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Spectral Mimetic Least-Squares Method for div-curl systems

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Abstract. In this paper the spectral mimetic least-squares method is applied to a two-dimensional div-curl system. A test problem is solved on an orthogonal and curvilinear and both h- and p-convergence is presented. The resulting solutions will be pointwise divergence-free for these test problems. For $N > 1$ optimal convergence rates on an orthogonal and a curvilinear mesh is observed. For $N = 1$ the method does not converge.

Keywords: div-curl system, spectral element method, mimetic methods

1 Introduction

Div-curl systems play an important role in static electromagnetic fields, [4, 8] and incompressible viscous flows, [8, Ch.5]. One of the first papers where mimetic discretization for div-curl problems is described, is by Nicolaides, [9]. Nicolaides introduces geometric degrees of freedom and incidence matrices for metric-free derivatives on dual grids. When homogeneous tangential boundary conditions, $\boldsymbol{n} \times \boldsymbol{u} = 0$, or homogeneous normal boundary conditions, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, are prescribed we have that $\mathcal{N}_0(\nabla\times) \perp \mathcal{N}(\nabla\cdot)$, where $\mathcal{N}(\mathsf{A})$ denotes the null space of the operator A. This orthogonality property is important for well-posedness of div-curl systems. Mimetic discretizations preserve this property at the finite dimensional level. The method described by Nicolaides, [9], is a mimetic method. In this paper mimetic spectral element methods are used in a *conforming least-squares formulation* as described in [2, Ch.6]. Application of the non-conforming approach described in [2, Ch.6] can be found in [3].

2 Div-curl system

Let Ω be a contractible domain $\mathbb{R}^d$, $d = 2, 3$ with Lipschitz continuous boundary $\partial\Omega$. The div-curl problem consists of finding $\boldsymbol{u} \in H_0(\nabla\times, \Omega, \boldsymbol{\Theta}_1) \cap H(\nabla\cdot, \Omega, \boldsymbol{\Theta}_1^{-1})$ which satisfies

$$\begin{cases} \nabla \times \boldsymbol{u} = \boldsymbol{g} & \text{in } \Omega \\ \Theta_0^{-1} \nabla \cdot \boldsymbol{\Theta}_1 \boldsymbol{u} = 0 & \text{in } \Omega \end{cases} \quad \text{and} \quad \boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0} \text{ along } \partial\Omega \,. \tag{1}$$

The construction of conforming finite dimensional subspaces for $H_0(\nabla\times,\Omega,\boldsymbol{\Theta}_1)\cap H(\nabla\cdot,\Omega,\boldsymbol{\Theta}_1^{-1})$ is non-trivial on arbitrary domains, therefore the a formulation in terms of $H_0(\nabla\times,\Omega,\boldsymbol{\Theta}_1)\times H(\nabla\cdot,\Omega,\boldsymbol{\Theta}_1^{-1})$ is preferred. See [5] for weak formulations based on (**??**).

Following the derivation in [2, Ch.6] the first order div-curl system is given by

$$\begin{cases} \nabla\times\boldsymbol{u}=\boldsymbol{g} & \text{in } \Omega \\ \boldsymbol{v}-\Theta_1\boldsymbol{u}=\boldsymbol{0} & \text{in } \Omega \\ \nabla\cdot\boldsymbol{v}=0 & \text{in } \Omega \end{cases} \quad \text{and} \quad \boldsymbol{n}\times\boldsymbol{u}=\boldsymbol{0} \text{ along } \partial\Omega\,. \tag{2}$$

There exists a solution if $\boldsymbol{g}\in\mathcal{R}(\nabla\times)$, which, due to Poincaré's Lemma, is equal to $\nabla\cdot\boldsymbol{g}=0$. Uniqueness follows from: Let $(\boldsymbol{u}_1,\boldsymbol{v}_1)$ and $(\boldsymbol{u}_2,\boldsymbol{v}_2)$ be two solutions of (1), then $(\boldsymbol{u}_2-\boldsymbol{u}_1,\boldsymbol{v}_2-\boldsymbol{v}_1)$ satisfies (1) with $\boldsymbol{g}=\boldsymbol{0}$, therefore $\boldsymbol{u}_2-\boldsymbol{u}_1\in\mathcal{N}_0(\nabla\times)$ and $\boldsymbol{v}_2-\boldsymbol{v}_1\in\mathcal{N}(\nabla\cdot)$. But since $\mathcal{N}_0(\nabla\times)\perp\mathcal{N}(\nabla\cdot)$, the second equation in (1) implies that $\boldsymbol{u}_1=\boldsymbol{u}_2$ and $\boldsymbol{v}_1=\boldsymbol{v}_2$, which proofs uniqueness.

Consider the least-squares functional

$$\begin{cases} \mathcal{J}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{g})=\|\nabla\times\boldsymbol{u}-\boldsymbol{g}\|^2_{0,\Theta_2}+\|\nabla\cdot\boldsymbol{v}\|^2_{0,\Theta_0^{-1}}+\|\boldsymbol{v}-\Theta_1\boldsymbol{u}\|^2_{0,\Theta_1^{-1}} \\ X=H_0(\nabla\times,\Omega,\Theta_1)\times H(\nabla\cdot,\Omega,\Theta_1^{-1}) \end{cases}, \tag{3}$$

The functional setting in terms of a two-dimensional double DeRham complex for the variables $(\boldsymbol{u},\boldsymbol{v})$ and the data $\boldsymbol{g}$ is shown in (**??**)

$$\begin{tikzcd} H_0(\nabla\times,\Theta_2^{-1},\Omega) \arrow[r, "\nabla\times"] \arrow[d, shift right, "\Theta_2^{-1}"'] & \boldsymbol{v}\in H_0(\nabla\cdot,\Theta_1^{-1},\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, shift right, "\Theta_1^{-1}"'] & L^2_0(\Theta_0^{-1},\Omega) \arrow[d, shift right, "\Theta_0^{-1}"'] \\ g\in H(\nabla\cdot,\Theta_2,\Omega) \arrow[u, shift right, "\Theta_2"'] & \boldsymbol{u}\in H(\nabla\times,\Theta_1,\Omega) \arrow[l, "\nabla\times"] \arrow[u, shift right, "\Theta_1"'] & H^1(\Theta_0,\Omega) \arrow[l, "\nabla"] \arrow[u, shift right, "\Theta_0"'] \end{tikzcd} \tag{4}$$

Theorem 6.5 in [2] asserts that the least-squares functional (2) is coercive with respect to the natural norm on X. This property is inherited on conforming subspaces of $H_0(\nabla\times,\Omega,\Theta_1)\times H(\nabla\cdot,\Omega,\Theta_1^{-1})$.

3 Spectral mimetic basis functions

On contractible domains, the horizontal operators in (**??**) form an exact sequence. The aim of mimetic spectral methods is form a sequence of finite dimensional subspaces which also form an exact sequence, see for instance [1, 6, ?]. Higher order methods for div-curl systems are also described in [10].

Let $L_N(\xi)$ the Legendre polynomial of degree N with derivative $L'_N(\xi)$. The $N+1$ roots, ξ_i, of $(1-\xi^2)L'_N(\xi)$ satisfy $-1=\xi_0<\xi_1<\ldots<\xi_{N-1}<\xi_N=1$ and are called the Gauss-Lobatto-Legendre (GLL) points. Next construct the Lagrange polynomials, $h_i(\xi)$ through the GLL points with

$$h_i(\xi_j)=\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}, \quad i,j=0,\ldots,N\,.$$

From the Lagrange polynomials, we can construct the so-called edge polynomials, [6], as

$$e_i(\xi) = -\sum_{k=0}^{i-1} \mathrm{d}h_k = -\sum_{k=0}^{i-1} \frac{dh_k}{d\xi}\mathrm{d}\xi \;, \quad i = 1, \ldots, N \;.$$

The edge polynomials have that property that

$$\int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \;, \quad i,j = 1, \ldots, N \;.$$

These polynomials were presented for the first time at 7^{th} International Conference on Large-Scale Scientific Computations in Sozopol. 2009, [7, 11]. If we expand a function in terms of Lagrange polynomials, then it derivative is naturally expanded in terms of edge polynomials

$$f(\xi) = \sum_{i=0}^{N} f_i h_i(\xi) \quad \Longrightarrow \quad f'(\xi) = \sum_{i=1}^{N} (f_i - f_{i-1})\, e_i(\xi) \;. \tag{5}$$

In multiple dimensions we use tensor products of Lagrange and edge functions. For instance, on $I^2 = [-1,1]^2$ vector fields $\boldsymbol{v} \in H(\nabla\cdot, I^2)$ are expanded as

$$\boldsymbol{v} = (p,q) = \left(\sum_{i=0}^{N}\sum_{j=1}^{N} p_{i,j} h_i(\xi) e_j(\eta), \sum_{i=1}^{N}\sum_{j=0}^{N} q_{i,j} e_i(\xi) h_j(\eta) \right) \;. \tag{6}$$

Then, using (3) we have

$$\nabla \cdot \boldsymbol{v} = \sum_{i=1}^{N}\sum_{j=1}^{N} \left[p_{i,j} - p_{i-1,j} + q_{i,j} - q_{i,j-1} \right] e_i(\xi) e_j(\eta) \;.$$

Since the $e_i(\xi)e_j(\eta)$ form a basis for $\mathbb{P}^{N-1,N-1}$, we have that

$$\nabla \cdot \boldsymbol{v} = 0 \quad \Longleftrightarrow \quad p_{i,j} - p_{i-1,j} + q_{i,j} - q_{i,j-1} = 0 \;. \tag{7}$$

Note that $\nabla \cdot \boldsymbol{v} = 0$ can be completely expressed in terms of the expansion coefficients $p_{i,j}$ and $q_{i,j}$ and the basis functions cancel from this equation. Secondly, the signs $(+1)$ and (-1) in the discrete divergence, (4), correspond to the incidence matrices used in [1, 9, **?**].

For $\boldsymbol{u} \in H(\nabla\times; I^2)$ we will use the expansion

$$\boldsymbol{u} = (u,v) = \left(\sum_{i=1}^{M}\sum_{j=0}^{M} u_{i,j} e_i(\xi) h_j(\eta), \sum_{i=0}^{M}\sum_{j=1}^{M} v_{i,j} h_i(\xi) e_j(\eta) \right) \;. \tag{8}$$

Using (3) again, we have

$$\nabla \times \boldsymbol{u} = \sum_{i=1}^{M}\sum_{j=1}^{M} \left[v_{i,j} - v_{i-1,j} - u_{i,j} + u_{i,j-1} \right] e_i(\xi) e_j(\eta) \;.$$

If the right hand side function $\boldsymbol{g}$ in (1) is projected onto $e_i(\xi)e_j(\eta)$ as

$$\boldsymbol{g}^h = \sum_{i=1}^{M}\sum_{j=1}^{M} g_{i,j} e_i(\xi) e_j(\eta) \ ,$$

then $\nabla \times \boldsymbol{u} = \boldsymbol{g}$ can be represented on the grid by the difference equation

$$v_{i,j} - v_{i-1,j} - u_{i,j} + u_{i,j-1} = g_{i,j} \ . \tag{9}$$

Note, that although we use high order polynomial expansions, the discrete equations (4) and (5) are very sparse. In fact, the sparsity of these two equations is independent of the polynomial degree.

It is in the equation $\boldsymbol{v} - \boldsymbol{\Theta}_1 \boldsymbol{u} = \boldsymbol{0}$ that the two different expansions are equated. Even when $\boldsymbol{\Theta}_1$ is the identity map, this will give a full matrix. The div and curl equations can be discretized independent of the particular choice of basis functions. The dependence on the basis functions only appears in the constitutive equation $\boldsymbol{v} - \boldsymbol{\Theta}_1 \boldsymbol{u} = \boldsymbol{0}$.

The variables $\boldsymbol{u}$ and $\boldsymbol{v}$ will be treated as contravariant vectors. If we transform the equation to curvilinear coordinates only the equation $\boldsymbol{v} - \Theta \boldsymbol{u} = \boldsymbol{0}$ is affected, the div and curl equations remain unchanged. In Section **??** the performance of this discretization in curvilinear coordinates is demonstrated.

4 Mapping to curvilinear coordinates

In Section 3 the expansion are given on the square $(\xi, \eta) \in I^2$. Consider the map

$$x = x(\xi, \eta) \ , \quad y = y(\xi, \eta) \ ,$$

then the components of $\boldsymbol{u}$ and $\boldsymbol{v}$ transform as

$$\boldsymbol{u}(x,y) = (p(x,y), q(x,y)) \ , \qquad \begin{cases} p(x,y) = \frac{1}{detJ}\left[p(\xi,\eta)\frac{\partial x}{\partial \xi} + q(\xi,\eta)\frac{\partial x}{\partial \eta}\right] \\ q(x,y) = \frac{1}{detJ}\left[p(\xi,\eta)\frac{\partial y}{\partial \xi} + q(\xi,\eta)\frac{\partial y}{\partial \eta}\right] \end{cases} ,$$

and

$$\boldsymbol{v}(x,y) = (u(x,y), v(x,y)) \ , \qquad \begin{cases} u(x,y) = \frac{1}{detJ}\left[u(\xi,\eta)\frac{\partial y}{\partial \eta} - v(\xi,\eta)\frac{\partial y}{\partial \xi}\right] \\ v(x,y) = \frac{1}{detJ}\left[-u(\xi,\eta)\frac{\partial x}{\partial \eta} + v(\xi,\eta)\frac{\partial x}{\partial \xi}\right] \end{cases} ,$$

where $detJ = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$. We use the expansions from Section 3 for $p(\xi,\eta)$, $q(\xi,\eta)$, $u(\xi,\eta)$ and $v(\xi,\eta)$.

5 Numerical results

Consider problem (1) on $\Omega = [-1,1]^2 \subset \mathbb{R}^2$ with right hand side function $\boldsymbol{g} = 2\pi^2 \cos(2\pi x)\cos(2\pi y)$. For $\boldsymbol{\Theta}_1 = \mathbb{I}$ the exact solution $\boldsymbol{u} = (u, v)$ for this test case is

$$\begin{cases} u = -\pi \cos(\pi x)\sin(\pi y) \\ v = \pi \sin(\pi x)\cos(\pi y) \end{cases} ; ,$$

which resembles the test case used in [10]. For the expansions of $\boldsymbol{u}$ and $\boldsymbol{v}$ we use $N = M$ in (**??**) and (**??**), respectively. Consider the map $\Phi : \Omega \to \Omega$ given by

Fig. 1. A 16×16 grid for $c = 0.0$ (left) and $c = 0.2$ (right).

$$\begin{cases} x = \xi + c\sin(\pi\xi)\sin(\pi\eta) \\ y = \eta + c\sin(\pi\xi)\sin(\pi\eta) \end{cases} .$$

For $c = 0.0$ this mapping maps the orthogonal coordinate system (ξ, η) in the orthogonal coordinate system (x, y), see the grid on the left in Figure 1, while for $c = 0.2$ the orthogonal coordinates (ξ, η) are mapped on the curvilinear coordinates (x, y) on the grid grid in Figure 1. Figure 2 displays h-convergence

Fig. 2. h-convergence of $\boldsymbol{u}$ (left) and $\boldsymbol{v}$ (right) for polynomial degrees $N = 1, \ldots, 6$ on the orthogonal grid corresponding to $c = 0.0$.

on a sequence of uniform, orthogonal grids. The corresponding convergence rates can be found in Table 1. Based on interpolation theory, we expect a convergence rate equal to N, which is confirmed for all polynomial degrees, except for $N = 1$ which does not seem to converge at all. Application of the least-squares to

Fig. 3. h-convergence of $\boldsymbol{u}$ (left) and $\boldsymbol{v}$ (right) for polynomial degrees $N = 1, \ldots, 6$ on the orthogonal grid corresponding to $c = 0.2$.

the curvilinear grid gives h-convergence plots for various polynomial degrees as shown in Figure 3. The observed convergence rates agree with the theoretical expected convergence rates, as shown in Table 1, except again for the piecewise linear-piecewise constant approximation corresponding to $N = 1$.

Table 1 also contains the L^∞-norm of $\boldsymbol{v}$ for all polynomial degrees, on all number of elements K on both the orthogonal grid, $c = 0.0$ and the curvilinear

Table 1. Convergence rates for the div-curl least-squares solution on orthogonal ($c = 0.0$) and curvilinear grids ($c = 0.2$).

	$c = 0.0$			$c = 0.2$		
N	$\boldsymbol{u}$	$\boldsymbol{v}$	$\Vert\nabla \cdot \boldsymbol{v}\Vert_\infty$	$\boldsymbol{u}$	$\boldsymbol{v}$	$\Vert\nabla \cdot \boldsymbol{v}\Vert_\infty$
1	0.2	0.2	0.0	0.1	0.1	0.0
2	2.0	2.0	0.0	2.0	2.0	0.0
3	3.0	3.0	0.0	3.0	3.0	0.0
4	4.0	4.0	0.0	4.0	4.0	0.0
5	5.0	5.0	0.0	5.0	5.0	0.0
6	6.0	6.0	0.0	6.0	6.0	0.0

grid, $c = 0.2$. In all cases the field $\boldsymbol{v}$ is exactly divergence-free. This conservation property (or involution constraint in time-dependent problems) which is essential for incompressible flows and electromagnetism, is a direct consequence of the topological property (4).

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