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# Generalised Spin Structures in General Relativity

Bas Janssens

**Abstract.** Generalised spin structures describe spinor fields that are coupled to both general relativity and gauge theory. We classify those generalised spin structures for which the corresponding fields admit an infinitesimal action of the space–time diffeomorphism group. This can be seen as a refinement of the classification of generalised spin structures by Avis and Isham (Commun Math Phys 72:103–118, 1980).

## 1. Introduction

In this paper, we study the space–time transformation behaviour of spinors that are coupled to general relativity (GR) as well as gauge theory.

In the absence of gauge fields, space–time transformations of spinors coupled to GR can be understood by considering pairs  $(g, \psi)$  of a metric  $g$ , together with a compatible spinor field  $\psi$ . The transformation behaviour is then governed not by a spin structure  $Q \rightarrow M$ , but rather by the principal  $\widetilde{\mathrm{GL}}^+(n, \mathbb{R})$ -bundle  $\widehat{Q} \rightarrow M$  associated to  $Q$  along the inclusion of the spin group in  $\widetilde{\mathrm{GL}}^+(n, \mathbb{R})$ . Since spinor fields acquire a minus sign upon a full rotation, the action of the space–time diffeomorphism group  $\mathrm{Diff}(M)$  does not lift from  $M$  to  $\widehat{Q}$ . It *does*, however, lift at the infinitesimal level, i.e. at the level of the Lie algebra  $\mathrm{Vec}(M)$  of vector fields. This implies that in the absence of gauge fields, the spinor fields carry an action of the universal cover of the connected component of unity of the space–time diffeomorphism group [8].

In the presence of gauge fields, the consistent description of spinors requires a so-called *generalised spin structure* or  $\mathrm{Spin}^G$ -structure [4, 16]. This is a natural generalisation of a  $\mathrm{Spin}^c$ -structure and reduces to this in the case  $G = \mathrm{U}(1)$  of electrodynamics. Generalised spin structures were classified in [2].

The aim of the present paper is to study the transformation behaviour of spinors in the presence of *both* gauge theory *and* GR. Just like in the case

of spin structures, the transformation behaviour of the fields is governed by the principal bundle  $\widehat{Q} \rightarrow M$  associated to a  $\text{Spin}^G$ -structure  $Q \rightarrow M$  along the inclusion of the spin group in  $\widetilde{\text{GL}}^+(n, \mathbb{R})$ . However, quite unlike in the case of spinors coupled to pure GR, the action of the Lie algebra  $\text{Vec}(M)$  of infinitesimal space–time transformations does *not* always lift from  $M$  to  $\widehat{Q}$ . The aim of this paper is to determine which generalised spin structures allow for such a lift, and which ones do not.

More precisely, a  $\text{Spin}^G$ -structure is called *infinitesimally natural* [19] if the  $\text{Vec}(M)$ -action can be lifted from  $M$  to  $\widehat{Q}$  in such a way that the induced transformation behaviour of the metric  $g$  is the usual one. The main result of this paper, Theorem 3, is the classification of these infinitesimally natural  $\text{Spin}^G$ -structures.

Let  $M$  be an orientable space–time manifold of dimension  $n \geq 3$ , and let  $G$  be a compact gauge group. We show that  $M$  admits an infinitesimally natural  $\text{Spin}^G$ -structure if and only if its universal cover is spin. To classify the infinitesimally natural  $\text{Spin}^G$ -structures on such a manifold  $M$ , note that the orbit map  $\iota: \text{GL}(n, \mathbb{R}) \rightarrow F$  for the frame bundle  $F$  induces an injective homomorphism  $\iota_*: \mathbb{Z}_2 \hookrightarrow \pi_1(F)$ . It is readily seen that every homomorphism  $\tau: \pi_1(F) \rightarrow G$  that maps the image of  $\mathbb{Z}_2$  to a central subgroup of  $G$  gives rise to an infinitesimally natural  $\text{Spin}^G$ -structure. We prove that every infinitesimally natural  $\text{Spin}^G$ -structure is isomorphic to one of this form.

From a technical point of view, the key to proving this ‘flat’ behaviour is showing that the lift of vector fields is a first-order differential operator. This is done by adapting results [19, 22] from the setting of principal bundles to the specific setting of  $\text{Spin}^G$ -structures, where Lie algebraic considerations allow one to exclude the possibility of higher derivatives.

Determining whether or not a  $\text{Spin}^G$ -structure is infinitesimally natural is important for the construction of stress–energy–momentum (SEM) tensors. The Lie algebra homomorphism  $\sigma: \text{Vec}(M) \rightarrow \mathfrak{aut}(\widehat{Q})$ , present only in the infinitesimally natural case, is needed if one wants to construct a SEM-tensor from Noether’s theorem [13, 14]. Essentially, by separating the infinitesimal space–time transformations from the infinitesimal gauge transformations, the homomorphism  $\sigma$  also separates the SEM-tensor from the conserved currents.

Although ordinary spin structures (the case  $G = \{\pm 1\}$ ) are always infinitesimally natural, this is no longer true for more general  $\text{Spin}^G$ -structures, not even in the case  $G = U(1)$  of  $\text{Spin}^c$ -structures. The requirement for a  $\text{Spin}^G$ -structure to be infinitesimally natural is quite restrictive and singles out a preferred class of  $\text{Spin}^G$ -structures.

For example, it was observed in the late 1970s that spinors on  $M = \mathbb{C}P^2$  are necessarily charged [16, 31, 38]. The reason for this is that  $\mathbb{C}P^2$  does not admit ordinary spin structures, but it does admit nontrivial  $\text{Spin}^c$ -structures. These are used in a variety of applications that involve spinors on  $\mathbb{C}P^2$ , such as spontaneous compactification [7, 37, 39] and fuzzy geometry [1, 6, 17]. Since  $\text{Im}(\iota_*) = \{1\}$  for  $\mathbb{C}P^2$ , our results show that *none* of the  $\text{Spin}^c$ -structures on  $\mathbb{C}P^2$  is infinitesimally natural. This means that in contrast to the case where

$M$  is a spin manifold, the space–time diffeomorphism group does *not* admit a natural action on the spinor fields of  $M = \mathbb{C}P^2$ , not even at the infinitesimal level.

## 2. Spinors Coupled to GR and Gauge Fields

In view of the central role of this notion in the present paper, we give a more detailed description of infinitesimally natural bundles in Sect. 2.1. In Sects. 2.2–2.4, we then formulate the kinematics of spinors coupled to GR in terms of fibre bundles over the space–time manifold  $M$ , the main point being that the relevant bundles are infinitesimally natural. In Sect. 2.5, we describe spinors coupled to *both* GR *and* gauge theory. In this setting, the relevant bundles are associated to  $\text{Spin}^G$ -structures rather than spin structures. In Sect. 2.6, we focus on the space–time transformation behaviour of these generalised spin structures and show that they are not necessarily infinitesimally natural.

### 2.1. Natural and Infinitesimally Natural Bundles

In a geometric setting, classical fields are sections of a fibre bundle  $\pi: Y \rightarrow M$  over the space–time manifold  $M$ . Such a bundle is called *natural* if (locally defined) diffeomorphisms  $\alpha$  of  $M$  lift to (locally defined) automorphisms  $\Sigma(\alpha)$  of  $Y \rightarrow M$ , in such a way that composition and inversion are preserved. More precisely, one requires that  $\Sigma(\alpha)^{-1} = \Sigma(\alpha^{-1})$  and  $\Sigma(\alpha \circ \beta) = \Sigma(\alpha) \circ \Sigma(\beta)$  for all composable local diffeomorphisms  $\alpha$  and  $\beta$  on  $M$ . The space–time diffeomorphism group then acts naturally on the space of fields: a diffeomorphism  $\alpha$  maps a field  $\phi: M \rightarrow Y$  to the field  $\Sigma(\alpha) \circ \phi \circ \alpha^{-1}$ .

Natural bundles are perfectly suited for GR, providing a geometric framework not only for (mixed) tensor fields, but also for the more complicated transformation behaviour of (Levi–Civita) connections. They first appeared under the name ‘geometric objects’ [28, 36, 41], although the modern definition is due to Nijenhuis [29, 30]. Natural bundles were fully classified by Palais and Terng [32], building on work of Salvioli [35] and Epstein and Thurston [12].

Unfortunately, the framework of natural bundles is quite unsuitable for field theories involving spinors. The reason is that a (local) full rotation of the space–time manifold  $M$  acts trivially on the (local) fields. Therefore, the minus sign associated to spinor rotation cannot be reproduced within the setting of natural bundles.

One way to deal with this is to give up on diffeomorphism invariance, and instead ask for invariance under the automorphism group of an underlying principal fibre bundle. This leads to the theory of *gauge natural bundles* [9, 21]. Because the distinction between space–time symmetries and gauge symmetries is lost, it is rather hard to recover the distinction between the SEM-tensor and the gauge currents in this formalism [25, 33].

In this paper, we propose a different solution. Rather than abandoning diffeomorphism invariance altogether, we require diffeomorphism invariance only at the *infinitesimal* level. A fibre bundle  $\pi: Y \rightarrow M$  is called *infinitesimally natural* if it comes with a lift of *infinitesimal* diffeomorphisms [19].

More precisely, an infinitesimally natural bundle is a smooth fibre bundle  $\pi: Y \rightarrow M$ , together with a Lie algebra homomorphism  $\sigma: \text{Vec}(M) \rightarrow \mathfrak{aut}(Y)$ . We require that the lift  $\sigma(v)$  of any vector field  $v \in \text{Vec}(M)$  projects down to  $v$  again,  $\pi_* \circ \sigma(v) = v$ .

*Remark 1.* If  $\pi: Y \rightarrow M$  is just a smooth fibre bundle, then  $\mathfrak{aut}(Y)$  is the Lie algebra of projectable vector fields on  $Y$ , and  $\mathfrak{aut}^V(Y)$  denotes the Lie algebra of *vertical* vector fields. However, if  $Y$  has additional structure, then we will take  $\mathfrak{aut}(Y)$  to be the corresponding subalgebra of infinitesimal automorphisms. For example, if  $Y$  is a principal  $H$ -bundle or a bundle of homogeneous spaces, then  $\mathfrak{aut}(Y) = \text{Vec}(Y)^H$  is the Lie algebra of *equivariant* vector fields, and  $\mathfrak{aut}^V(Y)$  is the *gauge algebra* of vertical, equivariant vector fields.

Rephrasing the above definitions, one can say that a *natural bundle* has a (local) splitting of the sequence

$$1 \rightarrow \text{Aut}^V(Y) \rightarrow \text{Aut}(Y) \rightarrow \text{Diff}(M) \rightarrow 1$$

of *groups*, whereas an *infinitesimally natural bundle* has a splitting of the corresponding exact sequence

$$0 \rightarrow \mathfrak{aut}^V(Y) \rightarrow \mathfrak{aut}(Y) \rightarrow \text{Vec}(M) \rightarrow 0$$

of *Lie algebras*.

Every natural bundle is of course infinitesimally natural, but the converse is not true. It turns out that the extra leeway provided by infinitesimally natural bundles is just enough to describe spin structures and certain types of generalised spin structures, while at the same time providing the extra structure needed to globally define a canonical SEM-tensor, cf. [13, p. 333], [14].

Throughout the paper, we assume that  $M$  is a smooth, connected, orientable manifold, and we fix a nondegenerate, bilinear form  $\eta: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ . Unless stated otherwise,  $M$  will be of dimension  $n \geq 3$ . The adaptations needed for the case  $n = 2$  will be briefly discussed in Remark 6. We denote the group of orientation preserving linear transformations of  $\mathbb{R}^n$  by  $\text{GL}^+(n, \mathbb{R})$ , and we denote by  $\text{SO}(\eta) \subseteq \text{GL}^+(n, \mathbb{R})$  the subgroup of transformations that preserve  $\eta$ . The principal  $\text{GL}^+(n, \mathbb{R})$ -bundle of oriented frames is denoted by  $F^+ \rightarrow M$ , or by  $F^+(M) \rightarrow M$  if we need to emphasise the manifold. If  $g$  is a pseudo-Riemannian metric on  $M$  of signature  $\eta$ , then the principal  $\text{SO}(\eta)$ -bundle of oriented,  $g$ -orthonormal frames is denoted  $OF_g^+ \rightarrow M$ . We assume that the gauge group  $G$  is a compact Lie group and denote its Lie algebra by  $\mathfrak{g}$ .

### 2.2. General Relativity

The fundamental degrees of freedom in general relativity are a pseudo-Riemannian metric  $g$  of signature  $\eta$  on space-time  $M$ , and a connection  $\nabla$  on  $TM$ . It will be convenient to describe both  $g$  and  $\nabla$  as sections of a bundle of homogeneous spaces. We identify the metric  $g$  with a section of the bundle

$$F^+/\text{SO}(\eta) \rightarrow M$$

in the usual way, namely by associating to  $g_x : T_x M \times T_x M \rightarrow \mathbb{R}$  the coset of all frames  $f : \mathbb{R}^n \rightarrow T_x M$  such that  $f^*g = \eta$ . We view the connection  $\nabla$  on  $TM$  as an equivariant connection on  $F^+ \rightarrow M$ . This in turn can be identified with a section of

$$J^1(F^+)/\text{GL}^+(n, \mathbb{R}) \rightarrow M.$$

Its value at  $x \in M$  is  $[j_x^1\phi]$ , where  $\phi : M \supset U \rightarrow F^+$  is a local section with  $\nabla_x\phi = 0$ .

These two, the metric  $g$  and the connection  $\nabla$ , are conveniently combined into a single section  $\Phi_{g,\nabla}$  of the fibre bundle

$$J^1(F^+)/\text{SO}(\eta) \rightarrow M.$$

By concatenating the section  $\Phi_{g,\nabla} : M \rightarrow J^1(F^+)/\text{SO}(\eta)$  with the projections  $J^1(F^+)/\text{SO}(\eta) \rightarrow F^+/\text{SO}(\eta)$  and  $J^1(F^+)/\text{SO}(\eta) \rightarrow J^1(F^+)/\text{GL}^+(n, \mathbb{R})$ , one recovers the metric  $g$  and the connection  $\nabla$  from the section  $\Phi_{g,\nabla}$ .

The fields  $\Phi_{g,\nabla}$  transform in a natural fashion under the group  $\text{Diff}^+(M)$  of orientation preserving diffeomorphisms. Indeed, any  $\alpha \in \text{Diff}^+(M)$  gives rise to an automorphism  $\Sigma(\alpha)$  of the bundle  $J^1(F^+)/\text{SO}(\eta) \rightarrow M$  of homogeneous spaces, defined by

$$\Sigma(\alpha)([j_m^1(\phi)]) = [j_{\alpha(m)}^1(\alpha_* \circ \phi \circ \alpha^{-1})].$$

The diffeomorphism  $\alpha$  then maps the field  $\Phi_{g,\nabla}$  to  $\Sigma(\alpha) \circ \Phi_{g,\nabla} \circ \alpha^{-1}$ , which is again a section of the bundle  $J^1(F^+)/\text{SO}(\eta) \rightarrow M$ .

Note that the group homomorphism

$$\Sigma : \text{Diff}^+(M) \rightarrow \text{Aut}(J^1(F^+)/\text{SO}(\eta))$$

splits the exact sequence of groups

$$1 \rightarrow \text{Aut}^V(J^1(F^+)/\text{SO}(\eta)) \rightarrow \text{Aut}(J^1(F^+)/\text{SO}(\eta)) \rightarrow \text{Diff}^+(M) \rightarrow 1. \quad (1)$$

The derived Lie algebra homomorphism

$$\sigma : \text{Vec}(M) \rightarrow \mathfrak{aut}(J^1(F^+)/\text{SO}(\eta))$$

therefore splits the corresponding exact sequence of Lie algebras

$$0 \rightarrow \mathfrak{aut}^V(J^1(F^+)/\text{SO}(\eta)) \rightarrow \mathfrak{aut}(J^1(F^+)/\text{SO}(\eta)) \rightarrow \text{Vec}(M) \rightarrow 0.$$

The bundle  $J^1(F^+)/\text{SO}(\eta) \rightarrow M$ , whose sections  $\Phi_{g,\nabla}$  describe a metric  $g$  together with a connection  $\nabla$ , is therefore an *infinitesimally natural bundle* in the sense of [19].

Needless to say, the lift  $\Sigma : \text{Diff}^+(M) \rightarrow \text{Aut}(J^1(F^+)/\text{SO}(\eta))$  is of central importance in GR, since diffeomorphism invariance

$$S(\Sigma(\alpha) \circ \Phi_{g,\nabla} \circ \alpha^{-1}) = S(\Phi_{g,\nabla})$$

is one of the guiding principles for finding the Einstein–Hilbert action.

### 2.3. Spin Structures

The description of spinors coupled to general relativity (GR) involves a twofold cover of  $SO(\eta)$ . In order to handle manifolds  $M$  that are oriented, but not necessarily time-oriented, we define  $\widetilde{SO}(\eta)$  to be the twofold cover  $\kappa^{-1}(SO(\eta))$  of  $SO(\eta)$  that arises as the restriction of the universal covering map  $\kappa : \widetilde{GL}^+(n, \mathbb{R}) \rightarrow GL^+(n, \mathbb{R})$ .

*Remark 2.* If  $\eta$  is positive definite, then  $\widetilde{SO}(\eta)$  is isomorphic to the spin group  $Spin(n)$ . Perhaps surprisingly, this is no longer the case if  $\eta$  is of indefinite signature. Suppose, for example, that  $\eta$  is of signature  $(3, 1)$ . Since  $SO(3, 1)$  has 2 connected components, there is no straightforward way to define a universal cover. If  $T$  denotes the time inversion and  $P$  the inversion of 3 space co-ordinates, then  $(PT)^2 = \mathbf{1}$  in  $\widetilde{SO}(3, 1)$ . However, in<sup>1</sup>  $Spin(3, 1)$ , we have  $(PT)^2 = -\mathbf{1}$  (cf. e.g. [3]). It follows that the preimage of  $\{\pm \mathbf{1}\} \subseteq SO(3, 1)$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  in  $\widetilde{SO}(3, 1)$ , and to  $\mathbb{Z}_4$  in  $Spin(3, 1)$ . So although the connected component of unity of  $\widetilde{SO}(3, 1)$  and that of  $Spin(3, 1)$  are both isomorphic to  $SL(2, \mathbb{C})$ , the groups  $\widetilde{SO}(3, 1)$  and  $Spin(3, 1)$  are not isomorphic.

Let  $M$  be an orientable manifold of dimension  $n \geq 3$  with a pseudo-Riemannian metric  $g$  of signature  $\eta$ . Then a spin structure is by definition an  $\widetilde{SO}(\eta)$ -bundle  $Q$  over  $M$ , equipped with a twofold cover  $u : Q \rightarrow OF_g^+$  of the oriented, orthogonal frame bundle, such that the following diagram commutes:

$$\begin{array}{ccc}
 \widetilde{SO}(\eta) & \xrightarrow{\kappa} & SO(\eta) \\
 \widehat{Q} & \xrightarrow{u} & \widehat{OF}_g^+ \\
 & \searrow & \swarrow \\
 & M &
 \end{array}$$

Recall that the twofold cover  $\kappa : \widetilde{SO}(\eta) \rightarrow SO(\eta)$  is the restriction of the universal covering map of  $GL^+(n, \mathbb{R})$ . A manifold is called *spin* if it admits a spin structure. We define the principal  $\widetilde{GL}^+(n, \mathbb{R})$ -bundle

$$\widehat{Q} := Q \times_{\widetilde{SO}(\eta)} \widetilde{GL}^+(n, \mathbb{R}), \tag{2}$$

and denote the induced map  $\widehat{Q} \rightarrow F^+$  by  $u$  as well. As any cover of  $F^+$  by a  $\widetilde{GL}^+(n, \mathbb{R})$ -bundle can be obtained in this way, there is a 1:1 correspondence between spin covers of  $OF_g^+(M)$  and  $F^+$ . In particular, whether or not  $M$  is spin depends neither on the metric nor on the signature.

For  $n \geq 3$ , we identify the fundamental group  $\pi_1(GL^+(n, \mathbb{R}))$  of  $GL^+(n, \mathbb{R})$  with  $\mathbb{Z}_2$ . The orbit map

$$\iota : GL^+(n, \mathbb{R}) \rightarrow F^+ \tag{3}$$

---

<sup>1</sup> Here,  $Spin(3, 1)$  is the double cover of  $SO(3, 1)$  generated by products  $v_1 \cdots v_{2r}$  of an even number of elements  $v_i \in \mathbb{R}^4 \subset Cl_{\mathbb{R}}(3, 1)$  with  $\eta(v_i, v_i) = \pm 1$ .

can be seen as fibre inclusion, so the Serre homotopy exact sequence gives rise to the exact sequence of groups

$$1 \rightarrow \mathbb{Z}_2/\text{Ker}(\iota_*) \rightarrow \pi_1(F^+) \rightarrow \pi_1(M) \rightarrow 1. \tag{4}$$

For orientable manifolds of dimension  $n \geq 3$ , the following proposition is well known.

**Proposition 1.** *A spin structure exists if and only if  $\iota_* : \mathbb{Z}_2 \rightarrow \pi_1(F^+)$  is injective and (4) splits as a sequence of groups. If spin structures exist, then equivalence classes of spin covers correspond to splittings of (4).*

*Proof.* See e.g. [26]. Alternatively, this criterion for  $M$  to be spin is equivalent to the vanishing of the second Stiefel–Whitney class [23].  $\square$

*Remark 3.* In terms of group cohomology, one can consider the sequence (4) as a cohomology class  $[\omega]$  in  $H^2(\pi_1(M), \mathbb{Z}_2/\text{Ker}(\iota_*))$ . Spin bundles exist if and only if both  $\text{Ker}(\iota_*)$  and  $[\omega]$  are trivial, in which case they are indexed by  $H^1(\pi_1(M), \mathbb{Z}_2)$ .

In the same vein, we have the following criterion for the universal cover of  $M$  to be spin.

**Proposition 2.** *The universal cover of  $M$  is spin if and only if the map  $\iota_* : \mathbb{Z}_2 \rightarrow \pi_1(F^+(M))$  is injective.*

*Proof.* The universal covering map  $p: \widetilde{M} \rightarrow M$  gives rise to the pushforward map  $Dp: F^+(\widetilde{M}) \rightarrow F^+(M)$  of oriented frame bundles. If we denote by  $\iota^M$  and  $\iota^{\widetilde{M}}$  the fibre inclusions for the oriented frame bundle of  $M$  and  $\widetilde{M}$ , respectively, we find the following commutative diagram:

$$\begin{array}{ccc} & \text{GL}^+(n, \mathbb{R}) & \\ \iota^{\widetilde{M}} \swarrow & & \searrow \iota^M \\ F^+(\widetilde{M}) & \xrightarrow{Dp} & F^+(M) \end{array} .$$

On the level of homotopy groups, this yields the commutative diagram

$$\begin{array}{ccc} & \mathbb{Z}_2 & \\ \iota_*^{\widetilde{M}} \swarrow & & \searrow \iota_*^M \\ \pi_1(F^+(\widetilde{M})) & \xrightarrow{Dp_*} & \pi_1(F^+(M)) \end{array} .$$

Since  $Dp: F^+(\widetilde{M}) \rightarrow F^+(M)$  is a covering map, it induces an injective homomorphism  $Dp_*: \pi_1(F^+(\widetilde{M})) \rightarrow \pi_1(F^+(M))$  of homotopy groups. From the above diagram, one then infers that  $\iota_*^M$  is injective if and only if  $\iota_*^{\widetilde{M}}$  is injective. By Proposition 1, this is the case if and only if  $\widetilde{M}$  is spin.  $\square$



### 2.4. Spinors Coupled to GR

Spinor fields are usually described as sections of a spinor bundle  $\mathbb{S}_g \rightarrow M$ , associated to a spin structure  $Q \rightarrow M$  along a unitary spinor representation<sup>2</sup>  $V$  of  $\widetilde{\text{SO}}(\eta)$ . This description is somewhat inconvenient to describe spinors coupled to GR, because variations in the metric  $g$  would change the very bundle  $\mathbb{S}_g \rightarrow M$  of which the spinors are sections.

Although it is possible to deal with this problem, we prefer to sidestep it by using the composite bundle

$$\widehat{Q} \times_{\widetilde{\text{SO}}(\eta)} V \rightarrow F^+/\text{SO}(\eta) \rightarrow M, \tag{5}$$

associated to the twofold cover  $u : \widehat{Q} \rightarrow F^+$  of Sect. 2.3 along the spinor representation  $V$  (cf. e.g. [18, p. 177] and [27]). From a section  $\tau : M \rightarrow \widehat{Q} \times_{\widetilde{\text{SO}}(\eta)} V$ , one recovers both the metric  $g$  and the spinor field  $\psi$ . Indeed, the concatenation of  $\tau$  with the projection  $\widehat{Q} \times_{\widetilde{\text{SO}}(\eta)} V \rightarrow F^+/\text{SO}(\eta)$  yields a section of  $F^+/\text{SO}(\eta) \rightarrow M$ , encoding the metric  $g$ . Using this metric  $g$ , one then defines the spin structure  $Q_g = u^{-1}(OF_g^+)$  inside  $\widehat{Q}$ . From this, one constructs the spinor bundle

$$\mathbb{S}_g = u^{-1}(OF_g^+) \times_{\widetilde{\text{SO}}(\eta)} V.$$

The section  $\psi$  of  $\mathbb{S}_g \rightarrow M$  is then obtained by simply restricting the image of  $\tau$ .

In the same vein, we will describe physical fields by sections of the fibre bundle  $J^1(\widehat{Q}) \times_{\widetilde{\text{SO}}(\eta)} V$ . This is equivalent to providing three sections: one of  $F^+/\text{SO}(\eta)$ , one of  $J^1(F^+)/\text{GL}^+(n, \mathbb{R})$ , and one of  $\mathbb{S}_g = u^{-1}(OF_g^+) \times_{\widetilde{\text{SO}}(\eta)} V$ . These correspond to the metric  $g_{\mu\nu}$ , the (Levi-Civita) connection  $\Gamma_{\mu\beta}^\alpha$ , and the spinor field  $\psi^a$ , respectively.

We investigate the transformation behaviour of this bundle. Note that it is *not* a natural bundle in the sense of [30] or [21]. As a spinor changes sign under a  $2\pi$ -rotation, there is no hope of finding an interesting group homomorphism  $\text{Diff}^+(M) \rightarrow \text{Aut}(J^1(\widehat{Q}) \times_{\widetilde{\text{SO}}(\eta)} V)$ . There is, however, a canonical homomorphism at the level of Lie algebras, making it an *infinitesimally natural bundle* in the sense of [19].

Because the twofold cover  $u : \widehat{Q} \rightarrow F^+$  has discrete fibres, it has a unique flat, equivariant connection, yielding a Lie algebra homomorphism

$$\nabla_{\text{can}} : \mathfrak{aut}(F^+) \rightarrow \mathfrak{aut}(\widehat{Q}). \tag{6}$$

This can be combined with the canonical Lie algebra homomorphism

$$D : \text{Vec}(M) \rightarrow \mathfrak{aut}(F^+) \tag{7}$$

<sup>2</sup> The indefinite article is appropriate since there is a choice involved here. The connected unit component of  $\widetilde{\text{SO}}(3, 1)$  is  $\text{Spin}^\uparrow(3, 1) \simeq \text{SL}(2, \mathbb{C})$ . A spinor representation for the connected component can then be unambiguously derived from a Clifford algebra representation [15]. But as  $\widetilde{\text{SO}}(3, 1)$  is not isomorphic to  $\text{Spin}(3, 1)$ , the action of the order 2 central elements covering  $PT$  will have to be specified ‘by hand’. This becomes relevant if  $M$  is orientable, but not time-orientable.

for the natural bundle  $F^+ \rightarrow M$ . At the point  $f \in F^+$ , it is defined by the first-order derivative of the pushforward map,

$$D(v)_f = \partial/\partial t|_0 \exp(tv)_* \circ f, \tag{8}$$

where  $t \mapsto \exp(tv)$  is the flow on  $M$  generated by the vector field  $v$ . The composition  $\sigma := \nabla_{\text{can}} \circ D$  is a Lie algebra homomorphism  $\sigma: \text{Vec}(M) \rightarrow \text{aut}(\widehat{Q})$  that splits the exact sequence of Lie algebras

$$0 \rightarrow \text{aut}^V(\widehat{Q}) \rightarrow \text{aut}(\widehat{Q}) \rightarrow \text{Vec}(M) \rightarrow 0. \tag{9}$$

This induces a splitting for  $J^1(\widehat{Q})$  by prolongation (see e.g. [13]), and consequently also one for  $J^1(\widehat{Q}) \times_{\widetilde{\text{SO}}(\eta)} V$ .

*Remark 4.* We would like to emphasise that even if a splitting at the level of groups does exist, it will not be physically relevant, since it cannot reproduce the minus sign under a full rotation that one expects in spinors. Take for example the spin structure  $Q = \mathbb{R}^n \times \widetilde{\text{SO}}(\eta)$  over  $\mathbb{R}^n$ , and lift  $\alpha \in \text{Diff}^+(M)$  to  $\text{Aut}(Q)$  by  $\Sigma(\alpha)(m, q) = (\alpha(m), q)$ . Restricting attention to  $\text{SO}(\eta) \subseteq \text{Diff}^+(M)$ , we see that sections of the spinor bundle  $\mathbb{S} = Q \times_{\widetilde{\text{SO}}(\eta)} V$  then transform under the trivial representation of the Lorentz group, producing Lorentz scalars rather than spin-1/2 particles. In general, using a different splitting results in an incorrect energy–momentum tensor [14].

The above remark shows that it is not only the bundle  $Q$  and the covering map  $u : Q \rightarrow OF_g$  that are relevant, but also the splitting

$$\sigma : \text{Vec}(M) \rightarrow \text{aut}(\widehat{Q}).$$

It must satisfy  $u_* \circ \sigma = D$  in order for the metric  $g \in \Gamma(F^+/\text{SO}(\eta))$  to transform properly.

Although a canonical splitting  $\sigma$  is naturally associated to any ordinary spin structure, this is no longer the case for the  $\text{Spin}^G$ -structures used to describe spinors coupled to gauge fields.

### 2.5. Generalised Spin Structures

In the presence of gauge fields, the topological conditions on  $M$  in order to support a spin structure are more relaxed. Roughly speaking, this is because the gauge group  $G$  can absorb some of the indeterminacy that stems from the 2:1 cover of the Lorentz group.

This is made more rigorous by the notion of a generalised spin structure or  $\text{Spin}^G$ -structure [2, 4, 16]. For  $n \geq 3$ , we identify the centre of  $\widetilde{\text{SO}}(\eta)$  with  $\mathbb{Z}_2 \simeq \pi_1(\text{GL}(n, \mathbb{R}))$ . If  $G$  is a Lie group with a central subgroup  $Z_2 \subseteq G$  isomorphic to  $\mathbb{Z}_2$ , then we define<sup>3</sup>

$$\text{Spin}^G := \widetilde{\text{SO}}(\eta) \times_{\mathbb{Z}_2} G. \tag{10}$$

We denote the map  $(x, g) \mapsto \kappa(x)$  by  $\kappa: \text{Spin}^G \rightarrow \text{SO}(\eta)$ .

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<sup>3</sup>This notation is convenient but slightly misleading. Beware that if  $\eta$  is of signature  $+++ -$ , then  $\text{Spin}^{\mathbb{Z}_2}$  is isomorphic to  $\widetilde{\text{SO}}(\eta)$ , not to  $\text{Spin}(3, 1)$ .

**Definition 1.** A  $\text{Spin}^G$ -structure is a  $\text{Spin}^G$ -bundle  $Q$  over  $M$ , together with a map  $u : Q \rightarrow OF_g^+$  that makes the following diagram commute:

$$\begin{array}{ccc}
 \text{Spin}^G & \xrightarrow{\kappa} & \text{SO}(\eta) \\
 \widehat{Q} & \xrightarrow{u} & \widehat{OF}_g^+ \\
 & \searrow & \swarrow \\
 & & M.
 \end{array}$$

An *isomorphism* of  $\text{Spin}^G$ -structures is an isomorphism  $\mu : Q \rightarrow Q'$  of principal fibre bundles with  $u' \circ \mu = u$ .

If  $G = \mathbb{Z}_2$ , we recover the spin structures of Sect. 2.4. Apart from spin structures, the best known examples of  $\text{Spin}^G$ -structures are  $\text{Spin}^c$ -structures. These are precisely the  $\text{Spin}^G$ -structures for the group  $G = U(1)$ , with central subgroup  $Z_2 = \{\pm 1\}$ .

The  $\text{Spin}^G$ -structure  $Q$  gives rise to the principal  $\widetilde{\text{GL}}(n, \mathbb{R}) \times_{\mathbb{Z}_2} G$ -bundle

$$\widehat{Q} := Q \times_{\widetilde{\text{SO}}(\eta)} \widetilde{\text{GL}}(n, \mathbb{R}). \tag{11}$$

Let  $V$  be a representation of  $\text{Spin}^G$ . The bundle of which the physical fields are sections is then the fibre bundle

$$J^1(\widehat{Q}) \times_{\text{Spin}^G} V \rightarrow M. \tag{12}$$

A single section of  $J^1(\widehat{Q}) \times_{\text{Spin}^G} V$  represents a metric  $g_{\mu\nu}$ , a Levi-Civita connection  $\Gamma_{\mu\beta}^\alpha$ , a gauge field  $A_\mu$ , and a spinor field  $\psi^a$ . The metric is the induced section of the bundle

$$F^+/\text{SO}(\eta) \rightarrow M, \tag{13}$$

and the Levi-Civita connection that of  $J^1(F^+)/\text{GL}^+(n, \mathbb{R}) \rightarrow M$ . One constructs the principal  $G/Z_2$ -bundle

$$P := \widehat{Q}/\widetilde{\text{GL}}^+(n, \mathbb{R}), \tag{14}$$

and the gauge field is the induced equivariant connection on  $P$ , a section of the bundle  $J^1(P)/(G/Z_2) \rightarrow M$ . The spinor field is the induced section of the spinor bundle  $\mathbb{S}_g = \pi^{-1}(OF_g^+) \times_{\text{Spin}^G} V$ .

### 2.6. Infinitesimally Natural Generalised Spin Structures

We now focus on the generalised spin structures that have an appropriate transformation law under infinitesimal space-time diffeomorphisms. We will call a  $\text{Spin}^G$ -structure  $Q \rightarrow M$  infinitesimally natural if the associated bundle  $\widehat{Q} \rightarrow M$  is infinitesimally natural in the sense of Sect. 2.1.

**Definition 2.** An *infinitesimally natural*  $\text{Spin}^G$ -structure is a  $\text{Spin}^G$ -structure  $u : Q \rightarrow OF_g^+$ , for which there exists a Lie algebra homomorphism

$$\sigma : \text{Vec}(M) \rightarrow \text{aut}(\widehat{Q})$$

that splits the exact sequence

$$0 \rightarrow \mathbf{aut}^V(\widehat{Q}) \rightarrow \mathbf{aut}(\widehat{Q}) \rightarrow \mathbf{Vec}(M) \rightarrow 0, \tag{15}$$

where  $\mathbf{aut}(\widehat{Q})$  is the Lie algebra of  $\text{Spin}^G$ -equivariant vector fields on  $\widehat{Q}$ . Moreover, we require that the composition  $u_* \circ \sigma$  of  $\sigma$  with the pushforward  $u_*$  is equal to the canonical splitting  $D$  of Eq. (7).

The splitting of (15) comes from the physical requirement that fields should have a well-defined transformation behaviour under infinitesimal coordinate transformations. The requirement  $u_* \circ \sigma = D$  corresponds to the fact that we need to interpret a section of  $\widehat{Q}/\text{GL}(n, \mathbb{R}) \times_{\mathbb{Z}_2} G \simeq F^+/\text{SO}(\eta)$  as a metric, and we know that its transformation behaviour is governed by  $D$ .

*Remark 5.* The usual boundary conditions at infinity (cf. [10,11]) will reduce the algebra of symmetries from  $\mathbf{Vec}(M)$  to some smaller Lie algebra  $L \subseteq \mathbf{Vec}(M)$ . This smaller algebra will still contain the Lie algebra  $\mathbf{Vec}_c(M)$  of compactly supported vector fields as a subalgebra,  $\mathbf{Vec}_c(M) \subseteq L \subseteq \mathbf{Vec}(M)$ . The natural requirement to impose on the  $\text{Spin}^G$ -structure is the existence of a lift of  $L$ . This directly implies existence of a lift of its subalgebra  $\mathbf{Vec}_c(M)$ . By [19, Proposition 4], however, every lift on  $\mathbf{Vec}_c(M)$  automatically extends to  $\mathbf{Vec}(M)$ . It, therefore, does not matter whether one requires a lift of  $\mathbf{Vec}_c(M)$ ,  $L$  or  $\mathbf{Vec}(M)$ .

The  $\text{Spin}^G$ -structures thus appear as the underlying principal fibre bundles in classical field theories combining gravity, spinors and gauge fields. If they are infinitesimally natural, then these fields have a well-defined transformation behaviour under infinitesimal space–time transformations. In particular, a stress–energy–momentum tensor corresponding to space–time transformations is then well defined by [13,14].

### 3. Classification

This raises the question which of the  $\text{Spin}^G$ -structures are infinitesimally natural, and which ones are not. This is answered by Theorem 3 in Sect. 3.1. The proof proceeds by adapting the classification theorem for infinitesimally natural *principal* bundles (Theorem 4.4 in [19]) to the specific case of  $\text{Spin}^G$ -structures. We review the necessary material in Sect. 3.2 and proceed with the proof of Theorem 3 in Sect. 3.3.

#### 3.1. The Classification Theorem

Let  $G$  be a Lie group with a central subgroup  $Z_2 \subseteq G$  isomorphic to  $\mathbb{Z}_2$ .

If the Lie algebra  $\mathfrak{g}$  of  $G$  does not contain any subalgebra isomorphic to  $\mathfrak{sl}(n, \mathbb{R})$ —a requirement that is automatically fulfilled if  $G$  is compact—then we shall prove the following classification theorem for infinitesimally natural  $\text{Spin}^G$ -structures.

**Theorem 3.** (Classification theorem) *An oriented manifold  $M$  of dimension  $n \geq 3$  admits infinitesimally natural  $\text{Spin}^G$ -structures if and only if its universal cover is spin. For every infinitesimally natural  $\text{Spin}^G$ -structure  $(Q, u)$ , there exists a homomorphism*

$$\tau : \pi_1(F^+) \rightarrow G$$

such that  $(Q, u)$  is isomorphic to the  $\text{Spin}^G$ -structure  $(Q', u')$ , where

$$Q' = \widetilde{OF}_g^+ \times_\tau G,$$

and  $u' : Q' \rightarrow OF_g^+$  is the canonical projection map. The composition  $\tau \circ \iota_*$  of  $\tau$  with the map  $\iota_* : \mathbb{Z}_2 \rightarrow \pi_1(F^+)$  induced by the orbit map (3) is an isomorphism onto  $Z_2 \subseteq G$ .

In other words, every infinitesimally natural  $\text{Spin}^G$ -structure is associated to the universal cover  $\widetilde{OF}_g^+$  of the oriented, orthogonal frame bundle, along a homomorphism  $\tau : \pi_1(F^+) \rightarrow G$  that identifies  $\iota_*(\mathbb{Z}_2) \subseteq \pi_1(F^+)$  with the central subgroup  $Z_2 \subseteq G$ .

*Remark 6.* For Riemannian manifolds of dimension  $n = 2$ , the classification theorem 3 continues to hold if one makes the necessary adaptations to account for the fact that  $\pi_1(\text{SO}(2)) = \mathbb{Z}$ . In this context, a  $\text{Spin}^G$ -structure can be defined as in Sect. 2.5 with  $\text{Spin}^G := (\mathbb{R} \times G)/\mathbb{Z}$ , where the action of  $\mathbb{Z}$  on  $\mathbb{R} = \widetilde{\text{SO}}(2)$  is by translation, and the action on  $G$  comes from the unique nontrivial homomorphism  $\mathbb{Z} \rightarrow Z_2 \subseteq G$ . The requirement is then that  $\tau \circ \iota_* : \mathbb{Z} \rightarrow G$  has image  $Z_2$ .

The classification theorem rather simplifies the data needed to construct the bundle of fields (12) in the infinitesimally natural case. Indeed, it suffices to have:

- An orientable manifold  $M$  whose universal cover is spin. The homomorphism  $\iota_* : \mathbb{Z}_2 \rightarrow \pi_1(F)$  is then injective, and its image  $\iota_*(\mathbb{Z}_2) \subseteq \pi_1(F)$  is a central subgroup.
- A representation  $(\rho, V)$  of  $\widetilde{\text{SO}}(\eta) \times_{\mathbb{Z}_2} \pi_1(F)$ , which is unitary when restricted to  $\pi_1(F)$ , and faithful on  $\mathbb{Z}_2$ .
- A subgroup  $G \subseteq U(V)$  that commutes with the image of  $\widetilde{\text{SO}}(\eta)$  under  $\rho$  and contains the image of  $\pi_1(F(M))$ .

One can then construct the  $\text{Spin}^G$ -structure  $\widehat{Q} = \widetilde{F}^+ \times_{\pi_1(F)} G$ , from which one recovers the bundle of fields

$$J^1(\widehat{Q}) \times_{\text{Spin}^G} V \rightarrow M.$$

As discussed in Sect. 2.5, a single section of this bundle provides the metric, Levi-Civita connection, gauge fields and spinors.

In particular, the bundle (5) describing spinors and metric is simply

$$\widetilde{F}^+ \times_R V \rightarrow M,$$

where  $R$  is the group  $R := \widetilde{\text{SO}}(\eta) \times_{\mathbb{Z}_2} \pi_1(F)$ . The principal  $G/\mathbb{Z}_2$ -bundle (14) describing the gauge fields is necessarily the trivial bundle  $P = M \times G/\mathbb{Z}_2$ .

According to Theorem 3, the above setting exhausts the possibilities in the infinitesimally natural case—at least under the natural assumption that  $V$  is a faithful, unitary representation for the group  $G$ , which is then automatically compact.

### 3.2. Infinitesimally Natural Principal Fibre Bundles

The proof of Theorem 3 relies on the result [19,22] that every infinitesimally natural *principal* fibre bundle is associated to the universal cover of a  $k$ th order frame bundle. The essential new ingredient in the proof of Theorem 3 is that in the particular case of  $\text{Spin}^G$ -structures, the order  $k$  must be equal to one. Before proceeding with this proof, we therefore briefly recall some results on infinitesimally natural principal bundles.

**3.2.1. The  $k$ th Order Frame Bundle and its Universal Cover.** A  $k$ th order frame  $f_x^k$  at a point  $x \in M$  is by definition the  $k$ -jet  $f_x^k = j_0^k \phi$  at zero of an orientation preserving local diffeomorphism  $\phi: \mathbb{R}^n \rightarrow M$  with  $\phi(0) = x$ . The oriented  $k$ th order frame bundle  $\pi: F^{k+} \rightarrow M$  is defined by

$$F^{k+} := \{j_0^k \phi; \phi \in \text{Diff}_{\text{loc}}^+(\mathbb{R}^n, M)\},$$

with projection  $\pi: F^{k+} \rightarrow M$  given by  $\pi(j_0^k \phi) = \phi(0)$ . It is a principal bundle with structure group

$$G(k, n) := \{j_0^k \phi; \phi \in \text{Diff}_{\text{loc}}^+(\mathbb{R}^n, \mathbb{R}^n), \phi(0) = 0\}.$$

In the trivial case  $k = 0$ , we have  $F^{0+} = M$ , and  $G(0, n) = \{\mathbf{1}\}$ . The first interesting example is  $k = 1$ , in which case the principal fibre bundle  $F^{1+} \rightarrow M$  is the oriented frame bundle  $F^+$ , with structure group  $G(1, n) = \text{GL}^+(n, \mathbb{R})$ . For  $k \geq 2$ , the natural projections  $F^{k+} \rightarrow F^+$  and  $G(k, n) \rightarrow \text{GL}^+(n, \mathbb{R})$  have contractible fibres, so that  $\pi_1(F^{k+}) \simeq \pi_1(F^+)$ , and

$$\pi_1(G(k, n)) \simeq \pi_1(\text{GL}^+(n, \mathbb{R})) \simeq \mathbb{Z}_2. \tag{16}$$

The universal cover  $\tilde{F}^{k+}$  of the oriented  $k$ -frame bundle is, therefore, essentially determined by the universal cover  $\tilde{F}^{+}$  of the ordinary frame bundle. To determine the structure group of the principal fibre bundle  $\tilde{F}^{k+} \rightarrow M$ , note that the orbit map  $\iota: G(k, n) \rightarrow F^{k+}$ , defined by  $\iota(g) = f_x^k g$ , gives rise to a group homomorphism

$$\iota_*: \mathbb{Z}_2 \rightarrow \pi_1(F^{k+}). \tag{17}$$

The structure group of the principal fibre bundle  $\tilde{F}^{k+} \rightarrow M$  is, therefore,

$$G(k, M) = (\tilde{G}(k, n) \times \pi_1(F^{k+})) / \mathbb{Z}_2, \tag{18}$$

where  $\mathbb{Z}_2$  is identified with the central subgroup  $\{(z, \iota_*(z^{-1})); z \in \mathbb{Z}_2\}$ .

**3.2.2. Classification Results for Principal Bundles.** Every principal fibre bundle  $P \rightarrow M$  gives rise to an exact sequence of Lie algebras

$$0 \rightarrow \mathbf{aut}^V(P) \rightarrow \mathbf{aut}(P) \rightarrow \mathbf{Vec}(M) \rightarrow 0, \tag{19}$$

where  $\mathbf{aut}(P)$  is the Lie algebra of *equivariant* vector fields on  $P$ , and  $\mathbf{aut}^V(P)$  is the Lie algebra of *vertical equivariant* vector fields. The latter is isomorphic to  $\Gamma(\mathbf{Ad}(P))$ , the Lie algebra of infinitesimal gauge transformations. A principal fibre bundle  $P \rightarrow M$  is called *infinitesimally natural* if it comes with a Lie algebra homomorphism

$$\sigma: \mathbf{Vec}(M) \rightarrow \mathbf{aut}(P)$$

that splits the exact sequence (19).

The  $k$ th order frame bundle  $F^{k+}$  is an infinitesimally natural principal fibre bundle, with section  $D^k: \mathbf{Vec}(M) \rightarrow \mathbf{aut}(F^{k+})$  defined by

$$D^k(v) \Big|_{j_0^k \phi} = \frac{d}{dt} \Big|_0 j_0^k(\exp(tv) \circ \phi),$$

where  $t \mapsto \exp(tv)$  is the flow on  $M$  generated by the vector field  $v$ .

*Remark 7.* A  $\text{Spin}^G$ -structure  $(Q, u)$  is infinitesimally natural if  $\widehat{Q} \rightarrow M$  is infinitesimally natural as a principal fibre bundle, and if  $u_* \circ \sigma = D^1$ . This additional compatibility condition expresses that the covering map  $u: \widehat{Q} \rightarrow F^+$  is a *morphism* of infinitesimally natural principal fibre bundles.

Since  $\widetilde{F}^{k+} \rightarrow F^{k+}$  is a discrete cover, it has a canonical flat equivariant connection  $\nabla_{\text{can}}: \mathbf{Vec}(F^{k+})^{G(k,n)} \rightarrow \mathbf{Vec}(\widetilde{F}^{k+})^{G(k,M)}$ . It follows that also  $\widetilde{F}^{k+} \rightarrow M$  is an infinitesimally natural bundle, with splitting  $\widetilde{D}^k = \nabla_{\text{can}} \circ D^k$ .

**Theorem 4.** *For every infinitesimally natural principal fibre bundle  $P \rightarrow M$  with structure group  $H$ , there exists a homomorphism  $\rho: \widetilde{G}(k, M) \rightarrow H$  such that  $P$  is associated to  $\widetilde{F}^{k+}$  along  $\rho$ , i.e.*

$$P \simeq \widetilde{F}^{k+} \times_{\rho} H.$$

The splitting  $\sigma$  is induced by the canonical splitting for  $\widetilde{F}^{k+}$ .

*Proof.* This is Theorem 4.4 in [19]. A version of this result was proven earlier by Lecomte in [22]. □

**3.3. Proof of the Classification Theorem**

By Theorem 4, we may assume that an infinitesimally natural  $\text{Spin}^G$ -structure takes the form

$$\widehat{Q} = \widetilde{F}^{k+} \times_{\rho} H, \tag{20}$$

where the group  $H$  is defined as

$$H := \widetilde{\text{GL}}(n, \mathbb{R}) \times_{\mathbb{Z}_2} G. \tag{21}$$

Accordingly, we denote elements of  $\widehat{Q}$  by an equivalence class  $[f^k, h]$  of an element  $\tilde{f}^k \in \widetilde{F}^{k+}$  and  $h \in H$ . The Lie algebra homomorphism  $\sigma: \mathbf{Vec}(M) \rightarrow \mathbf{aut}(\widehat{Q})$  is induced by the canonical splitting  $\widetilde{D}^k: \mathbf{Vec}(M) \rightarrow \mathbf{aut}(\widetilde{F}^{k+})$  of the infinitesimally natural bundle  $\widetilde{F}^{k+} \rightarrow M$ .

**3.3.1. Standard Form of the Covering Map.** It remains to determine the Lie algebra homomorphism  $\rho: G(k, M) \rightarrow H$  and the covering map  $u: \widehat{Q} \rightarrow F^+$ . From the requirement that  $u_* \circ \sigma = D$ , we obtain the following characterisation of the map  $u$ .

**Lemma 5.** *There exists an element  $c \in \text{GL}^+(n, \mathbb{R})$  such that  $u(\hat{q}) = f\kappa(h)$  for all  $\hat{q} = [\tilde{f}^k, h]$  in  $\widehat{Q}$ . If the group element  $\tilde{y} \in G(k, M)$  projects to  $y \in \text{GL}^+(n, \mathbb{R})$ , then  $\kappa\rho(\tilde{y}) = c^{-1}yc$ .*

*Proof.* Define the covering map  $u_0: \widehat{Q} \rightarrow F^+$  by  $u_0([\tilde{f}^k, h]) = f\kappa(h)$ , where  $f \in F^+$  is the image of  $\tilde{f}^k$  under the canonical projection  $\tilde{F}^{k+} \rightarrow F^+$ . Since both  $u_0$  and  $u$  satisfy the equivariance equation  $u(\hat{q}h) = u(\hat{q})\kappa(h)$  for  $h \in H$ , the two maps differ by a gauge transformation of  $F^+$ . We have  $u(\hat{q}) = u_0(\hat{q})g(\hat{q})$  for a smooth map  $g: \widehat{Q} \rightarrow \text{GL}^+(n, \mathbb{R})$  that satisfies  $g(\hat{q}h) = \kappa(h)^{-1}g(\hat{q})\kappa(h)$ .

Since  $u_*\sigma(v) = u_{0*}\sigma(v) = D(v)$  for every  $v \in \text{Vec}(M)$ , we find that the logarithmic derivative  $g^{-1}L_{\sigma(v)}g$  of  $g$  along any lift  $\sigma(v)$  vanishes, and  $g$  is constant along  $\sigma(v)$ . The lift  $\sigma(v)$  of  $v \in \text{Vec}(M)$  is induced by the canonical lift  $\tilde{D}^k(v)$  of  $v$  from  $M$  to  $\tilde{F}^{k+}$ . If  $v$  ranges over  $\text{Vec}(M)$  and  $\tilde{f}^k$  over  $\tilde{F}^{k+}$ , then  $\tilde{D}^k(v)|_{\tilde{f}^k}$  ranges over the full tangent bundle  $T\tilde{F}^{k+}$ . It follows that  $g$  is constant on the image of  $\tilde{F}^{k+}$  in  $\widehat{Q}$ , that is,  $g([\tilde{f}^k, \mathbf{1}]) = c$  for all  $\tilde{f}^k \in \tilde{F}^{k+}$ .

By  $H$ -equivariance, one sees that  $g([\tilde{f}^k, h]) = \kappa(h^{-1})c\kappa(h)$  for all  $[\tilde{f}^k, h]$  in  $\widehat{Q}$ . Since  $u(\hat{q}) = u_0(\hat{q})g(\hat{q})$ , it thus follows that  $u([\tilde{f}^k, h]) = f\kappa(h)$ . If the group element  $\tilde{y} \in G(k, M)$  projects to  $y \in \text{GL}^+(n, \mathbb{R})$ , then  $u([\tilde{f}^k\tilde{y}, \mathbf{1}]) = fyc$  equals  $u([\tilde{f}^k, \rho(\tilde{y})]) = f\kappa\rho(\tilde{y})$ . From this, we deduce that  $\kappa\rho(\tilde{y}) = c^{-1}yc$ .  $\square$

Using this, we can bring the infinitesimally natural  $\text{Spin}^G$ -structures in the following standard form.

**Lemma 6.** (Standard form) *The homomorphism  $\rho: G(k, M) \rightarrow H$  of Eq. (20) can be chosen in such a way that  $\kappa \circ \rho: G(k, M) \rightarrow \text{GL}^+(n, \mathbb{R})$  is the canonical projection, and the covering map  $u: \widehat{Q} \rightarrow F^+$  satisfies  $u([\tilde{f}^k, h]) = f\kappa(h)$ , where  $\tilde{f}^k \in \tilde{F}^{k+}$  projects to  $f \in F^+$ .*

*Proof.* Suppose that a  $\text{Spin}^G$ -structure is isomorphic to  $\widehat{Q}_\rho := \tilde{F}^{k+} \times_\rho H$  with  $u$  and  $\rho$  as in Lemma 5. Choose  $\tilde{c} \in \widetilde{\text{GL}}(n, \mathbb{R}) \subseteq H$  such that  $\kappa(\tilde{c}) = c$ , and define  $\rho' = \tilde{c}\rho\tilde{c}^{-1}$ . Then  $\kappa \circ \rho'(\tilde{y}) = y$  for all  $\tilde{y} \in G(k, M)$  that project to  $y \in \widetilde{\text{GL}}^+(n, \mathbb{R})$ . Define the  $\text{Spin}^G$ -structure  $\widehat{Q}_{\rho'} := \tilde{F}^{k+} \times_{\rho'} H$  with the standard covering map  $u_0: \widehat{Q} \rightarrow F^+$  given by  $u_0([\tilde{f}^k, h]) = f\kappa(h)$ . Then the isomorphism  $\widehat{Q}_\rho \rightarrow \widehat{Q}_{\rho'}$  defined by  $[\tilde{f}^k, h]_\rho \mapsto [\tilde{f}^k, \tilde{c}h]_{\rho'}$  intertwines  $u$  with  $u_0$ .  $\square$

**3.3.2. Standard Form of the Homomorphism.** From Lemma 6, it follows that not only the principal bundle  $\widehat{Q} \rightarrow M$ , but also the covering map  $u: \widehat{Q} \rightarrow F^+$  is entirely determined by the homomorphism  $\rho: G(k, M) \rightarrow H$ . We proceed by deriving a standard form for  $\rho$ .

Recall from (18) that  $G(k, M) \simeq G(k, n) \times_{\mathbb{Z}_2} \pi_1(F^+)$ . Further, we have  $G(k, n) \simeq \widetilde{\text{GL}}^+(n, \mathbb{R}) \ltimes \text{GL}^{>1}$ , where  $\text{GL}^{>1}$  denotes the subgroup of  $k$ -jets



that are the identity to first order. Decomposing  $\widetilde{\text{GL}}(n, \mathbb{R}) \simeq \widetilde{\text{SL}}(n, \mathbb{R}) \times \mathbb{R}^+$ , we may thus consider  $\rho$  as a map

$$\widetilde{\text{SL}}(n, \mathbb{R}) \times \mathbb{R}^+ \times \text{GL}^{>1} \times_{\mathbb{Z}_2} \pi_1(F^+) \rightarrow \widetilde{\text{SL}}(n, \mathbb{R}) \times \mathbb{R}^+ \times_{\mathbb{Z}_2} G.$$

If the infinitesimal  $\text{Spin}^G$ -structure is in the standard form of Lemma 6, then  $\rho$  takes the following form.

**Lemma 7.** *If  $\mathfrak{sl}(\mathbb{R}^n) \not\subseteq \mathfrak{g}$ , then  $\widehat{Q}$  is associated to a bundle  $\widetilde{F}^{k+}$  with order  $k = 1$ . Furthermore,  $\rho$  is completely determined by a homomorphism  $\tau : \pi_1(F^+) \rightarrow G$  that identifies  $\mathbb{Z}_2 \subseteq \pi_1(F^+)$  with  $Z_2 \subseteq G$ , a homomorphism  $\gamma : \pi_1(F^+) \rightarrow \mathbb{R}^+$ , and a scaling element  $\Lambda \in \mathfrak{g}$  such that  $\exp(\mathbb{R}\Lambda) \subseteq G$  commutes with  $\text{Im}(\tau)$ . We have*

$$\rho(\tilde{x}, e^t, g, [p]) = (\tilde{x}, e^t \gamma([p]), e^{t\Lambda} \tau([p])).$$

*Proof.* Consider the derived Lie algebra homomorphism

$$\dot{\rho} : \mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R} \times \mathfrak{gl}^{>1} \rightarrow \mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R} \times \mathfrak{g},$$

and let  $\dot{\rho}_{ij}$  be its  $(i, j)$  component for  $i, j \in \{1, 2, 3\}$ . From Lemma 6, we find that  $\dot{\rho}_{12}$  and  $\dot{\rho}_{21}$  are zero, whereas  $\dot{\rho}_{11} = \text{Id}_{\mathfrak{sl}(n, \mathbb{R})}$  and  $\dot{\rho}_{22} = \text{Id}_{\mathbb{R}}$ . Since  $\mathfrak{sl}(n, \mathbb{R})$  is a simple Lie algebra which is not contained in  $\mathfrak{g}$ , we have  $\dot{\rho}_{13} = 0$ .

We now show that  $\dot{\rho}(\mathfrak{gl}^{>1}) = 0$ , so that  $\dot{\rho}_{31} = \dot{\rho}_{32} = \dot{\rho}_{33} = 0$ . First of all, as  $[\mathfrak{sl}(n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{R}) + \mathfrak{gl}^{>1}]$  equals  $\mathfrak{sl}(n, \mathbb{R}) + \mathfrak{gl}^{>1}$  if  $n$  is at least 2 (cf. [19, Lemma 8]), we have

$$\dot{\rho}(\mathfrak{gl}^{>1}) \subset [\dot{\rho}(\mathfrak{sl}(n, \mathbb{R})), \dot{\rho}(\mathfrak{sl}(n, \mathbb{R}) + \mathfrak{gl}^{>1})] \subset \mathfrak{sl}(n, \mathbb{R}) \oplus 0 \oplus 0.$$

But on the other hand, we have  $[\mathbb{R}, \mathfrak{gl}^{>1}] = \mathfrak{gl}^{>1}$ , since  $\mathbb{R}$  represents the multiples of the Euler vector field. This yields

$$\dot{\rho}(\mathfrak{gl}^{>1}) = [\dot{\rho}(\mathbb{R}), \dot{\rho}(\mathfrak{gl}^{>1})] \subset 0 \oplus 0 \oplus \mathfrak{g}.$$

As the intersection is zero, we have  $\dot{\rho}(\mathfrak{gl}^{>1}) = \{0\}$ .

It follows that the only nonzero components of  $\dot{\rho}$  are  $\dot{\rho}_{11} = \text{Id}_{\mathfrak{sl}(n)}$ ,  $\dot{\rho}_{22} = \text{Id}_{\mathbb{R}}$  and the map  $\dot{\rho}_{23} : \mathbb{R} \rightarrow \mathfrak{g}$  defined by the scaling element  $\Lambda := \dot{\rho}_{23}(1)$ . Since the groups  $\widetilde{\text{SL}}(n, \mathbb{R})$ ,  $\mathbb{R}^+$  and  $\text{GL}^{>1}$  are simply connected, we have

$$\rho(\tilde{x}, e^t, g, 1) = (\tilde{x}, e^t, e^{t\Lambda}).$$

As the image  $\rho(\pi_1(F^+))$  commutes with  $(\tilde{x}, e^t, e^{t\Lambda})$ , it is a subgroup of  $\mathbb{R}^+ \times G$  commuting with  $\exp(\mathbb{R}\Lambda) \subseteq G$ . The restriction of  $\rho$  to  $\pi_1(F^+)$  is thus determined by two homomorphisms  $\gamma : \pi_1(F^+) \rightarrow \mathbb{R}^+$  and  $\tau : \pi_1(F^+) \rightarrow G$ , where the image of the latter commutes with the scaling group  $\exp(\mathbb{R}\Lambda)$ .  $\square$

*Remark 8.* Let  $\tilde{z}$  be a generator of the centre of  $\widetilde{\text{SL}}(n, \mathbb{R})$ . Since  $\widetilde{\text{SL}}(n, \mathbb{R})$  is simply connected, we have  $\rho(\tilde{z}, 1, 1, 1) = (\tilde{z}, 1, 1)$ . Using the equivalence relations on both sides, we find  $\rho(1, 1, 1, \iota_*(\tilde{z})) = (1, 1, z)$ , where  $z \in G$  is a generator of  $Z_2 \subseteq G$ . This shows that  $\tau \circ \iota_*$  is an isomorphism between  $Z_2$  and  $Z_2 \subseteq G$ , as required by Theorem 3. Moreover, we find that  $\gamma \circ \iota_*$  is trivial, so that  $\gamma$  factors through a homomorphism  $\gamma : \pi_1(M) \rightarrow \mathbb{R}^+$ . Since  $\iota_*$  is injective, the universal cover of  $M$  is spin by Proposition 2.

**3.3.3. Conclusion of the Proof of Theorem 3.** The proof of the classification theorem is completed by combining the standard form of  $\widehat{Q}$  (Lemma 6) with that of the homomorphism  $\rho$  (Lemma 7).

*Proof of Theorem 3.* Since  $\rho$  is trivial on the group  $GL^{>1}$  of  $k$ -frames that agree with the identity to order 1, one deduces from Lemma 6 that

$$\widehat{Q} \simeq \widetilde{F}^+ \times_{\rho} (\widetilde{GL}^+(n, \mathbb{R}) \times_{\mathbb{Z}_2} G).$$

By Lemma 7 and Remark 8, the homomorphism  $\rho: G(1, M) \rightarrow H$  depends only on the homomorphism  $\tau: \pi_1(F^+) \rightarrow G$ , the homomorphism  $\gamma: \pi_1(M) \rightarrow \mathbb{R}^+$ , and the element  $\Lambda$  of  $\mathfrak{g}$ . Recall that  $Q \subseteq \widehat{Q}$  is the preimage under  $u: \widehat{Q} \rightarrow F^+$  of the bundle  $OF_g^+$  of oriented, orthonormal frames for the metric  $g$ . Since  $u([\tilde{f}, (\tilde{x}, y)]) = fx$ , one can fix the representatives  $\tilde{f} \in \widetilde{F}^+$ ,  $\tilde{x} \in \widetilde{GL}^+(n, \mathbb{R})$ , and  $y \in G$  of the class  $[\tilde{f}, (\tilde{x}, y)] \in u^{-1}(OF_g^+)$  so that  $\tilde{x} = \mathbf{1}$  and  $f \in OF_g^+$ . We thus find  $Q = \{[\tilde{f}, (\mathbf{1}, y)]; \tilde{f} \in \widetilde{OF}_g^+, y \in G\}$ , and hence  $Q \simeq \widetilde{OF}_g^+ \times_{\tau} G$ .  $\square$

**3.3.4. Classification of the Splittings.** The classification theorem ensures that every infinitesimally natural  $\text{Spin}^G$ -structure is isomorphic to a  $\text{Spin}^G$ -structure of the form  $Q_{\tau} := \widetilde{OF}_g^+ \times_{\tau} G$ . Note that  $Q_{\tau}$  is itself an infinitesimally natural  $\text{Spin}^G$ -structure. Indeed, the principal  $H$ -bundle  $\widehat{Q}_{\tau} = \widetilde{F}^+ \times_{\tau} G$  comes with a canonical splitting  $\sigma: \text{Vec}(M) \rightarrow \text{aut}(\widehat{Q}_{\tau})$ , induced by the splitting  $\widetilde{D}$  for  $\widetilde{F}^+$ .

This splitting, however, is not necessarily identical to the one induced by  $\widehat{Q}$ . To obtain a model for  $\widehat{Q}$  that yields the correct natural splitting  $\sigma$  as well as the correct covering map  $u$ , one proceeds as follows. The metric  $g$  on  $M$  gives rise to a volume form  $\lambda$ . Denote by  $F_{\lambda} \subseteq F^+$  the principal  $SL(n, \mathbb{R})$ -bundle of frames with volume 1. Identifying  $F_{\lambda}$  with the quotient of  $F^+$  by  $\mathbb{R}^+$ , we obtain a principal  $\mathbb{R}^+ \times \pi_1(F)$ -bundle  $\widetilde{F}^+ \rightarrow F_{\lambda}$ .

Given a homomorphism  $\tau: \pi_1(F^+) \rightarrow G$  that identifies  $\mathbb{Z}_2 \subseteq \pi_1(F^+)$  with  $Z_2 \subseteq G$ , a homomorphism  $\gamma: \pi_1(F^+) \rightarrow \mathbb{R}^+$ , and an element  $\Lambda$  of  $\mathfrak{g}$  such that  $\exp(\mathbb{R}\Lambda)$  commutes with  $\text{Im}(\tau)$ , one constructs the homomorphism  $\rho: \mathbb{R}^+ \times \pi_1(F) \rightarrow \mathbb{R}^+ \times G$  by

$$\rho(e^t, [p]) = (e^t \gamma(\pi_*[p]), e^{t\Lambda} \tau([p])).$$

Since  $\gamma$  maps into an abelian group, it factors through the quotient  $H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$  of  $\pi_1(M)$  by its commutator subgroup.

The desired bundle  $\widehat{Q}_{\rho}$  is then obtained by associating  $\mathbb{R}^+ \times G$  to  $\widetilde{F}^+ \rightarrow F_{\lambda}$  along the homomorphism  $\rho$ , that is,

$$\widehat{Q}_{\rho} := \widetilde{F}^+ \times_{\rho} (\mathbb{R}^+ \times G). \tag{22}$$

From the proof of Theorem 3, we then obtain the following corollary.

**Corollary 8.** *Under the assumptions of Theorem 3, the bundle  $\widehat{Q}$  is isomorphic to  $\widehat{Q}_{\rho}$ , with covering map  $u: \widehat{Q}_{\rho} \rightarrow F^+$  given by  $u([\tilde{f}, (e^t, g)]) = fe^t$ , and splitting  $\sigma: \text{Vec}(M) \rightarrow \text{aut}(\widehat{Q}_{\rho})$  induced by the splitting  $\widetilde{D}$  for  $\widetilde{F}^+$ .*

In short, an infinitesimally natural  $\text{Spin}^G$ -structure  $(Q, u)$  is determined by a homomorphism  $\tau: \pi_1(F) \rightarrow G$  that identifies  $\mathbb{Z}_2 \subseteq \pi_1(F)$  with  $Z_2 \subseteq G$ . For a given  $\text{Spin}^G$ -structure  $(Q, u)$ , the splittings are determined by an element  $\Lambda \in \mathfrak{g}^{\text{Im}(\tau)}$  and a class  $\log(\gamma) \in H^1(M, \mathbb{R})$ .

### 4. Applications

It was already recognised by Hawking and Pope [16] that the existence of generalised spin structures may place restrictions on the space–time manifold  $M$ . When generalised spin structures were classified by Avis and Isham [2], it was found that if the Lie group  $G$  contains  $\text{SU}(2)$ , then ‘universal spin structures’ in the sense of [4] exist, irrespective of the topology of  $M$ . In particular, there are no topological obstructions to the existence of a  $\text{Spin}^G$ -structure as soon as  $\text{SU}(2) \subseteq G$ .

This is no longer the case for *infinitesimally natural* generalised spin structures. In this setting, universal spin structures exist only for certain non-compact groups. For compact  $G$ , the requirement that there exist a homomorphism  $\pi_1(F^+) \rightarrow G$  that maps  $\mathbb{Z}_2 \subseteq \pi_1(F^+)$  onto  $Z_2 \subseteq G$  provides an obstruction on the space–time manifold  $M$  in terms of the group  $G$  of internal symmetries.

In this section, we work out these obstructions for a number of specific gauge theories. For concreteness, we assume that  $M$  is an oriented, time-oriented, Lorentzian manifold of dimension 4. The time-orientability allows us to replace  $\widetilde{\text{SO}}(\eta)$  by  $\text{SL}(2, \mathbb{C})$ .

#### 4.1. Weyl and Dirac Spinors

Consider a single, massless, charged Weyl spinor coupled to a  $U(1)$  gauge field. In this setting, the gauge group  $G$  is  $U(1)$ , and  $V = \mathbb{C}^2 \otimes \mathbb{C}_q$  is the two-dimensional defining representation of  $\text{SL}(2, \mathbb{C})$  tensored with the one-dimensional defining representation of  $U(1)$ . This representation descends to  $\text{Spin}^c = \text{SL}(2, \mathbb{C}) \times_{\mathbb{Z}_2} U(1)$ . Given a  $\text{Spin}^c$ -structure  $Q$ , the configuration space consists of sections of the bundle  $J^1(\widehat{Q}) \times_{\text{Spin}^c} V \rightarrow M$ .

If  $Q$  is infinitesimally natural, then Theorem 3 yields a homomorphism  $\tau: \pi_1(F^+) \rightarrow U(1)$  that sends the image of  $\pi_1(\text{GL}^+(n, \mathbb{R}))$  in  $\pi_1(F^+)$  to  $\{\pm \mathbf{1}\}$ . If  $\pi_1(M)$  is finitely generated, then  $\text{Im}(\tau) \subseteq U(1)$  is a finitely generated subgroup containing  $\{\pm \mathbf{1}\}$ , hence  $\text{Im}(\tau) \simeq \mathbb{Z}^n \times (\mathbb{Z}/2m\mathbb{Z})$  for certain  $n, m \in \mathbb{N}$ . In particular, there exists a homomorphism  $\pi_1(F^+) \rightarrow \mathbb{Z}/2m\mathbb{Z} \subseteq U(1)$  that maps the image of  $\pi_1(\text{GL}^+(n, \mathbb{R}))$  in  $\pi_1(F^+)$  to  $\{\pm \mathbf{1}\}$ . Since every such homomorphism yields an infinitesimally natural  $\text{Spin}^c$ -structure by the procedure outlined in Sect. 3.1, we arrive at the following conclusion.

**Corollary 9.** *Suppose that  $\pi_1(M)$  is finitely generated. Then  $M$  admits infinitesimally natural  $\text{Spin}^c$ -structures if and only if there exists a homomorphism  $\pi_1(F^+) \rightarrow \mathbb{Z}/2m\mathbb{Z} \subseteq U(1)$  that identifies the image of  $\pi_1(\text{GL}^+(n, \mathbb{R}))$  in  $\pi_1(F^+)$  with  $\{\pm \mathbf{1}\}$ .*

The topological requirements on  $M$  for admitting infinitesimally natural  $\text{Spin}^c$ -structures are more restrictive than those for admitting ordinary  $\text{Spin}^c$ -structures. However, they are less restrictive than those for admitting spin structures. Indeed, if  $m$  is odd, then the sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2m\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 1$$

of groups is split. Every homomorphism  $\pi_1(F^+) \rightarrow \mathbb{Z}/2m\mathbb{Z}$  then induced a homomorphism  $\pi_1(F^+) \rightarrow \mathbb{Z}/2\mathbb{Z}$ , and hence a spin structure on  $M$ . If  $m$  is even, then this sequence does not split. In that case,  $M$  may admit infinitesimally natural  $\text{Spin}^c$ -structures without admitting ordinary spin structures.

**4.1.1. Dirac Spinors.** For Dirac spinors,  $V$  is the 4-dimensional representation  $\mathbb{C}^4 \otimes \mathbb{C}_q$ , where the Clifford representation  $\mathbb{C}^4$  splits into two identical irreducible representations  $\mathbb{C}^2 \oplus \mathbb{C}^2$  under  $\text{SL}(2, \mathbb{C})$ , the left-handed and right-handed spinors.

Note that the unitary commutant of  $\text{SL}(2, \mathbb{C})$  in  $V$  is  $U(2)$  rather than  $U(1)$ . For a discrete subgroup  $H \subseteq U(2)$ , we can, therefore, form the group  $U(1)_H$  generated by  $H$  and the gauge group  $U(1)$  and consider  $\text{Spin}^G$ -structures  $Q$  with structure group  $G = U(1)_H$ . The generic fibre of the bundle  $J^1(\hat{Q}) \times_{\text{Spin}^G} V$  is the same for  $G = U(1)$  as it is for  $G = U(1)_H$ , so adding  $H$  will not change the space of local sections.

If  $H$  is a discrete group of global symmetries of the Lagrangian, then the action is well defined for sections of this bundle. Indeed, the action is invariant under constant  $H$ -valued transformations because  $H$  is a global symmetry group, and the part of the transition functions involving  $H$  will be constant since  $H$  is discrete.

For a massive Dirac spinor, where the Lagrangian contains a term of the form  $m\bar{\psi}\psi$ , the subgroup of  $U(2)$  which preserves the Lagrangian is precisely the diagonal  $U(1)$ . This means that the infinitesimally natural  $\text{Spin}^G$ -structures are precisely the infinitesimally natural  $\text{Spin}^c$ -structures classified above, and there is no possibility to add a discrete subgroup  $H$ .

For massless Dirac spinors, where the term  $m\bar{\psi}\psi$  is absent, the left and right Weyl spinors decouple, so that the relevant symmetry group is  $U_L(1) \times U_R(1)$ . Although the requirement on a manifold to carry a  $\text{Spin}^G$ -structure does not change, this does give us more  $\text{Spin}^G$ -structures for the same manifold.

More generally, we may enlarge the gauge group  $G$  by any group  $H$  of discrete symmetries of the Lagrangian in order to obtain infinitesimally natural  $\text{Spin}^G$ -structures.

**4.2. The Standard Model**

In the standard model, the gauge group  $G$  is  $(\text{SU}(3) \times \text{SU}(2)_L \times \text{U}(1)_Y)/N$ , with  $N$  the cyclic subgroup of order 6 generated by  $(e^{2\pi i/3}\mathbf{1}, -\mathbf{1}, e^{2\pi i/6})$ . It is isomorphic to  $S(U(3) \times U(2))$ , a subgroup of  $\text{SU}(5)$ , and it has a unique central subgroup of order 2 generated by  $\text{diag}(1, 1, 1, -1, -1)$ .

The fermion representation  $V$  for a single generation can be conveniently described (see e.g. [5]) by  $\mathbb{C}^2 \otimes \wedge^{\bullet} \mathbb{C}^5$ , the tensor product of the defining representation of  $SL(2, \mathbb{C})$  and the exterior algebra of the defining representation of  $SU(5)$ . Under  $SL(2, \mathbb{C}) \times S(U(3) \times U(2))$ , this decomposes into 12 irreps corresponding to left- and right-handed electrons, neutrinos, up and down quarks and their antiparticles.

Unfortunately,  $\text{diag}(1, 1, 1, -1, -1) \in G$  acts by  $+1$  on right-handed fermions, whereas  $-\mathbf{1} \in SL(2, \mathbb{C})$  acts by  $-1$ . This means that  $V$  does *not* define a representation of  $SL(2, \mathbb{C}) \times_{\mathbb{Z}_2} S(U(3) \times U(2))$  if one were to identify the central order 2 elements on both sides.

As the gauge group alone is of no use when trying to find a  $\text{Spin}^G$ -structure, one has to involve the group of global symmetries of the standard model Lagrangian. It contains the gauge group  $G$ , but also (at least on the classical level) the global  $U(1)_B \times U(1)_L$ -symmetries that rotate quarks and leptons independently (these are connected to baryon and lepton number).

We conclude that the only infinitesimally natural  $\text{Spin}^G$ -structures relevant to the standard model are the ones associated to homomorphisms

$$\pi_1(F^+) \rightarrow \hat{G} \tag{23}$$

that preserve  $Z_2$ , the subgroup of  $U(1)_B \times U(1)_L$  generated by  $(-1, -1)$ . In this expression,  $\hat{G}$  is the group of global symmetries of the standard model Lagrangian, which at least contains  $S(U(3) \times U(2)) \times U(1)_B \times U(1)_L$ .

*Remark 9.* For three generations of fermions, there is some additional freedom. The relevant representation  $V \oplus V \oplus V$  then admits for an extra  $U(3)$ -symmetry commuting with both space–time and gauge transformations.

**4.2.1. Spherical Space Forms.** For  $G = S(U(3) \times U(2)) \times U(1)_B \times U(1)_L$ , any manifold which possesses an infinitesimally natural  $\text{Spin}^G$ -structure automatically permits an infinitesimally natural  $\text{Spin}^c$ -structure. On the other hand, there do exist  $\text{Spin}^G$ -structures for the standard model which are not  $\text{Spin}^c$ . We construct an example.

Consider de Sitter space  $H = \{\vec{x} \in \mathbb{R}^5 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ , which has a pseudo-Riemannian metric  $g$  with constant curvature induced by the Minkowski metric in the ambient  $\mathbb{R}^5$ . Its group of orientation preserving isometries is  $SO(1, 4)$ , and  $H \simeq SO(1, 4)/SO(1, 3)$ . Denote by  $OF_g^{+\uparrow}(H)$  the bundle of orthogonal frames with positive orientation and time orientation. By viewing  $OF_g^{+\uparrow}(H)$  as a submanifold of  $\mathbb{R}^5 \times SO(1, 4)^0$ , one can see that  $SO(1, 4)^0$  acts freely and transitively by  $x : f \mapsto x_*f$ . Therefore,  $OF_g^{+\uparrow}(H)$  is diffeomorphic to  $SO(1, 4)^0$ .

Now let  $\Gamma \subseteq SO(4)$  be a discrete group which acts freely, isometrically and properly discontinuously on  $S^3$ . Manifolds of the type  $\Gamma \backslash S^3$  are called *spherical space forms* (see [40] for a complete classification). As  $\Gamma$  includes into  $SO(1, 4)^0$ , it acts on  $H$ , making  $M = \Gamma \backslash H$  into a pseudo-Riemannian manifold with constant curvature.

As  $H$  is simply connected, we see that  $\pi_1(M) = \Gamma$ . We calculate the homotopy group of the frame bundle. Because  $OF_g^{+\uparrow}(M)$  is just  $\Gamma \backslash OF_g^{+\uparrow}(H)$ ,

it is isomorphic to  $\Gamma \backslash \text{SO}(1,4)^0$ . Going to the universal cover, we see that  $OF_g^+(M) = \tilde{\Gamma} \backslash \tilde{\text{SO}}(1,4)^0$ . As  $\Gamma \subseteq \text{SO}(4)$ , we may consider  $\tilde{\Gamma}$  to be the preimage of  $\Gamma$  in  $\text{Spin}(4)$ . As the universal cover is simply connected, it is now clear that  $\pi_1(OF_g^{+1}(M)) = \tilde{\Gamma}$ . We get for free a homomorphism  $\tilde{\Gamma} \rightarrow \text{Spin}(4) \simeq \text{SU}(2)_l \times \text{SU}(2)_r$ , which maps the noncontractible loop in the fibre to  $(-\mathbf{1}, -\mathbf{1})$ .

Triggered by the WMAP-data on cosmic background radiation, it has been proposed that space may carry the topology of  $I^* \backslash S^3$ , where  $\Gamma = I^*$  is the binary icosahedral group [24, 34]. Although these views are far from universally accepted [20], it is nonetheless interesting in this connection to note that  $M = I^* \backslash H$ , which has spacelike hypersurfaces  $I^* \backslash S^3$ , allows for infinitesimally natural  $\text{Spin}^G$ -structures which do not stem from  $\text{Spin}^c$ -structures.

Under the identification  $\text{Spin}(4) \simeq \text{SU}(2)_l \times \text{SU}(2)_r$ , we see that  $\Gamma = I^* \times 1$  lives only in  $\text{SU}(2)_l$ , so that  $\tilde{\Gamma}$  is the direct product of  $I^* \times 1$  and the  $\mathbb{Z}_2$  generated by  $(-\mathbf{1}, -\mathbf{1})$ . One can, therefore, define a homomorphism (23) by identifying  $\text{SU}(2)_l$  with  $\text{SU}(2)_L \subseteq G$ , and mapping  $(-\mathbf{1}, -\mathbf{1})$  to  $(-\mathbf{1}, -\mathbf{1}) \in U(1)_B \times U(1)_L$ . This yields an infinitesimally natural  $\text{Spin}^G$ -structure which uses the noncommutativity of the gauge group in an essential fashion. This means that  $M = I^* \backslash H$  carries more infinitesimally natural  $\text{Spin}^G$ -structures than just the ‘ordinary’  $\text{Spin}^c$ -structures.

### 4.3. Extensions of the Standard Model

The fact that  $S(U(3) \times U(2))$  does not contribute to the obstruction of finding infinitesimally natural  $\text{Spin}^G$ -structures on  $M$  is due to the fact that it never acts by  $-1$  on  $V$ . This is not true for some GUT-type extensions of the standard model, such as the Pati–Salam  $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(4)$  model and anything which extends it, for example  $\text{Spin}(10)$ .

If  $N$  is the group of order 2 generated by  $(-\mathbf{1}, -\mathbf{1}, -\mathbf{1})$ , then infinitesimally natural  $\text{Spin}^G$ -structures in the Pati–Salam model correspond, neglecting global symmetries, to homomorphisms

$$\pi_1(F^+) \rightarrow \text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(4)/N$$

which take  $Z_2$  to  $\langle(-\mathbf{1}, -\mathbf{1}, \mathbf{1})\rangle$ . It is therefore possible that a space–time manifold  $M$  admits infinitesimally natural  $\text{Spin}^G$ -structures for the Pati–Salam model, but not for the standard model. Indeed,  $M$  has this property if the smallest quotient of  $\pi_1(F(M))$  containing  $l_*(Z_2)$  is a nonabelian subgroup of  $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(4)$  containing  $(-\mathbf{1}, -\mathbf{1}, \mathbf{1})$ .

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