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A Framework to Assess Multi-Class Continuum Traffic Flow Models

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ABSTRACT
Since the beginning of this millennium many multi-class continuum traffic flow models have been proposed. We present a set of qualitative requirements for this type of models, including nonincreasing density-speed relations and anisotropy. The requirements are cast into a framework that applies a generalised deterministic multi-class kinematic wave traffic flow model. A step-by-step plan is developed to apply the framework to models that fit into the generalised model. The plan could only be developed using the Lagrangian formulation of the generic model, but it can also be applied to models in the traditional Eulerian formulation. We conclude that only few models known from literature satisfy all requirements unconditionally. The step- by-step plan can furthermore be applied in the development of new models, the adaptation of existing models and the calibration of model parameters.

INTRODUCTION

Multi-class kinematic wave traffic flow models are an extension of the LWR-model [1, 2] in the sense that they include heterogeneity among vehicles and drivers. For example, vehicles have different lengths and drivers have different preferred (maximum) speeds. Other multi-class models are mostly either car-following models cf. [3–5]. (Multi-class) traffic flow models are applied, for example, for traffic state estimation and prediction, for traffic management and for long term planning. Continuum models are especially well-suited for fast simulation such as in online traffic management systems. Their accuracy is improved by including different types of vehicles and drivers.

Since the first multi-class kinematic wave traffic flow model [6], many multi-class models have been proposed and extensions and adaptations have been developed. See e.g. [7] for an overview and [8, 9] for more detailed comparisons. However, the analyses are limited to a small set of criteria and not all currently known models are included. Moreover, a framework for the qualitative assessment of multi-class continuum traffic flow models is lacking. Such a qualitative assessment would include analysis of important criteria such as whether speeds are nonincreasing with increasing density and anisotropy (information does not travel faster than vehicles). The qualitative assessment is important because it shows which models have desirable properties and can be developed further. This would save effort in trying to improve and calibrate models with intrinsically undesirable properties. In this sense such a framework is similar to the qualitative analysis of car-following models [10, 11] and the criteria for fundamental relations [12].

We develop a framework, including a generalised model, for the mathematically rigorous qualitative assessment of deterministic multi-class kinematic wave traffic flow models. Our main contribution is the application of the framework to all such models known to the author. For the development of the framework, we apply the Lagrangian coordinate system. However, it can also be used to assess models in the traditional Eulerian formulation. The focus of our previous work [9] was on a subset of these models which describe traffic flow as a single-pipe flow (i.e. no explicit consideration of different lanes). Our current analysis extends the set of models [6, 8, 13–17] to also include multi-pipe models [18] and porous flow models [19– 21]. Furthermore, the reformulation [22] of the Fastlane model [17] is included. The multi-pipe model [18] has led to the need to redefine the conditions for the anisotropy requirement. The porous flow models by [19–21] led to the need to relax the assumption that there is a fastest class. Furthermore, the introduction of the new models has led to a further generalisation of the generalised model [9] allowing for the speed to directly depend on the class specific densities without the need to convert class specific densities into effective densities.

Still, some multi-class continuum traffic flow models are excluded from our analysis because they currently do not fit the generalised model. Ngoduy [23] adds stochasticity to the earlier space occupancy based model [16]. For example, Chanut [24] extends his earlier model to include moving bottlenecks which drive slowly on only one lane. A different approach to include heterogeneity in the LWR model is taken by Leclercq and Laval [25]. They introduce classes only after discretisation of the model and therefore this model does not fit the generalised continuum model we propose.

The outline of this article is as follows. The next section introduces the generalised deterministic multi-class kinematic wave traffic flow model and reviews models from literature that fit into the generalised model. Then, we define the requirements and show under which conditions the generalised model satisfies the requirements. Our main contribution can be found in the next section that describes and applies a step- by-step plan to assess the models. Finally, the last section includes a discussion and conclusion.

MULTI-CLASS CONTINUUM TRAFFIC FLOW MODELLING

In this section, we develop the generalised model and show how other models fit into it. It consists of a class specific conservation of vehicles equations and class specific fundamental relations. All deterministic multi-class kinematic wave traffic flow models known to the authors fit into this generalised model.

Generalized Model

The original (mixed-class) LWR model [1, 2] contains a continuum equation which expresses the conservation of vehicles: $\partial\rho/\partial t + \partial q/\partial x = 0$, with ρ the density in vehicles per length unit, q the flow in vehicles per time unit and x and t the space and time coordinates, respectively. In the multi-class model, we need an expression for the conservation of vehicles per class:

$$\frac{\partial \rho_u}{\partial t} + \frac{\partial q_u}{\partial x} = 0, \text{ for all classes } u \in \{1, \dots, U\} \tag{1}$$

with U the number of classes. ρ_u is the density of class u in number of vehicles of class u per length unit and q_u is the flow of class u in number of vehicles of class u per time unit. The class specific conservation of vehicles equation (1) is the same for each model, except for the value of U. Some models only have 2 classes ($U = 2$), other models are more generic and can deal with any number of classes.

The fundamental relation expresses the relation between number of vehicles on the road and their speed. The shape of the fundamental relation differs per model but can always be expressed as:

$$v_u = V_u(\rho_1, \dots, \rho_U) \tag{2}$$

Many models can be cast in the form $v_u = V_u(\rho)$, with ρ the effective density. In some models, the effective density ρ is a summation of all class specific densities: $\rho = \sum_u \rho_u$. In other models, the effective density is a weighted summation of all class specific densities: $\rho = \sum_u \eta_u \rho_u$, with weight η_u the passenger car equivalent (pce)-value. As we see later, the pce-value may be state dependent: $\eta_u = \eta_u(\rho_1, \dots, \rho_U)$.

Different modelling principles and assumptions, lead to different fundamental relations. Therefore, the exact form of the fundamental relation (2) is what distinguishes the models from each other. This is set out in Table 1 and detailed below.

Models with 3 Regimes

Most multi-class kinematic wave traffic flow models include two regimes: free flow and congestion. How- ever, the models byLogghe and Immers [8], Daganzo [18] include an extra regime: semi-congestion, see Figure 1(a). In free flow, the vehicles of both classes maintain their maximum speed. In semi-congestion, the slowest class still travels at its maximum speed, while

the other class slows down. Once the fastest class has reached the maximum speed of the slowest class, both classes maintain the same speed (which decreases for increasing densities) and traffic is in the congestion regime. The assumptions behind both models are different.
Daganzo [18] presents a model for two lane roads with two classes (fast ‘slugs’ and slow ‘rabbits’). The slugs only use the shoulder lane, while rabbits may choose to use both lanes, resulting in a single-pipe flow or, alternatively, the rabbits only use the median lane, resulting in a two-pipe flow. (In fact, the model is presented in more general terms, with possibly more lanes than two, and slugs possibly using all but one lane. However, in this more general model, it is not trivial to find the actual lane distribution [26] and therefore, in our analysis we limit ourselves to the two-lane model.) The rationale behind the lane choice is 1. certain vehicles (e.g. trucks, ‘slugs’) are not allowed to use the median lane, and 2. other vehicles (e.g. passenger cars, ‘rabbits’) will choose the lane in which they can maintain the highest speed. As a result, a user equilibrium will be obtained. Similarly, Logghe and Immers [8] also apply the concept of user-equilibrium, assuming that slow vehicles influence the speed of fast vehicles, but not vice versa. However, unlike [18], Logghe and Immers [8] model traffic flow as a single-pipe flow.
Both models use the bi-linear (or Daganzo, or triangular) fundamental relation with class specific parameters:

$$V_u(\rho_1,\rho_2) = \begin{cases} v_{u,\max} & u=1,2, \text{ free flow and } u=2, \text{ semi-congestion} \\ w_u\left(\frac{\alpha_u \rho_{u,\text{jam}}}{\rho_u} - 1\right) & u=1, \text{ semi-congestion and } u=1,2, \text{ congestion} \end{cases} \tag{3}$$

with α_u the fraction of road taken by class u (with $\alpha_1 + \alpha_2 = 1$). The parameter $v_{u,\max}$ is the maximum speed of class u, $\rho_{u,\text{jam}}$ is the jam density of class u and the congestion wave speed:

$$w_u = \frac{\rho_{u,\text{crit}} v_{u,\max}}{\rho_{u,\text{jam}} - \rho_{u,\text{crit}}} \tag{4}$$

with $\rho_{u,\text{crit}}$ the critical density of class u.
Logghe and Immers [8] define the fraction of road taken by class 1 as follows:

$$\alpha_1 = \begin{cases} 1 - \frac{\rho_2}{\rho_{2,\text{crit}}} & \text{semi-congestion: if } 1 - \frac{\rho_2}{\rho_{2,\text{crit}}} \le \frac{\rho_1}{\rho_{1,\text{crit}}} \text{ and } v_1^{\text{sc}} \ge v_{2,\max} \\ \frac{w_1 + w_2\left(\frac{\rho_{2,\text{jam}}}{\rho_2} - 1\right)}{w_1 \frac{\rho_{1,\text{jam}}}{\rho_1} + w_2 \frac{\rho_{2,\text{jam}}}{\rho_2}} & \text{congestion: if } v_1^{\text{sc}} < v_{2,\max} \end{cases} \tag{5}$$

Daganzo [18] apply fundamental diagram parameters that are mostly equal for both classes: $\rho_{u,\text{crit}} = \rho_{1,\text{crit}} = \rho_{2,\text{crit}}, \rho_{u,\text{jam}} = \rho_{1,\text{jam}} = \rho_{2,\text{jam}}$. And the fraction of road taken by class 1 is:

$$\alpha_1 = \begin{cases} \frac{1}{2} & \text{semi-congestion: if } \rho_1 + \rho_2 > \rho_{\text{crit}} \text{ and } \frac{\rho_1}{\rho_1+\rho_2} \le \frac{1}{2} \\ \frac{\rho_1}{\rho_1+\rho_2} & \text{congestion: if } \frac{\rho_1}{\rho_1+\rho_2} > \frac{1}{2} \end{cases} \tag{6}$$

We note that the formulation in [18] does not allow the slower class 2 to take more than half of the road space, i.e. $\alpha_2 \le 1/2$.

Models Using Space Occupancy
The models introduced by Ngoduy and Liu [16], Ngoduy [27] and the Fastlane model Van Lint et al. [17], Van Wageningen-Kessels et al. [22] use some form of assigning a fraction of the road to certain classes. The model in [27] is an extension of the model in [16] only in the sense that it

considers road inhomogeneities such as lane drops. Therefore, our analysis for the model in [16] also holds for the model in [27] and we only consider [16] in the following. [16] assume that the fraction of the road $\alpha_u = \rho_u / \sum_u \rho_u$ is available to class u. In Fastlane, each class has a (state-dependent) space occupancy: the length of road used by exactly one vehicle of that class.
Other important differences with the models discussed before [8, 18] are: 1. the option to model any number of classes, instead of only 2, 2. the application of the Smulders fundamental diagram (Figure 1(b)) allowing for decreasing speed in free flow, while speeds are equal in congestion, and 3. the absence of a semi-congestion state.
In the model by Ngoduy and Liu [16], the class specific fundamental diagram is scaled according to the road fraction assigned to that class:

$$V_u(\rho_1, \dots, \rho_U) = \begin{cases} v_{u,\max} - \frac{v_{u,\max} - v_{\text{crit}}}{\tilde{\rho}_{\text{crit}}} \rho_u & \text{free flow: } \rho_u \leq \tilde{\rho}_{\text{crit}} \\ w \left(\frac{\tilde{\rho}_{\text{jam}}}{\sum_u \rho_u} - 1 \right) & \text{congestion: } \rho_u > \tilde{\rho}_{\text{crit}} \end{cases} \tag{7}$$

with v_{crit} the critical speed and all other parameters as defined before. Congestion wave speed is:

$$w = \frac{\tilde{\rho}_{\text{crit}} v_{\text{crit}}}{\tilde{\rho}_{\text{jam}} - \tilde{\rho}_{\text{crit}}} \tag{8}$$

The scaling of the fundamental diagram parameters is as follows:

$$\tilde{\rho}_{\text{crit}} = \rho_{\text{crit}} \sum_u \frac{\alpha_u}{\eta_u}, \qquad \tilde{\rho}_{\text{jam}} = \rho_{\text{jam}} \sum_u \frac{\alpha_u}{\eta_u} \tag{9}$$

η_u is pce value of class u and authors refer to the Highway Capacity Manual [28] to look them up. The pce value can be constant or depend on traffic state (piecewise constant). We note that with the scaling (9), the congestion wave speed (8) can be rewritten as a constant parameter: $w = \rho_{\text{crit}} v_{\text{crit}} / (\rho_{\text{jam}} - \rho_{\text{crit}})$.

The Fastlane model [17, 22] introduces an effective density, as an intermediate step between the class specific densities and speeds. The effective density is a weighted summation of all class specific densities:

$$\rho = \sum_u \eta_u \rho_u \tag{10}$$

with pce values:

$$\eta_u = \frac{L_u + T_u v_u}{L_1 + T_1 v_1} \tag{11}$$

with L_u average gross vehicle length of class u and T_u the minimum time headway. The speed follows a multi-class Smulders fundamental diagram:

$$V_u(\rho_1, \dots, \rho_U) = \begin{cases} v_{u,\max} - \frac{v_{u,\max} - v_{u,\text{crit}}}{\rho_{\text{crit}}} \rho & \text{free flow: } \rho \leq \rho_{\text{crit}} \\ w \left(\frac{\rho_{\text{jam}}}{\rho} - 1 \right) & \text{congestion: } \rho > \rho_{\text{crit}} \end{cases} \tag{12}$$

In [22], the following conditions were added to the parameter values: $v_{\text{crit}} \leq v_{\text{u,max}} \leq v_{1,\max} \leq 2v_{\text{crit}}$ and $w \leq L_1/T_1 \leq L_\text{u}/T_\text{u}$. The reason is as follows. (10)–(12) form an implicit set of equations, possibly with two solutions. However, there is only one physically relevant solution, that can be found using a reformulation of (10)–(12), expressing the effective density only as a function of the class specific densities, without including the pce-values. This reformulation can only be done if the parameters satisfy the above criteria.

Porous Flow Models

Porous flow models are based on the assumption that at low speeds and high densities smaller, more agile, vehicles (e.g. bikes, tricycles) move in between other vehicles such as cars and trucks which are faster in low densities, see Figure 1(c). The concept was first introduced by Nair et al. [19, 20], and later applied by Fan and Work [21]. [19, 20] explain this behaviour by making an analogy with porous flow: the 'pores' between vehicles and between vehicles and the road side which may be accessible for small and agile vehicles, while being inaccessible for large vehicles. The later model [21] is, in fact, the simplest one. It only includes two classes and uses a Greenshields fundamental relation:

$$V_u(\rho) = v_{u,\max}\left(1 - \frac{\rho}{\rho_{u,\text{jam}}}\right) \tag{13}$$

with $\rho = \rho_1 + \rho_2$ the total density and $\rho_{u,\text{jam}}$ the class specific jam density. To obtain the porous flow or creeping effect, class 1 has a higher speed in low densities, but a lower jam density than class 2 ($v_{2,\max} < v_{1,\max}$ and $\rho_{1,\text{jam}} < \rho_{2,\text{jam}}$). In [19, 20] the speed is prescribed by:

$$V_u(p) = g_u(p)v_u^{\text{rs}}(p) + (1 - g_u(p))v_u^{\text{ur}}(p) \tag{14}$$

with $v_u^{\text{rs}} = v_{u,\max}(1-g_u(p))^{c^{\text{rs}}}$ and $v_u^{\text{ur}} = v_{u,\max}(1-g_u(p))^{c^{\text{ur}}}$ the speed of restricted and unrestricted vehicles of class u, respectively. $v_{u,\max}$, c^{rs} and c^{ur} are parameters of the model, with $c^{\text{ur}} \leq c^{\text{rs}}$. $g_u(p) = e^{-pr_{u,\text{crit}}}$ is the fraction of pores that is accessible to class u. $r_{u,\text{crit}}$ is the parameter indicating the class specific critical pore size. Finally, p is the mean pore space:

$$p = (b_{\max} - b_{\min})\left(1 - \sum_u a_u \rho_u\right) + b_{\min} \tag{15}$$

with $b_{\min}$ and $b_{\max}$ bounds on the mean of the distribution, a_u a constant and ρ_u is the class specific density. We note that this formulation only makes sense if the fraction of pores accessible to class u is between 0 and 1: $0 \leq g_u(p) \leq 1$. Therefore, the mean pore space must be nonnegative and only traffic states with $\sum_u a_u \rho_u \leq b_{\max}/(b_{\max} - b_{\min})$ are to be considered.

Basic Models

The models by Benzoni-Gavage and Colombo [13], Chanut and Buisson [14] include fundamental diagrams using different vehicle lengths and maximum speed. In the models by Wong and Wong [6], Zhang et al. [15] the classes only differ in speeds. These are the most basic multi class kinematic wave models.

[14] only include two classes and apply the Smulders fundamental diagram (7), (8), like the models based on space occupancy [16, 17, 22]. However, the scaling of the density parameters is different again:

$$\tilde{\rho}_{\text{crit}} = \beta\tilde{\rho}_{\text{jam}}, \qquad \tilde{\rho}_{\text{jam}} = \frac{\rho_1 + \rho_2}{L_1\rho_1 + L_2\rho_2} \tag{16}$$

with β ∈ [0.2, 0.5] a parameter and $L_1 = 1/\tilde{\rho}_{\text{jam}}(\rho_2, 0)$ and $L_2 = 1/\tilde{\rho}_{\text{jam}}(0, \rho_2)$ are the gross vehicle lengths of class 1 and 2, respectively. This model can be reformulated such that the speed can be expressed as a function of the effective density as in Fastlane (11) but with a constant weight (pce value) $\eta_u = L_u/L_1$. Furthermore, after reformulation, the parameters of the fundamental diagram are constant, like in Fastlane (12), with:

$$\rho_{\text{jam}} = 1/L_1, \qquad \rho_{\text{crit}} = \tilde{\rho}_{\text{crit}}(\rho_1, 0) = \beta/L_1 \tag{17}$$

Benzoni-Gavage and Colombo [13] define the effective density as the weighted summation of all class specific densities as in Fastlane (10) with a constant weight (pce value) $\eta_u = L_u/L_1$. The models in Wong and Wong [6], Zhang et al. [15] are a special case with $\eta_u = 1$, leading to the effective density being an unweighted summation of the class specific densities: ρ $\rho = \sum_u \rho_u$. To find the class specific speed, Wong and Wong [6], Benzoni-Gavage and Colombo [13], Zhang et al. [15] apply a scaled version of the fundamental diagram for class 1 (Figure 1(d)):

$$v_u = \frac{v_{u,\max}}{v_{1,\max}} v_1 \tag{18}$$

Zhang et al. [15] leave the shape of the fundamental relation open. Wong and Wong [6] propose to use the Drake fundamental relation:

$$V_u(\rho) = v_{u,\max} e^{-\frac{1}{2}\left(\frac{\rho}{\rho_{\text{crit}}}\right)^2} \tag{19}$$

and Benzoni-Gavage and Colombo [13] propose to use either the Drake or Greenshields fundamental relation:

$$V_u(\rho) = v_{u,\max}\left(1 - \frac{\rho}{\rho_{\text{jam}}}\right) \tag{20}$$

REQUIREMENTS AND MODEL REFORMULATION

We introduce qualitative requirements for continuum traffic flow models. We argue that any deterministic multi-class kinematic wave traffic flow model should satisfy the following requirements:

1. When the density reaches a certain threshold (which may depend on the traffic composition), all class specific vehicle speeds are zero.
2. When a single vehicle of any class is added to the flow, neither of the class specific speeds will increase.
3. Information travels at finite speed.
4. Information travels at a velocity not larger than that of vehicles.

In future research, the set of requirements will be extended, as discussed shortly below.

Fundamental Relation Requirements

The first two Requirements put conditions on the shape of the fundamental relation and the interaction between the classes. After a certain density threshold (jam density) has been reached, all vehicles come to a complete standstill and their speed is zero (Requirement 1). The actual value of the jam density may depend on the composition, e.g. with many trucks the jam density in number of vehicles per unit road length may be lower than with only passenger cars.

Requirement 2 may seem trivial ('if it gets busier, vehicles drive slower'), but we will show in the next section that this does not hold for all models unconditionally.

Del Castillo [12] formulates a wider set of requirements for fundamental relations, including a concavity requirement, which bounds the characteristic speeds. He also discusses that, the entropy solution (the solution at which flow is maximised) of mixed class kinematic wave traffic flow models is uniquely defined if the fundamental relation is concave. In multi-class models concavity is not a sufficient condition for an entropy solution, because flow maximisation needs to be defined as well. For example, is the solution maximising the flow in number of vehicles considered to be the entropy solution, or is it the solution that maximises the flow in pce-equivalent number of vehicles? Therefore, the concavity or entropy requirement is not included in our set of requirements as such.

Model Dynamics Requirements
The last two requirements relate to the model dynamics: how do traffic states change over time? They prescribe in which direction and at which speed information propagates. Information propagates over characteristics (also known as characteristic curves or characteristic waves). Along a characteristic a certain property (e.g. density or composition) is constant. The question on the direction and speed of characteristics can thus be interpreted as whether and how quickly vehicle-driver units react on each other. We know that drivers do not react instantaneously to changes and their vehicles also need some time to react on any actions by the driver. Therefore, characteristics can not travel at infinite speeds (Requirement 3).

Furthermore, we assume that drivers only react on their leaders and not on their followers. Therefore, characteristics can not travel faster than the fastest vehicles (Requirement 4). If a traffic flow model satisfies Requirement 4 it is said to be anisotropic [29]. Other authors argue that due to overtaking on multi- lane roads, characteristics may travel faster than the average vehicle speed [30]. Therefore, we only require that the characteristics are not faster than the fastest class, they may be faster than other classes. Require- ment 4 does not make Requirement 3 redundant, because with only Requirement 4 characteristic velocities may be $-\infty$.

Model Reformulation
We first apply the model dynamics requirements to the generalised model. A crucial step is the reformulation of the generalised model in the Lagrangian coordinate system Van Wageningen-Kessels et al. [9, 31]. After this reformulation, a relatively simple eigenvalue analysis can be done because we only need to determine whether eigenvalues are bounded and nonnegative, instead of determining their exact values. We adapt the formulation in Van Wageningen-Kessels et al. [9, 31] slightly by introducing a dummy class $u = 0$. The speed and density of the dummy class equal those of the fastest class $\hat{u}$:

$$v_0 = v_{\hat{u}} = \max_u(v_u), \quad \rho_0 = \rho_{\hat{u}} = \rho_{\{\arg\max_u(v_u)\}} \tag{21}$$

Moreover, the speed and spacing of the dummy class, do not influence those of the other classes and thus the fundamental relation $v_u = V_u(\rho_1, \dots \rho_U)$ is not changed by the introduction of the dummy class. The coordinates now move with the speed of the dummy class, instead of with the speed of class $u = 1$ as in [9, 31]. The position of vehicles is traced using vehicle numbering n, with the dummy class as reference.

This leads to the following conservation equations:

$$\frac{\partial \vec{s}}{\partial t} + \mathbf{J}(\vec{s})\frac{\partial \vec{s}}{\partial n} = \vec{0} \tag{22}$$

with the vector of class specific spacings $\vec{s} = (s_1,\, s_2,\, \ldots,\, s_U)^T$ and Jacobian matrix:

$$\mathbf{J} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,U} \\ \vdots & \ddots & \vdots \\ a_{U,1} & \cdots & a_{U,U} \end{pmatrix} \quad \text{with} \quad a_{i,j} = \begin{cases} \frac{s_i}{s_0}\frac{\partial v_i^*}{\partial s_i} + \frac{v_0 - v_i}{s_0} & \text{for } i = j \text{ (on the diagonal)} \\ \frac{s_i}{s_0}\frac{\partial v_i^*}{\partial s_j} & \text{for } i \neq j \text{ (off the diagonal)} \end{cases} \tag{23}$$

$s_u = 1/\rho_u$ class specific spacing. n is the vehicle number of the dummy class u = 0. $v_u^* = V_u^*(s_1, \ldots, s_U)$ is the Lagrangian fundamental relation which can be derived from its Eulerian equivalent: $V_u^*(s_1, \ldots, s_U) = V_u(1/s_1, \ldots, 1/s_U) = V_u(\rho_1, \ldots, \rho_U)$. To simplify notation, we omit the * in the following, unless there may be confusion. Note that we only introduce the dummy class for our analysis, and we would not recommend it, for example, when building a simulation tool.

For the following analysis, we assume that the model and initial conditions are 'well-formulated'. This implies that, initially, the class specific densities are nonnegative and such that jam density is not exceeded and that densities remain within these bounds. Furthermore, we assume that road conditions are homogeneous in space and time, implying that the fundamental diagram does not change. Finally, we assume no inflow or outflow, i.e. the right-hand side of (22) only consists of zeros.

Preliminaries

For the reformulation of the model requirements, we need some preliminaries. Results from theory of partial differential equations show that the eigenvalues of the Jacobian of a conservation equation like (22) are equal to the characteristic velocities (cf. any textbook on partial differential equations, e.g. LeVeque [32]). In the general case, with $U > 4$, the eigenvalues of the Jacobian J can not be computed analytically. However, we are able to determine whether they are real, finite and nonnegative, using the following preliminaries. The first preliminary is proven by Hille and Phillips [33]. The other preliminaries are results from linear algebra and can be found in many textbooks, such as Strang [34].

Preliminary 1. Any bounded continuous function has a series approximation that converges to the value of the function itself.

Preliminary 2. *The matrices* **A** *and* $\mathbf{SAS}^{-1}$ *have the same eigenvalues for any invertible matrix S of appropriate size.*

Preliminary 3. The eigenvalues of a real and symmetric matrix are real.

Preliminary 4 (Gershgorin's circle theorem). *Suppose* **A** *is an* $n \times n$ *matrix. Each eigenvalue of* **A** *lies in one of the circles* $C_1, \ldots, C_n$*, where* C_i *is a circle in the complex plane with the center of the circle at the diagonal entry* $a_{i,i}$ *and its radius* $r_i = \sum_{j \neq i} |a_{i,j}|$ *is equal to the absolute sum of the rest of the row.*

Preliminary 5. *Suppose* **A** *is a symmetric matrix. Pivots are the entries at the main diagonal of the triangular matrix that is obtained from* **A** *with Gaussian elimination. If all pivots are* *nonnegative, then all eigenvalues of* **A** *are nonnegative.*

Finally, we define two matrices that will be used later. First, the diagonal matrix **D** has zeros everywhere except on the main diagonal, where the element in the *i*-th row in the *i*-th column is defined by:

$$d_i = \sqrt{\frac{1}{s_i}\frac{\partial v_i}{\partial s_i}} \tag{24}$$

Secondly, the matrix **M** is defined by:

$$\mathbf{M} = \mathbf{D}\mathbf{J}\mathbf{D}^{-1} \tag{25}$$

with **J** the Jacobian (23).

Reformulation of Model Dynamics Requirements

We show that the model dynamics requirements hold under the following conditions.

Condition 1. *Take all class specific densities fixed, except for class* i. *Now For each class j* ∈ {1, . . . , *U* } *the fundamental relation* $v_j = V_j(\rho_1, \ldots, \rho_i, \ldots, \rho_U) = V_j(\rho_i)$ *is either continuously differentiable or it is bounded and continuous.*

Condition 2. *There is one class that is as least as fast as all other classes in any feasible traffic state: for any given set of class specific densities* $\{\rho_1, \ldots, \rho_U\}$, *there is a class i such that* $v_i(\rho_1, \ldots, \rho_U) \geq v_j(\rho_1, \ldots, \rho_U)$ *for all classes j.*

Condition 3. *There are only 2 classes, u* = 1 *and u* = 2.

Lemma 1 (Characteristic velocities and eigenvalues of matrix **M**). *The eigenvalues of matrix* **M** (25) *correspond to the characteristic velocities of the system* (22)*,*

Proof. Recall from the Preliminaries section that the characteristic velocities of (22) equal the eigenvalues of the Jacobian **J**. Furthermore, we conclude from Preliminary 2 in the same section that the eigenvalues of the matrix **M** (25) equal those of the Jacobian **J**. □

Lemma 2. *If and only if Requirement 2 holds (i.e. the Eulerian fundamental relation* (2) *does not increase:* $\partial v_i / \partial \rho_j \leq 0$, *for all combinations of classes i and j), the Lagrangian fundamental relation* $V_u^*(s_1, \ldots, s_U)$ *does not decrease:*

$$\frac{\partial v_i}{\partial s_j} \geq 0, \textit{ for all combinations of classes } i \textit{ and } j \tag{26}$$

Proof. Rewriting the left hand side yields

$$\frac{\partial v_i}{\partial s_j} = \frac{\partial v_i}{\partial \rho_j}\frac{\mathrm{d}\rho_j}{\mathrm{d}s_j} = \frac{\partial v_i}{\partial \rho_j}\frac{\mathrm{d}}{\mathrm{d}s_j}\left(\frac{1}{s_j}\right) = -\frac{\partial v_i}{\partial \rho_j}\frac{1}{s_j^2} \tag{27}$$

This shows that the signs of $\frac{\partial v_i}{\partial \rho_j}$ and $\frac{\partial v_i}{\partial s_j}$ are opposite. □

Lemma 3. *If Requirement 2 holds, then the matrix* **M** (25) *is real and symmetric.*

Proof. By substituting the elements of the Jacobian **J** (23) and the diagonal matrix **D** (24) into matrix **M** (25), we find its elements:

$$m_{i,j} = \frac{d_i}{d_j} a_{i,j} = \begin{cases} a_{i,i} & \text{for } i = j \text{ (on the diagonal)} \quad (28a) \\ \frac{\sqrt{s_i s_j}}{s_0} \sqrt{\frac{\partial v_j}{\partial s_i} \frac{\partial v_i}{\partial s_j}} & \text{for } i \neq j \text{ (off the diagonal)} \quad (28b) \end{cases}$$

The elements on the main diagonal $a_{i,i}$ are real because the Jacobian with elements $a_{i,j}$ is real. From Lemma 2 we conclude that the term under the second square root sign in (28b) is nonnegative. Therefore, also the elements of **M** not on the diagonal are real. Furthermore, we note that $m_{i,j} = m_{j,i}$ and thus matrix **M** is symmetric. □

Lemma 4. *If Condition 1 holds, then the partial derivatives* $\partial v_j / \partial s_i$ *exist for all classes i and j.*

Proof. If the Eulerian fundamental relation $v_j = V_j(\rho_1, \dots, \rho_i, \dots, \rho_U) = V_j(\rho_i)$ is continuously differentiable, then $|\partial v_i / \partial \rho_j| < \infty$, i.e. the partial derivative exists for all classes i and j. We recall that $\partial v_j / \partial s_i$ can be rewritten as in (27) and note that the second term $(1/s_j^2)$ is finite because $s_j > 0$. Therefore, also the Lagrangian fundamental relation is continuously differentiable and $\partial v_j / \partial s_i$ exists for all classes i and j.

If, however, the fundamental relation is not continuously differentiable but it is bounded and continuous, then Preliminary 1 applies. The fundamental relation can be approximated arbitrarily closely by a continuously differentiable function and the above arguments apply to the approximated fundamental relation. □

Theorem 1. *If both Requirement 2 and Condition 1 hold, then Requirement 3 holds.*
Proof. From Lemma 1 we conclude that we only need to show that the eigenvalues of matrix **M** are finite. Combining Lemma 3 with Preliminary 3 shows that **M** has real eigenvalues. Applying Preliminary 4 shows that matrix **M** has finite eigenvalues if all its elements are finite. Therefore, what is left to show, is that all elements $m_{i,j}$ (28) are finite. This readily follows from Lemma 4. □

Theorem 2. *If Requirement 2 and Conditions 1 and 2 hold, then Requirement 4 holds.*
Proof. We start with reshuffling the classes such that class 1 is the fastest class. We note that the eigenvalues of the Jacobian **J** represent the characteristic velocities in the Lagrangian coordinate system, i.e. the velocity of information relative to the velocity of the dummy class. Therefore, all we need to show is that the characteristic velocity is nonnegative. From Lemma 1 we conclude that we need to show that the eigenvalues of matrix **M** are nonnegative. Since matrix **M** is symmetric (Lemma 3), this can be shown using Gaussian elimination (Preliminary 5). The first step of Gaussian elimination consists of subtracting $m_{i,1}/m_{1,1}$ times row 1 from each row $i > 1$. This gives matrix $\tilde{\mathbf{M}}$:

$$\tilde{m}_{i,j} = \begin{cases} \tilde{m}_{1,j} = m_{1,j} & \text{if } i = 1 \text{ (row 1)} \quad \text{(29a)} \\ \tilde{m}_{i,1} = 0 & \text{if } j = 1, i \neq 1 \text{ (col 1, except row 1)} \quad \text{(29b)} \\ \tilde{m}_{i,i} = a_{i,i} - \dfrac{a_{i,1}a_{1,i}}{a_{1,1}} = \dfrac{s_i}{s_0} \dfrac{\frac{\partial v_i}{\partial s_i}(v_0 - v_1)}{s_1 \frac{\partial v_1}{\partial s_1} + (v_0 - v_1)} + \dfrac{v_0 - v_i}{s_0} & \\ & \text{if } i = j, i \neq 1 \text{ (diag, except row 1)} \quad \text{(29c)} \\ \tilde{m}_{i,j} = \dfrac{d_i}{d_j}\left(a_{i,j} - \dfrac{a_{i,1}a_{1,j}}{a_{1,1}}\right) = \dfrac{d_i}{d_j} \dfrac{s_i}{s_0} \dfrac{\frac{\partial v_i}{\partial s_j}(v_0 - v_1)}{s_1 \frac{\partial v_1}{\partial s_1} + (v_0 - v_1)} & \\ & \text{if } i \neq 1, j \neq 1, i \neq j \text{ (everywhere else)} \quad \text{(29d)} \end{cases}$$

Since class 1 is the fastest class, $v_0 = v_1$ and $\tilde{m}_{i,j}$ in (29d) becomes zero. Therefore, only one step of Gaussian elimination is enough to get zeros in the lower triangular (except for on the diagonal) and the Gaussian elimination is terminated. The pivots are the elements on the main diagonal of matrix $\tilde{\mathbf{M}}$, i.e. $\tilde{m}_{i,i}$ in (29c). Again substituting $v_0 = v_1$ and $s_0 = s_1$ and applying Preliminary 5, shows that the pivots and thus the eigenvalues are nonnegative: $\tilde{m}_{1,1} = a_{1,1} \geq 0$ and for all classes $u > 1, \tilde{m}_{u,u} = (v_1 - v_u)/s_1 \geq 0$. □

Theorem 3. *If Requirement 2 and Conditions 1 and 3 hold, then Requirement 4 holds. The essence of this theorem was already proven in [13], using the Eulerian model formulation. However, by using the Lagrangian formulation, the proof can be greatly simplified:*

Proof. The Jacobian $\mathbf{J}$ is a 2×2 matrix with eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left(a_{1,1} + a_{2,2} \pm \sqrt{(a_{1,1} + a_{2,2})^2 - 4(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})}\right) \tag{30}$$

We can apply Lemma 3 to shown that the eigenvalues are real. The lowest eigenvalue (with a minus sign before the square root term) is nonnegative only if the second term under the square root sign is nonnegative: $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \geq 0$. This is indeed the case because:

$$\begin{aligned} a_{1,1}a_{2,2} - a_{1,2}a_{2,1} &= \left(\frac{s_1}{s_0}\frac{\partial v_1}{\partial s_1} + \frac{v_0 - v_1}{s_0}\right)\left(\frac{s_2}{s_0}\frac{\partial v_2}{\partial s_2} + \frac{v_0 - v_2}{s_0}\right) - \frac{s_1}{s_0}\frac{\partial v_1}{\partial s_2}\frac{s_2}{s_0}\frac{\partial v_2}{\partial s_1} \\ &= \frac{s_1}{s_0}\frac{\partial v_1}{\partial s_1}\frac{v_0 - v_2}{s_0} + \frac{s_2}{s_0}\frac{\partial v_2}{\partial s_2}\frac{v_0 - v_1}{s_0} + \frac{v_0 - v_2}{s_0}\frac{v_0 - v_1}{s_0} \geq 0 \end{aligned} \tag{31}$$

For the last equality we use $\frac{\partial v_1}{\partial s_1}\frac{\partial v_2}{\partial s_2} = \frac{\partial v_1}{\partial s_2}\frac{\partial v_2}{\partial s_1}$. The inequality is true because all terms are nonnegative. □

STEP-BY-STEP PLAN FOR MODEL ASSESSMENT

The analysis from the previous section in now recast as a step-by-step plan, which makes it easier to apply the framework. Furthermore, we note that even though we applied the Lagrangian coordinate system in the framework development, for the application of the framework, we can use models in their traditional Eulerian formulation. We apply the plan to assess any deterministic multi-class kinematic wave traffic flow model with respect to the requirements set out in the previous section.

1. (Re)formulate the fundamental relation such that it expresses the class specific speeds only as a function of the class specific densities, as the fundamental relation of the generalised model (2).
2. Check whether there is a finite jam density at which the speed of all classes is zero. This can for example be done by finding a value for ρ_i for each class $i \in \{1,...,U\}$ for which all speeds are zero, even if all other densities are zero:
   $v_j = V_j(0, \ldots, 0, \rho_i, 0, \ldots, 0) = 0$.
3. Check whether the fundamental relation is nonincreasing for each pair of classes:
   $\partial v_i / \partial \rho_j \leq 0$ for all pairs of $i \in \{1,...,U\}$ and $j \in \{1,...,U\}$.
4. Check whether the fundamental relations $v_u = V_u(\rho_1, \ldots, \rho_U)$ are continuously differentiable functions or, alternatively, whether they are bounded and continuous.
5. If the model contains more than two classes, check whether there is a class that is not slower than any other class in all permissible traffic states.

If the model passes all tests (step 2–5), then all requirements are satisfied. If the test in step 2 is not passed, then Requirement 1 (zero speed at a finite jam density) is not satisfied. If the test in step 3 is not passed, then Requirement 2 (nonincreasing fundamental relation) is not satisfied. Furthermore, failing this test also has implications for Requirements 3 and 4 because the characteristic speeds can not be assessed using the proposed method. If the test in step 3 is passed, but not the one in step 4, then Requirement 3 is not satisfied and characteristics may travel at infinite speed. If the test in step 3 is passed, but not the ones in step 4 and 5, then Requirement 4 is not satisfied and characteristic velocity may be larger than vehicle velocity.

Application of Step-by-step Plan

We apply the step-by-step plan to all models discussed in Section 3. Step 1 was already done in that section, so we only discuss steps 2–5 here. The results are summarised in Table 2. Some highlights and results that are not trivial to obtain are detailed below.

Finite Jam Density (Requirement 1)

The models by Logghe and Immers [8], Chanut and Buisson [14], Ngoduy and Liu [16], Van Lint et al. [17], Daganzo [18], Fan and Work [21], Van Wageningen-Kessels et al. [22] all have a finite jam density at which the speed is zero and thus Requirement 1 is satisfied. In the models with 3 regimes [8, 18] zero speed is obtained by setting $\rho_1/\rho_{1,\text{jam}} + \rho_2/\rho_{2,\text{jam}} = 1$, which can be checked by substituting these values into the fundamental relation (3). In the other models [14, 16, 17, 21, 22] the jam density is explicitly given in the fundamental relation by ρ_{jam} or $\rho_{u,\text{jam}}$. The model by Wong and Wong [6] includes the Drake fundamental relation without a finite jam density, just as one of the variants of the model by Benzoni-Gavage and Colombo [13]. Zhang et al. [15] does not explicitly include a fundamental relation and therefore this model satisfies Requirement 1 conditionally.

Nonincreasing Fundamental Relation and Other Requirements (2-4)
All models assessed here pass the test in step 5. However, the model by Ngoduy and Liu [16] is only continuous if the pce-values are continuous (step 4). Furthermore, not all models have nonincreasing fundamental relations (step 3).

The fundamental relation of the models with 3 regimes [8, 18] is nonincreasing. This is trivial in free flow and for class 2 in semi-congestion. In congestion, class 2 has the same speed as class 1. Therefore, we only consider class 1 in (semi-)congestion:

$$\frac{\partial v_1}{\partial \rho_u} = w_1 \frac{\rho_{1,\text{jam}}}{\rho_1} \left(\frac{\partial \alpha_1}{\partial \rho_u} - \frac{\alpha_1}{\rho_1} \right) \tag{32}$$

In both models, the sign of the term between brackets is found by substituting the congestion branch of (5) or (6) for both $\rho_u = \rho_1$ and $\rho_u = \rho_2$ separately. It is relatively straightforward that this term is nonpositive and thus $\partial v_1 / \partial \rho_u \leq 0$ for all classes u. It can easily be seen that the fundamental relation is bounded and continuous. Finally, the models only contain 2 classes. Since the models also pass the tests in step 4 and 5, we can conclude that the 3 regime models [8, 18] satisfy all requirements.

The model by Ngoduy and Liu [16] has a nonincreasing fundamental relationship for all relevant traffic states if and only if the pce-values η_u are not 'too far' apart, i.e. if:

$$\eta_i \leq 2\eta_j, \text{ for all combinations of classes } i \text{ and } j \tag{33}$$

This can be shown by finding the partial derivative:

$$\frac{\partial v_i}{\partial \rho_j} = \begin{cases} -\dfrac{v_{i,\max} - v_{\text{crit}}}{\rho_{\text{crit}}} \dfrac{\rho_i \sum_u \frac{\rho_u}{\eta_u} + \sum_u \rho_u \sum_u \frac{\rho_u}{\eta_u} - \frac{\rho_i}{\eta_j} \sum_u \rho_u}{\left(\sum_u \frac{\rho_u}{\eta_u} \right)^2} & \text{in free flow} \quad (34\text{a}) \\[2ex] w \rho_{\text{jam}} \dfrac{\frac{1}{\eta_u} \left(\sum_u \rho_u \right)^2 - 2 \sum_u \frac{\rho_u}{\eta_u} \sum_u \rho_u}{\left(\sum_u \rho_u \right)^4} = \dfrac{w \rho_{\text{jam}}}{\left(\sum_u \rho_u \right)^3} \sum_u \left(\dfrac{\eta_u - 2\eta_j}{\eta_j \eta_u} \rho_u \right) & \text{in congestion} \quad (34\text{b}) \end{cases}$$

In free flow, the partial derivative is largest when $\rho_u = 0$ for all classes except for $u = i$. Substituting this yields that the free flow branch (34a) is nonnegative only if (33) holds, or if $v_{i,\max} = v_{\text{crit}}$. In congestion, only if the nominator in the second fraction of (34b) is nonnegative, the partial derivative is nonnegative for all possible combinations of class specific densities and thus (33) must hold.

The fundamental relation in the model by Ngoduy and Liu [16] is bounded but it is only continuous if the pce values are continuous, otherwise Condition 1 is not satisfied. Therefore, we conclude that Requirement 2 is satisfied only if $\eta_i \leq 2\eta_j$ for all combinations of classes i and j and that Requirements 3 and 4 are only satisfied if furthermore the pce-values are continuous.

In Fastlane Van Lint et al. [17], Van Wageningen-Kessels et al. [22], the porous flow model by Fan and Work [21] and the basic models by Wong and Wong [6], Benzoni-Gavage and Colombo [13], Chanut and Buisson [14] the fundamental diagram is nonincreasing. This can be proven by showing that for all classes u, both the effective density is increasing $\mathrm{d}\rho/\mathrm{d}\rho_u > 0$ and the fundamental relation is nonincreasing $\partial v_u/\partial \rho \leq 0$. Combining both gives $\partial v_i/\partial \rho_j = (\partial v_i/\partial \rho)(\mathrm{d}\rho/\mathrm{d}\rho_j) \leq 0$. Zhang et al. [15] does not explicitly include a fundamental relation and therefore this model satisfies Requirements 2–4 only if the applied fundamental relation is nonincreasing.

The fundamental relation in the model by Nair et al. [19, 20] (14) is nonincreasing if for all classes *u*:

$$\frac{1-c^{\mathrm{ur}}}{1+c^{\mathrm{rs}}} \leq \left(1-e^{-b_{\max} r_{u,\mathrm{crit}}}\right)^{c^{\mathrm{rs}}-c^{\mathrm{ur}}} \tag{35}$$

To show this we first reformulate the fundamental relation as: $v_i = v_{i,\max}\left[g_i(1-g_i)^{c^{\mathrm{rs}}} + (1-g_i)^{c^{\mathrm{ur}}+1}\right]$. The partial derivative to class specific density ρ_j is:

$$\frac{\partial v_i}{\partial \rho_j} = v_{i,\max}\frac{\partial g_i}{\partial \rho_j}\left[\left(1-g_i-g_i c^{\mathrm{rs}}\right)\left(1-g_i\right)^{c^{\mathrm{rs}}-c^{\mathrm{ur}}-1} - \left(c^{\mathrm{ur}}+1\right)\right]\left(1-g_i\right)^{c^{\mathrm{ur}}} \tag{36}$$

The second term in this partial derivative is nonnegative: $\partial g_i/\partial \rho_j = -p r_{u,\mathrm{crit}} e^{-p r_{u,\mathrm{crit}}}\, \partial p/\partial \rho_j \geq 0$. The term between square brackets in (36) is nonpositive if (35) holds. We show this by noting that $g_i \geq e^{-b_{\max} r_{i,\mathrm{crit}}}$ and rewriting the term as:

$$\begin{aligned}(1-g_i+c^{\mathrm{ur}}g_i)(1-g_i)^{c^{\mathrm{ur}}-1-c^{\mathrm{rs}}} - (1+c^{\mathrm{rs}}) &\leq (1-c^{\mathrm{ur}})(1-g_i)^{c^{\mathrm{ur}}-c^{\mathrm{rs}}} - (1+c^{\mathrm{rs}}) \\ &\leq (1-c^{\mathrm{ur}})(1-e^{-b_{\max} r_{i,\mathrm{crit}}})^{c^{\mathrm{ur}}-c^{\mathrm{rs}}} - (1+c^{\mathrm{rs}})\end{aligned} \tag{37}$$

The right hand side is nonpositive only if the condition (35) holds.

DISCUSSION AND CONCLUSION

We have developed a framework to assess whether deterministic multi-class kinematic wave traffic flow models satisfy certain important criteria. The Lagrangian coordinate system and reformulation of a generic model into this system was applied to establish the framework. However, to apply it, only a simple step-by- step plan needs to be followed. In this contribution, the plan was applied to assess 10 models. An important step in the plan is to fit the model into a generalised model, only consisting of a system of conservation equations and a generic formulation of a class specific fundamental relation. All multi-class models known to the author that fit the generalised model were included in the analysis. Only 5 of the models passed all of the tests unconditionally. Others do not have a finite jam density at which the speed is zero, or they satisfy the model dynamics requirements only if the parameters of their fundamental relation are within certain bounds.

The step-by-step plan was set up in such a way that it can readily be applied to other models as well. This is for example helpful when developing a new model or when adapting an existing model to fit the criteria. The framework can support the selection of models to be applied in simulation, research and other applications by dismissing models with intrinsically undesirable properties in an early stage. When applying a model with criteria on the fundamental diagram parameters, calibration is simplified by applying the bounds on the parameters.

Future research can focus on an even more complete set of requirements. For example, we did not include the concavity and entropy condition in our set of requirements, which could be a valuable addition as well. Furthermore, in previous work [35] we have seen a strange phenomena in the model by Logghe and Immers [8] that is not caught with the current set of requirements. It is shown that, under certain conditions, adding a truck to the density, while at the same time removing a passenger car, will increase the speed of passenger cars.

Future research also includes further development of the framework to include a wider range of models such as those with moving bottlenecks [24], stochasticity Ngoduy [23], other multi-lane models and higher order models Hoogendoorn and Bovy [36], Bagnerini and Rascle [37]. Furthermore, newly developed models not presently included can be assessed. Existing models that do not satisfy the requirements can be adapted such that they do.

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LIST OF TABLES

LIST OF FIGURES

TABLE 1 Multi-class models and how they fit into the generalised model

Model	Principle/ assumption	Nr of classes	Fundamental relation	
Logghe and Immers [8]	3 regimes	2	$V_u(\rho_1, \rho_2) = \begin{cases} v_{u,\text{max}} & \text{free, semi } u = 2 \\ w_u \left(\frac{\alpha_u(\rho_1, \rho_2)\rho_{u,\text{jam}}}{\rho_u} - 1 \right) & \text{semi } u = 1\text{, cong} \end{cases}$	α_1 as in (5), $\alpha_2 = 1 - \alpha_1$
Daganzo [18]				α_1 as in (6), $\alpha_2 = 1 - \alpha_1$
Ngoduy and Liu [16]	Space occupancy	U	$V_u(\rho_1, \ldots, \rho_U) = \begin{cases} v_{u,\text{max}} - \frac{v_{u,\text{max}} - v_{\text{crit}}}{\tilde{\rho}_{\text{crit}}} \rho_u & \text{free flow: } \rho_u \leq \tilde{\rho}_{\text{crit}} \\ w \left(\frac{\tilde{\rho}_{\text{jam}}}{\sum_u \rho_u} - 1 \right) & \text{congestion: } \rho_u > \tilde{\rho}_{\text{crit}} \end{cases}$	$\tilde{\rho}_{\text{jam}}$ and $\tilde{\rho}_{\text{crit}}$ as in (9)
Van Lint et al. [17], Van Wageningen-Kessels et al. [22]			$V_u(\rho_1, \ldots, \rho_U) = \begin{cases} v_{u,\text{max}} - \frac{v_{u,\text{max}} - v_{u,\text{crit}}}{\rho_{\text{crit}}} \rho & \text{free flow: } \rho \leq \rho_{\text{crit}} \\ w \left(\frac{\rho_{\text{jam}}}{\rho} - 1 \right) & \text{congestion: } \rho > \rho_{\text{crit}} \end{cases}$	$\rho = \sum_u \eta_u \rho_u$, $\eta_u = \frac{L_u + T_u v_u(\rho)}{L_1 + T_1 v_1(\rho)}$
Fan and Work [21]	Porous flow	2	$V_u(\rho) = v_{u,\text{max}} \left(1 - \frac{\rho}{\rho_{u,\text{jam}}} \right)$	$\rho = \rho_1 + \rho_2$
Nair et al. [19, 20]			$V_u(p) = g_u(p) v_u^{\text{rs}}(p) + (1 - g_u(p)) v_u^{\text{ur}}(p)$	$g_u(p)$, $v_u^{\text{rs}}(p)$ and $v_u^{\text{ur}}(p)$ as in (14)–(15)
Chanut and Buisson [14]	Different speeds, different lengths	2	$V_u(\rho_1, \rho_2)$ as Ngoduy and Liu [16]	$\tilde{\rho}_{\text{crit}} = \beta \tilde{\rho}_{\text{jam}}$, $\tilde{\rho}_{\text{jam}} = \frac{\rho_1 + \rho_2}{L_1 \rho_1 + L_2 \rho_2}$
Benzoni-Gavage and Colombo [13]		U	$V_u(\rho)$ as Fan and Work [21], or $V_u(\rho) = v_{u,\text{max}} e^{-\frac{1}{2}\left(\frac{\rho}{\rho_{\text{crit}}}\right)^2}$ (Drake)	$\rho = \sum_u \eta_u \rho_u$, $\eta_u = \frac{L_u}{L_1}$, i.e. as Fastlane with $T_u = 0$
Wong and Wong [6]	Different speeds		$V_u(\rho)$ as Drake in Benzoni-Gavage and Colombo [13]	$\rho = \sum_u \rho_u$, i.e. as Fastlane with $\eta_u = 1$
Zhang et al. [15]			Undefined	$V_u = \frac{v_{u,\text{max}}}{v_{1,\text{max}}} v_1$

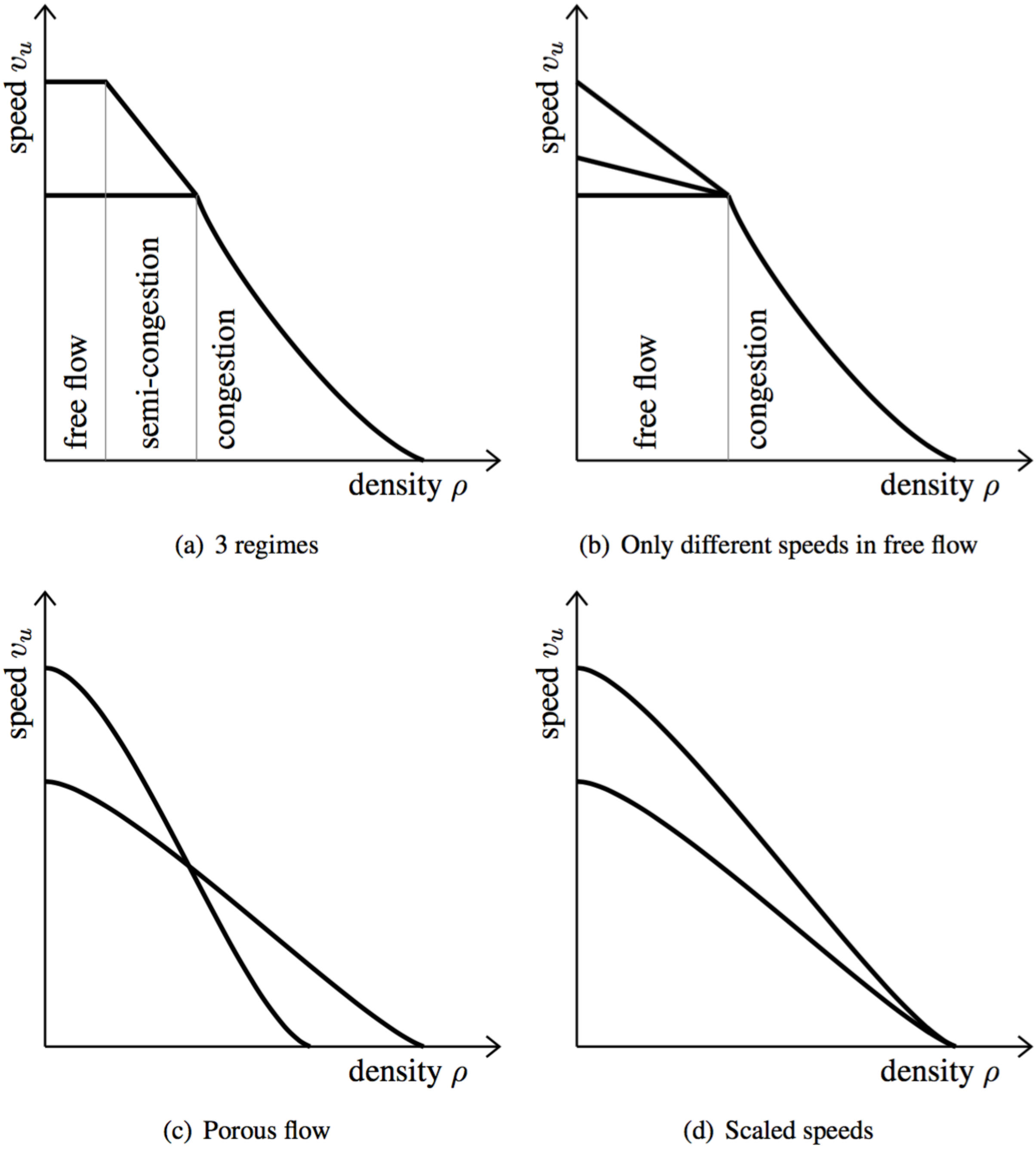
TABLE 2 Results of the model assessment

	1. Finite jam density	2. Nonincreasing fundamental relation	3. Finite characteristic speed	4. Characteristics not faster than vehicles
Logghe and Immers [8]	✓	✓	✓	✓
Daganzo [18]	✓	✓	✓	✓
Ngoduy and Liu [16]	✓	C^1	C^1, C^2	C^1, C^2
Van Lint et al. [17], Van Wageningen-Kessels et al. [22]	✓	✓	✓	✓
Fan and Work [21]	✓	✓	✓	✓
Nair et al. [19, 20]	–	C^3	C^3	C^3
Chanut and Buisson [14]	✓	✓	✓	✓
Benzoni-Gavage and Colombo [13]	C^4	✓	✓	✓
Wong and Wong [6]	–	✓	✓	✓
Zhang et al. [15]	C^5	C^5	C^5	C^5

✓ satisfied, – not satisfied, C satisfied conditionally:

C^1	$\forall i, \forall j: \quad \eta_i \leq 2\eta_j$
C^2	$\forall u: \quad \eta_u$ continuous
C^3	$\forall u: \quad \frac{1-c^{\text{ur}}}{1+c^{\text{rs}}} \leq \left(1 - e^{-b_{\max} r_{u,\text{crit}}}\right)^{c^{\text{rs}}-c^{\text{ur}}}$
C^4	✓, unless Drake fundamental relation
C^5	Depends on fundamental relation

FIGURE 1 Example fundamental diagrams related to the different types of models.



(a) 3 regimes

(b) Only different speeds in free flow

(c) Porous flow

(d) Scaled speeds