline{p}}^4\right) \gamma_{i\overline{p}} + \beta_{i\overline{p}} v_{i\overline{p}}, \tag{12b}$$

$$\text{with } \hat{\theta}_{ip}(t_0) > 0, \ \gamma_{i\overline{p}}(t_0) > v_{i\overline{p}}, \tag{12c}$$

where $\overline{p} \in \mathscr{N}(p)$; $\alpha_{ip}, \beta_{i\overline{p}}, \nu_{i\overline{p}}, \overline{\nu}_{i\overline{p}} \in \mathbb{R}^+$, $i = 0, 1, 2$ are static design scalars and t_0 is the initial time. Investigating the adaptive laws (12a)-(41) and the initial gain conditions (12c), it can be verified that there exists a positive fixed scalar $\underline{\gamma}_{i\overline{p}}$ such that

$$\hat{\theta}_{ip}(t) \geq 0 \text{ and } \gamma_{i\overline{p}}(t) \geq \underline{\gamma}_{i\overline{p}} > 0 \;\; \forall t \geq t_0. \tag{13}$$

The above condition is later exploited during the stability analysis. The following remark illustrates the major differences between (11)-(12) and state-of-the-art robust adaptive laws for uncertain switched systems.

Remark 3. *In [26], bounded stability requires that the gains for the inactive systems (corresponding to $\hat{\theta}_{i\overline{p}}$ in our case) vanish exponentially, as an effect of leakage. This implies that, if a system remains inactive for sufficiently long time, its gains drop to zero, generating a new transient at switch-on times. This vanishing-gain scenario is avoided by (41) where the adaptive gains of inactive subsystems are kept at the same value before switch-off time. In [27] bounded stability requires the adaptive laws for all active and inactive subsystem to be constantly active as the tracking error drives all of them simultaneously. A more preferable situation arises in (11)-(12), where only a limited set of the adaptive laws is actively driven by the tracking error.*

We define $\zeta_{Mp} \triangleq \lambda_{\max}(\mathbf{P}_p)$, $\zeta_{mp} \triangleq \lambda_{\min}(\mathbf{P}_p)$, $\bar{\zeta}_M \triangleq \max_{p\in\Omega} \zeta_{Mp}$ and $\underline{\zeta}_m \triangleq \min_{p\in\Omega} \zeta_{mp}$. Following Definition 1 of ADT [2], the switching law is proposed as

$$\vartheta > \vartheta^* = \ln \mu / \kappa, \tag{14}$$

where $\mu \triangleq \bar{\zeta}_M / \underline{\zeta}_m$; κ is a scalar defined as $0 < \kappa < \zeta$ where $\zeta_p \triangleq (\lambda_{\min}(\mathbf{Q}_p)/\lambda_{\max}(\mathbf{P}_p))$ and $\zeta \triangleq \min_{p\in\Omega}\{\zeta_p\}$. Note that the proposed leakage terms in (12) and switching law in (14) are independent of system parameters.

4. Stability Analysis of The Proposed Switched Controller

From the definitions of Λ_σ and ξ we have $\Lambda_\sigma \xi = \mathbf{K}_{1\sigma}\mathbf{e} + \mathbf{K}_{2\sigma}\dot{\mathbf{e}}$. Using this relation, the following error dynamics is obtained from (9)

$$\dot{\xi} = \mathbf{A}_\sigma \xi + \mathbf{B}\left(\Psi_\sigma - \Delta\tau_\sigma - \mathbf{E}_\sigma \Delta\tau_\sigma\right). \tag{15}$$

Before presenting the closed-loop stability result, let us recall the stability concept sought in switched robust adaptive control [26]:

Definition 2 (Uniform Ultimate Boundedness (UUB))**.** *The switched system (15) under switching signal $\sigma(\cdot)$ is uniformly ultimately bounded if there exists a convex and compact set $\mathscr{C}$ such that for every initial condition $\xi(t_0) = \xi_0$, there exists a finite time $T(\xi_0)$ such that $\xi(t) \in \mathscr{C}$ for all $t \geq T(\xi_0)$. Further, a constant b, independent of initial time t_0, is said to be the ultimate bound if $||\xi(t)|| \leq b$ for all $t \geq T(\xi_0)$.*

Theorem 1. *Under Assumptions 1-3, the closed-loop trajectories of system (4) employing the control laws (8) and (11) associated with adaptive law (12) and switching law (14) are UUB if the gains α_{ip} and $\beta_{i\overline{p}}$ are designed as $\alpha_{ip} > \zeta_p/2$ and $\beta_{i\overline{p}} > \zeta_{\overline{p}}/2$.*

Proof. The closed-loop stability analysis follows similar lines as the proof of Theorem 1 of [31], with the difference that the following Lyapunov-like candidate is considered

$$V(\xi(t), \tilde{\theta}_{ip}(t), \gamma_{ip}(t), t) = (1/2)\xi^T(t)\mathbf{P}_{\sigma(t)}\xi(t) + (1/2)\sum_{p=1}^{N}\sum_{i=0}^{2}\{(\hat{\theta}_{ip}(t) - \theta^*_{ip})^2 + \gamma^2_{ip}(t)\}, \tag{16}$$

where $\tilde{\theta}_{ip}(t) = (\hat{\theta}_{ip}(t) - \theta^*_{ip})$. For brevity, we will sometimes express $V(\xi(\cdot), \tilde{\theta}_{ip}(\cdot), \gamma_{ip}(\cdot), \cdot) = V(\cdot)$ to highlight the time evolution of (16). Note that the function (16) depends explicitly on time, due to the active subsystem $\sigma(t)$. This implies that (16) is a multiple Lyapunov-like function, popular in switched systems literature [2]. In fact, $\mathbf{P}_p$ can be designed differently for different subsystems due to the requirements, indicating that $V(\cdot)$ might be discontinuous at the switching instants and only remains continuous during the time interval of two consecutive switchings. Such a switched framework requires to study the behaviour of (16) at and in between switching instants, as carried out subsequently.

Analysis of the Lyapunov function at switching instants: We denote with $\sigma(t^-_{l+1})$ the active subsystem when $t \in [t_l \ \ t_{l+1})$ and with $\sigma(t_{l+1})$ the active subsystem when $t \in [t_{l+1} \ \ t_{l+2})$. Then, although the Lyapunov-like candidate is different, one can still follow similar lines as the proof of Theorem 1 in [31], obtaining that the following behaviour of $V(\cdot)$ is true at the switching instant $t_{l+1},\ l \in \mathbb{N}^+$:

$$\begin{aligned} V(t_{l+1}) - V(t^-_{l+1}) &\leq \frac{\bar{\zeta}_M - \underline{\zeta}_m}{\underline{\zeta}_m} V(t^-_{l+1}) \\ \Rightarrow V(t_{l+1}) &\leq \mu V(t^-_{l+1}), \end{aligned} \tag{17}$$

with $\mu = \bar{\zeta}_M / \underline{\zeta}_m \geq 1$ and t^-_{l+1} denotes the time instant right before switching at $t = t_{l+1}$ (i.e. the limit from the left of t_{l+1}).

Analysis of the Lyapunov function in between switching instants: This analysis refers to the behaviour of $V(\cdot)$ when $t \in (t_l \ \ t_{l+1})$. Note that $V(\cdot)$ is piecewise differentiable, and differentiable for $t \in (t_l \ \ t_{l+1}),\ l \in \mathbb{N}^+$, so that its time-derivative is well defined.

Using (7), (15) and the Lyapunov equation $\mathbf{A}^T_\sigma \mathbf{P}_\sigma + \mathbf{P}_\sigma \mathbf{A}_\sigma = -\mathbf{Q}_\sigma$, the time derivative of (16) yields

$$\begin{aligned} \dot{V}(t) &= (1/2)\xi^T(t)(\mathbf{A}^T_{\sigma(t^-_{l+1})}\mathbf{P}_{\sigma(t^-_{l+1})} + \mathbf{P}_{\sigma(t^-_{l+1})}\mathbf{A}_{\sigma(t^-_{l+1})})\xi(t) + \xi^T(t)\mathbf{P}_{\sigma(t^-_{l+1})}\mathbf{B}\left(\Psi_{\sigma(t^-_{l+1})} - (\mathbf{I} + \mathbf{E}_{\sigma(t^-_{l+1})})\Delta\tau_{\sigma(t^-_{l+1})}\right) \\ &\quad + \sum_{p=1}^{N}\sum_{i=0}^{2}\left\{(\hat{\theta}_{ip}(t) - \theta^*_{ip})\dot{\hat{\theta}}_{ip}(t) + \gamma_{ip}(t)\dot{\gamma}_{ip}(t)\right\} \\ &= -(1/2)\xi^T(t)\mathbf{Q}_{\sigma(t^-_{l+1})}\xi(t) + \mathbf{r}^T_{\sigma(t^-_{l+1})}\left(\Psi_{\sigma(t^-_{l+1})} - (\mathbf{I} + \mathbf{E}_{\sigma(t^-_{l+1})})\Delta\tau_{\sigma(t^-_{l+1})}\right) \\ &\quad + \sum_{p=1}^{N}\sum_{i=0}^{2}\left\{(\hat{\theta}_{ip}(t) - \theta^*_{ip})\dot{\hat{\theta}}_{ip}(t) + \gamma_{ip}(t)\dot{\gamma}_{ip}(t)\right\} \\ &\leq -(1/2)\xi^T(t)\mathbf{Q}_{\sigma(t^-_{l+1})}\xi(t) + ||\Psi_{\sigma(t^-_{l+1})}||||\mathbf{r}_{\sigma(t^-_{l+1})}|| - (1 - \bar{E}_{\sigma(t^-_{l+1})})\rho_{\sigma(t^-_{l+1})}\frac{||\mathbf{r}_{\sigma(t^-_{l+1})}||^2}{\sqrt{||\mathbf{r}_{\sigma(t^-_{l+1})}||^2 + \varepsilon}} \\ &\quad + \sum_{p=1}^{N}\sum_{i=0}^{2}\left\{(\hat{\theta}_{ip}(t) - \theta^*_{ip})\dot{\hat{\theta}}_{ip}(t) + \gamma_{ip}(t)\dot{\gamma}_{ip}(t)\right\}. \end{aligned} \tag{18}$$

Notice that, here and in the following, the time index t is kept only for $\xi(t)$, $\hat{\theta}_{ip}(t), \gamma(t)$ and otherwise omitted for brevity. For the ease of analysis, we define a region such that

$$\omega \frac{||\mathbf{r}_\sigma||^2}{\sqrt{||\mathbf{r}_\sigma||^2+\varepsilon}} \geq ||\mathbf{r}_\sigma|| \Rightarrow ||\mathbf{r}_\sigma|| \geq \sqrt{\frac{\varepsilon}{\omega^2-1}} \triangleq \varphi. \tag{19}$$

The condition (19) implies that one needs to select $\omega > 1$, which is always possible since ω is a user defined scalar.

Establishing exponential decrease of the Lyapunov function: Up to now we have established (17) at switching instants, and (18) in between switching instants. A crucial mechanism for establishing stability of a switched system is that the possible jump of the Lyapunov function at (17) is compensated by some exponential decrease of the Lyapunov function via (18). Therefore, in the following we will rewrite (18) to highlight the exponential decrease. Subsequently, we proceed with the stability analysis for two scenarios:

S1: $||\mathbf{r}_\sigma|| \geq \varphi$ and

S2: $||\mathbf{r}_\sigma|| < \varphi$.

We study the behaviour of the Lyapunov function for these two scenarios as below.

Scenario $S1$**:** The adaptive law (12) reveals that the gains $\hat{\theta}_{i\overline{p}}$ and γ_{ip} remain constant during inactive and active intervals, respectively. Therefore, utilizing these observations and the upper bound structure (10) of uncertainty, (18) is simplified for $t \in (t_l \;\; t_{l+1})$ as

$$\begin{aligned}\dot{V}(t) \leq & -(1/2)\xi^T(t)\mathbf{Q}_{\sigma(t_{l+1}^-)}\xi(t) - \mathbf{Y}^T_{\sigma(t_{l+1}^-)}(\hat{\Theta}_{\sigma(t_{l+1}^-)} - \Theta^*_{\sigma(t_{l+1}^-)})||\mathbf{r}_{\sigma(t_{l+1}^-)}|| \\ & + \sum_{i=0,p=\sigma(t_{l+1}^-)}^{2} (\hat{\theta}_{ip}(t) - \theta^*_{ip})\dot{\hat{\theta}}_{ip}(t) + \sum_{\overline{p}\in\mathscr{N}(p)} \sum_{i=0}^{2} \gamma_{i\overline{p}}(t)\dot{\gamma}_{i\overline{p}}(t). \end{aligned} \tag{20}$$

Using (12a) we have for $p = \sigma(t_{l+1}^-)$

$$\begin{aligned}\sum_{i=0}^{2}(\hat{\theta}_{ip} - \theta^*_{ip})\dot{\hat{\theta}}_{ip} &= \sum_{i=0}^{2}(\hat{\theta}_{ip} - \theta^*_{ip})(||\mathbf{r}_p||||\xi||^i - \alpha_{ip}\hat{\theta}_{ip}) \\ &= \sum_{i=0}^{2} ||\mathbf{r}_p||(\hat{\theta}_{ip} - \theta^*_{ip})||\xi||^i + \alpha_{ip}\hat{\theta}_{ip}\theta^*_{ip} - \alpha_{ip}\hat{\theta}^2_{ip} \\ &= \mathbf{Y}^T_p(\hat{\Theta}_p - \Theta^*_p)||\mathbf{r}_p|| + \sum_{i=0}^{2}\{\alpha_{ip}\hat{\theta}_{ip}\theta^*_{ip} - \alpha_{ip}\hat{\theta}^2_{ip}\}. \end{aligned} \tag{21}$$

Similarly, (41) leads to

$$\gamma_{i\overline{p}}\dot{\gamma}_{i\overline{p}} = -\left(\beta_{i\overline{p}} + \overline{\nu}_{i\overline{p}}\hat{\theta}^4_{i\overline{p}}\right)\gamma^2_{i\overline{p}} + \beta_{i\overline{p}}\nu_{i\overline{p}}\gamma_{i\overline{p}}. \tag{22}$$

From (13) we have $\gamma_{i\overline{p}} \geq \underline{\gamma}_{i\overline{p}} \;\; \forall t \geq t_0$. Applying this relation to the second term of (22) yields

$$\gamma_{i\overline{p}}\dot{\gamma}_{i\overline{p}} \leq -\beta_{i\overline{p}}\gamma^2_{i\overline{p}} - \underline{\gamma}^2_{i\overline{p}}\overline{\nu}_{i\overline{p}}\hat{\theta}^4_{i\overline{p}} + \beta_{i\overline{p}}\nu_{i\overline{p}}\gamma_{i\overline{p}}. \tag{23}$$

Note that the following equality holds

$$-\underline{\gamma}^2_{i\overline{p}}\overline{\nu}_{i\overline{p}}\hat{\theta}^4_{i\overline{p}} + (\zeta_{\overline{p}}/2)\hat{\theta}^2_{i\overline{p}} = -(\underline{\gamma}_{i\overline{p}}\sqrt{\overline{\nu}_{i\overline{p}}}\hat{\theta}^2_{i\overline{p}} - (\zeta_{\overline{p}}/(4\underline{\gamma}_{i\overline{p}}\sqrt{\overline{\nu}_{i\overline{p}}})))^2 + \zeta^2_{\overline{p}}/(16\overline{\nu}_{i\overline{p}}\underline{\gamma}^2_{i\overline{p}}). \tag{24}$$

Substituting (21), (23) and (24) in (20) yields for $t \in (t_l \;\; t_{l+1})$

$$\dot{V}(t) \leq -(1/2)\lambda_{\min}(\mathbf{Q}_{\sigma(t_{l+1}^-)})||\xi(t)||^2 + \sum_{i=0}^{2} \alpha_{ip}\hat{\theta}_{ip}(t)\theta_{ip}^* - \alpha_{ip}\hat{\theta}_{ip}^2(t) - \left(\sum_{\overline{p}\in\mathscr{N}(p)} \sum_{i=0}^{2} \beta_{i\overline{p}}\gamma_{i\overline{p}}^2 + \hat{\theta}_{i\overline{p}}^2 - \beta_{i\overline{p}}\underline{\gamma}_{i\overline{p}}\gamma_{i\overline{p}} \right)$$
$$-(1/2)\hat{\theta}_{i\overline{p}}^2 + \zeta_{\overline{p}}^2/(16\overline{\nu}_{i\overline{p}}\underline{\gamma}_{i\overline{p}}^2). \tag{25}$$

Since $\hat{\theta}_{ip} \geq 0$ by design (13), the definition of Lyapunov function (16) yields

$$V(\xi(t), \tilde{\theta}_{ip}(t), \gamma_{ip}(t), t) \leq \frac{1}{2}\lambda_{\max}(\mathbf{P}_\sigma)||\xi||^2 + \frac{1}{2}\sum_{p=1}^{N}\sum_{i=0}^{2}(\hat{\theta}_{ip}^2 + \theta_{ip}^{*\,2} + \gamma_{ip}^2), \;\; \forall t. \tag{26}$$

The definitions of $\zeta, \zeta_{Mp}, \alpha_{ip}, \beta_{i\overline{p}}$ and and the use of (26), allows to simplify the condition (25) into

$$\dot{V}(t) \leq -\zeta V(t) + \sum_{i=0,\; p=\sigma(t_{l+1}^-)}^{2} \alpha_{ip}\hat{\theta}_{ip}(t)\theta_{ip}^* - \bar{\alpha}_{ip}\hat{\theta}_{ip}^2(t) + \gamma_{ip}^2(t) + \sum_{p=1}^{N}\sum_{i=0}^{2}\frac{\zeta_p}{2}\theta_{ip}^{*\,2}$$
$$+ \sum_{\overline{p}\in\mathscr{N}(p)}\sum_{i=0}^{2}\{\beta_{i\overline{p}}\underline{\gamma}_{i\overline{p}}\gamma_{i\overline{p}}(t) - \bar{\beta}_{i\overline{p}}\gamma_{i\overline{p}}^2(t) + \zeta_{\overline{p}}^2/(16\overline{\nu}_{i\overline{p}}\underline{\gamma}_{i\overline{p}}^2)\}, \;\; t \in (t_l \;\; t_{l+1}) \tag{27}$$

where $\bar{\alpha}_{ip} \triangleq (\alpha_{ip} - \frac{\zeta_p}{2}) > 0$ and $\bar{\beta}_{i\overline{p}} \triangleq (\beta_{i\overline{p}} - \frac{\zeta_{\overline{p}}}{2}) > 0$. Note that the following equality holds

$$\alpha_{ip}\hat{\theta}_{ip}\theta_{ip}^* - \bar{\alpha}_{ip}\hat{\theta}_{ip}^2 = -\bar{\alpha}_{ip}\left(\hat{\theta}_{ip} - \frac{\alpha_{ip}\theta_{ip}^*}{2\bar{\alpha}_{ip}}\right)^2 + \frac{\left(\alpha_{ip}\theta_{ip}^*\right)^2}{4\bar{\alpha}_{ip}}. \tag{28}$$

The adaptive laws (12) reveals that $\gamma_{i\overline{p}}$ decreases for the inactive systems and remains unchanged for the active system. Together with inequality $\gamma_{ip} \geq \underline{\gamma}_{ip} \;\; \forall t \geq t_0$, we obtain that $\gamma_{ip} \in \mathscr{L}_\infty \; \forall p \in \Omega$. Therefore, there exists $\bar{\gamma}_{ip} \in \mathbb{R}^+$ such that $\gamma_{ip}(t) \leq \bar{\gamma}_{ip}$. Using κ such that $0 < \kappa < \zeta$ and using (28), we have that $\dot{V}(t)$ in (27) simplifies to

$$\dot{V}(t) \leq -\kappa V(t) - (\zeta - \kappa)V(t) + \varsigma + \varsigma_2, \;\; t \in (t_l \;\; t_{l+1}) \tag{29}$$

with $\varsigma \triangleq \sum_{p=1}^{N}\sum_{i=0}^{2}\frac{\zeta_p}{2}\theta_{ip}^{*\,2} + \beta_{ip}\underline{\gamma}_{ip}\bar{\gamma}_{ip} + (\zeta_{\overline{p}}^2/(16\overline{\nu}_{i\overline{p}}\underline{\gamma}_{i\overline{p}}^2))$ and $\varsigma_2 \triangleq \sum_{i=0,p=\sigma(t_{l+1}^-)}^{2}\frac{\left(\alpha_{ip}\theta_{ip}^*\right)^2}{4\bar{\alpha}_{ip}} + \bar{\gamma}_{ip}^2$. Note that (29) highlights, for Scenario 1, the exponential decrease of the Lyapunov function in a region around the origin.

Scenario $S2$**:** In this scenario we have $||\mathbf{r}_\sigma|| < \varphi$. Therefore, we have for $t \in (t_l \;\; t_{l+1})$

$$\dot{V}(t) \leq -(1/2)\xi^T(t)\mathbf{Q}_{\sigma(t_{l+1}^-)}\xi(t) - (1 - \bar{E}_{\sigma(t_{l+1}^-)})\rho_{\sigma(t_{l+1}^-)}\frac{||\mathbf{r}_{\sigma(t_{l+1}^-)}||^2}{\sqrt{||\mathbf{r}_{\sigma(t_{l+1}^-)}||^2 + \varepsilon}} + \mathbf{Y}_{\sigma(t_{l+1}^-)}^T\Theta_{\sigma(t_{l+1}^-)}^*||\mathbf{r}_{\sigma(t_{l+1}^-)}||$$
$$+ \sum_{i=0,p=\sigma(t_{l+1}^-)}^{2}(\hat{\theta}_{ip}(t) - \theta_{ip}^*)\dot{\hat{\theta}}_{ip}(t) + \sum_{\overline{p}\in\mathscr{N}(p)}\sum_{i=0}^{2}\gamma_{i\overline{p}}(t)\dot{\gamma}_{i\overline{p}}(t)$$
$$\leq -(1/2)\xi^T(t)\mathbf{Q}_{\sigma(t_{l+1}^-)}\xi(t) + \mathbf{Y}_{\sigma(t_{l+1}^-)}^T\Theta_{\sigma(t_{l+1}^-)}^*||\mathbf{r}_{\sigma(t_{l+1}^-)}||$$
$$+ \sum_{i=0,p=\sigma(t_{l+1}^-)}^{2}(\hat{\theta}_{ip}(t) - \theta_{ip}^*)\dot{\hat{\theta}}_{ip}(t) + \sum_{\overline{p}\in\mathscr{N}(p)}\sum_{i=0}^{2}\gamma_{i\overline{p}}(t)\dot{\gamma}_{i\overline{p}}(t). \tag{30}$$

Then, following similar lines as in Scenario $S1$, we have for $t \in (t_l \;\; t_{l+1})$

$$\dot{V}(t) \leq -\kappa V(t) - (\zeta - \kappa)V(t) + \mathbf{Y}^T_{\sigma(t^-_{l+1})}\Theta^*_{\sigma(t^-_{l+1})}||\mathbf{r}_{\sigma(t^-_{l+1})}|| + \varsigma + \varsigma_2. \tag{31}$$

From (7) one can verify $||\mathbf{r}|| < \varphi \Rightarrow ||\xi|| \in \mathscr{L}_\infty$ and consequently, the adaptive law (12a) implies $||\mathbf{r}||, ||\xi|| \in \mathscr{L}_\infty \Rightarrow \hat{\theta}_{ip}(t) \in \mathscr{L}_\infty$. Therefore, $\exists \varsigma_1 \in \mathbb{R}^+$ such that $\mathbf{Y}^T_{\sigma(t^-_{l+1})}\Theta^*_{\sigma(t^-_{l+1})} \leq \varsigma_1 \;\forall \sigma \in \Omega$ when $||\mathbf{r}_\sigma|| < \varphi$. Hence, replacing this relation in (32) yields

$$\dot{V}(t) \leq -\kappa V(t) - (\zeta - \kappa)V(t) + \varphi\varsigma_1 + \varsigma + \varsigma_2, \;\; t \in (t_l \;\; t_{l+1}). \tag{32}$$

Note that (32) highlights, for Scenario 2, the exponential decrease of the Lyapunov function in a region around the origin. Further, define the scalar

$$\mathscr{B} \triangleq \frac{\varphi\varsigma_1 + \varsigma + \varsigma_2}{(\zeta - \kappa)}. \tag{33}$$

From the two possible scenarios $S1$ and $S2$, it can be concluded that $\dot{V}(t) \leq -\kappa V(t)$ when $V(t) \geq \mathscr{B}$ for $t \in (t_l \;\; t_{l+1})$.

Overall behavior of the Lyapunov function: In light of this, further analysis is needed to observe the overall behaviour of $V(t)$, $t \in (t_l \;\; t_{l+1})$ for two possible cases:

(i) when $V(t) \geq \mathscr{B}$, we have $\dot{V}(t) \leq -\kappa V(t)$ from (29) implying exponential decrease of $V(t)$;

(ii) when $V(t) < \mathscr{B}$, $V(t)$ may increase.

The analysis of the overall behaviour of $V(t)$ (i.e. the combined behaviour at and in between switching instants) follows similar steps as the analysis of cases (i), (ii) in the proof of Theorem 1 in [31]. Therefore, we do repeat the analysis to avoid repetitions. At the end of such analysis, one obtains that once $V(t)$ enters the interval $[0, \mathscr{B}]$, it cannot exceed the bound $c\mu\mathscr{B}$ any time later with the ADT switching law (14), where $c \triangleq \exp(N_0 \ln \mu)$ is a constant.

Ultimate bound on tracking error: Further, based on this analysis, we have

$$V(t) \leq \max\{cV(t_0), c\mu\mathscr{B}\}, \;\; \forall t \geq t_0. \tag{34}$$

Again, the definition of the Lyapunov function (16) yields

$$V(\xi(t), \tilde{\theta}_{ip}(t), \gamma_{ip}(t), t) \geq (1/2)\lambda_{\min}(\mathbf{P}_{\sigma(t)})||\xi(t)||^2 \geq (\underline{\zeta}_m/2)||\xi(t)||^2, \;\; \forall t. \tag{35}$$

Using (34) and (35) we have

$$||\xi(t)||^2 \leq (2/\underline{\zeta}_m)\max\{cV(t_0), c\mu\mathscr{B}\}, \;\; \forall t \geq t_0. \tag{36}$$

Therefore, using the expressions of μ, $\mathscr{B}$ and c from (14), (33) and (30), an ultimate bound b on the tracking error ξ can be found from (36) as

$$b = \sqrt{\frac{2\bar{\zeta}_M^{(N_0+1)}(\varphi\varsigma_1 + \varsigma + \varsigma_2)}{\underline{\zeta}_m^{(N_0+2)}(\zeta - \kappa)}}. \tag{37}$$

Observing the stability arguments, it can be concluded that the closed-loop system is UUB with the control laws (8) and (11) in conjunction with the adaptive law (12) and switching law (14). □

Remark 4. *The importance of the auxiliary gain $\gamma_{i\overline{p}}$ in system stability can be realized from the following two observations: first, the term $\hat{\theta}^2_{i\overline{p}}$ on the right hand side of (26) was specifically cancelled by the similar term on the right side of (25), a consequence of the relations (22)-(24), to arrive at (27). In absence of $\gamma_{i\overline{p}}$ this would not have been achieved and, system stability could not be ensured. The second observation concerns the selection of a positive lower bound for $\gamma_{i\overline{p}}$ in (13). Note that the second term on the right side of (23) comes from the corresponding term of (22), by utilizing the condition $\gamma_{i\overline{p}} \geq \underline{\gamma}_{i\overline{p}}$. This validates the utility of the selection of a positive lower bound for $\gamma_{i\overline{p}}$ while the lower bounds for the other gains $\hat{\theta}_i$ $i = 0, 1, 2$ are designed as zero.*

Remark 5. *The gain (11) reveals that if $\bar{E}_\sigma$ is taken close to 1, then the gain ρ_σ increases (as $\frac{1}{1-\bar{E}_\sigma}$ increases); higher values of ω in (8b) also increase the effective weight of ρ_σ in $\Delta\tau_\sigma$: these higher gain conditions lead to faster robust adaptation (via $\Delta\tau_\sigma$) at the cost of higher control input τ_σ (cf. (8a)). Therefore, these parameters should be tuned according to the trade-off between tracking performance and control effort depending on application requirements.*

5. Simulation Results

This section studies the effectiveness of the proposed controller using a simplified scenario with pick-and-place robotic manipulator, often modelled via switched EL dynamics with two subsystems with different system parameters (one for the pick phase and one for the place phase) [34]:

$$\mathbf{M}_\sigma(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}_\sigma(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}_\sigma(\mathbf{q}) + \mathbf{F}_\sigma(\dot{\mathbf{q}}) + \mathbf{d}_\sigma = \tau_\sigma, \tag{38}$$

$$\begin{aligned}
\mathbf{M}_\sigma &= \begin{bmatrix} M_{\sigma_{11}} & M_{\sigma_{12}} \\ M_{\sigma_{12}} & M_{\sigma_{22}} \end{bmatrix}, \mathbf{q} = \begin{bmatrix} q_l \\ q_u \end{bmatrix}, \\
&M_{\sigma_{11}} = (m_{\sigma_l} + m_{\sigma_u}) l^2_{\sigma_u} + m_{\sigma_u} l_{\sigma_l}(l_{\sigma_l} + 2 l_{\sigma_u}\cos(q_u)), \\
&M_{\sigma_{12}} = m_{\sigma_u} l_{\sigma_u}(l_{\sigma_u} + l_{\sigma_l}\cos(q_u)), M_{\sigma_{22}} = m_{\sigma_u} l^2_{\sigma_u}, \\
\mathbf{C}_\sigma &= \begin{bmatrix} -m_{\sigma_u} l_{\sigma_l} l_{\sigma_u} \sin(q_u)\dot{q}_u & -m_{\sigma_u} l_{\sigma_l} l_{\sigma_u} \sin(q_u)(\dot{q}_l + \dot{q}_u) \\ 0 & m_{\sigma_u} l_{\sigma_l} l_{\sigma_u} \sin(q_u)\dot{q}_u \end{bmatrix}, \\
\mathbf{G}_\sigma &= \begin{bmatrix} m_{\sigma_l} l_{\sigma_l} g\cos(q_l) + m_{\sigma_u} g(l_{\sigma_u}\cos(q_l + q_u) + l_{\sigma_l}\cos(q_l)) \\ m_{\sigma_u} g l_{\sigma_u}\cos(q_l + q_u) \end{bmatrix}, \\
\mathbf{F}_\sigma &= \begin{bmatrix} f_{\sigma_{vl}} \frac{\dot{q}_l}{\sqrt{\dot{q}_l^2 + 0.1}} \\ f_{\sigma_{vu}} \frac{\dot{q}_u}{\sqrt{\dot{q}_u^2 + 0.1}} \end{bmatrix}, \mathbf{d}_\sigma = \begin{bmatrix} 0.5\sin(0.05t) \\ 0.5\sin(0.05t) \end{bmatrix},
\end{aligned}$$

where (m_{p_l}, l_{p_l}, q_l) and (m_{p_u}, l_{p_u}, q_u) denote the mass, length and position of link 1 and 2 respectively for subsystem p with $p = [1, 2]$. Note that the term $\mathbf{F}_\sigma$ approximates an unknown static friction. The actual (and unknown) parametric values of the manipulator subsystems are taken as

1. $m_{1_l} = m_{1_u} = 2.4$kg, $l_{1_l} = l_{1_u} = 1$m, $f_{1_{vl}} = f_{1_{vu}} = 0.6$,
2. $m_{2_l} = m_{2_u} = 3.6$kg, $l_{2_l} = l_{2_u} = 1$m, $f_{2_{vl}} = f_{2_{vu}} = 0.8$,

with $g = 9.8$m/sec^2 for both subsystems. Note that each of the subsystems satisfies Properties 1-2 in Assumption 1, whereas the objective is to track a desired trajectory defined as $\{q^d_l, q^d_u\} = \{\sin(0.5t), 0.5\sin(0.5t)\}$rad.

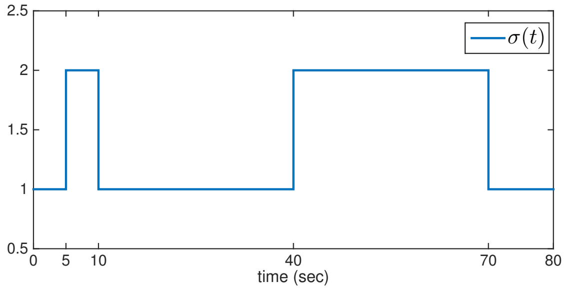

Figure 1: The switching signal.

Selection of $\mathbf{K}_{11} = 150\mathbf{I}, \mathbf{K}_{21} = 90\mathbf{I}, \mathbf{K}_{12} = 200\mathbf{I}, \mathbf{K}_{22} = 100\mathbf{I}, \mathbf{Q}_1 = \mathbf{Q}_2 = 0.2\mathbf{I}$ yields the ADT $\vartheta^* =$ 7.70sec according to (14) when $\kappa = 0.9\zeta$. Therefore, a switching law $\sigma(t)$ is designed as in Fig. 1 (fast switching is compensated by slower switching).

To design $\hat{\mathbf{M}}_p$ in (8a), we select the nominal parameter as $m_{1_l} = m_{1_u} = 2.0$kg, $l_{1_l} = l_{1_u} = 0.9$m and $m_{2_l} = m_{2_u} = 3.0$kg, $l_{2_l} = l_{2_u} = 0.9$m: for such nominal values, Assumption 2 is satisfied with $\bar{E}_1 = \bar{E}_2 = 0.7$. The terms $\mathbf{C}_\sigma, \mathbf{F}_\sigma, \mathbf{G}_\sigma$ and $\mathbf{d}_\sigma$ are considered to be completely unknown. Other control design parameters are selected as $\varepsilon = 0.1, \omega = 2, \alpha_{ip} = \beta_{i\overline{p}} = 0.5, \bar{v}_{i\overline{p}} = 1, v_{i\overline{p}} = 10^{-4}$ with $i = 0, 1, 2$. The initial gain and link positions are selected as $\hat{\theta}_{0p}(0) = 1.5 \times 10^{-4}, \hat{\theta}_{1p}(0) = 5 \times 10^{-5}, \hat{\theta}_{2p}(0) = 3 \times 10^{-5}, \gamma_{i\overline{p}}(0) = 1.5 \times 10^{-4}$ and $q_l(0) = q_u(0) = 0.5$rad, respectively.

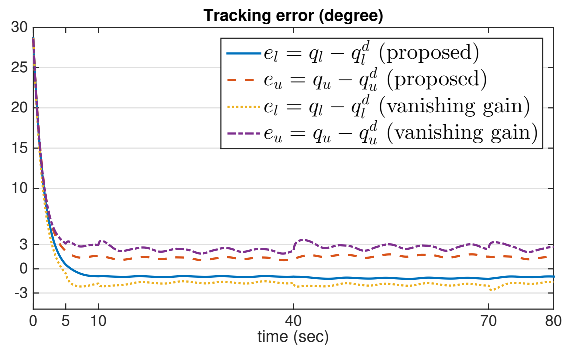

Figure 2: Tracking performance comparison.

To properly judge the performance of the new leakage mechanism against state-of-the-art vanishing gain mechanism, we have compared the proposed controller with the following one, inspired from [26]

$$\rho_\sigma = \frac{1}{1 - \bar{E}_\sigma}\{\hat{\theta}_{0\sigma} + \hat{\theta}_{1\sigma}||\xi|| + \hat{\theta}_{2\sigma}||\xi||^2\} \tag{39}$$

$$\dot{\hat{\theta}}_{ip} = ||\mathbf{r}_p||||\xi||^i - \alpha_{ip}\hat{\theta}_{ip}, \tag{40}$$

$$\dot{\hat{\theta}}_{i\overline{p}} = -\alpha_{i\overline{p}}\hat{\theta}_{i\overline{p}}, \tag{41}$$

$$\text{with } \hat{\theta}_{ip}(t_0), \hat{\theta}_{i\overline{p}}(t_0) > 0, \; i = 0, 1, 2, \tag{42}$$

while (8) remains unchanged. Note that (39)-(42) is a vanishing gain scheme, where the inactive gains $\hat{\theta}_{i\overline{p}}$ decrease exponentially. For parity in comparison, the same control parameters and initial gain conditions are selected for both controllers with $\alpha_{\bar{ip}} = 0.5$.

The tracking performance of the proposed controller is depicted in Fig. 2, in comparison with the vanishing gain scheme. It is clear that, beside having worse steady-state performance, the vanishing gain scheme has worse transient behaviour at each switching instants. This can be clearly explained by the evolutions of control gains for these two control schemes, as shown in Figs. 3-5. For all these figures it is worth mentioning that $\hat{\theta}_{0\sigma}, \hat{\theta}_{1\sigma}, \hat{\theta}_{2\sigma}$ have different orders of magnitude but similar trends ($\hat{\theta}_{0\sigma}$ is the largest gain and one should zoom in to better see the trends of $\hat{\theta}_{1\sigma}$ and $\hat{\theta}_{2\sigma}$).

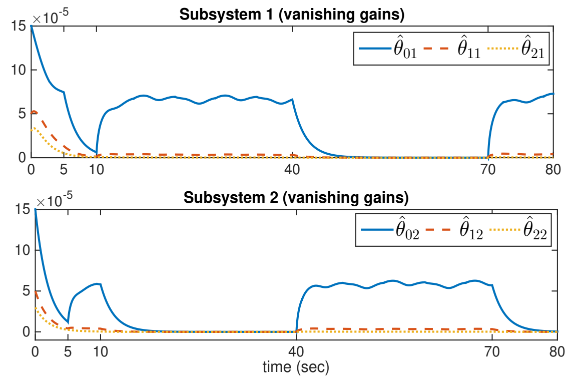

Figure 3: Evolution of gains under 'vanishing gain' scheme.

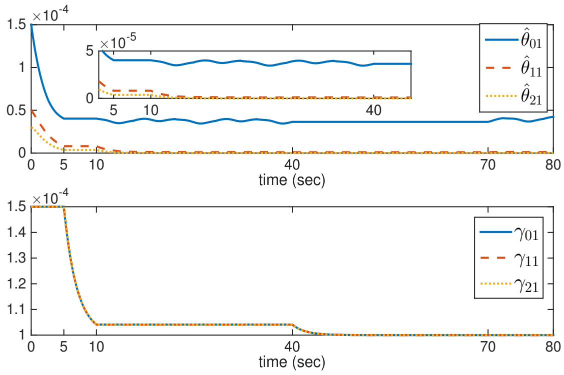

Figure 4: Gains for subsystem 1 with the proposed controller.

Fig. 3 clearly shows the state-of-the-art vanishing gain mechanism during inactive times: for example, during $t = [5-10)$sec and $t = [40-70)$sec when subsystem 1 was switched-off, $\hat{\theta}_{01}$ kept decreasing (i.e., kept vanishing); as a result, when it was again switched on at $t = 10$sec and $t = 70$sec, it had to adapt itself again, causing a transient at every switch-on instances. Similar situation can be noticed for $\hat{\theta}_{02}$ as well. Control gains dropping to zero essentially means no control, which is in general not desirable. Whereas, for the proposed scheme, Figs. 4 and 5 highlight that the vanishing trend has been removed. The different orders of magnitude also show that the upper bound structure in (5) is adaptively shaped in such a way to give more weight to low power coefficients (i.e. $\hat{\theta}_{0\sigma}$). To further demonstrate robustness against noise, the tracking performance of the proposed controller is depicted in Fig. 6 when a Gaussian noise of variance 0.001 is inserted in the feedback path for both $\mathbf{q}$ and $\dot{\mathbf{q}}$.

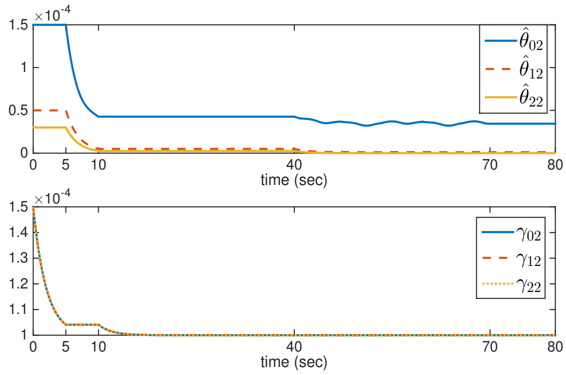

Figure 5: Gains for subsystem 2 with the proposed controller.

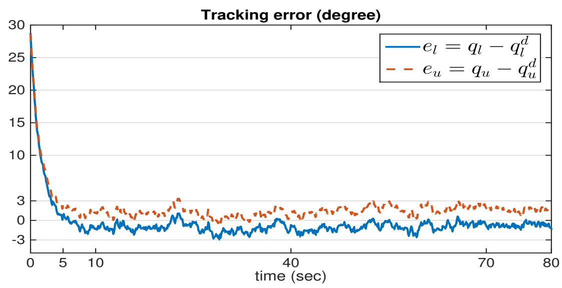

Figure 6: Tracking performance of the proposed controller with noise.

6. Conclusions

A new concept of robust adaptation with leakage mechanism for uncertain switched EL systems was presented in this paper. The issue of vanishing gains of inactive subsystems was completely eliminated by virtue of properly designed auxiliary gains. At the same time, unmodelled dynamics and uncertainties with no a priori bounds could be handled by a quadratic state-dependent upper bound structure that reduces conservativeness as compared to state-of-the-art structures. Bounded stability analysis and simulations with a pick-and-place robotic manipulator example have been provided.

Relevant future work would be to consider state-dependent switching or impulsive behaviour, or larger classes of nonlinear systems such as underactuated or nonholonomic systems [35, 36].

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