Structurally quotient fixed modes

Ramos, Guilherme; Aguiar, A. Pedro; Pequito, Sérgio

DOI
10.1016/j.sysconle.2021.104914

Publication date
2021

Document Version
Final published version

Published in
Systems and Control Letters

Systems & Control Letters 151 (2021) 104914

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Structurally quotient fixed modes[☆]

Guilherme Ramos[a,*], A. Pedro Aguiar[a], Sérgio Pequito[b]

[a] Department of Electrical and Computer Engineering, Faculty of Engineering, University of Porto, Portugal
[b] Center for Systems and Control, Delft University of Technology, Netherlands

ARTICLE INFO

Article history:
Received 27 November 2020
Received in revised form 15 February 2021
Accepted 15 March 2021
Available online 10 April 2021

Keywords:
Structural systems
Decentralized control
Large-scale systems
Quotient fixed modes

ABSTRACT

We provide a necessary and sufficient graph-theoretical characterization of quotient fixes modes occurring in parametric decentralized control systems. Specifically, we introduce the notion of structurally quotient fixed modes (SQFMs) that generically captures the quotient fixed modes and only depends on the system's structure. Additionally, we provide an efficient polynomial-time algorithm for the verification of this graph-theoretical condition. We show that this algorithm can be parallelized, and linear-time computational complexity approximation algorithms can be considered to attain a sub-optimal solution. Lastly, we discuss the implications of the actuation–sensing–communication capabilities and the systems' interconnections on the existence of SQFM.

1. Introduction

Autonomy is at the heart of automation of increasingly large-scale dynamical systems such as chemical processes, smart grid, smart cities, cyber–physical systems, multi-agent systems, and the Internet of Things (IoT) [1–5]. To address the inherent problem of restricted communications and spatially distributed sensors and actuators associated with these systems, *decentralized control* is often the setting used to perform their design [6]. In this work, we consider the term *decentralized control* as is stated in [7]. The decentralization of the control law involves an independent implementation of local control laws for each control subsystem in interconnected systems. In other words, the set of control inputs of each subsystem only depend on the set of outputs of the same subsystem, yet some may not be considered for the implementation of a specific control law.

An important property of decentralized control systems is their performance. The performance capabilities of a decentralized controller are dictated by the actuation and sensing capabilities, as well as by the *information pattern*, that describes what (sensor) data is available to which actuator. Under certain constraints (that depend on the information patterns), the decentralized controllers may not be able to change all the modes of the overall system. Specifically, the modes of a linear dynamical system may not be changed in closed-loop using time-invariant controllers under a given information pattern — such modes are known as *fixed modes*. Notwithstanding, such characterization only enables the consideration of time-invariant decentralized controllers. Indeed, as pointed out in [8,9], if we lift the requirement of linear time invariant controllers, then even for systems with fixed modes, it may be possible to find periodically time-varying decentralized controllers or other general nonlinear decentralized controller such that the closed-loop system does not have fixed modes. In particular, a system with unstable fixed modes, may become a closed-loop system that is stable. Alternatively, some fixed modes can be eliminated by vibrational control [10], or by sampling techniques [11]. However, if those fixed modes have the property of being also fixed modes of a *quotient system*, then it is shown in [12] that the closed-loop system will always have fixed modes, that is, there is no decentralized controller that can "remove" the fixed modes of the system. These fixed modes are referred to as *quotient fixed modes* [12–14].

Fixed and quotient fixed modes often occur due to the canceling of terms associated with the accurate representation of the system's parameter values. Some of these values are in many cases unknown, specially in large scale systems, and when it is not the case, there is always some uncertainty associated to them that arises for example from the system identification tools that may produce numerical errors. To cope with the parametric uncertainty, we propose to leverage structural systems theory [15]. Structural systems theory provides a framework to study systems properties under the assumption that the system parameters are either arbitrary independent unknown scalars or fixed zeros due to the non-existence of physical dependency between variables [15].

In particular, the notion of *structurally fixed modes* plays a key role in unveiling vital information when a system (under a given information pattern) has fixed modes that are solely due

☆ This work was partially supported by the FCT, Portugal project POCI-01-0145-FEDER-031411-HARMONY.

∗ Corresponding author.
E-mail addresses: gramos@fe.up.pt (G. Ramos), pedro.aguiar@fe.up.pt (A.P. Aguiar), sergio.pequito@tudelft.nl (S. Pequito).

https://doi.org/10.1016/j.sysconle.2021.104914

to its structure [16,17], and subsequently, plays a crucial role in designing actuation–sensing–communication for decentralized control [18–21]. In this paper, we introduce the notion of *structurally quotient fixed modes* that enables us to characterize, from a structural perspective, the quotient fixed modes emerging in the context of decentralized control. Furthermore, we provide necessary and sufficient graph-theoretical conditions for their non-existence. Additionally, we render a thorough study on the computational approaches to verify such conditions and the implications of the actuation–sensing–communication capabilities and the systems' interconnections on the existence of SQFM.

In summary, the main contributions of this paper are as follows: *(i)* we introduce the notion of *structurally quotient fixed modes*; *(ii)* we provide necessary and sufficient graph-theoretical conditions that guarantee the non-existence of structurally quotient fixed modes; *(iii)* we provide computationally efficient polynomial algorithms to verify the necessary and sufficient conditions; *(iv)* we explain how the proposed algorithms can be distributed using parallelization schemes, and also approximate while attaining linear-time computational complexity; and *(v)* we discuss how different actuation–sensing–communication capabilities, as well as interconnections between subsystems (also known as control stations [22]), lead to the existence (or non-existence) of structurally quotient fixed modes.

2. Structurally quotient fixed modes

In this section, we first do a brief overview of key concepts required to introduce the notion of structurally quotient fixed modes (SQFMs). Next, we provide necessary and sufficient graph-theoretical conditions that are easy to verify. In fact, we show that they can be verified in polynomial-time and are suitable to deploy in the context of large-scale decentralized control systems since they admit parallel implementations and approximation algorithms with nearly linear-time computational complexity. Lastly, we leverage the aforementioned characterizations to unveil new insights into how the actuation–sensing–communication (ASC) capabilities and interconnections between subsystems may lead to the existence of SQFMs.

In what follows, we represent an element that is different from zero by $\star$. Moreover, we refer to the *structural pattern* (or structure) of a matrix A as $\bar{A}$, where $\bar{A}_{ij} = 0$ if $A_{ij} = 0$ and $\bar{A}_{ij} = \star$, otherwise.

Let us start by considering a large-scale continuous-time system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{N} B_i u_i(t), \\ y_i(t) &= C_i x(t), \qquad\qquad i = 1, \ldots, N, \end{aligned} \tag{1}$$

where N denotes the number of subsystems, $x(t) \in \mathbb{R}^n$ is the system's state vector, $u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^{r_i}$ are the input and the output vectors of the ith subsystem, respectively.

The sensor measurements available to different actuators, and used for *feedback* in the context of closed-loop decentralized control, are described by the *information pattern* $\bar{K} \in \{0, \star\}^{m^* \times r^*}$ of the system (A, B, C), with $m^* = \left(\sum_{i=1}^{N} m_i\right)$ and $r^* = \left(\sum_{i=1}^{N} r_i\right)$, where $B = [B_1, \ldots, B_N]$ and $C = [C_1^\intercal, \ldots, C_N^\intercal]^\intercal$. The information pattern is a matrix that describes what (sensor) data is available to which actuator. Specifically, $\bar{K}_{j,k} = \star$ if the data of sensor k is available to the actuator j, and 0 otherwise.

The capability of changing the system performance, measured in terms of the closed-loop modes, is dictated by the information pattern. In particular, the decentralized controller may lack the capability to change some of the systems' modes referred to *fixed modes*, formally described as follows [23]:

$$\sigma_{\bar{K}} = \bigcap_{K \in [\bar{K}]} \sigma(A + BKC), \tag{2}$$

where $[\bar{K}] = \{K \in \mathbb{R}^{m^* \times r^*} : K_{j,k} = 0 \text{ if } \bar{K}_{j,k} = 0\}$ and $\sigma(M)$ denotes the *spectrum* (i.e., the set of eigenvalues) of a square matrix $M \in \mathbb{R}^{n \times n}$. Note that from (2) it follows that if a given eigenvalue of the closed-loop system (which has dynamics $A + BKC$) is a fixed mode, then there is no $K \in [\bar{K}]$ that will make this fixed mode to disappear.

Fixed modes often occur due to a *perfect canceling* of the system numerical parameters. These are not the ones that we are interested in, but the ones that are intrinsic to the structure of the system. Therefore, to avoid such scenarios, structural systems theory considers the systems' *structural patterns* $(\bar{A}, \bar{B}, \bar{C})$ to assess the possible systems parameters in $([\bar{A}], [\bar{B}], [\bar{C}])$, where we define for a matrix structural $\bar{D}$ the set $[\bar{D}] = \{E : \bar{E} = \bar{D}\}$. When we consider only the system's matrices' structural pattern, we say that the triple $(\bar{A}, \bar{B}, \bar{C})$ denotes a *structural linear system*. A structural linear system, $(\bar{A}, \bar{B}, \bar{C})$ is said to have *structurally fixed modes* (SFMs) [24] with respect to (w.r.t.) an information pattern $\bar{K}$ if, for all $A \in [\bar{A}]$, $B \in [\bar{B}]$ and $C \in [\bar{C}]$, we have that

$$\bigcap_{K \in [\bar{K}]} \sigma(A + BKC) \neq \emptyset. \tag{3}$$

Conversely, $(\bar{A}, \bar{B}, \bar{C})$ does not have SFMs w.r.t. an information pattern K if there are instances of $A \in [\bar{A}]$, $B \in [\bar{B}]$ and $C \in [\bar{C}]$ such that

$$\bigcap_{K \in [\bar{K}]} \sigma(A + BKC) = \emptyset. \tag{4}$$

Moreover, if $(\bar{A}, \bar{B}, \bar{C})$ does not have SFMs with respect to $\bar{K}$, then almost all systems with the same sparsity do not have fixed modes [16].

Nonetheless, fixed modes are inherently associated with the use of static output feedback (possibly with controllers with memory). That said, we can eliminate some fixed modes by resorting to nonlinear output feedback, whereas others that are not possible are referred to as *quotient fixed modes* [12]. The notion of quotient fixed modes (QFMs) is associated with the *quotient system*, which is built upon a graph created using the systems transfer function between different inputs and outputs of (1).

As such, let us recall that a *digraph* $\mathcal{D} = \langle \mathcal{V}, \mathcal{E} \rangle$ is described by a set of vertices (or nodes) $\mathcal{V} = \{1, \ldots, n\}$ and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. A *path* p of size $k \in \mathbb{N}$ starting in v_1 and ending in v_k is a sequence of edges $p = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$, such that $v_i \neq v_j$ for any $i \neq j$, and if $v_1 = v_k$, then p is a *cycle*. A path p passes in vertex v if p has an edge of the form (u, v) and/or (v, u).

Subsequently, we can create a *quotient system digraph* whose vertices are the number of subsystems, and there is an edge from subsystem k to j if and only if the transfer function $C_j(sI - A)^{-1}B_k \neq 0$ [12]. If for two distinct subsystems $k, j \in \{1, \ldots, N^*\}$ we have that $C_j(sI - A)^{-1}B_k \neq 0$ and $C_k(sI - A)^{-1}B_j \neq 0$, then the subsystems are called *strongly connected*. A maximal set of strongly connected subsystems form a *strongly connected subsystem* (SCS).

If we decompose the system (1) into N^* SCS, the overall dynamics of the *quotient system* can be represented as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{N^*} B_i^* u_i^*(t), \\ y_i^*(t) &= C_i^* x(t), \qquad\qquad i = 1, \ldots, N^*, \end{aligned} \tag{5}$$

where $x(t) \in \mathbb{R}^n$ is the system's state vector, $u_i^*(t) \in \mathbb{R}^{m_i^*}$ and $y_i^*(t) \in \mathbb{R}^{r_i^*}$ are the input and the output vectors of the systems' partition ith, respectively. Simply speaking, the input and output vectors of each partition corresponds to those in the subsystems belonging to the same SCS in the quotient digraph.

Subsequently, a *quotient fixed mode* is a fixed mode of the quotient system. In other words, the system described in (1) has quotient fixed modes if and only if

$$\sigma^*_{\bar{K}_1,\ldots,\bar{K}_{N^*}} = \bigcap_{K_1\in[\bar{K}_1],\ldots,K_{N^*}\in[\bar{K}_{N^*}]} \sigma\left(A + \sum_{i=1}^{N^*} B_i^* K_i C_i^*\right) \neq \emptyset. \qquad (6)$$

In what follows, we seek to introduce the notion of SQFMs, in the same spirit that the notion of SFMs. Towards this goal, we first need to introduce the *structural* version of the quotient system digraph.

Definition 1 (*Structural Quotient System Digraph*). Let $(\bar{A}, \bar{B}, \bar{C})$, with $\bar{B} = [\bar{B}_1, \ldots, \bar{B}_N]$ and $\bar{C} = [\bar{C}_1^\intercal, \ldots, \bar{C}_N^\intercal]^\intercal$, denote the structural pattern of the system matrices associated with (1). The structural quotient system digraph is a digraph composed by:

(1) as many nodes as the number of subsystems labeled by their indices;
(2) the set of edges defined are such that there is an edge from subsystem k to j if and only if $C_j(sI - A)^{-1}B_k \neq 0$ for some $C_j \in [\bar{C}_j]$, $B_k \in [\bar{B}_k]$, and $A \in [\bar{A}]$. ∘

Remarkably, we will be able to create the structural quotient system digraph by just considering the digraph representation of the system presented in (1). Specifically, the *system digraph* $\mathcal{D}(\bar{A}, \bar{B}, \bar{C})$, given by $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}) = \langle \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}}\rangle$, where $\mathcal{X}$ is the set of state variables, $\mathcal{U}$ is the set of input variables, and $\mathcal{Y}$ is the set of output variables, and the set of edges between these are described as $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(j, i) : \bar{A}_{ij} \neq 0\}$, $\mathcal{E}_{\mathcal{X},\mathcal{Y}} = \{(j, i) : \bar{C}_{ij} \neq 0\}$, and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(j, i) : \bar{B}_{ij} \neq 0\}$. Henceforth, the following holds.

Lemma 1. *Let $(\bar{A}, \bar{B}, \bar{C})$, with $\bar{B} = [\bar{B}_1, \ldots, \bar{B}_N]$ and $\bar{C} = [\bar{C}_1^\intercal, \ldots, \bar{C}_N^\intercal]^\intercal$, denote the structural pattern of the system matrices associated with (1). Then, the structural quotient system digraph is constructed as follows: (1) the nodes' set is composed by as many nodes as the number of subsystems, labeled by their indices, (2) the set of edges are such that there is an edge from subsystem k to j if and only if there is a path from an input to an output in $\mathcal{D}(\bar{A}, \bar{B}_k, \bar{C}_j)$.* ∘

To illustrate the simplicity of invoking Lemma 1, we provide a pedagogical example next.

Example 1. Consider the structural plant $(\bar{A}, \bar{B}, \bar{C})$ with two subsystems (i.e., $N = 2$), where

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 \\ 0 & \star & 0 & \star \\ 0 & 0 & \star & 0 \end{bmatrix},\ \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \hline \bar{B}_2 \end{bmatrix} = \begin{bmatrix} \star & 0 \\ \star & 0 \\ \hline 0 & \star \\ 0 & \star \end{bmatrix}, \text{ and}$$

$$\bar{C} = \left[\begin{array}{c|c} \bar{C}_1 & \bar{C}_2 \end{array}\right] = \left[\begin{array}{cc|cc} \star & \star & 0 & 0 \\ 0 & 0 & \star & \star \end{array}\right].$$

In Fig. 1 (a), we depict the digraph representation of the structural plant $\mathcal{D}(\bar{A}, \bar{B}, \bar{C})$. In Fig. 1 (b), we present the corresponding structural quotient system digraph, which corresponds to the structural quotient subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_2$ circumscribed by the dashed gray boxes of Fig. 1 (a). We can see that there is an edge from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ in Fig. 1 (b) because there is a path from u_1 to y_2 in $\mathcal{D}(\bar{A}, \bar{B}_1, \bar{C}_2)$, as per Lemma 1. ⋄

Now, we can introduce the definition of *structurally quotient fixed modes (SQFMs)*.

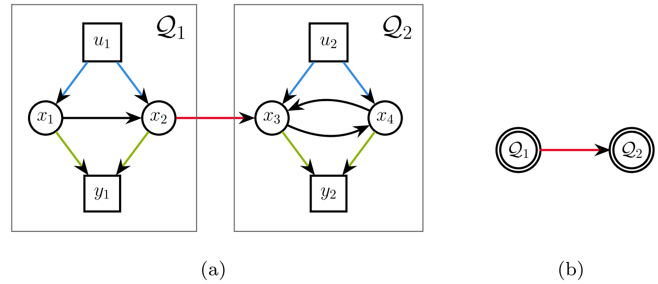

Fig. 1. In (a), we have a digraph representation of the structural plant $(\bar{A}, \bar{B}, \bar{C})$, with four states, two subsystems (one with input u_1 and output y_1, and the other with input u_2 and output y_2), with quotient subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_2$, delimited by the gray boxes. The obtained quotient system digraph by Lemma 1 is depicted in (b), where the red edge in (b) corresponds to the red edge in (a) connecting the subsystem $\mathcal{Q}_1$ with the subsystem $\mathcal{Q}_1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Definition 2 (*Structurally Quotient Fixed Modes*). Let $(\bar{A}, \bar{B}, \bar{C})$, be as in Definition 1 such that $\bar{B}^* = [\bar{B}_1^*, \ldots, \bar{B}_{N^*}^*]$ and $\bar{C}^* = [(\bar{C}_1^*)^\intercal, \ldots, (\bar{C}_{N^*}^*)^\intercal]^\intercal$, denote the structural pattern of the system matrices associated with (5), and obtained using the structural quotient system digraph with N^* partitions associated with the SCSs, where $(\bar{K}_1^*, \ldots, \bar{K}_{N^*}^*)$ denote the information patterns of each partition. We say that the triple $(\bar{A}, \bar{B}, \bar{C})$ has no SQFMs with respect to $(\bar{K}_1, \ldots, \bar{K}_N)$ when $(\bar{A}, \bar{B}^*, \bar{C}^*)$ has no SFMs with respect to $(\bar{K}_1^*, \ldots, \bar{K}_{N^*}^*)$. ∘

Subsequently, we have that the following generic property holds.

Lemma 2 (*Necessary Condition for the Non-existence of QFMs*). *Let $(\bar{A}, \bar{B}, \bar{C})$, with $\bar{B} = [\bar{B}_1, \ldots, \bar{B}_N]$ and $\bar{C} = [\bar{C}_1^\intercal, \ldots, \bar{C}_N^\intercal]^\intercal$, denote the structural pattern of the system matrices associated with (1). If $(\bar{A}, \bar{B}, \bar{C})$ has no SQFMs w.r.t. information patterns $\bar{K}_1, \ldots \bar{K}_N$, then almost all (A, B, C), where $A \in [\bar{A}]$, $B \in [\bar{B}]$, and $C \in [\bar{C}]$, have no QFMs w.r.t. to information patterns $\bar{K}_1, \ldots \bar{K}_N$.* □

Subsequently, we provide a graph-theoretical characterization of systems without SQFMs, which will require the following additional definitions. A subdigraph $\mathcal{D}' \equiv \langle \mathcal{V}', \mathcal{E}' \rangle$ of a digraph $\mathcal{D} \equiv \langle \mathcal{V}, \mathcal{E} \rangle$ is a digraph such that $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{E}' \subset \mathcal{V}' \times \mathcal{V}' \subset \mathcal{E}$. The *strongly connected components* (SCCs) of a digraph $\mathcal{D} \equiv \langle \mathcal{V}, \mathcal{E} \rangle$ are the set of maximal subgraphs which are strongly connected.

Theorem 1 (*Necessary and Sufficient Condition for the Non-existence of SQFMs*). *Consider the same setting as in Definition 2. Additionally, consider the* closed-loop *system digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}) = \langle \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \cup \mathcal{E}_{\mathcal{Y},\mathcal{U}} \rangle$, where $\mathcal{E}_{\mathcal{Y},\mathcal{U}} = \{(j, i) : \bar{K}_{ij} \neq 0\}$. Furthermore, consider the subsystems digraphs associated with each SCS of the structural quotient system digraph, denoted by $\mathcal{D}(\mathcal{Q}_i) \equiv \mathcal{D}(\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{K}_i)$, $i = 1, \ldots, N^*$, where $\bar{A}_i$ denotes the submatrix of $\bar{A}$ corresponding to the coupling between states that belong to a path starting at one of the inputs associated with $\bar{B}_i$ and ending at one of the sensors associated with $\bar{C}_i$.*

Then, $(\bar{A}, \bar{B}, \bar{C})$ has no SQFMs with respect to $(\bar{K}_1, \ldots, \bar{K}_N)$ if and only if the following conditions hold:

(i) each state variable $x \in \mathcal{X}_i$ is a vertex of an SCC of $\mathcal{D}(\mathcal{Q}_i)$, and the SCC includes an edge of $\mathcal{E}_{\mathcal{Y}_i,\mathcal{U}_i}$; and
(ii) $\mathcal{D}(\mathcal{Q}_i)$ has a finite disjoint union of cycles (say k_i), $\mathcal{C}_{k_i} = (\mathcal{V}_{k_i}, \mathcal{E}_{k_i})$ such that $\mathcal{X}_i \subset \bigcup_{j=1}^{k_i} \mathcal{V}_j$. □

Remark 1. From Lemma 2, it follows that the condition holds generically, and thus, provide us with a necessary condition to assess QFM of systems with known parameters. The conditions in Theorem 1 are easier to verify than the ones previously explored in [22,25] that depend on the system's exact parameters, which conclusions hold almost surely. ◇

Besides, both conditions presented in Theorem 1 can be efficiently verified, as it requires three main steps: (*i*) form the structural quotient system digraph and identify the SCSs; (*ii*) determine the SCCs of the different SCSs' closed-loop system digraphs and verify if they contain a feedback edge (i.e., to guarantee condition Theorem 1-(i)); and (*iii*) determine if there is a collection of cycles that contain all the state of the system digraph (i.e., to guarantee condition Theorem 1-(ii)) — see Algorithm 1.

Algorithm 1 Verification of Theorem 1

1: **input**: The tuple of structural matrices $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$, where $\bar{B} = [\bar{B}_1^\intercal, \ldots, \bar{B}_N^\intercal]^\intercal$, $C = [\bar{C}_1, \ldots, \bar{C}_N]$ and $\bar{K} = diag(\bar{K}_1, \ldots, \bar{K}_N)$
2: **output**: True if $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ has no SQFMs and False otherwise

3: **build** $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}) = \langle \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \cup \mathcal{E}_{\mathcal{Y},\mathcal{U}} \rangle$
4: **compute** the quotient system digraph $\mathcal{D}_Q$, with quotient subsystems $\mathcal{Q}_1, \ldots, \mathcal{Q}_{N^*}$
5: **for** i from 1 to N^* **do**
6: **build** $\mathcal{D}(\mathcal{Q}_i) = \langle \mathcal{X}_i \cup \mathcal{U}_i \cup \mathcal{Y}_i, \mathcal{E}_{\mathcal{X}_i,\mathcal{X}_i} \cup \mathcal{E}_{\mathcal{X}_i,\mathcal{Y}_i} \cup \mathcal{E}_{\mathcal{U}_i,\mathcal{X}_i} \cup \mathcal{E}_{\mathcal{Y}_i,\mathcal{U}_i} \rangle$
7: **compute** the SCCs of $\mathcal{Q}_i$ using the Tarjan's SCCs algorithm [26]
8: **if** $\mathcal{D}(\mathcal{Q}_i)$ does not have an edge of $\mathcal{E}_{\mathcal{Y}_i,\mathcal{U}_i}$ **then**
9: **return** False
10: **else**
11: **compute** a decomposition in paths $\mathcal{P}$ and cycles $\mathcal{C}$ of $\mathcal{D}(\mathcal{Q}_i)$
12: **if** $\mathcal{P} \neq \emptyset$ (there are paths) **then**
13: **return** False
14: **end if**
15: **end if**
16: **end for**
17: **return** True

Therefore, Algorithm 1 has the following computational complexity.

Theorem 2 (*Computational Complexity of Verifying the Conditions in Theorem 1*)**.** *The time-complexity of verifying the conditions in Theorem 1 is of order* $\mathcal{O}\left(\sqrt{N_\alpha}M_\alpha N^*\right)$, *with* $\alpha = \underset{i=1,\ldots,N^*}{\operatorname{argmax}} \sqrt{N_i}M_i$, *where* N_i *and* M_i *are the number of vertices and edges of* $\mathcal{D}(\bar{A}_i, \bar{B}_i, \bar{C}_i)$, *respectively.* □

Despite the polynomial computational complexity for the verification of Theorem 1, it might still be prohibitive to determine the non-existence of SQFM in the context of large-scale decentralized control systems. Therefore, we now discuss how the proposed algorithm can be adapted to be computed in parallel or to achieve an approximated almost linear-time solution. Therefore, we propose two ways of optimizing the algorithm's running-time performance.

A parallel computational version of Algorithm 1

Observe that the for-loop starting in step 5 can be executed in a parallel fashion since each subsystem i can compute $\mathcal{D}(Q_i)$. Subsequently, each subsystem can do steps 7–15 independently in parallel.

Observe that, when the computation is done in parallel by N^* processes, as in the setting described, the computation complexity becomes $\frac{\mathcal{O}\left(\sqrt{N_\alpha}M_\alpha N^*\right)}{N^*} = \mathcal{O}\left(\sqrt{N_\alpha}M_\alpha\right)$, where $\alpha = \underset{i=1,\ldots,N^*}{\operatorname{argmax}} \sqrt{N_i}M_i$, and N_i and M_i are the number of vertices and edges of $\mathcal{D}(\mathcal{Q}_i)$, respectively.

Approximated solution with almost linear-time complexity of Algorithm 1

We can obtain an approximated solution to the verification of Theorem 1 in almost linear-time (in the number of vertices and edges of the associated system's digraph), if we allow obtaining approximated maximum matching (MM), see more details of where the MM emerges in Algorithm 1 in the proof of Theorem 2. For example, we may use [27] which allows us to obtain a $(1-\varepsilon)$-approximation of the solution (for any specified $\varepsilon > 0$), with time complexity that depend on ε of $\mathcal{O}\left(M\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ (i.e., linear time), where M is the number of edges of $\mathcal{D}(\bar{A}, \bar{B}, \bar{C})$. Subsequently, the total time-complexity cost of Algorithm 1 becomes $\mathcal{O}\left([N+M] + \left[M\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right]\right)$, where N is the number of vertices of $\mathcal{D}(\bar{A}, \bar{B}, \bar{C})$ and $\mathcal{O}(N+M)$ comes from the computation of $\mathcal{D}(Q_i)$ with an algorithm similar to the Tarjan's strongly connected components algorithm [26] (step 7 of Algorithm 1, more detailed in the proof of Theorem 2).

Remark 2. We can further combine the parallel and the approximation computation versions of Algorithm 1 to improve the overall computational complexity. ◇

Structurally quotient fixed modes in interconnected subsystems. Lastly, we discuss under what conditions the interconnections between subsystems impact the existence of SQFMs based on graphical conditions, in Theorem 1, given that each subsystem may have or not SFM. Specifically, consider that the structural plant in (5) has, without loss of generality, two subsystems, $N^* = 2$. Suppose that $\mathcal{Q}_1 = (\bar{A}_1, \bar{B}_1, \bar{C}_1, \bar{K}_1)$ and $\mathcal{Q}_2 = (\bar{A}_2, \bar{B}_2, \bar{C}_2, \bar{K}_2)$ are the subsystems associated to each subsystem. There are, generically, three possible cases: [*Case 1*] Subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have no SFMs; [*Case 2*] One subsystem has SFMs and the other does not have SFMs, for instance $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively; and [*Case 3*] Subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have SFMs.

[*Case 1*] Regardless of the interconnections between the subsystems, the resultant system has no SQFMs. This yields since there is one instance, corresponding to assigning the value zero to all such interconnections, which would lead to a structural quotient system digraph where each partition yield both conditions (i) and (ii) of Theorem 1 since they match the ones for the SFMs in this scenario. Furthermore, we can invoke measure theoretical arguments to guarantee that if there is one such realization that yields a system without SQFMs, then almost all possible realizations of the systems' parameters (including the interconnections) yield a system without QFM; hence, no SQFM by definition.

[*Case 2*] First, notice that $\mathcal{Q}_1$ can only have SFMs if and only if it is not *structurally controllable*, not *structurally observable*, or both. This happens because at the subsystem-level the information patterns are full (i.e., all sensors are available to all actuators) [15]. Briefly, the conclusion follows from invoking measure theoretical arguments in the context of the pole-placement theorem that ensures that an arbitrary pole-placement is available if a system is both controllable and observable [28].

A system described by the pair $(\bar{A}, \bar{B})$ is structurally controllable if and only if its digraph representation $\mathcal{D}(\bar{A}, \bar{B})$ has a disjoint union of cycles and paths, such that: the paths start from an input vertex; there is (at least) one input assigned to a state variable of each source SCC (i.e., a digraph can be uniquely decomposed into SCCs and the *source* SCCs are those with no incoming edges into their states originated in the states of other SCCs) [18]. By invoking duality, a system described by $(\bar{A}, \bar{C})$ is structurally observable if and only if its digraph representation $\mathcal{D}(\bar{A}, \bar{B})$ has a disjoint union of cycles and paths, such that: the paths end in an output vertex; there is one output assigned to a state variable of each *target* SCC (i.e., the SCCs without outgoing edges originated in their states and ending at states of another SCC).

As such, the solutions to yield an interconnected system without SQFMs are as follows: (*i*) ensure that the subsystem with SFM becomes structurally controllable and/or structurally observable. This property can be ensured using minimum actuation and sensing capabilities [18,29]; and (*ii*) by adding connections between the state variables within each subsystem or in-between the states of two subsystems to ensure the overall system is structurally controllable and/or observable [30] — see illustrative example in Fig. 2. Remarkably, both solutions involve the use of a crafted *weighed* maximum matching problem [26]. This problem can be efficiently solved in polynomial time and can be accomplished in parallel and using approximation algorithms, as considered in Remark 1.

[*Case 3*] Similarly to the previous case, it can occur that subsystem $\mathcal{Q}_1$ is not structurally controllable or structurally observable (or both), and $\mathcal{Q}_2$ also is not structurally controllable or structurally observable (or both). We can remove the SQFMs combining the approaches for [*Case 2*] in both subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Remark 3. The discussion above can be easily generalized (by induction) for the case where we have more than two subsystems. Hence, it provides a constructive analysis of the existence of SQFMs, as well as methods to mitigate their existence. ◇

3. Illustrative example

Consider the plant (A, B, C) used in [12], where the matrices are parameterized as follows:

$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & a_{35} \\ 0 & 0 & a_{43} & 0 & 0 \\ 0 & 0 & a_{53} & a_{54} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} | & | & | & | & | \\ B_1 & B_2 & B_3 & B_4 & B_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & b_{55} \end{bmatrix}, \text{ and}$$

$$C = \begin{bmatrix} - & C_1 & - \\ - & C_2 & - \\ - & C_3 & - \\ - & C_4 & - \\ - & C_5 & - \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & c_{55} \end{bmatrix}.$$

In Fig. 3 (a), we depict the digraph associated with the structural plant $(\bar{A}, \bar{B}, \bar{C})$, which has two subsystems, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, as depicted in the gray boxes of Fig. 3 (a), and with quotient subsystems represented in Fig. 3 (b).

Additionally, consider a full information pattern for each of the subsystems, as depicted in Fig. 4. Using Algorithm 1, we obtain a decomposition in paths and cycles for each subsystem that does not contain paths. The decomposition in paths and cycles for the subsystem digraph $\mathcal{D}(\mathcal{Q}_1)$ is depicted by the red edges, in Fig. 4. Analogously, the decomposition in paths and cycles for the subsystem digraph $\mathcal{D}(\mathcal{Q}_1)$ is depicted by the blue edges, in Fig. 4. Since for each subsystem digraph the decomposition only has cycles ($\mathcal{P} = \emptyset$), the algorithm outputs `True`, and the system does not have SQFMs.

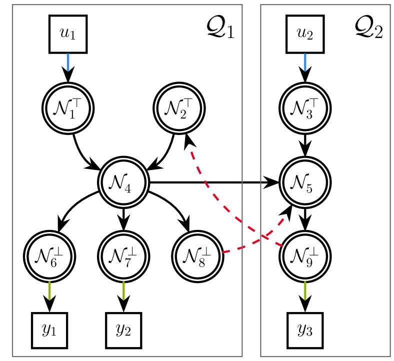

Fig. 2. Example of how to add edges between two subsystems' subsystems digraph representations to remove SQFMs, illustrating the case where one subsystem has SFMs and the other subsystems does not. To simplify the visualization and the analysis, we use meta-nodes that represent an SCC of the system's digraph representation given as follows: the source SCCs of the structural plant digraph representation are denoted by $\mathcal{N}_1^{\top}, \mathcal{N}_2^{\top}$, and $\mathcal{N}_3^{\top}$; the target SCCs are denoted by $\mathcal{N}_6^{\perp}\mathcal{N}_7^{\perp}, \mathcal{N}_8^{\perp}$, and $\mathcal{N}_9^{\perp}$; and the other SCCs are denoted by $\mathcal{N}_4$, and $\mathcal{N}_5$. For simplicity, here we consider that both subsystems are such that all the states belong to a disjoint union of cycles. Thus, the dashed red edges indicate that there is one edge from and to the state variables in different SCCS. Notice that after considering these edges, the structurally quotient system digraph changes into a single partition that is now structurally controllable and observable, and therefore, the subsystem does not have SFM, which implies that the original information pattern does lead to the existence of SQFM. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

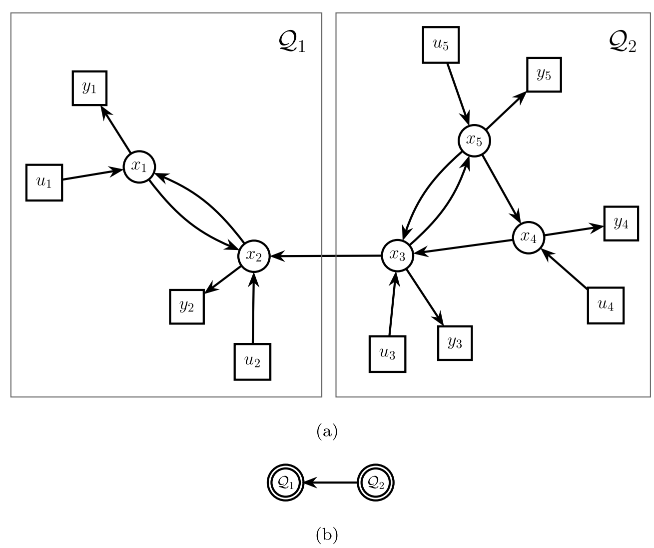

Fig. 3. In (a), we have a digraph representation of the structural plant $(\bar{A}, \bar{B}, \bar{C})$, with five states, two subsystems (one with inputs u_1, u_2 and outputs y_1, y_2, and the other with inputs u_3, u_4, u_5 and output y_3, y_4, y_5), with quotient subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_2$, delimited by the gray boxes. The obtained quotient system digraph by Lemma 1 is depicted in (b).

4. Conclusions

In this paper, we characterized quotient fixed modes in the scope of decentralized control of large-scale dynamical systems. Due to the parameter uncertainty, we introduced the novel concept of structurally quotient fixed modes (SQFMs), which generically captures the quotient fixed modes that depend on the system's structure. If we can ensure that the system does not have quotient fixed modes, then it is possible to design decentralized controllers (possibly nonlinear) to shape the system's performance through the reassignment of its modes. Therefore,

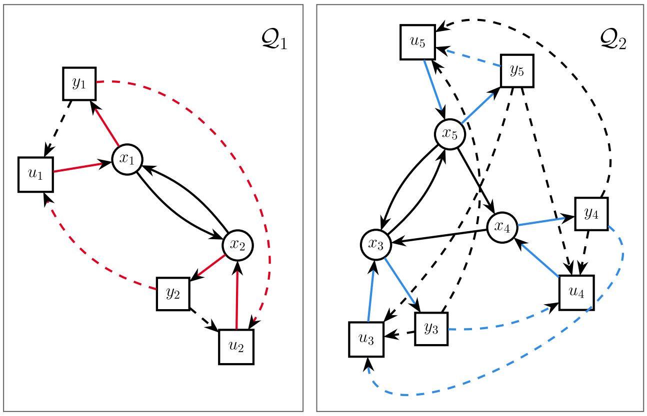

Fig. 4. Decomposition in paths and cycles of each subsystem's digraph of the structural plant $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$, i.e., $\mathcal{D}(\mathcal{Q}_1)$ and $\mathcal{D}(\mathcal{Q}_2)$, obtained with Algorithm 1. In red, we have this decomposition for $\mathcal{D}(\mathcal{Q}_1)$ with only once cycle. In blue, we have this decomposition for $\mathcal{D}(\mathcal{Q}_1)$ with only two cycles. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

we render a graph-theoretical necessary and sufficient conditions for the non-existence of SQFMs. Further, we presented an efficient polynomial-time algorithm for the verification of this graph-theoretical condition. Moreover, we described how a parallelized version of the algorithm can be implemented, and how we can consider an almost linear-time computational complexity approximation algorithm to produce approximated solutions. These two time-complexity optimizations make the proposed algorithm suitable for designing large-scale dynamical systems. Finally, we discussed the implications of the actuation–sensing–communication capabilities, as well as the systems' interconnections, on the existence of SQFM that rely solely on the graph-theoretical necessary and sufficient conditions.

CRediT authorship contribution statement

Guilherme Ramos: Conceptualization, Writing - original draft, Investigation, Methodology, Software, Writing - review & editing, Validation, Formal analysis. **A. Pedro Aguiar:** Conceptualization, Supervision, Validation, Writing - review & editing, Funding acquisition. **Sérgio Pequito:** Conceptualization, Writing - original draft, Investigation, Methodology, Writing - review & editing, Validation, Formal analysis, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Proof of Lemma 1. When we build the structural quotient system digraph as being composed by all the vertices that correspond to the labels of the different subsystems in (1), and edges such that if for each vertex k and j there is a path from the inputs to the outputs in the digraph $\mathcal{D}(\bar{A}, \bar{B}_k, \bar{C}_j)$. This ensures, by [31], that $C_j(sI - A)^{-1}B_k \neq 0$ for some $C_j \in [\bar{C}_j]$, $B_k \in [\bar{B}_k]$, and $A \in [\bar{A}]$. $\square$

Proof of Lemma 2. Consider the system in (5) such that the system has no SQFMs w.r.t. $\bar{K}_1, \ldots, \bar{K}_N$. On one hand, we have that almost all set of parameters of structural quotient system digraph lead to a quotient system digraph such that the transfer function is nonzero [31]. On the other hand, at each of the partition subsystem of the quotient system digraph it readily follows that it does not have structurally fixed modes $\bar{K}_1, \ldots, \bar{K}_{N^*}$. This implies that, for almost all parameters, the partition subsystem does not have fixed modes w.r.t. $\bar{K}_1, \ldots, \bar{K}_{N^*}$ [16]. Hence, the system does not have QFM w.r.t. $\bar{K}_1, \ldots, \bar{K}_N$, for almost all parameters. $\square$

Proof of Theorem 1. First, we fix a parameterization of the system's structure such that the entries of $\bar{A}$ corresponding to edges between state variables of different subsystems to 0. By doing this, we can write $\bar{A} = diag(\bar{A}_1, \ldots, \bar{A}_2)$, where $\bar{A}_i$ contains the state variables of subsystems i. The system $(\bar{A}_i, \bar{B}_i, \bar{C}_i)$ does not have SFMs w.r.t. the structural pattern $\bar{K}_i$ if and only if (iff) the following two conditions hold [32]: (i) each state vertex $x \in \mathcal{X}_i$ ($\mathcal{X}_i$ is the set of state variables of $\bar{A}_i$) is in an SCC of $\mathcal{D}(\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{K}_i)$ that includes an edge of $\mathcal{E}_{\mathcal{Y}_i, \mathcal{U}_i}$ ($\mathcal{Y}_i$ and $\mathcal{U}_i$ are the set of output and input variables of $\bar{B}_i$ and $\bar{C}_i$, respectively); and (ii) $\mathcal{D}(\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{K}_i)$ has a disjoint union of k cycles $\{\mathcal{C}_k = (\mathcal{V}_k, \mathcal{E}_k)\}$ such that $\mathcal{X}_i \subset \bigcup_{j=1}^{k} \mathcal{V}_j$. So, we notice that we have only to ensure that the subsystem corresponding to each subsystem does not have SFMs, which are the conditions (i) and (ii) of the theorem, to ensure that, for that parameterization of the system's structure, the system does not have SQFMs.

Next, we observe that if the aforementioned conditions hold for a parameterization of the system's structure, then the conditions hold for almost all parameterizations of the system's structure. Therefore, $(\bar{A}, \bar{B}, \bar{C})$ has not SQFMs w.r.t. $(\bar{K}_1, \ldots, \bar{K}_N)$ if and only if conditions (i) and (ii) hold. $\square$

Proof of Theorem 2. The time-complexity of verifying the condition of Theorem 1 is the sum of the time-complexity of verifying each condition.

We can build the SCSs, $\mathcal{D}(\mathcal{Q}_i)$, from the input matrices in $\mathcal{O}(M_i)$, where M_i is the number of edges of $\mathcal{D}(\mathcal{Q}_i)$ (number of non zeros in matrices $(\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{K}_i)$) supposing that the system's matrices are represented as sparse matrices, and where $\bar{A}_i$ is the submatrix of $\bar{A}$ containing the state variables accessible from inputs of $\bar{B}_i$.

We can compute the SCCs of (i) using the Tarjan's strongly connected components algorithm [26]. This can be done in $\mathcal{O}(N_i + M_i)$, where N_i is the number of vertices and M_i the number of edges of $\mathcal{D}(\bar{A}_i)$. Subsequently, we can compute condition (ii) by building N^* problems of decomposing the state digraph representation of $\mathcal{Q}_i$ (only considering the state vertices and respective edges) into paths and cycles.

This step is less straightforward (i.e., step 11), and it can be executed by considering *maximum matching* (MM) (i.e., the maximum number of edges without common end-points) on a *bipartite graph* $\mathcal{B} \equiv \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2, \mathcal{E}_{\mathcal{V}_1, \mathcal{V}_2})$. $\mathcal{B}$ is a graph where the set of vertices may be split in two, $\mathcal{V}_1$ and $\mathcal{V}_2$, such that the there are no edges between vertices in $\mathcal{V}_1$ or between vertices in $\mathcal{V}_2$. Specifically, Algorithm 1 relies on the relationship between the MM on a system bipartite graph $\mathcal{B}(\bar{A}) \equiv \mathcal{B}(\mathcal{V}, \mathcal{V}, \mathcal{E}_{\mathcal{V}, \mathcal{V}})$ (with some abuse of notation for the vertices labels) and the decomposition of the system digraph $\mathcal{D}$ into a disjoint union of cycles and paths — see [18]. In particular, for the case of a *perfect matching* (i.e., the number of edges in the maximum matching equals the number of vertices in $\mathcal{V}$), we obtain a decomposition into a disjoint union of cycles (having no paths), where all vertices are part of such cycles. These N^* independent maximum matching problems can be solved using the Hopcroft–Karp algorithm [26] in time $\mathcal{O}(\sqrt{N_i} M_i)$.

Therefore, the total time-complexity is the sum for each subsystem of the two previous quantities: $\mathcal{O}\left(\sum_{i=1}^{N^*} [(N_i + M_i) + \sqrt{N_i} M_i]\right) = \mathcal{O}\left(\sum_{i=1}^{N^*} \sqrt{N_i} M_i\right)$. Let $\alpha = \underset{i=1,\ldots,N^*}{\text{argmax}} \sqrt{N_i} M_i$, then $\mathcal{O}\left(\sum_{i=1}^{N^*} \sqrt{N_i} M_i\right) = \mathcal{O}\left(\sqrt{N_\alpha} M_\alpha N^*\right)$. $\square$

References

[1] S. Skogestad, Control structure design for complete chemical plants, Comput. Chem. Eng. 28 (1–2) (2004) 219–234.
[2] D. Šiljak, Large-Scale Dynamic Systems: Stability and Structure, in: Dover Civil and Mechanical Engineering Series, Dover Publications, 2007.
[3] M. Egerstedt, Degrees of control, Nature 473 (7346) (2011) 158–159.
[4] G. Ramos, D. Silvestre, C. Silvestre, A general discrete-time method to achieve resilience in consensus algorithms, in: 2020 59th IEEE Conference on Decision and Control, CDC, IEEE, 2020, pp. 2702–2707.
[5] G. Ramos, D. Silvestre, C. Silvestre, General resilient consensus algorithms, Internat. J. Control (2020) 1–27.
[6] A. Mahajan, N.C. Martins, M.C. Rotkowitz, S. Yüksel, Information structures in optimal decentralized control, in: 2012 IEEE 51st Conference on Decision and Control, CDC, IEEE, 2012, pp. 1291–1306.
[7] L. Bakule, Decentralized control: An overview, Annu. Rev. Control 32 (1) (2008) 87–98.
[8] J. Lavaei, Decentralized implementation of centralized controllers for interconnected systems, IEEE Trans. Automat. Control 57 (7) (2011) 1860–1865.
[9] H. Xu, J. Wang, Distributed observer-based control law with better dynamic performance based on distributed high-gain observer, Internat. J. Systems Sci. 51 (4) (2020) 631–642.
[10] S. Lee, S. Meerkov, T. Runolfsson, Vibrational feedback control: Zeros placement capabilities, IEEE Trans. Automat. Control 32 (7) (1987) 604–611.
[11] U. Ozguner, E. Davison, Sampling and decentralized fixed modes, in: 1985 American Control Conference, IEEE, 1985, pp. 257–262.
[12] Z. Gong, M. Aldeen, Stabilization of decentralized control systems, J. Math. Syst. Estimation Control 7 (1997) 111–114.
[13] H. Kobayashi, H. Hanafusa, T. Yoshikawa, Controllability under decentralized information structure, IEEE Trans. Automat. Control 23 (2) (1978) 182–188.
[14] H. Kobayashi, T. Yoshikawa, Graph-theoretic approach to controllability and localizability of decentralized control systems, IEEE Trans. Automat. Control 27 (5) (1982) 1096–1108.
[15] G. Ramos, A.P. Aguiar, S. Pequito, Structural Systems Theory: an overview of the last 15 years, 2020, arXiv preprint arXiv:2008.11223.
[16] M. Sezer, D. Šiljak, Structurally fixed modes, Syst. Control Lett. 1 (1) (1981) 60–64.
[17] K. Reinschke, Graph-theoretic characterization of fixed modes in centralized and decentralized control, Internat. J. Control 39 (4) (1984) 715–729.
[18] S. Pequito, S. Kar, A.P. Aguiar, A framework for structural input/output and control configuration selection in large-scale systems, IEEE Trans. Autom. Control 61 (2) (2015) 303–318.
[19] S. Pequito, G. Ramos, S. Kar, A.P. Aguiar, J. Ramos, The robust minimal controllability problem, Automatica 82 (2017) 261–268.
[20] G. Ramos, S. Pequito, C. Caleiro, The robust minimal controllability problem for switched linear continuous-time systems, in: 2018 Annual American Control Conference, ACC, IEEE, 2018, pp. 210–215.
[21] S. Moothedath, P. Chaporkar, M.N. Belur, Minimum cost feedback selection for arbitrary pole placement in structured systems, IEEE Trans. Autom. Control 63 (11) (2018) 3881–3888.
[22] M. Abdolmaleki, M. Aldeen, Graph theoretic approach to characterisation of quotient fixed modes in LTI systems, Eur. J. Control 31 (2016) 41–51.
[23] S.-H. Wang, E. Davison, On the stabilization of decentralized control systems, IEEE Trans. Autom. Control 18 (5) (1973) 473–478.
[24] C.H. Papadimitriou, J. Tsitsiklis, A simple criterion for structurally fixed modes, Syst. Control Lett. 4 (6) (1984) 333–337.
[25] S. Sojoudi, J. Lavaei, A.G. Aghdam, Characterization of decentralized and quotient fixed modes via graph theory, in: Structurally Constrained Controllers, Springer, 2011, pp. 9–21.
[26] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein, Introduction to Algorithms, MIT Press, 2009.
[27] R. Duan, S. Pettie, Linear-time approximation for maximum weight matching, J. ACM 61 (1) (2014) 1–23.
[28] W. Wood, Linear Multivariable Control: A Geometric Approach, Springer, New York, 1986.
[29] S. Pequito, S. Kar, A.P. Aguiar, Minimum cost input/output design for large-scale linear structural systems, Automatica 68 (2016) 384–391.
[30] X. Chen, S. Pequito, G.J. Pappas, V.M. Preciado, Minimal edge addition for network controllability, IEEE Trans. Control Netw. Syst. 6 (1) (2018) 312–323.
[31] J. Van der Woude, A graph-theoretic characterization for the rank of the transfer matrix of a structured system, Math. Control Signals Syst. 4 (1) (1991) 33–40.
[32] V. Pichai, M.E. Sezer, D. Šiljak, A graph-theoretic characterization of structurally fixed modes, Automatica 20 (2) (1984) 247–250.