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DISTRIBUTED ANALYTICAL GRAPH IDENTIFICATION

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ABSTRACT

An analytical algebraic approach for distributed network identification is presented in this paper. The information propagation in the network is modeled using a state-space representation. Using the observations recorded at a single node and a known excitation signal, we present algorithms to compute the eigenfrequencies and eigenmodes of the graph in a distributed manner. The eigenfrequencies of the graph may be computed using a generalized eigenvalue algorithm, while the eigenmodes can be computed using an eigenvalue decomposition. The developed theory is demonstrated using numerical experiments.

Index Terms— Distributed graph-spectral decomposition, graph signal processing, system identification, spectrum analysis, topology identification.

1. INTRODUCTION

The field of signal processing has recently evolved into *graph signal processing* to represent, interpret, and analyze signals with hidden geometric structures or signals that are defined on graphs [1,2]. Some of the basic signal processing concepts for graphs, such as Fourier transformation, filtering, sampling, are built based on the knowledge of the underlying graph. Therefore, to know or identify the underlying graph is crucial to process structured data.

In this paper, algorithms for *graph identification* are presented. To do so, we borrow some classic algebraic tools from spectrum analysis and system identification. The main idea is to excite *all* the modes of the graph using "known" input signals (e.g., white noise) and monitor the information propagation in the network. The goal is to compute, in a *distributed* manner, all the eigenfrequencies and eigenmodes of the underlying graph based on the observations gathered at a single node. Here, the term distributed means that, each node should be able to compute the eigenfrequencies and eigenmodes of the graph from its own signal values collected during the propagation of the input signal.

Typically, graph or topology identification is studied as a learning problem [3–7], in which the graph-shift operator (e.g., the graph Laplacian) that characterizes the information propagation in the network is computed under some assumptions on the data with respect to the discovered graph. One such commonly used assumption is that the signals have smooth variations on the underlying graph. The graph learning problem is posed as a constrained optimization problem to recover a sparse graph-shift operator with the constraint set being the set of valid graph-shift operators, such as the set of valid graph Laplacian matrices; see [3–6] for details. In another related line of work, under the assumption that the graph signal is vertex stationary [8, 9] on the graph and that the eigenmodes are given, the graph learning problem is studied in [7]. The problem is again posed as a constrained optimization problem to recover a sparse graph-shift operator as well as the eigenfrequencies by searching over the space of all valid graph Laplacians. In contrast to the above prior works, we do not assume that the signal is smooth or stationary with respect to the graph. Instead we assume that the known input excites the network and the observations at the nodes follows a first-order difference equation. In a closely related work, [10] and [11] present algorithms to compute the eigenmodes using a consensus-like approach. When at steady state, the first-order difference equation may be viewed as a structural equation model; see [12] for an overview of centralized topology identification algorithms based on structural equation models. In this paper, we present an *analytical* approach to compute all the eigenmodes and eigenfrequencies without any exchange of the observations gathered.

2. GRAPH THEORY

Consider an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, which consists of a finite set of vertices $\mathcal{V}$ with cardinality $|\mathcal{V}| = N$ and a set of edges $\mathcal{E}$. If there is an edge connecting vertices i and j, then $(i, j) \in \mathcal{E}$. A signal or function $x : \mathcal{V} \to \mathbb{R}$ defined on the vertices of the graph can be collected in a length-N vector $\boldsymbol{x}$, where the nth element of $\boldsymbol{x}$ represents the function value at the nth vertex in $\mathcal{V}$. Since $\boldsymbol{x}$ resides on the graph, we refer to the function $\boldsymbol{x}$ as a *graph signal*.

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Let us introduce an operator $\boldsymbol{A}$ of size $N \times N$. For a signal $\boldsymbol{x}$, the signal $\boldsymbol{Ax}$ denotes the unit-shifted version of $\boldsymbol{x}$. In other words, the operator $\boldsymbol{A}$ basically represents the information propagation or diffusion in the graph, and is commonly referred to as the *graph-shift* operator [1,2]. Some candidates for the graph-shift operator are local-structure-preserving operators such as the adjacency matrix, the Laplacian matrix, or their energy-preserving variants [13].

We assume that $\boldsymbol{A}$ is normal and thus admits the following eigenvalue decomposition

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T \\ &= [\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N]\, \mathrm{diag}[\lambda_1, \cdots, \lambda_N]\, [\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N]^T, \end{aligned} \tag{1}$$

where the eigenvectors $\{\boldsymbol{u}_n\}_{n=1}^N$ and the eigenvalues $\{\lambda_n\}_{n=1}^N$ of $\boldsymbol{A}$, respectively, are the *eigenmodes* and *eigenfrequencies* of the graph.

3. STATE-SPACE REPRESENTATION

The information propagation to a node from its neighbors is modeled using the following state-space representation

$$\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{b}u_{k-1} \tag{2}$$

$$y_k = \boldsymbol{c}^T\boldsymbol{x}_k \tag{3}$$

for shifts (or propagation steps) $k = 0, 1, \ldots, K-1$. Here, $\boldsymbol{A}$ is the graph-shift operator, $\boldsymbol{b} \in \mathbb{R}^N$ is the input vector (i.e., the excitation signal), $u_{k-1} = \delta[k]$ is the control input with $\delta[\cdot]$ being the Kronecker delta function (i.e., $\delta[k] = 1$ when $k = 0$ and $\delta[k] = 0$ when $k \neq 0$), $\boldsymbol{c} \in \mathbb{R}^N$ is the sampling vector, and in the above description, $\boldsymbol{x}_{-1}$ is $\boldsymbol{0}$.

For observations at *node* i, the sampling vector $\boldsymbol{c} = \boldsymbol{e}_i$ with $\boldsymbol{e}_i$ being the ith column of the identity matrix of size $N \times N$. In this case, the observations $\{y_0, y_1, \cdots, y_K\}$ may be interpreted as the observations gathered at a single node by sequentially applying the graph-shift operator leading to *aggregation sampling* [14].

Suppose $M \geq K$ snapshots of the *input* signal $\boldsymbol{b}$, denoted by, $\{\boldsymbol{b}[1], \boldsymbol{b}[2], \cdots, \boldsymbol{b}[M]\}$, are available. Then the observation at propagation step k and snapshot m is $y_k[m] = \boldsymbol{c}^T\boldsymbol{A}^k\boldsymbol{b}[m]$. Collecting M snapshots of the input signal in an $N \times M$ matrix $\boldsymbol{B} = [\boldsymbol{b}[1], \boldsymbol{b}[2], \ldots, \boldsymbol{b}[M]]$ and all the observations in a $K \times M$ matrix $\boldsymbol{Y}$ with (i,j)th entry $y_i[j]$, we can formally state the problem of interest. *Given $\boldsymbol{Y}$ and $\boldsymbol{B}$, how can we estimate "all" the eigenmodes and eigenfrequencies of the graph?*

Since we focus on algorithms that utilize only the observations at a single node the proposed scheme amounts to computing the eigenmodes and eigenfrequencies of the graph-shift operator in a distributed manner. To do so, we will require that the known seed signal $\boldsymbol{B}$ excites all the modes of the underlying graph, as described more formally in the next section.

4. COMPUTING THE EIGENFREQUENCIES

Let us stack the observed signal in (3) for K propagation steps in $\boldsymbol{y}$ as $\boldsymbol{y} = [y_1, y_2, \ldots, y_K]^T$, where we consider one of the available snapshots (hence, the snapshot index is dropped). Substituting (1), we have

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^T\boldsymbol{A}^{K-1} \end{bmatrix} \boldsymbol{b} = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T \\ \vdots \\ \boldsymbol{c}^T\boldsymbol{U}\boldsymbol{\Lambda}^{K-1}\boldsymbol{U}^T \end{bmatrix} \boldsymbol{b}. \tag{4}$$

Let us define the length-N vector $\underline{\boldsymbol{u}} = \boldsymbol{c}^T\boldsymbol{U}$, which for observations at *node* i will be the ith row of $\boldsymbol{U}$, and the $N \times N$ Vandermonde matrix $\boldsymbol{V}$ with the (i,j)th entry λ_j^{i-1}, i.e.,

$$\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_N^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{K-1} & \lambda_2^{K-1} & \cdots & \lambda_N^{K-1} \end{bmatrix}.$$

With these definitions, (4) can be equivalently expressed as

$$\boldsymbol{y} = \boldsymbol{V}\,\mathrm{diag}[\underline{\boldsymbol{u}}]\boldsymbol{U}^T\boldsymbol{b} = \boldsymbol{V}\boldsymbol{\theta}. \tag{5}$$

It is easy to see in (5) that, in order to observe all the modes, the graph Fourier transform [1,2] of the excitation signal, i.e., $\boldsymbol{b}_{\mathrm{f}} = \boldsymbol{U}^T\boldsymbol{b}$, should not be sparse and it should excite all the modes.

The Vandermonde structure of $\boldsymbol{V}$ can be exploited to determine *all* the modes using standard tools from spectrum analysis, such as the matrix pencil method or an annihilating filter; see [15, Ch. 9]. We next describe the matrix pencil method that we use in this work. Let us consider $\boldsymbol{Y}_0$ and $\boldsymbol{Y}_1$, two Toeplitz data matrices defined by

$$\begin{aligned} \boldsymbol{Y}_0 &= \begin{bmatrix} y_N & y_{N-1} & \cdots & y_1 \\ y_{N+1} & y_N & \cdots & y_2 \\ \vdots & \vdots & & \vdots \\ y_{K-2} & y_{K-3} & \cdots & y_{N-K-1} \end{bmatrix}; \\ \boldsymbol{Y}_1 &= \begin{bmatrix} y_{N-1} & y_{N-2} & \cdots & y_0 \\ y_N & y_{N-1} & \cdots & y_1 \\ \vdots & \vdots & & \vdots \\ y_{K-1} & y_{K-2} & \cdots & y_{N-K} \end{bmatrix}. \end{aligned} \tag{6}$$

The roots of the pencil of matrices $\boldsymbol{Y}_0 - \lambda\boldsymbol{Y}_1$ produce the roots of the Vandermonde matrix given by $\{\lambda_n\}_{n=1}^N$. This is in essence a generalized eigenvalue problem. That is, the eigenfrequencies of the graph are the generalized eigenvalues of the pair $(\boldsymbol{Y}_0, \boldsymbol{Y}_1)$ or equivalently, the eigenvalues of $\boldsymbol{Y}_1^{-1}\boldsymbol{Y}_0$.

Depending on the graph-shift operator and the size of the graph, the data matrices might be highly ill-conditioned, preventing an accurate computation of the generalized eigenvalues. Therefore, we require a numerically stable generalized eigenvalue algorithm, e.g., based on the *generalized Schur decomposition* (also referred to as the QZ decomposition) of the involved data matrices [16]. In the following, we use the QZ method for finding the roots of the Vandermonde matrix.

Given two square matrices $\boldsymbol{Y}_0$ and $\boldsymbol{Y}_1$, the generalized Schur decomposition factorizes both matrices as $\boldsymbol{Y}_0 = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Z}^H$ and $\boldsymbol{Y}_1 = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Z}^H$, where $\boldsymbol{Q}$ and $\boldsymbol{Z}$ are unitary, and $\boldsymbol{S}$ and $\boldsymbol{T}$ are upper triangular. Hence, the generalized eigenvalues of the pair $(\boldsymbol{Y}_0, \boldsymbol{Y}_1)$ are given by

$$\lambda(\boldsymbol{Y}_0, \boldsymbol{Y}_1) = \{[\boldsymbol{S}]_{nn}/[\boldsymbol{T}]_{nn} \,:\, [\boldsymbol{T}]_{nn} \neq 0\}.$$

When some of the eigenfrequencies are very close to each other certain diagonal entries of $\boldsymbol{T}$ will be extremely small. In such cases, we will have to restrict ourselves to a reduced rank approximation and compute only those eigenfrequencies, for which $[\boldsymbol{T}]_{nn} > \epsilon$, for some threshold ϵ (see the illustration in Section 6). This comprises the search-free (i.e., it does not involve a line search for λ) algorithm to determine the eigenfrequencies of the graph.

We conclude this section with the following remarks. Typically used graph-shift operators, such as the adjacency matrix and the Laplacian matrix, sometimes have real eigenvalues due to which the Toeplitz matrices in (6) become ill-conditioned very quickly with the number of shifts. In contrast, the energy-preserving graph-shift operators [13] lead to a reasonably stable system as the roots of the Vandermonde matrix are on the unit circle (something that is also typical in frequency or angle estimation problems in array signal processing).

5. COMPUTING THE EIGENMODES

In this section, we develop an algorithm to compute the eigenmodes $\boldsymbol{U}$ based on multiple realizations of the observations at a single node. It is easy to notice that the signal (5) for multiple snapshots is of the form

$$\boldsymbol{Y} = \boldsymbol{V}\,\mathrm{diag}[\underline{\boldsymbol{u}}]\boldsymbol{U}^T\boldsymbol{B}.$$

When the eigenfrequencies $\{\lambda_n\}_{n=1}^N$ are available, we can construct the Vandermonde matrix $\boldsymbol{V}$. Multiplying from the right with $\boldsymbol{B}^\dagger = \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-1}$ and from the left with $\boldsymbol{V}^\dagger = (\boldsymbol{V}^H\boldsymbol{V})^{-1}\boldsymbol{V}^H$, we obtain

$$\boldsymbol{H} = \boldsymbol{V}^\dagger\boldsymbol{Y}\boldsymbol{B}^\dagger = \mathrm{diag}[\underline{\boldsymbol{u}}]\boldsymbol{U}^T, \tag{7}$$

whose Gramian is given by

$$\boldsymbol{G} = \boldsymbol{H}^T\boldsymbol{H} = \boldsymbol{U}\mathrm{diag}^2[\underline{\boldsymbol{u}}]\boldsymbol{U}^T. \tag{8}$$

The matrix $\boldsymbol{G}$ on the left hand side is known from data, and the matrix product on the right hand side is recognized as an eigenvalue equation: $\boldsymbol{U}$ contains the eigenvectors of $\boldsymbol{G}$ and the entries of $\mathrm{diag}^2[\underline{\boldsymbol{u}}]$ on the diagonal are the eigenvalues. Hence, we can simply compute the eigenvalue decomposition of $\boldsymbol{G}$, and take the eigenvectors of $\boldsymbol{G}$ as the eigenmodes of the graph.

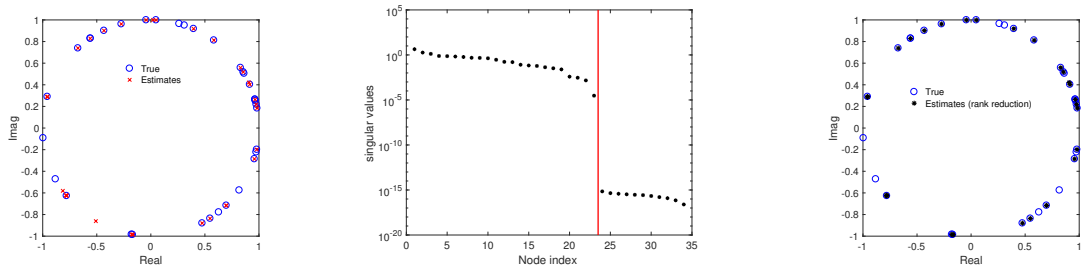
When the true eigenfrequencies are not readily available at the node, we can use the estimated eigenfrequencies from the previous section for constructing an estimate, $\hat{\boldsymbol{V}}$, of the Vandermonde matrix $\boldsymbol{V}$. This approach, in many practical instances, will produce a reduced rank approximation of $\boldsymbol{G}$. This is due to the fact that some of the eigenfrequencies might be very close to each other or might not be distinct, leading to a taller $\hat{\boldsymbol{V}}$. As a consequence, after the eigenvalue decomposition of (8), the representative modes of the graph can be recovered up to an orthonormal transformation.

6. NUMERICAL EXPERIMENTS

In this section, we present numerical simulations in order to illustrate the developed theory of distributed eigendecomposition of the graph-shift operator. To do so, we consider the Zachary's Karate club network [17]. This graph consists of 34 nodes representing members of the club and 78 undirected edges representing friendships among members. For the state-space description of the information propagation, we employ the energy-preserving graph-shift operator, $\boldsymbol{A}$, proposed in [13]. This graph-shift operator has the same eigenmodes, $\boldsymbol{U}$, as that of the graph adjacency matrix. Furthermore, its eigenvalues, $\{\lambda_n\}_{n=1}^N$, are arbitrary unit-modulus complex numbers. That is, the employed graph-shift operator is defined as $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $\lambda_n = e^{j\phi_n}$ for arbitrary phase $\phi_n \in [0, 2\pi]$ and $\phi_m \neq \phi_n$ for $m \neq n$.

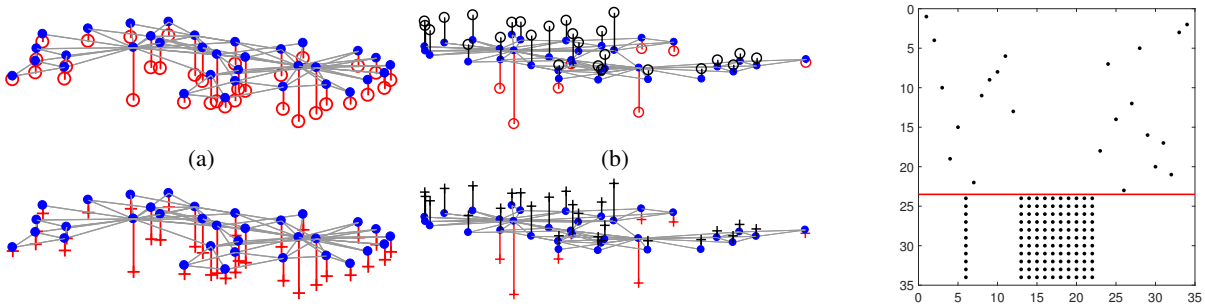
For the simulations, we assume that for a given node, the information flow for $K = 2N$ propagation steps is recorded. We generate the excitation signal $\boldsymbol{B} \in \mathbb{R}^{N\times 2N}$ with independent and identically distributed real Gaussian entries having zero mean and unit variance.

For estimating the eigenfrequencies, we use only a single realization of the process and construct the Toeplitz data matrices [cf. (6)] using the excitation signal $\boldsymbol{b}[1]$. The eigenfrequencies are estimated using the generalized Schur decomposition described in Section 4. These are shown in Fig. 1a along with the true eigenfrequencies. From this plot it is clear that despite the fact that most of the eigenfrequencies are properly estimated, some of them are reconstructed with large errors. This result can be explained by observing the singular values of the data matrix $\boldsymbol{Y}_1$ shown in Fig. 1b, where there exists a clear gap, depicted by a vertical red line, in the spectrum of $\boldsymbol{Y}_1$. This implies that some of the eigenfrequencies cannot be retrieved due to its weak presence in the available data. Therefore, to avoid numerical instabilities in the estimation process, a reduced-rank approximation might be considered. A reduced-order model may be obtained by truncating



(a) (b) (c)

Fig. 1: Eigenfrequencies of the graph. (a) True and estimated eigenfrequencies. (b) Spectrum of the Toeplitz data matrix $\boldsymbol{Y}_1$. (c) Estimated eigenfrequencies based on the reduced-rank model.



(a) (b) (c) (d) (e)

Fig. 2: Eigenmodes of the graph. (a) and (b) True eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_N$, respectively. (c) and (d) estimated eigenmodes $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_N$, respectively. (e) Orthogonal transformation between the true eigenvectors and the recovered eigenvectors illustrating that most of the eigenvectors are recovered up to a column permutation.

the entries $[\boldsymbol{T}]_{nn}$. In Fig. 1c, the eigenfrequencies are estimated for $[\boldsymbol{T}]_{nn} \geq \epsilon$ with $\epsilon = 10^{-6}$. The value of ϵ used for the truncation can easily be related to the gap in the singular values of the data matrix $\boldsymbol{Y}_1$.

After an estimate of the eigenvalues is obtained, we follow the method proposed in Section 5 to reconstruct the eigenmodes of the graph. A Vandermonde matrix, $\hat{\boldsymbol{V}}$, is constructed using the set of estimated frequencies $\{\hat{\lambda}_n\}_{n=1}^N$, i.e., we do not perform any rank reduction. The eigenmodes are retrieved by an eigenvalue decomposition of $\boldsymbol{G}$ [cf. (8)], after applying to $\boldsymbol{Y}$ the matrices $\hat{\boldsymbol{V}}^\dagger$ and $\boldsymbol{B}^\dagger$, by the left and right, respectively. In Fig. 2a and Fig. 2b, we show the first and last eigenmodes of the graph, i.e., $\boldsymbol{u}_1$ and $\boldsymbol{u}_N$. It is easy to observe that the true and reconstructed modes (Fig. 2c and Fig. 2d) have the same frequency variation over the graph, i.e., zero crossings are preserved.

To illustrate the quality of the reconstruction, we depict the orthogonal transformation between the true eigenmodes and the recovered eigenmodes in Fig. 2e. From this plot, it is possible to see that most of the eigenmodes are reconstructed up to a column permutation; see the sparse profile in the top most elements of the orthogonal transformation. However, some of the modes (modes below the red line in Fig. 2e) are retrieved as a mixture of a subset of the original modes. This is again due to the low singular values of the $\boldsymbol{Y}_1$ matrix, shown in Fig. 1b, which motivate a reduced-rank approximation of the graph-shift operator for describing the information propagation process.

7. CONCLUDING REMARKS

We presented algorithms to compute the eigenmodes and eigenfrequencies of a graph by observing at a single node the information propagation in the network. To do so, we assumed that the excitation signal is completely known and that it excites all the modes of the graph. Estimating the eigenfrequencies of the graph amounts to solving a generalized eigenvalue problem, and estimating the eigenmodes of the graph amounts to solving an eigenvalue problem. Although the excitation signal is assumed to be known in this work, it might be interesting to study, as future work, blind system identification approaches for graphs.

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