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Identification of Affinely Parameterized State-Space Models with Unknown Inputs

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***Abstract*—The identification of affinely parameterized state-space system models is quite popular to model practical physical systems or networked systems, and the traditional identification methods require the measurements of both the input and output data. However, in the presence of partial unknown input, the corresponding system identification problem turns out to be challenging and sometimes unidentifiable. This paper provides the identifiability conditions in terms of the structural properties of the state-space model and presents an identification method which successively estimates the system states and the affinely parameterized system matrices. The estimation of the system matrices boils down to solving a bilinear optimization problem, which is reformulated as a difference-of-convex (DC) optimization problem and handled by the sequential convex programming method. The effectiveness of the proposed identification method is demonstrated numerically by comparing with the Gauss-Newton method and the sequential quadratic programming method.**

***Index Terms*—Subspace identification, affinely parameterized state-space model, unknown system input.**

I. Introduction

Along with the extensive research on complex networked systems, the identification of affinely parameterized (or structured) state-space models has attracted interest from the identification community [1]–[3], which provides a foundation to the model-based controller design, such as robust H_∞ control [4] and model-based predictive control [5]. For the structured state-space models that are used for depicting complex physical systems or networked systems, the non-zero entries always have physical interpretations [6]–[10]. Relevant examples are compartmental systems [6], [11], hydraulic systems and power systems [12]. Due to the importance of structured state-space models, the corresponding identification problem is investigated in this paper when there are some unmeasurable inputs. The unmeasurable inputs can be some undesired excitations acting on a networked system model [13], such as the attack signal or the actuator failure [14]. As a result, the identification of structured system models with unknown inputs is practically meaningful for modeling complex networked systems in harsh environment.

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In this paper, the unknown input signal is restricted to be deterministic and persistently exciting. The system identification with deterministic unknown inputs mainly relies on the basic algebraic calculation, such as greatest common polynomial divisor extraction [7] and the low rank singular value decomposition [15]. For the deterministic blind identification, i.e., to identify the system model and system input from only the output measurements, it is usually impossible without exploiting the specific structural property of the system model or the specific pattern of the input signal. In the literature, there are mainly two approaches to deal with this challenging identification problem: one is to assume the unknown input signal to be modelled via a restricted number of known basis functions [16]–[19], and the other is to assume the system transfer function (or impulse response) to have single-input multi-output (SIMO) structures [13], [20], [21]. Note that it is usually not enough for the blind system identification by only assuming the transfer function to be linearly represented in terms of a finite number of known basis functions; as a compensation, both the transfer function and the unknown input were assumed to be modelled via a finite number of known basis functions in [22]–[24].

The blind identification problem has been intensively investigated by exploiting various specific patterns of the unknown input signal. For the blind identification of an IIR filter, by assuming the unknown input signal to be piecewise constant (or over-sampled) with a fixed period, the corresponding identification problem can be reformulated as a blind SIMO system identification problem which can then be addressed by the structured subspace factorization [16], [17]. More generally, by assuming that the unknown input lies in a known subspace, the blind identification of an ARX (autoregressive with external input) model can be formulated as a low-rank optimization problem, which was handled by solving the corresponding nuclear norm optimization problem [18], [23], [25]. The blind identification of a time-varying state-space model with fully observed states was investigated in [19]. By assuming that the unknown input is sparse and the same time-varying dynamic is excited by different batches of input data, the system matrices and the unknown input can be estimated even though we do not know on which states the unknown inputs act [19].

The blind identification of LTI systems without any restrictions on the unknown system inputs (other than the persistent excitation condition) mainly relies on the SIMO framework, i.e., the same input signal is filtered through different transfer functions [20]–[22]. By exploiting the correlations among different output signals, the unknown transfer functions can be directly identified from the output sequences up to a scalar ambiguity (or a matrix ambiguity for the blind MIMO system identification). From the multi-agent system perspective, the SIMO system model indicates that multiple agents are excited by the same input signal, where the multiple agents are totally decoupled; therefore, the SIMO system model represents only a special networked system. For the identification of local modules in a dynamic network identification, the immersion approach [26], the elimination approach [13], the abstraction approach [27] and the subspace intersection approach [28] were developed to eliminate/remove the unmeasurable (or selected) node signals through representing them in terms of other measured node signals in the network. Although some node signal in a network cannot be measured, its information is (partially) embedded in other measured node signal through the network representation. Therefore, the identification approaches in [13], [26], [27] inherently make use of some a priori knowledge of the unmeasurable node signals to some extent.

In this paper, the blind identification of LTI structured systems without any restrictions on the unknown input signal (other than the persistent excitation condition) is considered. Different from the SIMO framework [20], [21] which is inherently a networked system consisting of several decoupled LTI subsystems, the structured state-space model that represents a more general networked system is investigated. Identifiability conditions in terms of the system structural properties are provided, and a subspace-based blind identification method is presented. In the developed identification method, the system state sequence is firstly estimated up to a similarity transformation, and the structured system matrices are identified together with the similarity transformation matrix by solving a bilinear estimation problem. The above identification idea is similar to the classic N4SID subspace method [8], [9]; however, the structural constraints of the system matrices as well as the unknown input signal pose challenges for the system identification problem. The contributions of this paper are summarized as follows.

1) The considered structured state-space model in this paper can be used to represent a networked system that has a more general topological structure than the SIMO framework [20], [21], [29]; thus, it has a wider range of applications. In addition, the identifiability conditions for the structured state-space models with unknown inputs are provided.
2) The presented identification method does not need to know the basis of the unknown input sequence. This differs from the immersion approach [26], the elimination approach [13] and the abstraction approach [27] where the unmeasurable (or selected) node signals can be (partially) represented in terms of other measured node signals in the network.
3) Different from the subspace-based gray-box identification problem [3], [30], [31], when there exists some unknown input, the system matrices/Markov parameters/transfer functions cannot be straightforwardly identified using the classical identification methods. However, inspired from the N4SID method [32], the identification problem is dealt with by firstly estimating the system state in the presence of unknown input, following the identification of structured system matrices.
4) Many existing blind identification approaches require to solve a bilinear estimation problem, which are usually handled by the convex relaxed methods such as nuclear norm optimization [18], [23], [25]. In this paper, this bilinear estimation problem is reformulated as a difference-of-convex function which is then solved by the sequential convex programming technique. The initial condition is obtained by solving the convex part of difference-of-convex problem, which turns out to be a nuclear norm regularized least-squares optimization problem. This initial parameter estimate can be used to initialize the gradient methods such as the Gauss-Newton method.

The rest of this paper is organized as follows. In Section II, the identification problem for the structured state-space model with some unknown input is formulated. In Section III, identifiability conditions for the concerned identification problem are provided. In Section IV, a subspace identification method is presented which successively estimates the system state and the structured system matrices. In Section V, the performance of the proposed identification method is demonstrated through numerical simulations, following the conclusions in Section VI.

II. Problem formulation

In this paper, we consider the identification of a parameterized LTI state-space system model. Let $\theta \in \mathbb{R}^l$ be a parameter vector. The concerned state-space system model is given as

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B(\theta)u(k) + Hf(k) \\ y(k) &= C(\theta)x(k) + w(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n, y(k) \in \mathbb{R}^p$ and $w(k) \in \mathbb{R}^p$ are the system state, output, and measurement noise, respectively; $u(k) \in \mathbb{R}^m$ is the measurable input, while $f(k) \in \mathbb{R}^r$ represents the unknown input signal. The system matrices $A(\theta), B(\theta)$ and $C(\theta)$ are assumed to be affine with respect to $\theta \in \mathbb{R}^l$:

$$\begin{aligned} A(\theta) &= A_0 + A_1\theta_1 + \cdots + A_l\theta_l, \\ B(\theta) &= B_0 + B_1\theta_1 + \cdots + B_l\theta_l, \\ C(\theta) &= C_0 + C_1\theta_1 + \cdots + C_l\theta_l, \end{aligned} \tag{2}$$

where A_i, B_i, C_i for $i = 0, 1, \cdots, l$ are known coefficient matrices and θ_i for $i = 1, 2, \cdots, l$ is the i-th component of θ. For notational simplicity, A, B, C sometimes are adopted as the abbreviations of $A(\theta), B(\theta), C(\theta)$ in the context, respectively. The coefficient matrix $H \in \mathbb{R}^{n\times r}$ is assumed to be known; otherwise, the matrix H is not identifiable due to the coupling with the unknown input signal $f(k)$. Also, the matrix H is assumed to have full column rank; otherwise, the term $Hf(k)$ can be more parsimoniously parameterized. Since the parameters $\{\theta_i\}_{i=1}^l$ usually have physical interpretations for a structured system model, their values are assumed to be nonzero throughout the paper.

Remark 1. *By taking into account the unknown input signal, the state-space model in* (1) *can be used to model a wider*

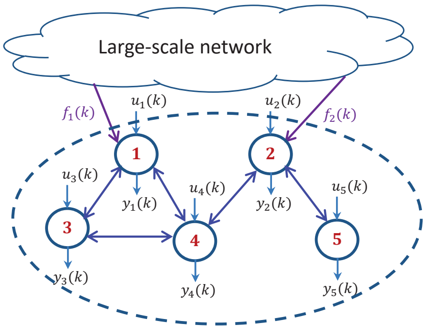

Fig. 1. A local system operating in a large-scale network. $f_1(k)$ and $f_2(k)$ are unmeasurable input signals. $\{u_i(k)\}_{i=1}^5$ and $\{y_i(k)\}_{i=1}^5$ are respectively the measurable input and output signals.

range of practical systems. For instance, it can be used to model the room temperatures in an air-conditioned building with the people who walk in/out to be considered as an unknown signal which may not possess any stochastic properties. Moreover, it can be exploited to model the local system of a large-scale network (see Fig. 1), where the local system can be described as a structured state-space model and the external unmeasurable interconnection signals can be regarded as the unknown inputs.

The *problem of interest* is to estimate the system parameter vector θ using the measurable input and output data $\{u(k), y(k)\}$ of the system model (1) in the presence of the unknown input $f(k)$. Different from the recently developed blind identification methods [18] [19], the proposed method does not require to know the basis of the unknown input $f(k)$ and only relies on the deterministic and persistent excitation conditions. In addition, the considered structured state-space model is more general than the traditional SIMO framework [20] [29]. In order to focus on the essence of the system identification scheme, we assume that the system order is known, and the following standard assumptions are adopted throughout the paper.

- A1 The system determined by the matrix tuple $(A, B, C, 0)$ is minimal and stable;
- A2 The input signal $[u^T(k) \;\; f^T(k)]^T$ is persistently exciting [8];
- A3 The sequence $w(k)$ is a zero-mean white noise with variance $\sigma_w^2 I$ and is uncorrelated with the input signal $u(k)$.

The main notations used throughout the paper are listed below. The horizontally concatenated state sequence is defined as

$$x(k_1 : k_2) = [x(k_1) \; x(k_1+1) \; \cdots \; x(k_2)] \, .$$

The row subspace of $x(k_1 : k_2)$ is denoted as $\mathrm{Row}[x(k_1 : k_2)]$. $\mathrm{Diag}(a_1, a_2, \cdots, a_n)$ denotes a $n \times n$ block diagonal matrix with $a_1, a_2, \cdots, a_n$ being its block diagonal entries. Given the nonnegative integers s and h such that $h \gg s$, we define the output related block Hankel matrix as

$$\mathcal{H}_{s,h}[y(1 : s+h-1)] = \begin{bmatrix} y(1) & y(2) & \cdots & y(h) \\ y(2) & y(3) & \cdots & y(h+1) \\ \vdots & \vdots & \iddots & \vdots \\ y(s) & y(s+1) & \cdots & y(s+h-1) \end{bmatrix}.$$

The extended observability matrix and the Markov parameter related block Toeplitz matrix are respectively defined as

$$\mathcal{O}_s(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{bmatrix}$$

$$\mathcal{T}_s(A, B, C) = \begin{bmatrix} 0 & & \\ CB & \ddots & \\ \vdots & \ddots & 0 \\ CA^{s-2}B & \cdots & CB \end{bmatrix}.$$

III. Identifiability analysis

In this section, the identifiability analysis for the system in (1) will be carried out under two different scenarios: (a) the system state $x(k)$ can be fully observed; (b) the state $x(k)$ cannot be directly measured. To facilitate the identifiability analysis in the sequel, the output observation is assumed to be noise-free, i.e., $\hat{y}(k) = Cx(k)$, but the measurement noise will be taken into account in the algorithm development and the numerical simulations.

A. Identifiability analysis based on full state observation

Suppose that the system state $x(k)$ can be fully observed. The noise-free state-space model in (1) can be equivalently written as

$$\begin{aligned} x(k+1) &= \underbrace{(A + HQ_1)}_{\hat{A}} x(k) + \underbrace{(B + HQ_2)}_{\hat{B}} u(k) \\ &\quad + H \underbrace{[f_k - Q_1 x(k) - Q_2 u(k)]}_{\hat{f}(k)} \\ \hat{y}(k) &= Cx(k) \end{aligned} \tag{3}$$

where $Q_1 \in \mathbb{R}^{r \times n}$ and $Q_2 \in \mathbb{R}^{r \times m}$ represent ambiguity matrices. Since both the models in (1) and (3) can fit the measured input and output data, it can be concluded that: without any prior information about the structural properties, the system matrices cannot be identified even if the states can be fully observed. It is interesting to remark that, when (A, H) is controllable, the eigenvalues of $A + HQ_1$ can be arbitrarily assigned; therefore, the system poles of (1) cannot be identified without any structural constraints on the system matrices. In other words, if the system model is represented in terms of the transfer function, it is impossible to identify both the numerator and denominator polynomials.

The following lemma is instrumental to dealing with the unknown input signal as well as the identifiability problem.

Lemma 1. *Let $\mathcal{P}_H^{\perp} = I - H(H^TH)^{-1}H^T$. Given the true values of $\{x(k), u(k), H\}$ and in the presence of the unknown signal $f(k)$, the state evolution equation $x(k+1) = Ax(k) + Bu(k) + Hf(k)$ and the modified state evolution equation $\mathcal{P}_H^{\perp}x(k+1) = \mathcal{P}_H^{\perp}Ax(k) + \mathcal{P}_H^{\perp}Bu(k)$ have the same solution set $\{A, B\}$.*

Proof: Denote

$$\begin{aligned} X_h &= x(1:h), \quad \bar{X}_h = x(2:h+1), \\ U_h &= u(1:h), \quad F_h = f(1:h). \end{aligned}$$

The solution $\{A, B\}$ to the equation $x(k+1) = Ax(k) + Bu(k) + Hf(k)$ can be obtained by solving the following least-squares optimization problem:

$$\min_{A,B,F_h} \quad \left\| \bar{X}_h - AX_h - BU_h - HF_h \right\|^2, \tag{4}$$

where the minimum of the above optimization is zero.

By setting the first-order derivatives of the above objective function with respect to the variables A, B, F_h to zero, we have that

$$\begin{aligned} &(AX_h + BU_h + HF_h)X_h^T - \bar{X}_hX_h^T = 0 \\ &(AX_h + BU_h + HF_h)U_h^T - \bar{X}_hU_h^T = 0 \\ &H^T(AX_h + BU_h + HF_h) - H^T\bar{X}_h = 0. \end{aligned} \tag{5}$$

It can be derived from the last equality that

$$F_h = (H^TH)^{-1}H^T(\bar{X}_h - AX_h - BU_h). \tag{6}$$

Substituting the above equation to the first two equalities in (5) yields that

$$\begin{aligned} &\mathcal{P}_H^{\perp}(\bar{X}_h - AX_h - BU_h)X_h^T = 0 \\ &\mathcal{P}_H^{\perp}(\bar{X}_h - AX_h - BU_h)U_h^T = 0, \end{aligned} \tag{7}$$

which are exactly the first-order optimality conditions to the following least-squares optimization problem

$$\min_{A,B} \quad \left\| \mathcal{P}_H^{\perp}\bar{X}_h - \mathcal{P}_H^{\perp}AX_h - \mathcal{P}_H^{\perp}BU_h \right\|^2. \tag{8}$$

In addition, by left-multiplying $\mathcal{P}_H^{\perp}$ to both hand sizes of the first equation in (3), we can see that the above least-squares problem can reach zero at its minimum. Therefore, the solution set of (A, B) to the equation $x(k+1) = Ax(k) + Bu(k) + Hf(k)$ is the same as that to $\mathcal{P}_H^{\perp}x(k+1) = \mathcal{P}_H^{\perp}Ax(k) + \mathcal{P}_H^{\perp}Bu(k)$. This completes the proof of the lemma. ■

Remark 2. *To illustrate Lemma 1, we can observe from equation (3) that the set*

$$\{\hat{A}, \hat{B} | \hat{A} = A + HQ_1, \hat{B} = B + HQ_2, Q_1 \in \mathbb{R}^{r \times n}, Q_2 \in \mathbb{R}^{r \times m}\}$$

is a solution set of (A, B) to the equation $x(k+1) = Ax(k) + Bu(k) + Hf(k)$, as well as a solution set to the equation $\mathcal{P}_H^{\perp}x(k+1) = \mathcal{P}_H^{\perp}Ax(k) + \mathcal{P}_H^{\perp}Bu(k)$.

Lemma 1 indicates that, when the state sequence can be fully observed, the unknown input signal $f(k)$ can be removed by modifying the state evolution equation, which does not affect the identification of the system matrices or the parameter vector θ. However, if the estimation of the unknown input $f(k)$ is desired, it can be obtained by deconvolution once the system model has been identified.

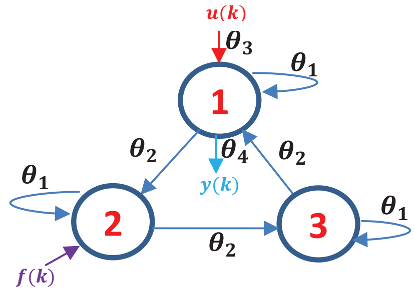

Fig. 2. Multi-agent system of Example 1: $u(k)$ and $y(k)$ are respectively the input and output of agent 1, while $f(k)$ is the unknown input signal added to agent 2.

As a result of Lemma 1, the identification of the state-space model in (1) can be reformulated as that in the following corollary.

Corollary 1. *Suppose that the state sequence $x(k)$ can be fully observed, the identification problem for* (1) *boils down to estimating the parameter vector θ from the following modified state-space model*

$$\begin{aligned} \mathcal{P}_H^{\perp}x(k+1) &= \mathcal{P}_H^{\perp}A(\theta)x(k) + \mathcal{P}_H^{\perp}B(\theta)u(k) \\ \hat{y}(k) &= C(\theta)x(k) \end{aligned} \tag{9}$$

where $\hat{y}(k) = y(k) - w(k)$ represents the noise-free system output.

Example 1. *Suppose that the system matrices $A(\theta), B(\theta)$, $C(\theta)$ and H are parameterized as follows*

$$\begin{aligned} &A(\theta) = \begin{bmatrix} \theta_1 & 0 & \theta_2 \\ \theta_2 & \theta_1 & 0 \\ 0 & \theta_2 & \theta_1 \end{bmatrix}, B(\theta) = \begin{bmatrix} \theta_3 \\ 0 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &C(\theta) = \begin{bmatrix} \theta_4 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

The corresponding networked system is depicted in Fig. 2, where the three agents are interconnected. The first agent is excited by the input signal $u(k)$ and generates the output signal $y(k)$; however, the second agent in the network is spoiled by the unmeasurable signal $f(k)$. It is desired to identify the parameter vector θ using the input and output measurements of the first agent.

The projection matrix $\mathcal{P}_H^{\perp}$ can be calculated, i.e., $\mathcal{P}_H^{\perp} = Diag(1, 0, 1)$. When the state sequence $x(k)$ can be fully observed, according to the results in Corollary 1, the estimate of θ is the solution to the following equation group

$$\begin{aligned} &\begin{bmatrix} x_1(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 & \theta_2 \\ 0 & \theta_2 & \theta_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} \theta_3 \\ 0 \end{bmatrix} u(k) \\ &\hat{y}(k) = \theta_4 x_1(k). \end{aligned}$$

It can be observed from this example that the state evolution equation associated with the second agent is removed due *to the unknown input signal. However, the whole parameter vector θ can still be estimated, since the parameters in the second state-evolution equation have duplicates in other state evolution equations. This idea is similar to our previous work on the local identification of networked systems [33]. Suppose that the coefficient matrix of the unknown input signal is set to $H = [1\ 1\ 1]^T$. By repeating the above process, it can be verified that the system parameters can be uniquely determined, indicating that the presented identification is more general than just removing some node signals.*

B. Identifiability analysis based on measured output

In practice, the state sequence of a state-space model cannot be directly measured, and the state sequence can be (at most) estimated up to a similarity transformation using the measured input and output data [9], [32]. In the sequel, the identifiability conditions of the concerned state-space model (1) shall be investigated in two steps. Firstly, in presence of some unknown input, the estimation of the state sequence (up to a similarity transformation) will be investigated. Secondly, given the estimated state sequence, the identifiability of the structured system matrices (or the system parameters) will be analyzed.

In presence of the unknown input signal, the state estimation using the classical subspace methods is not straightforward. Here, inspired from the N4SID and PO-MOESP methods [9], the state estimation for the concerned model in (1) will be handled using the subspace intersection approach. The data equations of (1) using the past and future data are respectively given as

$$\begin{aligned} \hat{Y}_p &= OX_p + T_u U_p + T_f F_p \\ \hat{Y}_f &= OX_f + T_u U_f + T_f F_f \end{aligned} \tag{11}$$

where

$\hat{Y}_p = \mathcal{H}_{s+1,h}[\hat{y}(1:s+h)],\ \hat{Y}_f = \mathcal{H}_{s+1,h}[\hat{y}(s+1:2s+h)],$
$U_p = \mathcal{H}_{s,h}[u(1:s+h-1)],\ U_f = \mathcal{H}_{s,h}[u(s+1:2s+h-1)],$
$F_p = \mathcal{H}_{s,h}[f(1:s+h-1)],\ F_f = \mathcal{H}_{s,h}[f(s+1:2s+h-1)],$
$X_p = x(1:h),\ X_f = x(s+1:s+h),\ O = \mathcal{O}_{s+1}(A,C),$
$T_u = \mathcal{T}_{s+1}(A,B,C),\ T_f = \mathcal{T}_{s+1}(A,H,C).$

Based on the past and future data equations above, the estimate of the state sequence can be obtained as shown in the following lemma.

Lemma 2. *Suppose that Assumptions A1-A2 hold. Assume that CH has full column rank and the state-space model described by the matrix of tuples $(A,H,C,0)$ is strongly observable [14], i.e., rank$[O\ \ T_f] = n+$rank$[T_f]$. Then, the row subspace of the state space sequence can be estimated as*

$$\textit{Row}[X_f] = \textit{Row}\begin{bmatrix} U_p \\ \hat{Y}_p \end{bmatrix} \cap \textit{Row}\begin{bmatrix} U_f \\ \hat{Y}_f \end{bmatrix} \tag{12}$$

Proof: In view of the lower-triangular pattern of T_f, under the assumption that CH has full rank, it is easy to verify that T_f has full column rank. In addition, by Assumption A1 and the strong observability condition, it can be established that the composite matrix $[O\ \ T_f]$ has full column rank. Then, it can be derived from equation (11) that [14, Theorem 3.1]:

$$\text{Row}\begin{bmatrix} X_p \\ F_p \end{bmatrix} \subseteq \text{Row}\begin{bmatrix} U_p \\ \hat{Y}_p \end{bmatrix} \tag{13}$$

and

$$\text{Row}\begin{bmatrix} X_f \\ F_f \end{bmatrix} \subseteq \text{Row}\begin{bmatrix} U_f \\ \hat{Y}_f \end{bmatrix}. \tag{14}$$

According to the state evolution equation (1), we can obtain that

$$\text{Row}\,[X_f] \subseteq \text{Row}\begin{bmatrix} X_p \\ U_p \\ F_p \end{bmatrix} = \text{Row}\begin{bmatrix} U_p \\ \hat{Y}_p \end{bmatrix}, \tag{15}$$

where the last equality follows from the equality

$$\text{Row}\begin{bmatrix} U_p \\ \hat{Y}_p \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ O & 0 & T_f \end{bmatrix} \begin{bmatrix} X_p \\ U_p \\ F_p \end{bmatrix} \tag{16}$$

and the full column rank property of the coefficient matrix $\begin{bmatrix} 0 & I & 0 \\ O & 0 & T_f \end{bmatrix}$.

By equations (14) and (15), it can be derived that

$$\begin{aligned} \text{Row}[X_f] &\subseteq \text{Row}\begin{bmatrix} U_p \\ \hat{Y}_p \end{bmatrix} \cap \text{Row}\begin{bmatrix} U_f \\ \hat{Y}_f \end{bmatrix} \\ &= \text{Row}\begin{bmatrix} X_p \\ U_p \\ F_p \end{bmatrix} \cap \text{Row}\begin{bmatrix} X_f \\ U_f \\ F_f \end{bmatrix}. \end{aligned} \tag{17}$$

By Assumption A2 and Lemma 10.4 in [9], it can be established that the augmented matrix $[\ X_p^T\ \ U_p^T\ \ F_p^T\ \ U_f^T\ \ F_f^T\ ]^T$ has full row rank. Therefore, we can obtain from the equation (17) that

$$\text{Row}[X_f] = \text{Row}\begin{bmatrix} X_p \\ U_p \\ F_p \end{bmatrix} \cap \text{Row}\begin{bmatrix} X_f \\ U_f \\ F_f \end{bmatrix}.$$

This completes the proof of the lemma. ■

For a multi-agent system, the strong observability condition in the above lemma can be verified via checking the associated topological characteristics without the rank computations [34]. After obtaining the state sequence up to a similarity transformation as shown in Lemma 2, the identifiability of the structured system matrices (or the system parameters) will be investigated.

Let $\hat{x}(k)$ be the estimated system state and Q the nonsingular similarity transformation matrix such that $x(k) = Q\hat{x}(k)$. Then, the state-space model in (1) can be rewritten as

$$\begin{aligned} Q\hat{x}(k+1) &= AQ\hat{x}(k) + Bu(k) + Hf(k) \\ \hat{y}(k) &= CQ\hat{x}(k). \end{aligned} \tag{18}$$

By Assumption A1, i.e., the concerned system model is minimal, the estimated state sequence should have full row rank; therefore, the matrix Q should be nonsingular.

Since the matrix Q is unknown, the identifiability problem for the above state-space model is whether the matrix Q and the parameter vector θ can be uniquely determined from equation (18). Following the results in Corollary 1, the estimation of the parameter vector θ can be transformed to identify the following system model

$$\begin{aligned}\mathcal{P}_H^{\perp} Q\hat{x}(k+1) &= \mathcal{P}_H^{\perp} A(\theta) Q\hat{x}(k) + \mathcal{P}_H^{\perp} B(\theta) u(k)\\ \hat{y}(k) &= C(\theta) Q\hat{x}(k).\end{aligned} \tag{19}$$

As a result, the identifiability of the state-space model in (18) can be determined by checking the uniqueness of the solution to equation (19).

Similar to the identifiability conditions for standard structured models [8, Chapter 4.6] [35, Theorem 3.7], the identifiability conditions for the state-space model in (19) are given in the following lemma.

Lemma 3. *Consider the state-space equations* (19) *with* $\hat{x}(k), u(k), \hat{y}(k)$ *and* $\mathcal{P}_H^{\perp}$ *being available. Let the low-rank factorization of the projection matrix* $\mathcal{P}_H^{\perp}$ *be given as*

$$\mathcal{P}_H^{\perp} = U_H V_H^T$$

where $U_H \in \mathbb{R}^{n\times(n-r)}$ *and* $V_H \in \mathbb{R}^{n\times(n-r)}$ *have full column rank. Under Assumptions A1-A2, the parameter* θ *in* (19) *is identifiable if and only if the following equations*

$$\begin{aligned}&\Pi V_H^T Q = V_H^T\\ &\Pi V_H^T A(\theta^*) Q = V_H^T A(\theta)\\ &\Pi V_H^T B(\theta^*) = V_H^T B(\theta)\\ &C(\theta^*) Q = C(\theta)\end{aligned} \tag{20}$$

yield that $\theta = \theta^*$, $Q = I$ *and* $\Pi = I$.

It is remarked that the ambiguity matrix Π is introduced in the above identifiability condition due to the fact that the state-space model in (19) is a descriptor system [36].

Proof: Given $u(k)$, $\hat{y}(k)$ and $\hat{x}(k)$, we denote by $\{\hat{E}, \hat{A}, \hat{B}, \hat{C}\}$ the solution set to the following equation group

$$\begin{aligned}\hat{E}\hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}u(k)\\ \hat{y}(k) &= \hat{C}\hat{x}(k).\end{aligned} \tag{21}$$

Define the following row sequences

$$\begin{aligned}&\bar{x}_f = \hat{x}(s+2:s+h+1), \quad x_f = \hat{x}(s+1:s+h),\\ &u_f = u(s+1:s+h), \quad f_f = f(s+1:s+h),\\ &y_f = \hat{y}(s+1:s+h).\end{aligned}$$

It can be obtained from Lemma 2 that the augmented matrix $\begin{bmatrix} x_f^T & u_f^T & f_f^T\end{bmatrix}^T$ has full row rank. It is straightforward from the second equality of (21) that $\hat{C} = C(\theta^*)Q^*$, with Q^* being the true similarity transformation of the state estimate $\hat{x}(k)$.

In addition, it can be derived from the following equality

$$\begin{bmatrix} \bar{x}_f \\ x_f \\ u_f \end{bmatrix} = \begin{bmatrix} Q^{*,-1}AQ^* & Q^{*,-1}B & Q^{*,-1}H \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_f \\ u_f \\ f_f \end{bmatrix}$$

that the matrix $\begin{bmatrix} \bar{x}_f^T & x_f^T & u_f^T \end{bmatrix}^T$ is rank deficient by $n-r$. By the first equation of (21), the matrix $[\hat{E} \quad -\hat{A} \quad -\hat{B}]$ should lie in the left null subspace of $\begin{bmatrix} \bar{x}_f^T & x_f^T & u_f^T \end{bmatrix}^T$ which has dimension $n-r$. In addition, the first equation of (19) indicates that, for the true parameter vector θ^*, the matrix $\mathcal{P}_H^{\perp}[Q^* \quad -A(\theta^*)Q^* \quad -B(\theta^*)]$ has rank $n-r$ and forms a basis for the left null subspace of $\begin{bmatrix} \bar{x}_f^T & x_f^T & u_f^T \end{bmatrix}^T$. As a result, we can obtain that

$$\begin{bmatrix}\hat{E} & \hat{A} & \hat{B}\end{bmatrix} = \Xi \mathcal{P}_H^{\perp}[Q^* \quad -A(\theta^*)Q^* \quad -B(\theta^*)] \tag{22}$$

where Ξ is an ambiguity matrix satisfying that

$$\text{rank}[\Xi\mathcal{P}_H^{\perp}] = \text{rank}[\mathcal{P}_H^{\perp}] = n-r.$$

As a result, the solution set $\{\hat{E}, \hat{A}, \hat{B}, \hat{C}\}$ has the form as follows:

$$\begin{aligned}\hat{E} &= \Xi\mathcal{P}_H^{\perp} Q^*\\ \hat{A} &= \Xi\mathcal{P}_H^{\perp} A(\theta^*) Q^*\\ \hat{B} &= \Xi\mathcal{P}_H^{\perp} B(\theta^*)\\ \hat{C} &= C(\theta^*) Q^*.\end{aligned} \tag{23}$$

By comparing the coefficient matrices in (19) and (21), the parameter set $\{\theta, Q\}$ to equation (19) is identifiable if and only if the following equations

$$\begin{aligned}\hat{E} &= \Xi\mathcal{P}_H^{\perp} Q^* = \mathcal{P}_H^{\perp} Q\\ \hat{A} &= \Xi\mathcal{P}_H^{\perp} A(\theta^*) Q^* = \mathcal{P}_H^{\perp} A(\theta) Q\\ \hat{B} &= \Xi\mathcal{P}_H^{\perp} B(\theta^*) = \mathcal{P}_H^{\perp} B(\theta)\\ \hat{C} &= C(\theta^*) Q^* = C(\theta) Q,\end{aligned} \tag{24}$$

with variables θ and Q can yield the true parameter vector and the similarity transformation matrix, i.e.,

$$\theta = \theta^*, \; Q = Q^*.$$

Using the matrix factorization $\mathcal{P}_H^{\perp} = U_H V_H^T$ and by regarding Q^*Q^{-1} as a new variable, the sufficient conditions in (20) are straightforward.

■

It is noted that the difference between the identifiability condition in (20) and that in [35] lies at the projection matrix $\mathcal{P}_H^{\perp}$ as well as the ambiguity matrix Π. For instance, when there is no unknown input or $H = 0$, it follows that $\mathcal{P}_H^{\perp} = I$; then, by some trivial algebraic manipulations, it can be verified that the equation group in (20) degenerates to the standard identifiability conditions for structured state-space model [35].

According to the state estimation result in Lemma 2 and the identifiability result of the state-space model (19) with known state information in Lemma 3, the identifiability conditions for the concerned state-space model in (1) are summarized in the following theorem.

Theorem 1. *Suppose that Assumptions A1-A2 hold and the state-space model described by* $(A, H, C, 0)$ *is strongly observable. Let*

$$\mathcal{P}_H^{\perp} = U_H V_H^T$$

with $U_H \in \mathbb{R}^{n\times(n-r)}$ *and* $V_H \in \mathbb{R}^{n\times(n-r)}$ *being of full column rank. Then, the parameter vector* θ *in the state-space model* (1) *is identifiable if and only if the following equations*

$$\begin{aligned}&\Pi V_H^T Q = V_H^T\\ &\Pi V_H^T A(\theta^*) Q = V_H^T A(\theta)\\ &\Pi V_H^T B(\theta^*) = V_H^T B(\theta)\\ &C(\theta^*) Q = C(\theta)\end{aligned} \tag{25}$$

yield that $\theta = \theta^*$, $Q = I$ and $\Pi = I$.

The flexibility of the factorization $\mathcal{P}_H^\perp = U_H V_H^T$ in the above theorem enables us to analyze the identifiability problem in a tractable manner. For example, if the matrix $H \in \mathbb{R}^3$ has the form $H = [h_1 \ \ h_2 \ \ h_3]^T$ with nonzero entries, then it is easy to find a low rank factorization of $\mathcal{P}_H^\perp$ such that V_H^T has the form

$$V_H^T = \begin{bmatrix} h_2 & -h_1 & 0 \\ 0 & h_3 & -h_2 \end{bmatrix}.$$

This can facilitate the identifiability analysis for the equations in (25).

Example 2. *The identifiability of the state-space model with the following structured system matrices is considered:*

$$A(\theta) = \begin{bmatrix} \theta_1 & 0 & \theta_2 \\ \theta_2 & \theta_1 & 0 \\ 0 & \theta_2 & \theta_1 \end{bmatrix}, \ B(\theta) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ H = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$C(\theta) = \begin{bmatrix} \theta_3 & 1 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}.$$

Due to the physical interpretations of the parameters in a structured system model, all the components of θ have nonzero values. Then, it can be verified that the matrix CH has full column rank and the system described by the matrix tuple $(A, H, C, 0)$ is generically strong observable [34]; therefore, the conditions of Theorem 1 are generically satisfied. The matrix V_H^T in Theorem 1 can be written as

$$V_H^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let the ambiguity matrix $Q \in \mathbb{R}^{3\times 3}$ be partitioned as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \end{bmatrix}$$

with $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R}^{3\times 1}$.

The equation group in (25) can be simplified as

$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_3^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} \theta_1^* & 0 & \theta_2^* \\ 0 & \theta_2^* & \theta_1^* \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 & \theta_2 \\ 0 & \theta_2 & \theta_1 \end{bmatrix}$$
$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \theta_3^* & 1 & 0 \\ 0 & 0 & \theta_3^* \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \end{bmatrix} = \begin{bmatrix} \theta_3 & 1 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}.$$

From the above equation group, it can be derived through trivial calculations that $\theta = \theta^*$, $Q = I$ and $\Pi = I$; hence, the system model is identifiable.

Also, it can be verified by Theorem 1 that, if the matrix $C(\theta)$ is parameterized as

$$C(\theta) = \begin{bmatrix} \theta_3 & 1 & 0 \\ 0 & 0 & \theta_4 \end{bmatrix}, \tag{26}$$

then the θ_1 and θ_3 are identifiable, but the signs of θ_2 and θ_4 cannot be determined.

IV. Subspace-based blind identification method

The previous section analyzes the identifiability of structured state-space model in the presence of some unknown input, from which we can obtain an identification procedure by firstly estimating the state sequence from the measurable input and output data, following the identification of structured system matrices. In the sequel, numerical methods for the system state estimation and system parameter identification will be presented.

A. Estimation of the system state

According to Lemma 2, under the strong observability condition, the state sequence can be estimated by the row-subspace intersection of the past noise-free data and future noise-free data (12). However, in the presence of measurement noise, the row subspace of the state sequence cannot be accurately estimated. Instead, a new computational approach will be developed which can achieve an unbiased estimate of the state sequence by compensating the noise influence, and the associated estimation accuracy will be analyzed.

To deal with the biased subspace estimation caused by the measurement noise, the variance of the measurement noise $w(k)$ will be estimated by exploiting the rank deficiency of the past-data matrix in (16).

Lemma 4. *Consider the equation* (16). *Suppose that Assumptions A2-A3 hold and the dimension parameter s satisfies that $s(p-r) - n > 0$, namely the matrix $[O \ \ T_f]$ is a tall matrix. Then, the variance of the measurement noise can be estimated as*

$$\sigma_w^2 = \lambda_{min}\left[\lim_{h\to\infty} \frac{1}{h} Y_p Y_p^T - Y_p U_p^T (U_p U_p^T)^{-1} U_p Y_p^T\right] \tag{27}$$

where $\lambda_{min}(\cdot)$ represents the least eigenvalue.

Proof: Denote by σ_w^2 the variance of $w(k)$. By Assumption A3, it can be established that

$$\lim_{h\to\infty} \frac{1}{h} \begin{bmatrix} U_p \\ Y_p - W_p \end{bmatrix} \begin{bmatrix} U_p \\ Y_p - W_p \end{bmatrix}^T = \begin{bmatrix} R_{uu} & R_{yu} \\ R_{uy} & R_{yy} - \sigma_w^2 I \end{bmatrix} \geq 0.$$

According to equation (16) and the tall matrix $[O \ \ T_f]$, it can be verified that the above correlation matrix is rank deficient.

By Assumption A2, the matrix R_{uu} is positive definite, i.e., $R_{uu} > 0$. Using the Schur Complement theorem, it can be derived that

$$\text{rank} \begin{bmatrix} R_{uu} & R_{yu} \\ R_{uy} & R_{yy} - \sigma_w^2 I \end{bmatrix} = \text{rank}\,[R_{uu}] + \text{rank}\left[R_{yy} - \sigma_w^2 I - R_{yu} R_{uu}^{-1} R_{uy}\right].$$

Due to the positive semi-definiteness and rank deficiency of the above matrix, it is obvious that the $R_{yy} - \sigma_w^2 I - R_{yu} R_{uu}^{-1} R_{uy}$ matrix is positive semi-definite and rank deficient. As a result, σ_w^2 is the least eigenvalue of $R_{yy} - R_{yu} R_{uu}^{-1} R_{uy}$. This completes the proof of the lemma. ∎

After obtaining the estimated noise variance σ_w^2, the subspace intersection shall be algebraically computed using the method provided in Appendix A [37]. The SVD decomposition of the following correlation matrix compensated by the noise influence is given as

$$\lim_{h\to\infty}\frac{1}{h}\begin{bmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{bmatrix}\begin{bmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{bmatrix}^T - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_w^2 I & 0 & \begin{bmatrix} 0 & 0 \\ \sigma_w^2 I & 0 \end{bmatrix} \\ 0 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 0 & \sigma_w^2 I \\ 0 & 0 \end{bmatrix} & 0 & \sigma_w^2 I \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}, \tag{28}$$

where Σ_1 is a diagonal matrix containing the n largest singular values. By partitioning the matrix U_2 into two sub-matrices of the same size $U_2 = [U_{21}^T \ \ U_{22}^T]^T$, the state estimate is given as

$$\hat{X}_f = U_{21}^T \begin{bmatrix} U_p \\ Y_p \end{bmatrix}. \tag{29}$$

When the data length tends to infinity, the SVD decomposition in (28) is the same as that of

$$\begin{bmatrix} U_p \\ Y_p - W_p \\ U_f \\ Y_f - W_f \end{bmatrix}\begin{bmatrix} U_p \\ Y_p - W_p \\ U_f \\ Y_f - W_f \end{bmatrix}^T.$$

Then, the true state estimation (up to a similarity transformation) can be expressed in terms of noise-free measurements as

$$X_f = U_{21}^T \begin{bmatrix} U_p \\ Y_p - W_p \end{bmatrix}. \tag{30}$$

As a result, the estimated state sequence in (29) and the true state sequence in (30) have the following relation:

$$\hat{X}_f = QX_f + U_{21}^T \begin{bmatrix} 0 \\ W_p \end{bmatrix} \tag{31}$$

where Q is an ambiguity matrix. By Assumption A3, it can be verified without any difficulty that the state estimation error $U_{21}^T \begin{bmatrix} 0 \\ W_p \end{bmatrix}$ is uncorrelated with the state X_f. This is crucial to the accurate identification of the concerned state-space model.

B. Identification of the structured system matrices

Given the estimated system state in (29), we define the following row sequences

$$\begin{aligned} &\bar{x}_h = \hat{x}(s+2 : s+h+1), \ \ x_h = \hat{x}(s+1 : s+h), \\ &u_h = u(s+1 : s+h), \ \ y_h = y(s+1 : s+h), \\ &w_h = w(s+1 : s+h). \end{aligned}$$

According to equation (19), the identification problem boils down to estimating the similarity transformation matrix and the parameter vector from the following equations

$$\begin{aligned} \mathcal{P}_H^{\perp} Q\bar{x}_h &= \mathcal{P}_H^{\perp} A(\theta)Qx_h + \mathcal{P}_H^{\perp} B(\theta)u_h + \eta_1 \\ y_h &= C(\theta)Qx_h + w_h + \eta_2, \end{aligned} \tag{32}$$

where η_1 and η_2 are perturbations caused by the state estimation error. Following the analysis in the previous subsection, as the data length tends to infinity, the induced error terms $\eta_1(k), \eta_2(k)$ and the measurement noise $w(k)$ are asymptotically uncorrelated with the state and input sequences. As a result, the following optimization problem is proposed for the state-space model identification

$$\begin{aligned} \min_{Q,\theta} \quad & \left\|\mathcal{P}_H^{\perp} Q\bar{x}_h - \mathcal{P}_H^{\perp} A(\theta)Qx_h - \mathcal{P}_H^{\perp} B(\theta)u_h\right\|_F^2 \\ & + \|y_h - C(\theta)Qx_h\|_F^2 . \end{aligned} \tag{33}$$

Due to the couplings among unknown variables, the above optimization problem is bilinear; thus, the traditional gradient-based optimization methods are sensitive to the initial parameter estimate. In order to handle this problem, the difference-of-convex programming framework [38] is adopted, which transforms the concerned bilinear estimation problem to a rank-one constrained optimization problem and then a difference-of-convex programming problem.

To deal with the parameter couplings among unknown variables, the following auxiliary parameters are introduced

$$\begin{aligned} \mathcal{A} &= A(\theta)Q = A_0 Q + \sum_{i=1}^{l} A_i Q_i \\ \mathcal{C} &= C(\theta)Q = C_0 Q + \sum_{i=1}^{l} C_i Q_i \end{aligned} \tag{34}$$

where $Q_i = Q\theta_i$ for $i = 1, \cdots, l$. It can be observed that the equations in (34) are affine with respect to the unknown (auxiliary) parameters. In addition, the bilinear constraints $Q_i = Q\theta_i$ can be equivalently formulated as a rank-one constraint

$$\text{rank}\begin{bmatrix} 1 & \theta_1 & \cdots & \theta_l \\ vec(Q) & vec(Q_1) & \cdots & vec(Q_l) \end{bmatrix} = 1. \tag{35}$$

As a result, the optimization problem in (33) can be reformulated as a rank-one constrained optimization problem

$$\begin{aligned} \min_{Q_i,\mathcal{A},\mathcal{C},\theta,\Gamma} \quad & \left\|\mathcal{P}_H^{\perp} Q\bar{x}_h - \mathcal{P}_H^{\perp}\mathcal{A}x_h - \mathcal{P}_H^{\perp} B(\theta)u_h\right\|_F^2 + \|y_h - \mathcal{C}x_h\|_F^2 \\ s.t. \quad & \mathcal{A} = A_0 Q + \sum_{i=1}^{l} A_i Q_i \\ & \mathcal{C} = C_0 Q + \sum_{i=1}^{l} C_i Q_i \\ & \Gamma = \begin{bmatrix} 1 & \theta_1 & \cdots & \theta_l \\ vec(Q) & vec(Q_1) & \cdots & vec(Q_l) \end{bmatrix} \\ & \text{rank}[\Gamma] = 1. \end{aligned} \tag{36}$$

Due to the rank-one constraint, the above optimization problem is NP-hard and difficult to handle. Here, the rank-one constraint is interpreted as that the nuclear norm (sum of all the singular values) of the matrix equals the largest singular value, i.e.,

$$\|\Gamma\|_* - \|\Gamma\|_2 = 0, \tag{37}$$

where the nuclear norm $\|\cdot\|_*$ and the 2-norm $\|\cdot\|_2$ are convex functions [39]. The above difference-of-convex function is non-convex; however, it is always non-negative. By treating the nonnegative difference-of-convex function as a penalty, we obtain the following difference-of-convex programming problem

$$\begin{aligned}
\min_{Q_i,\mathcal{A},\mathcal{C},\theta,\Gamma} \quad & \left\|\mathcal{P}_H^{\perp} Q\bar{x}_h - \mathcal{P}_H^{\perp}\mathcal{A}x_h - \mathcal{P}_H^{\perp}B(\theta)u_h\right\|_F^2 \\
& + \|y_h - \mathcal{C}x_h\|_F^2 + \lambda\left(\|\Gamma\|_* - \|\Gamma\|_2\right) \\
s.t. \quad & \mathcal{A} = A_0 Q + \sum_{i=1}^{l} A_i Q_i \\
& \mathcal{C} = C_0 Q + \sum_{i=1}^{l} C_i Q_i \\
& \Gamma = \begin{bmatrix} 1 & \theta_1 & \cdots & \theta_l \\ vec(Q) & vec(Q_1) & \cdots & vec(Q_l) \end{bmatrix}
\end{aligned} \tag{38}$$

where λ is a penalty parameter. In the absence of the measurement noise or when the equations in (19) hold exactly, the above optimization problem has the same global minimum as (36) for any positive value of λ. Furthermore, if the system model is identifiable, then the true parameters can be obtained by solving either (36) or (38).

Next, the sequential convex programming framework will be adopted to solve the difference-of-convex problem in (38). In each iteration, the concave term in the objective function is linearized based on the current parameter estimate. Denote by Γ^j the estimate at the j-th iteration, and its SVD decomposition is given as

$$\Gamma^j = \begin{bmatrix} U_1^j & U_2^j \end{bmatrix} \begin{bmatrix} \Sigma_1^j & \\ & \Sigma_2^j \end{bmatrix} \begin{bmatrix} V_1^{j,T} \\ V_2^{j,T} \end{bmatrix}, \tag{39}$$

where Σ_1^j is a scalar that is the largest singular value of Γ^j. Then, the term $\|\Gamma\|_2$ can be linearized as

$$\|\Gamma\|_2 \approx \|\Gamma^j\|_2 + \text{tr}\left(U_1^{j,T}(\Gamma - \Gamma^j)V_1^j\right). \tag{40}$$

Substituting it into (38) yields the $(j+1)$-th optimization problem

$$\begin{aligned}
\min_{Q_i,\mathcal{A},\mathcal{C},\theta,\Gamma} \quad & \left\|\mathcal{P}_H^{\perp} Q\bar{x}_h - \mathcal{P}_H^{\perp}\mathcal{A}x_h - \mathcal{P}_H^{\perp}B(\theta)u_h\right\|_F^2 \\
& + \|y_h - \mathcal{C}x_h\|_F^2 + \lambda\left(\|\Gamma\|_* - \text{tr}\left(U_1^{j,T}\Gamma V_1^j\right)\right) \\
s.t. \quad & \mathcal{A} = A_0 Q + \sum_{i=1}^{l} A_i Q_i \\
& \mathcal{C} = C_0 Q + \sum_{i=1}^{l} C_i Q_i \\
& \Gamma = \begin{bmatrix} 1 & \theta_1 & \cdots & \theta_l \\ vec(Q) & vec(Q_1) & \cdots & vec(Q_l) \end{bmatrix}.
\end{aligned} \tag{41}$$

For the above iterative optimization process, it is necessary to choose an initial condition; however, there are rarely any good initialization methods for the structured state-space (gray-box) model identification [40]. Here, the initialization is obtained by solving the convex part of the difference-of-convex problem (38) by setting $U_1^0 = 0$ and $V_1^0 = 0$. The associated convex part is a nuclear norm regularized convex optimization problem, which can yield a good initial parameter estimate such that the difference-of-convex programming method can yield more accurate parameter estimate than the nuclear norm method. Also, the initial parameter estimate obtained by solving the nuclear norm regularized optimization problem can also be used to initialize the gradient-based method, for which the performance will be shown in Section V.

By implementing the above iterative optimization procedure, the objective function decreases as the iterations continue. Let $\Theta = \{Q_i, \mathcal{A}, \mathcal{C}, \theta\}$. Denote by $g(\Theta, \Gamma)$ the objective function of (38). Then, it can be verified that

$$g(\Theta^{k+1}, \Gamma^{k+1}) \le g(\Theta^{k+1}, \Gamma^k) \le g(\Theta^k, \Gamma^k), \tag{42}$$

indicating that the presented iterative optimization is a descent algorithm. By the results of [41, Theorems 1-2], the presented algorithm can be interpreted as a re-weighted nuclear norm optimization algorithm, which can be shown that the estimated parameter sequence converges to a stationary point since the objective function (36) is coercive and its derivative is Lipschitz continuous in terms of those unknown variables.

V. Numerical simulation

In this section, the identification of the structured state-space model in Example 2 will be simulated to validate the theoretical results.

In order to demonstrate the performance of the proposed identification method against the noise perturbation, the normalized estimation error (NEE) criterion will be used. Denote by $\hat{\theta}^j$ the estimate of θ at the j-th Monte-Carlo trial. The NEE value at the j-th trial is defined as

$$\text{NEE}^j = \frac{\|\hat{\theta}^j - \theta^*\|}{\|\theta^*\|}, \tag{43}$$

where θ^* represents the true system parameter vector.

In the simulation, the parameter vector is set to

$$\theta = \begin{bmatrix} 0.5 & 0.3 & 1 \end{bmatrix}^T,$$

and the data length is set to 200. It has been shown in Example 2 that this parameter vector is identifiable. The stopping criterion of the proposed method, which is defined by the relative error of the parameter estimation, is set to 10^{-8}. The penalty coefficient λ in (38) is empirically set to 10^{-3} and the maximum number of iterations is set to 100.

For the comparison purpose, the following three method will be simulated: (i) the nuclear norm method which is to solve the convex part of (38) is abbreviated as NN method; (ii) the Gauss-Newton (GN) method which is initialized by solving the convex part of (38) is abbreviated as NN-GN method; (iii) the sequential quadratic programming (SQP) method with bound constraint $0 < \theta_i < 1$ is abbreviated as SQP method. In the simulation, the initial guess for the SQP method is chosen randomly according to the uniform distribution in the range

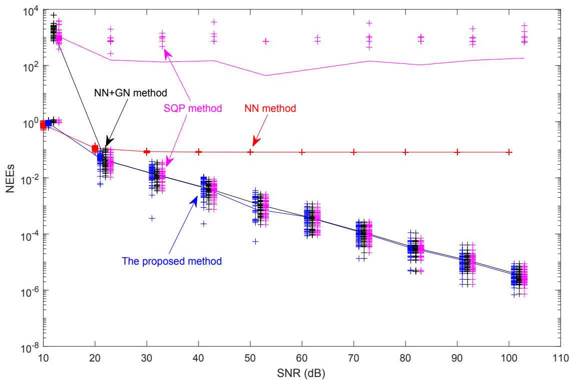

Fig. 3. Scatter plots of the NEEs obtained by the NN method, the NN-GN method, the SQP method and the proposed method at different SNRs. The red, black, magenta and blue crosses represent respectively the NEEs of the NN method, the NN-GN method, the SQP method and the proposed method at different Monte-Carlo trials. The solid curves denote the mean NEEs at different SNRs. Note that the NEEs of these four methods are computed at the same SNRs; however, they are slightly separated to clearly depict their individual performance.

$[0, 1]$. The maximum number of iterations for the NN-GN method and the SQP method is set to 100.

The scatter plots in terms NEEs obtained by the NN method, the NN-GN method, the SQP method and the proposed method are shown in Fig. 3, where the number of Monte-Carlo trials is set to 50. It can be observed that: (i) the proposed identification method can yield accurate estimation of the system parameters at large SNRs, indicating that the true system parameters can be recovered in the absence of measurement noise; (ii) comparing with the NN-GN method, the proposed method performs better at low SNRs; (iii) the proposed method outperforms the nuclear norm method, since the nuclear norm method is inherently a one-step implementation of the proposed method; (iv) although the SQP method uses the bound information of the system parameters, it is still sensitive to the initial parameter estimate. It can be found that the NEE values obtained by the SQP method at each SNR have two clusters which correspond to the local minima and the global minima of the optimization problem in (33).

The numerical simulations are run on a laptop with a 2.9 GHz 651 processor and a 8.0 GB RAM. The average computational times for the NN method, the NN-GN method, the SQP method and the proposed method are about 0.4216s, 0.4661s, 0.0950s and 2.172s, respectively.

VI. Conclusion

In this paper, the identifiability conditions for structured state-space models with unknown inputs have been provided, i.e., by exploiting the structural properties of the state-space model, the system model can be identified with deterministic unknown system inputs. This differs from the blind identification of SISO models with the input signal having specific patterns. In addition, a subspace-based identification method has been presented which firstly estimates the system state up to a similarity transformation, following the simultaneous estimation of the similarity transformation matrix and the system parameters through solving a bilinear estimation problem. For the state estimation, a noise-compensated subspace method has been provided which can yield an unbiased estimate for the row subspace of the state sequence. To deal with the bilinear estimation problem, it is formulated as a difference-of-convex optimization problem which is handled by the sequential convex programming method.

The concerned structured state-space models with unknown input signals can be confronted in many practical scenarios, such as the networked systems with node failures or adversarial attacks, infrastructure models with external wind/wave disturbance, Heating Venting and Air Conditioning (HAVC) systems with unknown occupancy, etc. In addition, after the identification of the system models, both the internal state sequence and the unknown input signal can also be recovered using the available measurements, indicating that the behavior of an agent operating in a network can be fully detected by the other agents using their input and output data. As a result, this work can be potentially implemented in the network security related research.

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