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The Symmetric Exclusion Process and the Gausian Free Field on compact Riemannian manifolds

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DOI
10.4233/uuid:0a0344dc-b98b-4539-8456-2c6de4843315

Publication date
2021

Document Version
Final published version

Citation (APA)
van Ginkel, G. J. (2021). *The Symmetric Exclusion Process and the Gausian Free Field on compact Riemannian manifolds.* [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:0a0344dc-b98b-4539-8456-2c6de4843315

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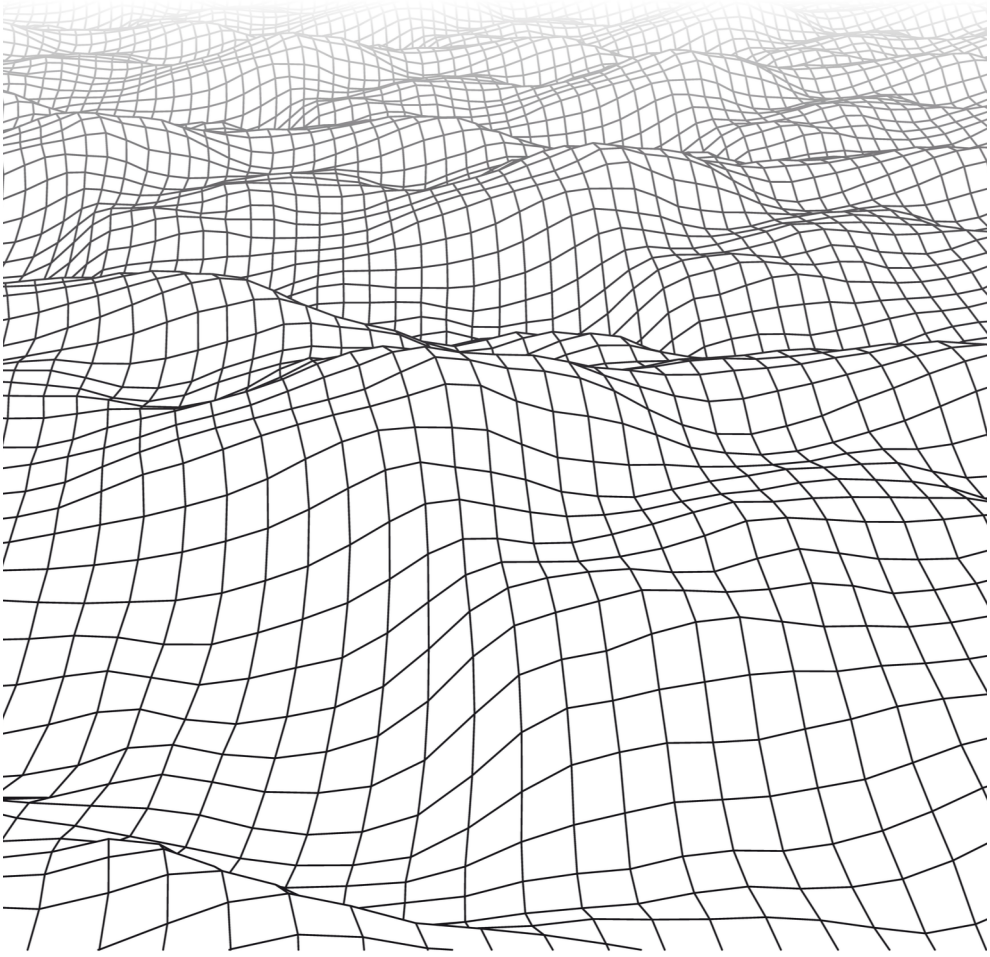
The Symmetric Exclusion Process and the Gaussian Free Field on compact Riemannian manifolds

Bart van Ginkel



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Bart van Ginkel
The Symmetric Exclusion Process and the Gaussian Free Field on compact Riemannian manifolds, Delft, 2021

Printed and cover design by: GVO Drukkers & Vormgevers
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The Symmetric Exclusion Process and the Gaussian Free Field on compact Riemannian manifolds

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus Prof.dr.ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op
donderdag 14 oktober 2021 om 12:30 uur

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Part I

Introduction

Chapter 1

Introduction

In this thesis we study models from statistical physics in curved space. The models that we treat are standard and their properties are well-known in flat spaces. However, the fact that we consider these models in curved space is new. The context of curved space leads to new challenges, both of conceptual and of technical nature.

The goal of this introduction is to present the main concepts that are dealt with in this thesis, to point to the relevant literature and to introduce the main research goals to which this thesis contributes[1].

We start by describing mathematical statistical physics, the subfield of mathematics (or, more precisely, probability theory) of which the research presented in this thesis is a part. Then we introduce the models that we will work with, along with some important properties and methods that are known and used in flat spaces. To be more precise, we introduce interacting particle systems, where we mainly discuss hydrodynamic limits and equilibrium fluctuations, and the Gaussian Free Field. Following that, we give an overview of how these notions have been studied outside of the standard spaces like $\mathbb{R}^d$ or the flat torus. Then we motivate why we study these models in curved space and we highlight which conceptual and technical challenges arise. At that point we can formulate the aim of this thesis and sketch the content of the following chapters.

In this chapter we stay on a conceptual level without going into mathematical details (a master student in mathematics, say, should be able to read it without much trouble). In Chapter 2 we give a brief mathematical introduction of the probabilistic and analytic tools that are used in this thesis and we describe some basic notions from differential geometry that are necessary to understand the upcoming chapters. All other mathematical definitions will be given in the chapters concerned.

[1] Note that we focus mainly on the topics that are presented in Part II. The content of Chapter 7 stands more on its own, so will be introduced separately.

The readers who are familiar with the field and mathematical background are advised to skip to Section 1.6 and 1.7 for the motivation and goal of this thesis and an overview of the upcoming chapters and then proceed to Chapter 3.

1.1 Mathematical statistical physics

1.1.1 Statistical physics

The key observation to make when one enters statistical physics is that many natural objects and phenomena can be described on different scales. For instance the motion of a liquid or the evolution of a gas can be described with partial differential equations. However, on a much smaller scale, a gas or fluid actually consists of an enormous amount of tiny particles that move around chaotically and interact with each other. Similarly, consider a ferromagnetic material. As a whole object it has magnetic properties, but these are determined by the magnetic moments of the many particles that it consists of. Another example is a porous material: whether or not water can percolate through the whole object depends on how the many tiny holes in the material are distributed.

In all of these examples, we distinguish (at least) two different levels of description. We call the large scale the macrolevel, this is the level of a system as a whole. The small scale is called the microlevel. This should be thought of as the level of individual particles or elements of a system. Note that the descriptions on these different scales show very different behaviour. For instance, even when a gas seems to stay still on a macrolevel (i.e. its density is constant in time), the particles that it consists of keep bouncing around at very high speeds on a microlevel. However, both levels of description describe the same gas, so they must be closely related to each other.

Therefore, one of the goals of statistical physics is to derive the macroscopic properties (density, temperature, entropy, magnetisation, percolation) of matter and the equations that govern them from the microscopic behaviour, properties and/or interactions of the particles that they are made of. In particular, one tries to understand how phenomena like phase transitions, shock waves, condensation and spontaneous magnetisation emerge. Special attention goes out to what happens around the critical points of the systems: around the states or parameters that lie between regions with different behaviour.

We want to point out that this idea of passing from microscopic to macroscopic descriptions is certainly not restricted to physics, but also occurs in other fields. For instance in genealogy, the microscopic scale is the scale of individuals in a population. From their interactions and reproduction events, one would like to understand on a macroscopic scale how allele frequencies evolve and which characteristics of species survive or die out. In neuroscience the brain can be regarded on a microscopic level as the firings of neurons through a complex network, from which macroscopic phenomena like the functions of the brain and even consciousness emerge. Finally, in epidemiology,

one of the goals is to understand the macroscopic spread of a virus or disease in a population from the microscopic interactions between individuals in that population.

1.1.2 Mathematical statistical physics

Statistical physics gave rise to a subfield of mathematics: mathematical statistical physics. The goal of the mathematician is to make mathematically solid definitions of the objects and quantities involved in statistic physics and to derive its statements in a mathematically rigorous way. This leads to an interesting interplay between mathematics and physics. On the one hand, mathematicians develop a framework to rigorously formulate and verify the physical results and find new results. On the other hand, physicists supply a lot of interesting objects, models, examples and phenomena that give direction to a better understanding of mathematics and even lead to new mathematical theories. These are of course rough characterisations, in practice the distinction is not so sharp.

The field of mathematical statistical physics is mostly part of probability theory. The reason is that on the microscopic level, we generally use probabilistic models. Therefore also the tools and methods to relate them to macroscopic models come from probability theory. This might seem surprising at first. After all, the movement and collisions of particles or the alignment of magnetic spins are chaotic but deterministic. However, there are good reasons to use probabilistic models. First, there is the empirical reason that some physical phenomena are described very well with stochastic models, think for example of the ubiquitous Brownian motion that famously describes the trajectory of pollen grains in water. Second, there is a pragmatic reason, namely that some randomness gives better ergodic properties of the microscopic system. The third reason is that understanding these models with random dynamics and interactions is an important step towards understanding a more deterministic setting where the only randomness is in the initial configuration (see for instance Bodineau et al. [19], Liverani [110], Bodineau et al. [20]).

The physical question of relating microscopic and macroscopic descriptions now becomes part of a broader probabilistic question: how does randomness of many (interacting) components of a system influence the behaviour of the bulk? Note that some classical theorems from probability theory, namely the Law of Large Numbers and the Central Limit Theorem, are of the same nature. Also in the cases of those theorems there are many random variables on a microscopic level and one studies the behaviour of their average, which is a macroscopic quantity. In fact, we will explain later in this introduction that results on hydrodynamic limits and equilibrium fluctuations can be interpreted as infinite-dimensional versions of the Law of Large Numbers and the Central Limit Theorem.

The main probabilistic framework in which we analyse the questions of statistical physics is interacting particle systems, which we will introduce shortly.

Remark 1.1 (Related models)**.** In addition to the models from statistical physics, there are many other probabilistic models for all kinds of physical phenomena. For instance random fields, random polymers, random graphs, sandpiles and many more are widely studied both in probability and physics. The reason to mention them here is first that more often than not, these models are related to models from statistical physics. The second reason is that they share the common trait with interacting particle systems that they are discrete, probabilistic models that usually have continuous limits. Therefore the methods to study them have a lot in common. Indeed, the Gaussian Free Field (that we will introduce later) is an example of a random field and we will see in this thesis how the methods of studying it are related to the methods used in certain problems in interacting particle systems. Finally, the active particles that we study in Chapter 7 are also a probabilistic model for a physical phenomenon. As we mentioned before, we will introduce these separately in Chapter 7.

1.2 Interacting particle systems

Now we arrive at one of the main classes of probabilistic models that we will deal with in this thesis: interacting particle systems (IPS). The main idea is that we want to model a space through which particles move randomly and where they influence each other. We already explained why we have randomness in the motion of the particles, but there is another key ingredient for interacting particle systems, namely discretisation. Even though we normally think of particles as moving through a continuum, it turns out that it is very helpful for our models to discretise the space into a collection of points (or a grid) between which the particles can jump. We will see soon how some grids can be rescaled to approximate a continuum.

1.2.1 Common interacting particle systems

Let us start by considering the following IPS. This will be our main example to illustrate the concepts that we encounter later. It is also the IPS that is studied in this thesis (although in a different space).

Example 1.2 (SEP)**.** Let the collection of possible particle positions (or sites) be given by $\mathbb{Z}$. At each site we put either one particle or no particle. This is the initial configuration. Now we need to specify the dynamics: how the particles move. Each particle has a random clock: it rings after exponential times with rate 1 (independent of the other particles). When the clock of a particle rings, it jumps to the left with probability $\frac{1}{2}$ or to the right otherwise. In case the site that it wants to jump to is already occupied, the jump is suppressed. This model is known as the Symmetric Exclusion Process (SEP) on $\mathbb{Z}$, see also Figure 1.1. The particles all perform symmetric random walks with the restriction of at most one particle per site. This restriction induces a repulsion between the particles.

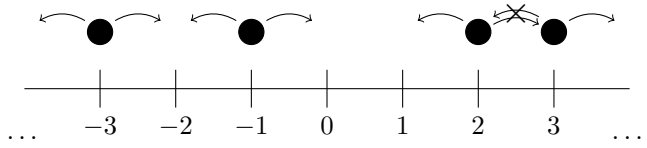


Figure 1.1: SEP on $\mathbb{Z}$ with the next possible jumps.

Starting from this basic model, we can make a lot of variations by changing one or more of the following aspects.

i) Initial configuration. We can influence how we initialise the particle system. For instance fix a parameter $\rho \in (0, 1)$ and at each position $x \in \mathbb{Z}$ put a particle with probability ρ (independent for different positions). One can also start with a Poisson distributed amount of points at each site or from a deterministic configuration (of course taking into account that in case of SEP there can be at most one particle per site).

ii) Jumping distributions. For instance we can let particles jump farther than one site or we let them jump to the right with higher probability than to the left. We could also let the jumping distribution depend on the position of the particle, for example by imposing a potential on the particles.

iii) Interaction. As the name of IPS suggests, the most interesting particle systems are those where the particles interact. The most basic interaction is to enforce a maximum of one particle per site as in the Symmetric Exclusion Process (SEP). As we said earlier, the exclusion induces a repulsion between particles. To create attraction, one can add the tendency to jump to sites with a lot of particles. This is done in the Symmetric Inclusion Process (SIP). Finally we mention the class of Zero Range Processes (ZRP), where the only interaction is with particles at the same site, which can lead to both attractive or repulsive behaviour (depending on parameters). Many more types of interaction have been studied.

iv) Space. In the example above, the particles jump on $\mathbb{Z}$. A first step would be to extend to $\mathbb{Z}^d$. This seems straightforward, but can seriously alter the properties of the system. We could also arrange a finite number of sites in a circle and obtain a (discrete) torus, again this can be done in multiple dimensions. In general, one can just specify a graph together with probabilities to jump from points to other points. We will elaborate on this in Section 1.6 and Section 2.1.3.

v) Boundary conditions. We can add points where particles enter or leave the system (reservoirs). Such points can model boundary conditions.

vi) Multiple types. Instead of having all particles behave the same, we could have different kinds of particles. For instance have blue particles that move to the

right and red particles that move to the left. Particles can also be allowed to switch colour. This is used to study reaction-diffusion equations.

vi) Internal state. Building on the previous point, we could even have particles that decide which jumps they prefer based on an internal state process. Such particles are called active particles and we will see more about those in Chapter 7.

This list is not exhaustive, many more variations can be considered. Of course for each variation one should check that it is well-defined mathematically (for instance to prevent that the numbers of particles at a fixed site goes to infinity in finite time).

The start of this field came through the work of Spitzer and Dobrushin around 1970. The Symmetric Exclusion Process was introduced with a variety of other interacting particle systems in Spitzer [139]. A good reference on some common IPS and the main techniques and results is for instance Liggett [109]. Note that [109] also contains a different class of systems, where at each site there is a spin (up/down) or other parameter that evolves randomly and is influenced by its neighbours (important examples include the Ising or Curie-Weiss model, the voter model and the contact process). A further elaboration is beyond the scope of this introduction.

All these models have in common that the future evolution of the process only depends on the current state (or configuration) of the process. In other words, they are Markov processes. Indeed, mathematically, we regard such systems as Markov processes in the space of possible particle configurations. For SEP, for instance, this state space is $\{0,1\}^{\mathbb{Z}}$. Here a 1 indicates the presence of a particle (and a 0 the absence). Note, however, that the trajectories of individual particles are in general not Markov.

In the next sections, we will explain the kind of results that we obtain for SEP. We want to mention here some other research questions that are typically investigated for interacting particle systems. We already mentioned that first of all it needs to be checked that a model is mathematically well-defined and does not blow up. The next goal is often to see if there is a stationary measure (see Section 2.1.2 for the definition). And if so, is it unique? And does the system converge to such measure? What are its properties? And how does the behaviour of the IPS depend on its parameters? In particular, are there any phase transitions? Can we discern multiple time scales on which the process evolves?
For the purposes of this thesis, the most interesting properties of interacting particle systems are what happens when we rescale them in an appropriate way.

1.2.2 Rescaling

As we mentioned earlier, the IPS are defined on grids that serve as a discretisation of an underlying continuum. To do this, space and time need to be rescaled in an appropriate way. We will illustrate this by defining SEP on a discretisation of the interval $[0,1]$. To avoid dealing with boundary conditions, we identify 0 and 1 and

work on the 1-dimensional torus (the circle). With a slight abuse of notation, we will still denote it as $[0,1]$.

Example 1.3 (SEP on a torus)**.** We define the discrete torus $G_N = \{1, 2, .., N\}$ by connecting the points 1 and N. Now SEP is defined on G_N in the same way as on $\mathbb{Z}$: at each site $i \in G_N$ we initially put either one particle or no particle. Then the particles jump after exponential times to a neighbour with equal probability. Note that now 1 and N are considered neighbours. See Figure 1.2 for SEP on $\frac{1}{N}G_N$.

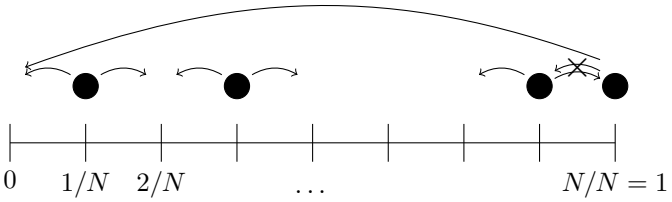


Figure 1.2: SEP on $\frac{1}{N}G_N$ with the next possible jumps.

If we rescale G_N with a factor $\frac{1}{N}$, we obtain

$$\frac{1}{N}G_N = \left\{ \frac{1}{N}, \frac{2}{N}, .., \frac{N-1}{N}, 1 \right\}. \tag{1.1}$$

The larger N is, the better this approximates the torus $[0,1]$. We will therefore use G_N as the microscopic space and $[0,1]$ as the macroscopic space by associating

$$[0,1] \ni x \longleftrightarrow \lceil Nx \rceil \in G_N. \tag{1.2}$$

We call the elements of $[0,1]$ macroscopic locations and the elements of G_N microscopic locations.

To pass from a microscopic description of a process on G_N to a macroscopic description on $[0,1]$, we should let N go to infinity. Now note that also the rate with which particles jump should depend on N. Indeed, when G_N is rescaled, the particles make jumps of size $\frac{1}{N}$. In the limit of N to infinity the particles would not move at all. Therefore we need to answer the following question.

> How many steps of size $\frac{1}{N}$ should on average be made per unit time, to have a macroscopic movement of $O(1)$?

For SEP, the answer is N^2. This comes from the symmetry of the random walks that the particles perform. For instance by using the Central Limit Theorem, one can see that after N i.i.d. steps with mean 0, a particle travels $O(\sqrt{N})$ distance. So after N^2 steps, it travels $O(N)$ distance on the microscopic grid, so $O(1)$ in the macroscopic space. Therefore we associate

$$t \longleftrightarrow N^2 t, \tag{1.3}$$

where t is the macroscopic time associated to the microscopic time $N^2 t$.

Remark 1.4. Note from (1.2) and (1.3) that when space is rescaled by a factor N, time should be rescaled with N^2. We will see in Section 1.3 that the limiting PDE that the rescaled SEP satisfies (in a way to be defined) is the heat equation. This equation has a second derivative with respect to space and a first derivative with respect to time. It is therefore invariant under the proposed scaling. That is another way to see why it is the right scaling. For other interacting particle systems the time scaling might be different. For instance with other kinds of interaction or (weak) asymmetries it is not always directly clear how strong the interaction should be on a microscopic level to be visible at a macroscopic level. The limiting equations can also be very different from the heat equation.

Summarising what we discussed so far, we conclude the following. To study SEP on a discretisation of the torus in such a way that it has a meaningful scaling limit, we should scale space with a factor $\frac{1}{N}$ and time with a factor N^2. This results in the following process. At the point $\frac{i}{N} \in \frac{1}{N}G_N$, a particle waits an exponential time with rate N^2 and then jumps to $\frac{i-1}{N}$ or $\frac{i+1}{N}$ with equal probability unless that site is occupied.

1.3 Hydrodynamic limit

As we discussed in Section 1.1, our goal with interacting particle systems is to go from a microscopic to a macroscopic description. We do so when we take the hydrodynamic limit of an IPS. We already saw in Section 1.2.2 how to rescale SEP to obtain a meaningful scaling limit. Now we will explain how to relate rescaled particle systems to limiting density profiles that satisfy a PDE.

1.3.1 The idea

When considering a gas or fluid, we do not wish to track the position of every particle but rather we would like to know how macroscopic quantities evolve. These are quantities that do not depend on individual particles but on (local) averages over groups of many particles. For the hydrodynamic limit, the quantity that we are interested in is the density of the fluid or gas (as a function of space). In a hydrodynamic limit result we prove that such limiting densities exist and how they behave.

We must first define how to relate a microscopic description to a macroscopic description. Again, let us consider G_N as the microscopic space and the torus $[0,1]$ as the macroscopic space. For each N, let η^N be a particle configuration on G_N, i.e. a function $G_N \to \{0,1\}$ that is 1 at a site iff there is a particle at that site. Note that there is actually not one microscopic particle configuration, but a sequence of configurations (depending on N). A macroscopic density profile is a (smooth, say) non-negative function ρ on the macroscopic space $[0,1]$. We now associate the sequence η^N of microscopic configurations to ρ if the following holds. For every macroscopic point

$x \in [0,1]$ there are in the limit on average $\rho(x)$ particles per site in η^N around the associated microscopic point $\lceil Nx \rceil$.

We will make this concept mathematically precise shortly. For now it is important to note two things. First, for each macroscopic profile there are very many different sequences of configurations that are associated to it, both random and deterministic ones. A useful way to define a sequence of configurations for a fixed density profile is as follows. Fix a macroscopic density profile ρ. Now for each N construct the configuration η^N by placing a particle at $i \in G_N$ with probability $\rho(i/N)$ independently for different sites. This way the configurations η^N have associated density profile ρ. We come back to this in Example 1.5.
Second, having an associated density profile is a special property that shows regularity of the particle configurations. It is easy to make examples where this property does not hold, i.e. sequences of particle configurations that do not have an associated density profile.

Now the idea of a hydrodynamic limit (formulated for SEP) is as follows (see also Figure 1.3). We start SEP on G_N from initial configurations that are associated to some initial density profile $\rho_0(\cdot)$. Then we rescale space and time and show that in the limit of N to infinity two things hold.

a) At every later macroscopic time point t, the particle configurations have an associated density profile $\rho_t(\cdot)$.

b) The density profiles $\rho_t(\cdot)$ satisfy the heat equation $\partial_t \rho_t = \Delta \rho_t$ with initial condition ρ_0.

We call the limiting equation the hydrodynamic equation or the hydrodynamic limit of SEP on $[0,1]$. A more mathematical formulation of this theorem will be given later.

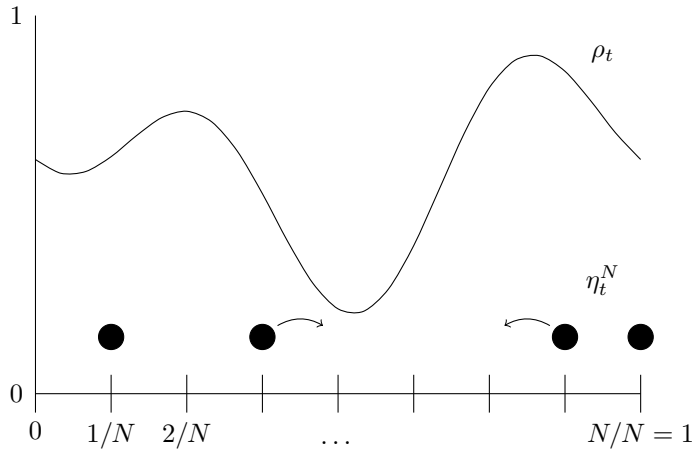


Figure 1.3: SEP and a density profile which evolves according to the heat equation.

Note that part a) is quite surprising. As we noted above there are many different sequences of particle configurations (either random or deterministic) associated to the same initial profile ρ_0. Apparently it does not matter from which of these configuration the system starts. Just having an associated density profile provides enough regularity to make sure that at every later point in time there is still an associated density profile, which is a very special property. Part b) now tells us that this density is actually uniquely determined by the initial density ρ_0 through the heat equation. This equation describes the behaviour of SEP when it is rescaled.

1.3.2 Mathematical framework

To define mathematically that a sequence of microscopic particle configurations is associated to a macroscopic density, we proceed as follows. Denote a particle configuration on G_N by η^N, where $\eta^N(i)$ is the number of particles at the point i. Place mass $\frac{1}{N}$ at each particle, by defining the empirical measure

$$\mu^N = \frac{1}{N}\sum_{i=1}^{N} \eta^N(i)\delta_{i/N}, \tag{1.4}$$

where for $x \in [0,1]$, δ_x denotes the Dirac measure at x. When η^N is random, μ^N is a random measure on $[0,1]$. Now we say that a sequence of particle configurations on G_N has associated density profile $\rho(\cdot)$ if

$$\mu^N \longrightarrow \rho(x)\mathrm{d}x \qquad (N \to \infty)$$

weakly in probability.

Example 1.5 (Local equilibrium)**.** As we mentioned before, as an example with a fixed density profile $\rho : [0,1] \to [0,1]$, one can let $\eta^N(i)$ be independent and Bernoulli distributed with parameter $\rho(i/N)$. This implies the required weak convergence in probability of the corresponding empirical measures to $\rho\mathrm{d}x$. The resulting random configurations η^N display what we call local equilibrium. When N is large, the sites around $i \in G_N$ have independent Bernoulli distributions with parameter very close to $\rho(i/N)$ (since ρ is smooth), so in the limit around every point the distribution looks like a product of i.i.d. Bernoullis. However, when moving to a different macroscopic point, the parameter changes. This is called a slowly varying parameter.

We illustrate the hydrodynamic limit result by formulating it for SEP on G_N. Denote the particle configuration on G_N at time t by η_t^N. Here $\eta_t^N(i)$ is the number of particles at position i at time t. Starting from some initial distribution, we run SEP. Together this defines the particle process for all $t \geq 0$. Write

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \eta_{N^2t}^N(i)\delta_{i/N} \tag{1.5}$$

and note the space rescaling in $\frac{i}{N}$ and the time rescaling in N^2t (as discussed in Section 1.2.2). Note that the trajectory $\mu^N = (\mu_t^N, t \geq 0)$ is a random measure-valued trajectory.

We can now give the statement of the hydrodynamic limit result. Note that a hydrodynamic limit is first of all about pointwise convergence of μ_t^N for each t. However, it is possible to show convergence for the whole trajectory, which is a bit stronger. Since we will sketch a proof for convergence of the trajectory, we formulate the result directly in that way. The pointwise convergence is a direct consequence.

Theorem 1.6. *Consider the Symmetric Exclusion Process η_t^N on G_N with corresponding empirical process μ_t^N. Let $\rho_0 : [0,1] \to [0,1]$ be a continuous function with $\rho_0(0) = \rho_0(1)$. Suppose that the initial configuration has associated density ρ_0, i.e.*

$$\mu_0^N \longrightarrow \rho_0 \mathrm{d}x \qquad (N \to \infty)$$

weakly in probability. Then:

i) The trajectory $(\mu_t^N, t \geq 0)$ converges weakly in path space to a deterministic trajectory $t \mapsto \rho_t \mathrm{d}x$.

ii) $t \mapsto \rho_t$ satisfies the heat equation.

This type of result for SEP was first proved in Galves et al. [67] and Ferrari et al. [61]. We will sketch a proof from Kipnis and Landim [96] in Section 1.3.3.

The result of Theorem 1.6 is in some sense a rather basic hydrodynamic limit results since the particles perform simple symmetric random walks, the interaction leads to closed equations (as will be explained later), there are no boundaries and the space (the circle) is easy to discretise. All of these things can be changed. Most importantly, we can add different interactions between the particles, like we discussed before, and obtain non-linear hydrodynamic equations. Further, macroscopic PDEs with boundary conditions can be obtained by letting particles enter or leave the system in an appropriate way. There are also results for particles with a drift, different kinds of particles and even active particles (we will mention some of them). In this thesis we consider a basic kind of interaction (SEP), no boundaries and no drift, because we change something else: the space. We will get back to this in Section 1.6.

1.3.3 Method of proof

To obtain a hydrodynamic limit result, one often proves the following three steps (as described in for instance Kipnis and Landim [96]).

1) The distributions of the random trajectories are tight (relatively compact) as measures on the path space. This ensures that every subsequence of μ^N has a convergent subsequence.

2) Every limiting point of μ^N satisfies in the weak sense a deterministic PDE with intial condition $\rho_0 \mathrm{d}x$.

3) The PDE has a unique weak solution of the form $\rho_t \mathrm{d}x$ where ρ_t strongly satisfies the PDE with initial condition ρ_0.

Step 2 and 3 together imply that all limiting points of μ^N are the same. Together with step 1, this implies convergence in distribution to the trajectory $t \mapsto \rho_t \mathrm{d}x$. Since this limiting point is deterministic (concentrated on a single trajectory), the weak convergence implies convergence in probability. Usually step 1 is mostly technical and step 3 is a result from analysis. Therefore we are usually mostly concerned with step 2.

For step 2 a common method (that is also used in this thesis) is the martingale approach, we will sketch it here.

Step 2 proof sketch. Using the Dynkin martingale (see Chapter 2), one can show that for every test function ϕ

$$\langle \mu_t^N, \phi \rangle - \langle \mu_0^N, \phi \rangle - \int_0^t N^2 L^N \langle \mu_s^N, \phi \rangle \mathrm{d}s = M_t^N. \tag{1.6}$$

Here $\langle \mu_t^N, \phi \rangle$ denotes integration of ϕ with respect to μ_t^N and M^N is a martingale. L^N is the generator of SEP (see Chapter 2) multiplied by N^2 that comes from the time scaling. For now it is only important that

$$\begin{aligned} N^2 L^N \langle \mu_t^N, \phi \rangle &= N \sum_{i \in G_N} \left[\eta_t^N(i)(1 - \eta_t^N(i+1)) \left(\phi\left(\tfrac{i+1}{N}\right) - \phi\left(\tfrac{i}{N}\right) \right) \right. \\ &+ \left. \eta_t^N(i)(1 - \eta_t^N(i-1)) \left(\phi\left(\tfrac{i-1}{N}\right) - \phi\left(\tfrac{i}{N}\right) \right) \right]. \end{aligned} \tag{1.7}$$

By manipulating the sum, we see that this equals

$$N \sum_{i \in G_N} (\eta_t^N(i) - \eta_t^N(i+1)) \left(\phi\left(\tfrac{i+1}{N}\right) - \phi\left(\tfrac{i}{N}\right) \right).$$

Note that there are only linear terms of η_t^N left, the second order terms cancelled each other out. A partial summation now shows that this equals

$$\frac{1}{N} \sum_{i \in G_N} \eta_t^N(i) N^2 \left(\phi\left(\tfrac{i+1}{N}\right) + \phi\left(\tfrac{i-1}{N}\right) - 2\phi\left(\tfrac{i}{N}\right) \right) = \langle \mu_t^N, \Delta_N \phi \rangle. \tag{1.8}$$

Here Δ_N is the discrete Laplace operator. Plugging this back into (1.6), we obtain

$$\langle \mu_t^N, \phi \rangle - \langle \mu_0^N, \phi \rangle - \int_0^t \langle \mu_s^N, \Delta_N \phi \rangle \mathrm{d}s = M_t^N.$$

Now one shows that the martingale M_t^N vanishes in the limit (to be precise, by studying its quadratic variation) and that Δ_N converges to Δ. The limiting equation then becomes

$$\langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle - \int_0^t \langle \mu_s, \Delta \phi \rangle \mathrm{d}s = 0. \tag{1.9}$$

This shows that the limiting density μ_t is a weak solution to the heat equation. □

There are two things to take away from this proof sketch. First, it really helped that the nonlinear terms with η_t^N in (1.7) cancelled each other out and that we were able to write (1.8) as a function of the empirical measure μ_t^N without any other dependence on η_t^N. Because of this we obtained a closed equation for the empirical measures μ_t^N. In general this is not directly the case, for instance for particle systems with other kinds of interaction. To close the equation in those cases, the inconvenient terms are replaced by averages over large blocks of sites. There is whole class of systems (the so called gradient systems) for which a method to do this has been developed. The crucial estimates to replace quantities by averages are known as the one block and two block estimates. We will not need these in this thesis (since we treat SEP), but we will comment on them in Section 4.5.
Second, note that in (1.8), the discrete Laplacian Δ_N appears. The reason is that this operator is the generator of the symmetric random walk that the particles perform (see Chapter 2 for more about generators). This tells us that to make such proof work on a manifold, we will need to have a discrete Laplace operator there that converges to the manifold Laplacian: the Laplace-Beltrami operator. We will get back to this in Chapter 3.

To prepare for the next topic, we finally want to note that a hydrodynamic limit result can be thought of as an infinite-dimensional version of the Law of Large Numbers (LLN). Indeed, this can be seen in two ways. First, the trajectory of empirical measures is the average of the (dependent) trajectories of the individual particles when they are represented by Dirac measures. In the limit, all the randomness averages out and a deterministic trajectory emerges. Second, we can look at it locally in space. When a sequence of configurations η_t^N has an associated density profile $\rho_t(\cdot)$, it means that around a macroscopic point x, the average of the random particle numbers in microscopic sites around $\lceil Nx \rceil$ is the deterministic real number $\rho(x)$. This can be seen as a local LLN around x.

1.3.4 Relevant literature

Among the first mathematical results about hydrodynamic limits was Morrey Jr [121]. Two important methods to obtain hydrodynamic limits are the entropy method and the relative entropy method, developed in Guo et al. [81] and Yau [159], respectively. An overview of the concepts and methods of hydrodynamic limits as well as an overview of the early literature can be found in Kipnis and Landim [96]. For exclusion processes also Seppäläinen [136] gives a proof for the hydrodynamic limit. Other relevant overviews are given in Spohn [141], De Masi et al. [37] and De Masi and Presutti [36].

For hydrodynamic limits with boundary conditions see for instance Gonçalves [75] and Franceschini et al. [64]. We will point to results in more general spaces in Section 1.6. For an asymmetric version of SEP see Kipnis et al. [98], Landim et al. [103] and Ferrari [62]. In the context of this thesis it is also interesting to mention the hydrodynamic limit for a system of active particles in Erignoux [54].

1.4 Equilibrium fluctuations

1.4.1 The idea

We have seen that a hydrodynamic limit result can be thought of as an infinite-dimensional version of the Law of Large Numbers (LLN). It is only natural to ask whether there exists a corresponding Central Limit Theorem (CLT). To go from a usual LLN to a CLT, one subtracts the limit given by the LLN and rescales in a different way to see the fluctuations around this limit. We can follow the same approach with a particle system: we subtract the solution to the hydrodynamic equation, rescale and quantify how the resulting object behaves. We will see that the limiting object can be interpreted as infinite-dimensional Gaussian process, so indeed we get a CLT-like result. The limiting object is again a trajectory, but this time it is a random trajectory. It formally solves a stochastic PDE (SPDE) instead of a PDE.

The fluctuations around the hydrodynamic limit are usually considered in equilibrium, i.e. when the microscopic process is started from a stationary distribution (see Chapter 2 for the definity of stationarity) and, as a consequence, the macroscopic density profile is constant in time. The reason is that the limiting Gaussian noise is so rough that it is not always clear how to define it in the non-equilibrium case. In equilibrium, the limiting object is a stationary Gaussian process, for which there are more techniques available to define them. Another more pragmatic advantage of stationarity is that the distribution of the microscopic process is known and equal at each point in time. There are also results for non-equilibrium fluctuations. We will mention some of them later.
In the case of SEP on the circle, the stationary distributions are independent Bernoulli random variables with constant parameter ρ. The hydrodynamic equation is the heat equation, so a density profile that is constant in time is also constant in space and equals ρ everywhere.

1.4.2 Mathematical framework

We will again illustrate the mathematical framework around equilibrium fluctuations by using the example of SEP on G_N. As we discussed above, to consider the process in equilibrium, the density profile must be constant. We set it equal to a constant $\rho > 0$. The limiting fluctuation field will be so rough (similar to white noise) that we need to interpret it as a random distribution. Therefore we define the fluctuation fields Y_t^N as acting on a smooth function f as follows:

$$Y_t^N(f) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f\left(\tfrac{i}{N}\right) (\eta_{N^2 t}^N(i) - \rho). \tag{1.10}$$

Here the scaling in space and time is present in the factors $\frac{i}{N}$ and $N^2 t$, respectively. Als note that we subtracted ρ, the constant macroscopic density, and that

the fluctuation field has a scaling of $\frac{1}{\sqrt{N}}$ instead of $\frac{1}{N}$, as usual with a CLT. Now $Y^N = (Y_t^N, t \geq 0)$ is a random distribution-valued trajectory.

When we start SEP on G_N from a product of Bernoulli distributions with parameter ρ, it is stationary. This implies that the distribution of η_t^N is also a product of Bernoullis for every t. Using Lindeberg's criterion, we can directly infer that

$$Y_t^N(f) \longrightarrow \mathscr{N}\left(0, \rho(1-\rho)\int_0^1 f(x)^2 \mathrm{d}x\right) \qquad (N \to \infty) \tag{1.11}$$

in distribution. This suggests that Y_t^N (for fixed t) should have a Gaussian distribution since the marginals are Gaussian.

To see what the correlations are for different points of time requires a bit more work. We sketch now how to do that. Note that some concepts that are explained in Chapter 2 are used here.

Sketch of correlations. We start with the formula

$$\mathbb{E}_\eta \eta_t(j) = \sum_{j \in G_N} p_t^N(i,j)\eta(i),$$

which holds for SEP and where $p_t^N(i,j)$ is the probability that a symmetric random walk on G_N starting from i is at j at time t. Using this and the Markov property (see Chapter 2), one can show that for $s < t$

$$\mathrm{Cov}(\eta_s^N(i), \eta_t^N(j)) = \rho(1-\rho)p_{t-s}^N(i,j).$$

This yields

$$\mathrm{Cov}(Y_s^N(f), Y_t^N(g)) = \frac{1}{N}\sum_{i,j \in G_N} f\left(\tfrac{i}{N}\right) g\left(\tfrac{j}{N}\right) \rho(1-\rho) p_{N^2(t-s)}^N(i,j),$$

which converges to

$$\rho(1-\rho)\int_0^1 f(x)\int_0^1 p_{t-s}(x,y)g(y)\mathrm{d}y\mathrm{d}x,$$

where p_t is the heat kernel on the circle. From this we conclude that the marginals of the limiting process satisfy

$$\mathrm{Cov}(Y_t(f), Y_s(g)) = \rho(1-\rho)\langle f, S_{|t-s|}g\rangle, \tag{1.12}$$

where $\langle f, g\rangle$ denotes the L^2 inner product on $[0,1]$ and S is the heat semigroup. □

It turns out that the limit Y of Y^N is a stationary Gaussian process that formally solves the SPDE

$$\mathrm{d}Y_t = \Delta Y_t \mathrm{d}t + \sqrt{2\rho(1-\rho)}\nabla \mathrm{d}W_t. \tag{1.13}$$

Here W is space-time white noise. (1.13) shows that the limiting fluctuation field satisfies a noisy version of the heat equation. It can be interpreted as an infinite dimensional version of the Ornstein-Uhlenbeck process (see Chapter 2).

The interpretation of the term $\sqrt{2\rho(1-\rho)}\nabla \mathrm{d}W_t$ in (1.13) is that if we let it act on a test function f and integrate it from 0 to t, the resulting process is a martingale $M_t^f(Y)$ with quadratic variation $2t\rho(1-\rho)\int(\nabla f)^2 \mathrm{d}x$ (see Section 2.1.4 for more on martingales and quadratic variation), and is therefore in fact a constant times Brownian motion. The fact that the noise is given by $\nabla \mathrm{d}W_t$ instead of $\mathrm{d}W_t$ is due to a (local) conservation law, i.e. the conservation of the number of particles.

This interpretation suggests that we can give meaning to the SPDE in (1.13) by evaluating both sides on test functions and integrating them. We then obtain the following definition of solutions of (1.13). For each test function f, the following two processes should be martingales:

$$\begin{aligned} M_t^f(Y) &:= Y_t(f) - Y_0(f) - \int_0^t Y_s(\Delta f)\mathrm{d}s \\ N_t^f(Y) &:= (M_t^f)^2 - 2t\rho(1-\rho)\int(\nabla f)^2 \mathrm{d}x. \end{aligned} \tag{1.14}$$

This requirement is called a martingale problem.

We summarise these results in the following theorem.

Theorem 1.7. *Consider the Symmetric Exclusion Process η_t^N on G_N with corresponding fluctuations fields Y_t^N. Let the initial configuration be a product of Bernoulli distributions with parameter $\rho \in (0,1)$. Then the trajectory $(Y_t^N, t \geq 0)$ converges in distribution in path space to the generalised Ornstein-Uhlenbeck process Y defined through* (1.14)*. As a consequence, the marginals of Y are Gaussian and its covariance structure is given by* (1.12)*.*

This type of result for SEP was first proved in Galves et al. [68]. We will sketch a proof from Kipnis and Landim [96] in Section 1.4.3.

1.4.3 Method of proof

To prove that the fluctuation fields converge to a generalised Ornstein-Uhlenbeck process, we can again proceed by proving the following three steps that are somewhat similar to the steps in Section 1.3.3.

1) The distributions of Y^N are tight (relatively compact). This ensures that every subsequence of Y^N has a convergent subsequence.

2) Every limiting point of Y^N satisfies the martingale problem (1.14).

3) The martingale problem in step 2 uniquely determines the distribution of the underlying process.

Together these steps imply the convergence in distribution of Y^N to some random trajectory Y, which satisfies the martingale problem. The martingale problem characterises Y as solving (1.13).

Again, usually we are mostly concerned with step 2. It can be handled with a Dynkin martingale approach that is similar to the approach in Section 1.3.3. Only this time the scaling is different, so the martingale part does not disappear and the limiting process is still random. As in Section 1.3.3, for processes other than SEP, the martingale equation is not closed: the integral part of the Dynkin martingale cannot be expressed directly in terms of Y_t^N but depends on the microscopic configuration in other ways. To overcome this, one again needs to replace the difficult terms by averages over large boxes. This replacement is called the Boltzmann-Gibbs principle. Since we will be dealing with SEP, we will not need that principle in this thesis.

1.4.4 Relevant literature

For the first results in this direction, including the Boltzmann-Gibbs principle, see Martin-Löf [115] or Rost [133]. The proof for equilibrium fluctuations that we sketched above can be found in Kipnis and Landim [96, Chapter 11], along with an overview of other literature. In Holley and Stroock [84], the authors prove existence and uniqueness of the generalised Ornstein-Uhlenbeck process. Non-equilibrium fluctuations were first obtained in De Masi et al. [39] and then for SEP in general dimensions in Ravishankar [128]. More recent results include Jara and Menezes [92] or Erhard et al. [53].

1.5 (Discrete) Gaussian Free Field

One of the main models that is used in this thesis is the Gaussian Free Field (GFF). We can think of the continuum GFF as a high-dimensional or spatial version of Brownian motion, not in the sense that it takes values in space, but that it is indexed by space. Like Brownian motion, it has a Gaussian distribution and it is centered. Moreover, we will see that it satisfies a Markov property. In fact, in one dimension the Gaussian Free Field is actually the same as a Brownian bridge. In higher dimensions, the GFF is so rough that it is not function-valued but should be modelled as a random distribution. We will first briefly discuss the idea of random fields and then introduce the Discrete Gaussian Free Field (DGFF) and its continuum counterpart the GFF. Finally we mention the steps to prove that the DGFF converges to the GFF.

1.5.1 Random fields

A stochastic process can be viewed as a collection of random variables indexed by time: the values randomly evolve as time passes. Similarly, we can consider a collection of random variables indexed by space: at every point in space there is a random variable. If we were to make a graph of this collection, we would obtain a random

surface or interface. To see why these are natural objects to study, consider the following example. Suppose two fluids do not mix and one floats on top of the other. The interface between the fluids is not smooth but goes up and down seemingly randomly. Such interface could be modelled as a random surface. This is just one example, random fields also appear in many other places such as material science, cosmology, noise in sensory data, machine learning, image processing, textures in computer graphics and more. See for instance Rue and Held [134] for an introduction to Gaussian Markov random fields and some applications. Also for instance Zhu et al. [161], Rangarajan and Chellappa [127], Diebel and Thrun [44], Cao and Worsley [25] and Marinucci and Peccati [114] feature applications of random fields.

1.5.2 Discrete Gaussian Free Field

To define the Discrete Gaussian Free Field, we fix a finite graph V (denoting both the graph and its vertex set by V). We assume that the edges have weights $c_{vw} = c_{wv} \geq 0$ for $v, w \in V$ (where $c_{vw} = 0$ means that there effectively is no edge between v and w). Further, we require that between any two vertices, there exists a path consisting of edges with weights greater than 0. Now we want to define a random variable φ_v in each of the vertices $v \in V$ in such a way that φ_v and φ_w tend to be close to each other when c_{vw} is high. In other words, $\sum_{v,w} c_{vw}(\varphi_v - \varphi_w)^2$ should typically be small. Since this only gives a restriction on the φ_v's relative to each other and not about their values themselves, we need to 'pin' the field somewhere. Therefore we let V_0 be a non-empty proper subset of V and require that $\varphi_v = 0$ for $v \in V_0$. Together this leads to the definition that φ is a random vector indexed by $v \in V$ with the following distribution:

$$\frac{1}{Z} \exp\left(-\frac{1}{2} \sum_{v,w} c_{vw}(\varphi_v - \varphi_w)^2 \right) \prod_{v \notin V_0} \lambda(\mathrm{d}\varphi_v) \prod_{v \in V_0} \delta_0(\mathrm{d}\varphi_v). \tag{1.15}$$

Here λ is the Lebesgue measure on $\mathbb{R}$, δ_0 is the Dirac measure at 0 and Z is a deterministic normalising constant. We see indeed from (1.15) that the field φ is equal to 0 at the points in V_0 because of the Dirac measures at those points. At the other points, it has a density (with respect to the Lebesgue measure) that penalises realisations with large differences between values at neighbouring points. See Figure 1.4 for a realisation of the DGFF on a square lattice.

Instead of $(\varphi_v - \varphi_w)^2$ we could have chosen $h(\varphi_v - \varphi_w)$ for some other convex function h that is symmetric around 0. However, for this particular choice of $h(x) = x^2$, the resulting distribution has surprisingly elegant properties. These are some of the properties of the Discrete Gaussian Free Field (DGFF), as this model is called.

i) Gaussianity. φ has a multivariate normal distribution.

ii) Covariance. The covariance function of φ is $\mathrm{Cov}(\varphi_v, \varphi_w) = G(v, w)$. Here G is the Green's function corresponding to the random walk on V with jump rates c_{vw} and that is killed at sites in V_0.

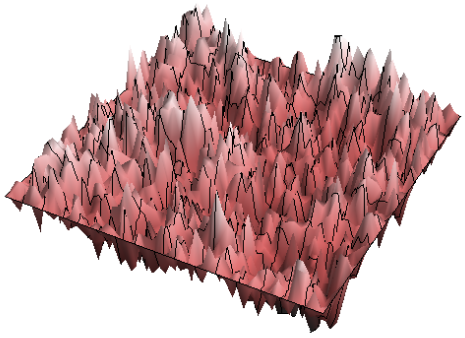


Figure 1.4: The DGFF on a 60×60 square grid (source: `https://en.wikipedia.org/wiki/Gaussian_free_field`).

iii) Mean. φ has mean 0. Moreover, when we let φ take other fixed values at V_0, the mean become the solution to the Poisson equation on V with those fixed boundary values at V_0 (and the covariance structure stays the same).

iv) Markov property. This property basically means that if the DGFF is conditioned on the values in a subset of the sites, the resulting field is again a DGFF. To be more precise, fix a non-empty proper subset U of V and fix values in U. Now let φ^U be φ conditioned on the values in U. Then

$$\varphi^U = h^U + \varphi_0^{V\setminus U}.$$

Here h^U is harmonic (i.e. the solution of the Poisson equation) with the fixed boundary values in U and $\varphi_0^{V\setminus U}$ is a DGFF with boundary values 0 in U. Moreover, $\varphi_0^{V\setminus U}$ is independent of the values in U. This property implies the following. Suppose one fixes a 'closed contour' of vertices somewhere in V. Then, conditioned on the values of φ on that ring, the values inside of the ring are independent of the values outside of the ring. Therefore it can be considered a spatial analogue of the Markov property for stochastic processes.

We see that many important properties of the DGFF can be described in terms of the Green's function and the Poisson equation. Both of these are directly related to the discrete Laplace operator on V. We will see that this Laplace operator turns out to be very important in defining a suitable DGFF on a manifold. For completeness, we mention here that one can add a constant multiple of the identity matrix to the Laplacian in this model and obtain the so-called massive free field. The Laplacian can also be replaced by its square. The resulting model is called the membrane model.

The nice structure of the DGFF allows for a thorough analysis of its properties. Common examples of such properties are its extreme values, level set percolation and

entropic repulsion.

1.5.3 Gaussian Free Field

To introduce the continuum counterpart of the DGFF, suppose we define the DGFF φ^N on the grid V^N given by

$$V^N = \left\{ \left(\frac{i}{N}, \frac{j}{N} \right) \middle| i, j = 0, 1, .., N \right\}$$

(with edge weight 1 between neighbours and 0 otherwise). We set $V_0 \subset V^N$ to be the natural boundary: the points of V^N on the boundary of $[0,1]^2$. This grid is a discretisation of the square $[0,1]^2$. Now we can let N tend to infinity and study the properties of the limiting field. A study of the Green's function on V^N shows that the variance of the fields (around a macroscopic point in $(0,1)^2$) blows up. In fact, the limiting field is so rough that it cannot be described as a random function. Therefore it should be viewed as a random distribution, acting on an appropriate class of test functions.

The limiting field φ is called the Gaussian Free Field (GFF) on $(0,1)^2$. Mathematically, it is a random element of the space of distributions acting on functions on $(0,1)^2$ with the following properties.

i) Gaussianity. The evaluation of φ on a suitable test function f, denoted by $\varphi(f)$, is a random variable with a Gaussian distribution.

ii) Covariance. When f, g are test functions, $\mathrm{Cov}(\varphi(f), \varphi(g)) = (f, Gg)$, where G is the Green's function on $[0,1]^2$.

iii) Mean. $\varphi(f)$ has mean 0 for each f.

iv) Markov property. As the DGFF, the GFF has a spatial Markov property. It states that if φ is restricted to orthogonal closed subspaces of the space of test functions, then the resulting random distributions are independent. This implies that if f and g are test functions with disjoint support, then $\varphi(f)$ and $\varphi(g)$ are independent. It also implies roughly that we can fix a domain $U \subset (0,1)^2$ and then write φ outside of U as the sum of a harmonic function with fixed values in U and a GFF that is independent of the values in U. This is similar to the Markov property of the DGFF.

Because of these properties, we view the GFF as a generalised Gaussian distribution indexed by smooth functions instead of by points in space. The previous properties were analogous to the DGFF properties. However, there are additional properties that the GFF satisfies.

v) Conformal invariance. The GFF property is invariant under conformal mappings (mappings that preserve angles), i.e. the image of φ under a conformal mapping is again a DGFF (although on a different domain).

vi) Circle averages. Fix a point on $(0,1)^2$ and consider the average of the GFF on a circle of radius ϵ around that point. Then these averages as a function of the circle radii ϵ can be rescaled to Brownian motion.

We mentioned at the start of this section that the GFF can be interpreted as a spatial version of Brownian motion. The circle averages show another way in which the Gaussian Free Field is related to Brownian motion.

One of the important applications of the GFF is in the construction of Liouville Quantum Gravity (LQG), a central object in random conformal geometry. In this construction the conformal invariance property of the GFF is used. In fact LQG is (formally) a random surface of which the Riemannian metric equals the exponential of a GFF. Important for the study of LQG are the so called thick points of the GFF, the points where the GFF is unusually high. The Gaussian Free Field also pops up as a limiting random field for various probabilistic models, including for instance dimer models.

We illustrated the GFF as a continuum limit of the DGFF in the 2-dimensional case on the unit square $(0,1)^2$. We can also consider the GFF on domains in other dimensions with analogous properties. It turns out that in dimension 1, it has enough regularity to be a random function and actually equals a Brownian bridge. In dimension 2 and higher, however, it cannot be described as a function. The 2-dimensional case is the most studied one, because of its use in other models in physics (like LQG). It is also the critical case between dimension 1 and dimension 3, where other scalings are sensible (see further Remark 1.8).

1.5.4 Convergence of the DGFF to the GFF

We sketch here how the convergence of the DGFF to GFF can be proved in arbitrary dimension d. Fix some domain D in $\mathbb{R}^d$. To prove convergence of the DGFF on grids approximating D to the GFF on D, we interpret both the DGFF and the GFF as elements of a Sobolev space of an appropriate negative index. Then there are often two important ingredients.

i) Tightness (relative compactness) of the sequences of DGFFs.

ii) Convergence of the Green's functions to the continuum Green's function.

Tightness implies that subsequences have convergent subsequences and then the convergence of the Green's functions yields the uniqueness of the limit and shows that it equals the GFF. See for instance Biskup [16] for more details.

In Chapter 6 we will generalise this result to compact manifolds of general dimensions. For technical reasons, it is more convenient to define the GFF as acting on smooth functions on a manifold (this will be explained in Chapter 6). The main advantage is that it is easier to show tightness in that case. The main challenge will be to show the convergence of the Green's functions. Further, it is not directly clear how

to define the DGFF when space cannot be rescaled. Additionally, since there is no natural boundary on a manifold without boundary, there is no natural way to 'pin' the DGFF. Therefore we will consider a zero-average DGFF.

Remark 1.8. In some dimensions there are other ways to take limits of the DGFF. We mention them briefly. In dimension $d \geq 3$, we can define the DGFF on a large box of length N in $\mathbb{Z}^d$ and let N go to infinity. Since the random walk is transient for $d \geq 3$, the Green's function and hence the covariance structure is bounded and has a well-defined limit. The limiting field is called the infinite-volume limit.

In dimensions 1 and 2, the Green's function is not bounded, so we cannot follow the same approach. In dimension 2 it grows like $\log(N)$ and in dimension 1 like N. We could rescale them to the unit square (in dimension 2, as above) or the unit interval (in dimension 1) and rescale the fields in such a way that the variance becomes $O(1)$. In dimension 1 this results in a Brownian bridge, which coincides with the GFF. However, in dimension 2 the limiting object is white noise (i.i.d. Gaussians), which does not have any spatial covariances and is therefore not an interesting limiting field.

1.5.5 Relevant literature

The GFF was first studied in the physics community, where it was called a Euclidean bosonic massless free field or sometimes it was used without a name (see [137, Remark 1.1]). A first more mathematical introduction of the GFF is Sheffield [137]. For an introduction of the GFF in connection with LQG see Berestycki [12]. For an introduction of the (D)GFF and its extreme values see Biskup [16]. The DGFF and its relation with the underlying random walk can be found in Sznitman [144]. We also mention the following related results. Kenyon [95] obtains the GFF as the limit of a dimer model. In Cipriani et al. [33], the membrane model is studied and it is shown that the discrete membrane model converges to its continuum counterpart.

1.6 Generalisation to other spaces: manifolds

Most results on scaling limits of interacting particle systems and random fields are set in (a domain in) $\mathbb{R}^d$ or on a d-dimensional (flat) torus. However a lot of processes do not take place in such flat and homogeneous spaces, but in more complex spaces. For this reason it is relevant to see if particle systems and random fields can be studied in those more complex spaces. Moreover, because of the symmetries of spaces like $\mathbb{R}^d$ (and grids like $\mathbb{Z}^d$) a lot of proofs, calculations and expressions simplify considerably. To understand what the essential ingredients are to make these models and proofs work, we study them in spaces where such symmetries are not available. In return, the behaviour of the models may help us to understand the geometry of these spaces better.

We will give a brief overview of different ways in which these models have been studied in more complex spaces. Then we introduce and motivate the main setting of this

thesis: manifolds. We will provide some available results on manifolds and describe the main issues that come up in this context.

1.6.1 Other underlying spaces

Hydrodynamic limits have been studied in a variety of spaces other than the standard $\mathbb{Z}^d$ or the torus.

i) Still on $\mathbb{Z}^d$. The first way to move to a more complicated environment is by staying on $\mathbb{Z}^d$ but adding inhomogeneity to the space. For instance by setting random conductances (Faggionato et al. [59]), random maximal particle numbers at sites (Floreani et al. [63]) or even by letting the environment itself be determined by an exclusion process (Avena et al. [6]).

ii) Other approximation of $\mathbb{R}^d$. The next step is to move away from $\mathbb{Z}^d$ as a discretisation of $\mathbb{R}^d$. This can be done by allowing more general grid structures, so called crystal lattices (triangular, hexagonal, etc), as in Tanaka [145], or by considering random approximations of $\mathbb{R}^d$ (Faggionato [57]).

iii) Away from $\mathbb{R}^d$. One can also consider macroscopic spaces other than $\mathbb{R}^d$ or the torus. For instance by studying hydrodynamic limits of particle systems on fractal structures, see Jara [90] and Chen and Gonçalves [26]. In Jara [91] and Faggionato [56] an approach can be found for obtaining hydrodynamic limits in the presence of a certain type of inhomogeneities under the assumption of homogenisation. Here the macroscopic spaces can be general metric spaces such as percolation clusters or fractals.

For some models above also equilibrium fluctuations have been obtained. For instance in a random environment (Farfan et al. [60], Gonçalves and Jara [76]) and on the Sierpinksi gasket (Jara [89]).

The convergence of the DGFF to the GFF was obtained in a model with random conductances in Biskup [15, Thm 6.7]. The DGFF with random conductances was also studied in Chiarini and Nitzschner [29]. Further, the DGFF was studied on a fractal structure in Chen and Ugurcan [27].

1.6.2 Manifolds: introduction and motivation

In this thesis we consider SEP and the DGFF in yet a different setting: on manifolds. We will introduce manifolds along with some important concepts from differential geometry in Section 2.2. However, since they are important to understand the goal of this thesis, we will quickly sketch the main idea here.

The idea of a manifold

A d-dimensional manifold is a geometric object that locally looks like $\mathbb{R}^d$, but globally can be very different from $\mathbb{R}^d$. A basic but instructive example to keep in mind is the sphere. When zooming in closely on a particular point on the sphere, its surroundings

start to look like a plane (similar to the daily-life experience of a person looking around from a point on earth). In this sense a sphere is locally like $\mathbb{R}^2$. However, its global properties are very different. For instance, on a sphere one can move along a straight line (a so-called geodesic) and still return to the initial location. Also, it is possible to make a triangle with three 90° angles, by connecting the north pole with two points on the equator. And, more mathematically, a sphere is compact (without boundaries). $\mathbb{R}^2$ itself certainly does not have these properties. A sphere can be thought of as many parts of planes that are smoothly patched together to create a surface, similar to how the charts in an atlas can be patched together to describe a globe. Of course this patching together can be done in many ways to create all kinds of smooth surfaces. Roughly speaking, these surfaces are the two-dimensional manifolds. This idea can be generalised to define d-dimensional manifolds by requiring that they locally look like $\mathbb{R}^d$ instead of $\mathbb{R}^2$.

Applications
As we said earlier, we will give some more mathematical definitions concerning manifolds in Section 2.2. For now we want to explain why it makes sense to study (processes on) manifolds. Manifolds are ubiquitous in physics, biology, astronomy and other fields. First, the shapes of smooth physical objects can be modelled by manifolds. This means that every process that moves along a non-flat shape or surface is essentially a process on a manifold. An example that comes up in bionanoscience is particles that move along cell membranes. To understand their behaviour, the geometric setting is essential and must be taken into account. Next, state spaces of physical systems can often be most naturally expressed as manifolds. See for instance Epstein [52] for an introduction to the use of manifold theory in continuum mechanics. In the theory of general relativity even space-time itself is modelled as a manifold. More practically, manifolds also find wide application in imaging (Pless and Souvenir [126]) and the creation of computer textures. Another example comes from data analysis. In high-dimensional data sets the dependence between variables can create the effect that all data points are actually on a submanifold of the whole space, see for instance Singer [138]. We will mention more references from the context of data analysis in Chapter 3.

In all of these contexts, when there is noise on anything that takes values in a manifold, we deal with a stochastic process or model on a manifold. When noise or disturbances emerge locally, we would like to know how this affects the global behaviour. For instance in material science there are so-called dislocations, local disturbances of patterns of atoms in a material. These dislocations move randomly and one would like to know the effect on the material as a whole. This kind of probabilistic question of how randomness on a microscopic scale affects the evolution on a macroscale has similarities with results like hydrodynamic limits.

Interplay between randomness and geometry
Besides the point of view of applications, there are mathematical reasons to study stochastic processes in general and particle systems in particular on manifolds. This

works two ways.

First, as we mentioned at the start of this section, removing structure of the underlying space helps to get to the essence of a proof. In the usual flat space, the lattice structure of the grids on which particle systems are defined is helpful in the proofs. They are homogeneous, translation invariant and symmetric. We will comment more in Section 1.6.4 on the difficulties that arise when those properties are not present. Studying these models in a context without such nice structure, helps us understand to what extent that structure is really essential for the results or merely helpful but not necessary.

Second, studying stochastic processes on manifolds may help us understand more about the manifolds themselves. For instance Brownian motion is completely determined by the Laplace-Beltrami operator (see Chapter 2 for the relation between Markov processes and their generators), which in turn contains a lot of information about the geometry of the manifold. An example of this is the question whether one can 'hear the shape of a drum' (Kac [94]). It turns out that knowing the eigenvalues of the Laplace operator gives information about the shape of the domain. Therefore the behaviour of Brownian motion might contain information about the manifold, for instance about its curvature. For another example see for instance Wang [156] where connections between diffusion semigroups and geometric properties of manifolds are studied. This interplay with geometry is also present for random fields. See for instance Adler and Taylor [2] for a detailed account of the connections between random fields and geometry. Because of these relations, it is reasonable to expect that also the properties of interacting particle systems contain information on the underlying manifold. If these connections are known, the particle systems could be used to explore manifolds. For instance, a direct consequence of the hydrodynamic limit results is that SEP can be used to approximate solutions to the heat equation on a manifold.

Additionally, studying particle systems can help the study of other stochastic processes on manifolds. We prove for instance the existence of a generalised Ornstein-Uhlenbeck process on a compact Riemannian manifold. It is normally not straightforward to prove such a result, but in Chapter 5 we obtain the generalised Ornstein-Uhlenbeck process as the limit of the fluctuation fields of SEP.

1.6.3 Available results

Before we describe our goals on a manifold, we give a brief overview of more or less related results that have already been obtained, either before the start of this project or during it.

i) Stochastic processes. Starting from general results, we point out that there is a well-established theory of stochastic processes on manifolds. This includes martingales and, in particular, Brownian motion. See for instance Hsu [86] or Émery [49] in general and Hsu [87] for Brownian motion specifically. In Nursul-

tanov et al. [122] the mean arrival times of a Brownian particle are studied. In Lessa [106] Brownian motion is defined on a random manifold.

ii) Random walks. Then there are results about the manifold analogue of a random walk: the geodesic random walk (more about this in Section 2.2.2). In Jørgensen [93] and Blum [17] it is shown that a rescaled version of this random walk converges to Brownian motion. More recently results have been obtained about the large deviations of the geodesic random walk, see Kraaij et al. [100] and Versendaal [154].

iii) Particles. Also systems of particles have been studied. Concentration inequalities and large deviations for Coulomb gasses on compact manifolds are obtained in García-Zelada [70], García-Zelada [71] and Berman [14]. Also Yavari and Marsden [160] study particles with pairwise interaction on a manifold. All of these models were studied on the manifold itself, not on a discretisation. We will come back to this shortly. Also note that not all of these models have a time component.

iv) Random fields. Random fields have been studied on general geometric structures including manifolds, for an overview see for instance Adler and Taylor [2]. Here a random field on a manifold is a collection of random variables taking values in $\mathbb{R}$, say, that is indexed by the manifold. However, in this thesis we will consider random fields on discretisations of a manifold. We do study the convergence to their continuum limit, but the limiting field (the GFF) is so rough that it is distribution-valued, so it is also different from random fields in the function-valued sense.

v) GFF. In Chen and Jakobson [28] a construction of the GFF on even-dimensional manifolds is discussed. Further, in David et al. [35], the GFF is studied on the two-dimensional sphere to construct Liouville Quantum Gravity on the sphere. There are also already results about the approximation of the GFF on a manifold (in two dimensions). For instance in Rivera [130] the GFF on a surface is approximated by random sums of eigenfunctions of the Laplacian and in Schramm and Sheffield [135] the GFF is discretised by projecting it on triangulations. Both are different from the approach that we take in this thesis, where we define the usual (zero-average) DGFF on grids that approximate the manifold.

1.6.4 New challenges

To study IPS and the DGFF on a manifold, there are some challenges that do not arise in a flat space.

Grids on a manifold

We have seen that it is important for the type of IPS that we study that they are defined on a discrete space. Indeed, we consider models where particles jump between

fixed locations. One of the reasons for this is to be able to define interaction. For instance for the exclusion process, the interaction is provided by the restriction of at most one particle in each position. Such interaction would not make sense in a continuum, since particles would never be in the same location. Of course, other types of interaction can be defined in a continuum, but these lead to other technical problems. Therefore it is most common in our field to define particle systems on grids: discrete approximations of the underlying space.

In a flat space it is easy to come up with a discretisation: just a lattice (like $\mathbb{Z}^d$) will do. Moreover, such lattice can approximate the continuum by simply rescaling it (so for instance by considering $\frac{1}{N}\mathbb{Z}^d$ and letting N go to infinity). On a manifold this is not so easy. One cannot simply impose an equally spaced, regular lattice on a manifold. And even then it cannot be 'rescaled', multiplying a manifold position with a constant does not mean anything. Therefore such grids must in the limit get finer in a different way. We will do this by adding points (instead of rescaling the points that are already there), for instance by sampling points uniformly from the manifold.

Jumping probabilities

The next problem is how to define the particle jumps on a manifold. In a lattice one can simply say that particles jump to one of their neighbours with equal probability. In a grid on a manifold, it is not clear what a neighbour means. And even then the neighbours will not be positioned symmetrically around them, so it is not clear with what probability the particles should jump to which points. We will see that a useful requirement is that the graph Laplacians corresponding to the random walks converge to the Laplace-Beltrami operator (the Laplace operator on a manifold). We will show that such graph Laplacians can be constructed by defining jumping probabilities depending on the distance between points.

An overview of different ways in which manifolds have been discretised will be given in Chapter 3. There we also describe which properties the grids should have to study the hydrodynamic limit and equilibrium fluctuations of SEP on it. It will turn out that these properties also make it possible to define the DGFF on such grids and prove convergence to the GFF.

Lack of translation invariance

The fact that a grid on a manifold will look different everywhere (as opposed to a lattice that looks the same around each point) has another consequence, namely the lack of translation invariance. A lot of proofs and results involving interacting particle systems depend in some way or another on translation invariance. For instance, they use that expressions are equal around each lattice point or that lattices or particle configurations can be shifted. All of these actions are not possible on a manifold.

It is not only the lack of translation invariance but even of translation itself. For instance in the case of random conductances on $\mathbb{Z}^d$ (which make that the environment is not translation invariant), one can still translate the whole environment and use

its ergodicity properties under translation. In the case of a manifold, this is also not possible.

Non-metrisable path space
A final challenge that we mention here comes up when we study equilibrium fluctuations on a manifold. We will regard the fluctuation process as a random trajectory of distributions acting on the smooth functions on a manifold. The space of these trajectories is a Skorokhod space. Equipped with the Skorokhod topology, this is a topological space that is not metrisable, so we have to find a smart way to deal with it.

1.7 Goal and overview of the thesis

The goal of this thesis is to make the first steps in studying interacting particle systems and discrete random fields like the DGFF on manifolds. We have seen that an essential ingredient to do this is a good discretisation of the manifold. Therefore we proceed as follows.

In Chapter 2 we introduce some mathematical tools and concepts that are used later in the thesis.

In Chapter 3 we start with the construction of a class of grids with conductances on a compact Riemannian manifold and we show that such grid can be obtained by sampling points uniformly from the manifold.

Then in Chapter 4, we use this grid to define the Symmetric Exclusion Process on a compact Riemannian manifold and we prove that the hydrodynamic limit satisfies the heat equation on the manifold.

These two chapters are based on

[149] B. van Ginkel and F. Redig. Hydrodynamic limit of the Symmetric Exclusion Process on a compact Riemannian manifold. *Journal of Statistical Physics*, 178 (1):75–116, 2020

Following the derivation of the hydrodynamic limit, we study the equilibrium fluctuations of the Symmetric Exclusion Process in Chapter 5. We prove that the fluctuation fields converge to a generalised Ornstein-Uhlenbeck process.
This chapter is based on

[150] B. van Ginkel and F. Redig. Equilibrium fluctuations for the Symmetric Exclusion Process on a compact Riemannian manifold. *arXiv preprint arXiv:2003.02111*, 2020

In Chapter 6 we use the grids from Chapter 3 again to define the (zero-average) Discrete Gaussian Free Field on a compact Riemannian manifold. We prove that it

converges to the continuum Gaussian Free Field.
This chapter is based on

[32] A. Cipriani and B. van Ginkel. The discrete Gaussian free field on a compact manifold. *Stochastic Processes and their Applications*, 130(7):3943–3966, 2020

The final chapter of the thesis stands more on its own. In Chapter 7 we study a class of active particles (or persistent random walks). Their limiting diffusion coefficient and large deviations rate functions are derived. Then we investigate the role of reversibility of the underlying state process and prove that the diffusion coefficient and free energy functions are maximal for reversible state processes.
This chapter is based on

[152] B. van Ginkel, B. van Gisbergen, and F. Redig. Run-and-tumble motion: the role of reversibility. *Journal of Statistical Physics*, 183(3):1–31, 2021

Chapter 2

Mathematical background

In this chapter we introduce a range of mathematical concepts and tools that are used in this thesis. The goal is to give the reader an idea of their definitions and properties. The reader who is familiar with any of these topics can skip them. For details and proofs, we will point to some main references throughout the text.

2.1 Probabilistic tools

2.1.1 Markov processes, generators and semigroups

We first introduce Markov processes and their associated semigroup and generator. These concepts are critical for all of this thesis, since almost all of the processes that we encounter are Markov processes. Some good references include Blumenthal and Getoor [18], Ethier and Kurtz [55] and Liggett [108]. Here we will just sketch some basic elements of the theory.

Markov processes

The main property that makes a stochastic process a Markov process (the so-called Markov property) is that the distribution of the future of the process given the past only depends on the current state. Indeed, let $X = (X_t, t \geq 0)$ be a stochastic process taking values in a state space $\mathscr{S}$ and let $\mathscr{F}_t = \sigma(X_s, 0 \leq s \leq t)$ be the corresponding natural filtration ($\mathscr{F}_t$ can be viewed as all the information on the process X until time t). Then we call X a Markov process if for every bounded and measurable function f and for all $s < t$,

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = \mathbb{E}[f(X_t)|X_s].$$

This means that the distribution at time t conditioned on all the information we have at time s ($\mathscr{F}_s$) only depends on the state at time s itself (X_s). We will restrict to the

time-homogeneous case, so that the evolution of the process also does not depend on the current time. This implies additionally that

$$\mathbb{E}\left[f(X_t)|X_s\right] = \mathbb{E}_{X_s}\left[f(X_{t-s})\right],$$

where $\mathbb{E}_x[f(X_t)] = \mathbb{E}[f(X_t)|X_0 = x]$ denotes the expectation of the process started from $X_0 = x$. As examples, we mention two important classes of Markov processes.

Example 2.1 (Jump processes)**.** A jump process is a stochastic process on a possibly infinite state space $\mathscr{S}$ with the following dynamics. At each point $x \in \mathscr{S}$, the process waits for an exponential time with rate $\gamma_x > 0$ and then independently jumps to a point $y \in \mathscr{S}$ with probability $p(x, y)$. An alternative formulation is that the process jumps from point x to y with rate $\gamma_x p(x, y)$. Now the Markov property comes from the fact that γ_x and $p(x, \cdot)$ only depend on x (and hence not on the past of the process) and that the exponential distribution is memoryless. In fact, the exponential distribution is the only waiting time distribution that makes a jump process a Markov process. In Section 2.1.3 we will see some properties of jump processes in the case where $\mathscr{S}$ is a weighted graph.
A special case that we will analyse further in this chapter is the Simple Symmetric Random Walk (SSRW) on $\mathbb{Z}$. At each site, the random walk waits an exponential time with rate 1 and then jumps to one of its neighbours with equal probability. Therefore for the SSRW, $\gamma_x = 1$ for all x and $p(x, y) = \frac{1}{2}$ for $y \in \{x-1, x+1\}$ and 0 otherwise.

Example 2.2 (Diffusion processes)**.** A second class of processes is the diffusion processes: stochastic processes with almost surely continuous sample paths and independent and stationary increments. These properties give the processes the Markov property. The canonical example in this case is Brownian motion on $\mathbb{R}$. This is a continuous stochastic process B_t with increments $B_t - B_s \sim N(0, t-s)$ for $t > s$ such that increments over non-overlapping time intervals are independent. We will see later that it can be obtained as the limit of a Simple Symmetric Random Walk.

Semigroup
Now we can associate some mathematical objects to a Markov process X that are very useful to study its behaviour. It will turn out that the Markov property gives these objects a nice structure. First we introduce the operator S_t. It works on a suitable normed space of functions $(C, \|\cdot\|)$ as follows. For $f \in C$

$$S_t f(x) = \mathbb{E}_x[f(X_t)].$$

Recall that $\mathbb{E}_x$ denotes the expectation of X_t started from $X_0 = x$. The classes of functions C that are usually considered include the continuous functions, the bounded continuous functions, the continuous functions vanishing at infinity or the functions that are square-integrable with respect to a stationary measure (see Section 2.1.2 for more on stationary measures). The operator S_t is like a flow, it describes how the expectation of a function applied to the process evolves through time. If we know $S_t f(x)$ for a large enough class of functions f, we obtain information on the

distribution of X_t starting from $X_0 = x$. Now note that $S_0 f(x) = f(x)$ trivially, so $S_0 = I$. Also the Markov property, together with time-homogeneity, implies that

$$\begin{aligned} S_{s+t}f(x) &= \mathbb{E}_x[f(X_{s+t})] = \mathbb{E}_x[\mathbb{E}[f(X_{s+t})]|X_s] = \mathbb{E}_x[\mathbb{E}_{X_s}[f(X_t)]] \\ &= \mathbb{E}_x[S_t f(X_s)] = S_s(S_t f)(x). \end{aligned}$$

This shows that $S_{s+t} = S_s S_t$. A trajectory $t \mapsto S_t$ in the bounded linear operators on C with these properties is called a semigroup. A semigroup corresponding to a Markov process has additional properties, it satisfies the following for all $f \in C$ and $s, t \geq 0$:

i) Identity at 0: $S_0 = I$.

ii) Semigroup property: $S_t S_s = S_{s+t} = S_s S_t$.

iii) Positivity: if $f \geq 0$, then $S_t f \geq 0$.

iii) Contractivity: $\|S_t f\| \leq \|f\|$.

If additionally for all $f \in C$, $t \mapsto S_t f$ is right-continuous at 0 in the norm of the function space C, then we call $t \to S_t$ a strongly continuous semigroup. There is a well-established functional analytic theory about strongly continuous semigroups, see for instance Engel and Nagel [50]. For more details on semigroups associated to Markov processes, see Liggett [109].

We have seen so far that we can associate a semigroup to a Markov process. In fact, we can also go the other way: we can construct the finite dimensional distributions of the process from the semigroup by using the Markov property. This property is very useful in probability theory, since it allows us to obtain information on Markov processes by studying the corresponding semigroups.

Generator

An important property of a strongly continuous semigroup is that it is determined completely by its 'derivative at time 0'. This is called the generator $\mathscr{L}$. It is an operator that is defined on the set

$$D(\mathscr{L}) = \left\{ f \in C : \lim_{t \downarrow 0} \frac{S_t f - f}{t} \text{ exists in } C \right\}, \tag{2.1}$$

which is called the domain of $\mathscr{L}$. Here the limit is taken with respect to the norm of C. If the semigroup is associated to a Markov process, $D(\mathscr{L})$ is dense in C. $\mathscr{L}$ acts on functions $f \in D(\mathscr{L})$ as follows

$$\mathscr{L} f = \lim_{t \downarrow 0} \frac{S_t f - f}{t}.$$

The generator encodes the instantaneous rate of change of the semigroup and hence of the underlying Markov process.

In general the domain $D(\mathscr{L})$ is a complicated set that is not easy to determine. To deal with this, we usually define a generator first on a smaller set which we call a core. A much used example of a core is the set of smooth functions. Then we extend the definition by taking the closure of the generator defined on the core with respect to an appropriate norm. Of course one needs to prove that the resulting operator is indeed the generator of a semigroup, we will mention an important result on that shortly. Also for other results (like the convergence result that we mention later) it is often sufficient to reduce the computations and analysis to a core.

Relation with the semigroup
For $f \in D(\mathscr{L})$ the generator and semigroup commute, i.e. $\mathscr{L} S_t f = S_t \mathscr{L} f$. Moveover, for such f the trajectory $t \mapsto S_t f$ satisfies

$$\partial_t S_t f = \mathscr{L} S_t f, \qquad S_0 f = f.$$

This is a differential equation with formal solution

$$S_t f = \mathrm{e}^{\mathscr{L} t} f. \tag{2.2}$$

To interpret (2.2), one needs to define the exponential function applied to operators in an appropriate way. For Markov processes with a finite state space, $\mathscr{L}$ is a matrix and the exponential function is well-defined (for instance as a power series). However, in general $\mathscr{L}$ is an unbounded operator and it is not straightforward how to give meaning to (2.2). This is solved by the Hille-Yosida theorem, which characterises operators that generate a strongly continuous semigroup and provides a construction of the semigroup.

Example 2.3 (SSRW continued)**.** Returning to the example of the SSRW on $\mathbb{Z}$ as described in Example 2.1, we want to compute $\mathscr{L}$. Note that to obtain this generator we need to define an appropriate underlying function space, take limits with respect to the corresponding norm and determine the domain of the generator. This is more than just a formal task and it is not always clear which function space and norm are suitable. However, since the goal of this example is to get an idea of the expression of the generator, we will avoid these details and just perform a pointwise calculation. Indeed, let f be a bounded, compactly supported function on $\mathbb{Z}$ and let N_t be the amount of jumps of the particle up to time t. Then N_t has a Poisson distribution with parameter t. We compute:

$$\begin{aligned}
\mathbb{E}_x[f(X_t)] &= \mathbb{E}_x[f(X_t)|N_t = 0]\mathbb{P}(N_t = 0) + \mathbb{E}_x[f(X_t)|N_t = 1]\mathbb{P}(N_t = 1) \\
&+ \mathbb{E}_x[f(X_t)|N_t \geq 2]\mathbb{P}(N_t \geq 2) \\
&= f(x)\mathrm{e}^{-t} + \frac{1}{2}(f(x-1) + f(x+1))t\mathrm{e}^{-t} + \mathbb{E}_x[f(X_t)|N_t \geq 2]O(t^2) \\
&= f(x)(1 - t + O(t^2)) + \frac{1}{2}(f(x-1) + f(x+1))(t + O(t^2)) + O(t^2) \\
&= f(x) + \frac{t}{2}(f(x+1) + f(x-1) - 2f(x)) + O(t^2).
\end{aligned}$$

Now we see that

$$\mathscr{L}f(x) = \lim_{t\downarrow 0} \frac{1}{t}\left(\mathbb{E}_x[f(X_t)] - f(x)\right) = \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1) - f(x). \tag{2.3}$$

We can interpret (2.3) as follows. At each point in time the particle jumps with rate $\frac{1}{2}$ to the left (to $x-1$) and with rate $\frac{1}{2}$ to the right (to $x+1$). It also jumps away from x with rate 1. We will see this more generally in Section 2.1.3. It shows that the generator $\mathscr{L}$ indeed encodes the instantaneous rate of change of the stochastic process.

Example 2.4 (Brownian motion)**.** We give a heuristic pointwise computation to obtain the expression for the generator of Brownian motion. Using a Taylor expansion around $x \in \mathbb{R}$, we see

$$\begin{aligned}\mathbb{E}_x[f(B_t)] &= \mathbb{E}_x\Big[f(x) + (B_t - x)f'(x) + \frac{1}{2}(B_t-x)^2 f''(x)\\ &+ \frac{1}{6}(B_t-x)^3 f^{(3)}(x) + O((B_t-x)^4)\Big]\\ &= f(x) + 0 + \frac{t}{2}f''(x) + 0 + O(t^2),\end{aligned}$$

so

$$\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}_x[f(B_t)] - f(x)\right) = \frac{1}{2}f''(x).$$

We see that in $\mathbb{R}$, the generator of Brownian motion is $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{2}\Delta$, a half times the Laplace operator. In $\mathbb{R}^n$ a similar calculation can be made to obtain $\frac{1}{2}\Delta$ as the generator of Brownian motion. In fact, we will see that also Brownian motion on a manifold has the manifold analogue of the Laplace operator as its generator.

Generator convergence

For us the semigroup and generator are important for the following reason. From a time-homogeneous Markov process X, we can calculate its semigroup and the corresponding generator. Conversely, the semigroup corresponding to X determines the distribution of X and the generator determines the semigroup. Therefore X, $(S_t, t \geq 0)$ and $\mathscr{L}$ are all in a one-to-one relationship with each other. For this reason we can call $\mathscr{L}$ the generator of X and X the process generated by $\mathscr{L}$.

This one-to-one correspondence stays valid when taking limits. Indeed, we can use the convergence of semigroups or generators to prove convergence of Markov processes, see for instance the Trotter-Kurtz theorem (Kurtz [102]). Of course a lot of technical details play a role, but we focus here on the main ideas.

Example 2.5 (SSRW continued)**.** We can see the use of this again in the SSRW on $\mathbb{Z}$. If we rescale space by a factor $\frac{1}{N}$ and time by a factor N^2, we obtain the generator

$$\mathscr{L}^N f(x) = \frac{N^2}{2}\left(f\left(x+\tfrac{1}{N}\right) + f\left(x-\tfrac{1}{N}\right) - 2f(x)\right).$$

Now if we let N go to infinity, we obtain

$$\lim_{N\to\infty} \mathscr{L}^N f(x) = \lim_{N\to\infty} \frac{N^2}{2}\left(f\left(x+\tfrac{1}{N}\right)+f\left(x-\tfrac{1}{N}\right)-2f(x)\right) = \frac{1}{2}f''(x).$$

Recall from Example 2.4 that $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is the generator of Brownian motion. We conclude that the generator of the SSRW converges pointwise to the generator of Brownian motion. If we strengthen this convergence to convergence in the norm of the underlying function space, the Trotter-Kurtz theorem allows us to conclude that the corresponding semigroups converge to the semigroup of Brownian motion. This then implies that the finite-dimensional distributions of the SSRW converge to those of Brownian motion.

2.1.2 Invariant measures and reversibility

We will describe some properties that Markov processes and associated probability measures can satisfy. As we will see, the semigroup and generator machinery provides us with elegant ways to write down these properties. For more details we refer to the books on Markov processes that were mentioned before or, for a shorter overview, to Redig [129].

Stationary measure

We can start a Markov process from a deterministic point x (as in the definition of the semigroup), but it can also start from a probability measure μ. This means that at time 0 we pick a point from the state space according to μ and then start the process from that point. Now some of these measures μ have a special property: when we start the process from $X_0 \sim \mu$ (so the distribution of X_0 is μ), at every later point t in time still $X_t \sim \mu$. Such measure is called invariant or stationary for the process X. In semigroup language this means that for every function $f \in C$,

$$\int S_t f \mathrm{d}\mu = \int f \mathrm{d}\mu.$$

This implies that for all $f \in D(\mathscr{L})$,

$$\int \mathscr{L} f \mathrm{d}\mu = 0.$$

In fact this is also a sufficient condition for stationarity. When we start the process from an invariant measure, we say that the process is stationary. There can be multiple invariant measures associated to the same Markov process. In fact, any convex combination of invariant measures is again invariant, which implies that the space of invariant measures is convex. The extreme points of this set (so the invariant measures that cannot be obtained as a convex combination of other invariant measures) have a special property that we will get to now.

Ergodic measure

To introduce ergodic measures we first need to define invariant sets. A measurable subset A of the state space $\mathscr{S}$ is called invariant if for all $t > 0$,

$$S_t \mathbf{1}_A = \mathbf{1}_A.$$

Here $\mathbf{1}_A$ is the indicator function of the set A. The idea is that the state space of the Markov process is split into two or more parts in such a way that if the process starts in a part, it always stays in that part. These parts are the invariant sets. Now an ergodic measure is a stationary measure of which the support is contained in exactly one invariant set. To make this precise, we say that a measure μ is ergodic if every invariant set has μ-measure either 0 or 1. Alternatively, we can define a function f to be invariant for the Markov process if

$$S_t f = f$$

for all t and call a stationary measure μ ergodic if all invariant functions are μ-a.s. constant.

We already mentioned that the stationary measures form a convex set. It turns out that the ergodic measures are exactly the extreme points of this convex set, i.e. the stationary measures that cannot be obtained as a convex combination of other stationary measures.

Example 2.6. As an example consider the state space $\mathscr{S} = \{a, b, c, d\}$ and a process X that either jumps with contant rate γ between a and b or between c and d, depending on where it started (see Figure 2.1). Now note that the Dirac measure $\mu_1 = \delta_a$ is not stationary. Indeed, if we start from this measure, for every $t > 0$ there is a positive probabiliy of being in state b, so X_t does not have distribution δ_a. The uniform measure μ_2 is stationary. Indeed it is not hard to see that if you start in each state with probability $\frac{1}{4}$, then at every later point in time the probability to be in each state is $\frac{1}{4}$. However, μ_2 is not ergodic, since $\mathscr{S}_1 = \{a, b\}$ and $\mathscr{S}_2 = \{c, d\}$ are both invariant sets and μ_2 has support in both of them. The uniform measure $\mu_3 = \frac{1}{2}(\delta_a + \delta_b)$ on $\mathscr{S}_1$ is ergodic, it is both invariant and has support in exactly one invariant set. In fact, $\mu_4 = \frac{1}{2}(\delta_c + \delta_d)$ is the only other ergodic measure and the convex combinations $\mu = \lambda\mu_3 + (1 - \lambda)\mu_4$ for $\lambda \in [0, 1]$ form all the invariant measures.

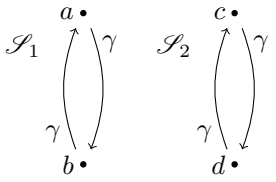


Figure 2.1: The situation of Example 2.6.

Reversibility
Another possible property of Markov processes that we will use in this thesis is reversibility with respect to an invariant measure μ. The idea is that the process started from μ at time 0 and run for time t has the same distribution as the process started from μ at time t that is played backward till time 0. More precisely, we call a measure μ reversible if for all f, g

$$\int (S_t f) g \mathrm{d}\mu = \int f (S_t g) \mathrm{d}\mu. \tag{2.4}$$

When a process has a reversible measure μ, we call the process reversible or symmetric with respect to μ. Sometimes we will simply say that the process is reversible. If we set g in (2.4) to be the constant function $\mathbf{1}$ and note that always $S_t\mathbf{1} = \mathbf{1}$, then (2.4) directly implies that $\int S_t f \mathrm{d}\mu = \int f \mathrm{d}\mu$. In other words: every reversible measure is invariant.

It follows from (2.4) that for $f, g \in D(\mathscr{L})$

$$\int (\mathscr{L} f) g \mathrm{d}\mu = \int f (\mathscr{L} g) \mathrm{d}\mu,$$

so $\mathscr{L}$ is a self-adjoint operator on $L^2(\mu)$. This is also a sufficient condition for reversibility.

Example 2.7 (RW on torus)**.** As an example, consider a random walk on the torus $\frac{1}{N}G_N = \{\frac{1}{N}, \frac{2}{N}, .., 1\}$ that jumps to neighbouring points (and where 0 and 1 are identified, so a particle can jump between 1 and $\frac{1}{N}$). Let p be the probability to jump to the right and $(1-p)$ the probability to jump to the left. Whatever the value of p is, the uniform distribution μ on G_N is invariant and ergodic. However, the reversibility depends on p. If $p > \frac{1}{2}$, we will observe a drift to the right. In that case if we were to let time run backwards, we would see a drift to the left and recognise that time runs backward. Therefore μ is not reversible in that case. Similarly, μ is not reversible for $p < \frac{1}{2}$. Only for the value $p = \frac{1}{2}$, the process is symmetric and we would not see the difference between moving forward or backward through time. In that case the process is reversible with respect to μ.

Example 2.8 (Ornstein-Uhlenbeck process)**.** As a final example for Section 2.1.2, we discuss the Ornstein-Uhlenbeck process. This process can be viewed as Brownian motion that is attached to the origin with a spring. It is the solution to the following Stochastic Differential Equation:

$$\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \sigma \mathrm{d}W_t.$$

Here $\theta > 0$ is the spring constant, W_t is a standard Brownian motion and σ determines the variance of the Brownian motion and hence the strength of the noise. By first solving the equation for $Y_t = \mathrm{e}^{-\theta t} X_t$ and then rewriting, we obtain the explicit solution

$$X_t = \mathrm{e}^{-\theta t} X_0 + \sigma \int_0^t \mathrm{e}^{\theta(s-t)} \mathrm{d}W_s. \tag{2.5}$$

Since the second term on the RHS of (2.5) is a deterministic function integrated with respect to Brownian motion, it is Gaussian. Denoting X_t started from $x \in \mathbb{R}$ by X_t^x, we can write

$$X_t^x = \mathrm{e}^{-\theta t}x + \sigma \mathscr{N}\left(0, \tfrac{1-\exp(-2\theta t)}{2\theta}\right), \tag{2.6}$$

where the equality is in distribution. From this we can derive the following formula for the semigroup corresponding to the Ornstein-Uhlenbeck process:

$$S_t f(x) = \int_{-\infty}^{\infty} f\left(\mathrm{e}^{-\theta t}x + \sigma y\sqrt{\frac{1-\exp(-2\theta t)}{2\theta}}\right)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-y^2}{2}}\mathrm{d}y.$$

The corresponding generator $\mathscr{L}$ acts on smooth functions f as follows:

$$\mathscr{L}f(x) = -\theta x\frac{\mathrm{d}f}{\mathrm{d}x}(x) + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x). \tag{2.7}$$

Letting t go to infinity in (2.6), the initial value x vanishes and we are left with a Gaussian distribution with mean 0 and variance $\frac{\sigma^2}{2\theta}$. We will call this distribution

$$\mu := \mathscr{N}\left(0, \tfrac{\sigma^2}{2\theta}\right).$$

μ is the unique stationary measure of the Ornstein-Uhlenbeck process. Indeed when we set $X_0 \sim \mu$, a computation shows that for each $t > 0$ also $X_t \sim \mu$. When started from μ, the Ornstein-Uhlenbeck process is a Gaussian process. This implies that it is uniquely determined by its covariance structure

$$\mathbb{E}[X_t] = 0, \qquad \mathrm{Cov}(X_s, X_t) = \frac{\sigma^2}{2\theta}\mathrm{e}^{-\theta|t-s|}.$$

μ is also reversible and ergodic (for more about this see Section 7.3.2). Accordingly, the generator (2.7) can be extended to a self-adjoint operator on $L^2(\mu)$. We conclude from the above that the Ornstein-Uhlenbeck process started from μ is a stationary, time-homogeneous and Gaussian Markov process. In fact, it is the only stochastic process with these properties.

We can also consider the Ornstein-Uhlenbeck process in $\mathbb{R}^d$, where the positive constant θ is replaced by a positive definite matrix. In that case similar methods can be used to reach similar results. A difference that will be important in Chapter 7 is that in the multidimensional case, the stationary distribution is not necessarily reversible. In Example 7.12 we consider a one-parameter family of Ornstein-Uhlenbeck processes with the same invariant measure, but only for one trivial choice of the parameter the process is reversible.

Finally, there are infinite dimensional versions of the Ornstein-Uhlenbeck process, where the constant θ is formally replaced by a positive operator. We will encounter such process in Chapter 5. It is so rough that it takes values in the space of generalised functions or distributions and cannot be treated in the same way as the finite-dimensional versions. It will be defined through a martingale problem instead.

2.1.3 Random walks on graphs

We will now consider a context that we use a lot in this thesis and that is in some sense a generalisation of the SSRW example of Section 2.1.1, namely random walks on graphs.

Weighted graph
Let V be a finite edge-weighted graph on n vertices. With slight abuse of notation we also denote by V the set of vertices of the graph. We denote the edge weight between $v, w \in V$ by $c_{vw} = c_{wv} \geq 0$. These edge weights are also called conductances. The reason for this is that we can interpret such a graph as an electric network with connections between nodes and where the edge weights have the physical meaning of conductance (i.e. the reciprocal of resistance) between the nodes. For a very nice introduction to this point of view and a lot of useful connections, see Gaudillière [72]. We assume that between any two nodes there exists a path of edges with positive conductance, so the graph is connected.

Random walk
Using these conductances, we can define a random walk on the graph as follows. The walk starts from a (possibly random) node in V. Then it jumps to other points with the rate of the conductance between them. More precisely, set for $v \in V$,

$$d_v = \sum_w c_{vw},$$

the total rate with which the random walk jumps away from v (note that $d_v < \infty$ for all v since V is finite). When the random walk arrives at v, it waits an exponential time with rate d_v, then chooses a neighbour w with probability $\frac{c_{vw}}{d_v}$ and jumps to that neighbour. This defines a random walk $X = (X_t, t \geq 0)$ on V.

Graph Laplacian
As in the examples in Section 2.1.1, we can determine the generator $\mathscr{L}$ of this random walk. Using a computation that is similar to the computation for the SSRW case, we obtain that $\mathscr{L}$ acts on functions f as follows:

$$\mathscr{L}f(v) = \sum_w c_{vw}(f(w) - f(v)).$$

This operator is called the graph Laplacian. Since we consider a finite graph, $\mathscr{L}$ can be represented as a matrix and its domain is all of $\mathbb{R}^n$. Because of the symmetry requirement that $c_{vw} = c_{wv}$ for all $v, w \in V$, $\mathscr{L}$ is a symmetric matrix. It also follows from this requirement that the uniform measure is reversible and hence invariant for X. Indeed, in the context of countable state spaces a measure μ is reversible if and only if for all sites $v, w \in V$,

$$\mu_v c_{vw} = \mu_w c_{wv}.$$

This is called the detailed balance condition. In the present case with symmetric conductances $c_{vw} = c_{wv}$, we see directly from this detailed balance condition that the unique reversible measure is the uniform measure.

Since $\mathscr{L}$ is symmetric, it has real eigenvalues and there exists an orthogonal eigenbasis for $\mathbb{R}^n$. Moreover, it can be shown that $-\mathscr{L}$ is positive semidefinite, so the eigenvalues of $\mathscr{L}$ are non-positive. There is at least one eigenvalue 0 (since $\mathscr{L}\mathbf{1} = 0$) and because of the connectedness of V this is the only eigenvalue 0. Therefore we can write the eigenvalues of $\mathscr{L}$ as

$$0 = \lambda_1 > \lambda_2 \geq .. \geq \lambda_n$$

with corresponding orthogonal eigenvectors $\phi_1 = \mathbf{1}, \phi_2, .., \phi_n$.

Green's function

We will now introduce the Green's function on V and collect some of its properties. For more details on Discrete Green's functions, see Chung and Graham [31].

Fix a proper, non-empty subset V_0 of V. Now define for $v, w \in V$ the Green's function (matrix) as

$$G(v, w) = \mathbb{E}_v \left[\int_0^\tau \mathbf{1}_w(X_t) \mathrm{d}t \right], \tag{2.8}$$

where τ is the first time X enters V_0. In other words, $G(v, w)$ is the total time that X started from v spends in w before entering V_0. Since our graph is finite, the Green's function would be infinite if the domain of integration in (2.8) was all of $(0, \infty)$. Therefore the set V_0 can be interpreted as the boundary set where the random walk is killed, the walk stops when it enters that set. τ is the killing time that makes (2.8) finite. A further analysis shows that G is the inverse of $-\mathscr{L}$ when restricted to $V \setminus V_0$.

The reason to study the Green's function is that it is the covariance function of the Discrete Gaussian Free Field on V, which we will consider in Chapter 6. On the grids in Chapter 6, there is no natural choice for V_0. Therefore, we will define the Green's function as the inverse of $-\mathscr{L}$ on the space of zero-average functions, i.e. the functions f with $\sum_v f(v) = 0$. Indeed, since the only eigenvalue 0 of $\mathscr{L}$ is on the constant functions and the eigenspaces are orthogonal, $\mathscr{L}$ is invertible on the orthogonal complement of the constant function, which is the set of zero-average functions. We set the Green's function to be 0 on the constant functions. Therefore the Green's function will equal

$$G = -\sum_{i=2}^{n} \frac{1}{\lambda_i} P_i, \tag{2.9}$$

where the λ_i's are the (ordered) eigenvalues of $\mathscr{L}$ and P_i is the projection on the subspaced spanned by ϕ_i, the eigenvector corresponding to λ_i.[1] The advantage of this definition is that we do not need a boundary set V_0. The drawback, however, is that there is no corresponding random walk interpretation as in (2.8).

[1] Note the difference of sign compared to (6.5). This comes from the extra minus in the definition of the graph Laplacian in Chapter 6.

2.1.4 Dynkin martingales

In this thesis we will frequently encounter martingales associated to Markov processes, so-called Dynkin martingales. Also the generalised Ornstein-Uhlenbeck process in Chapter 5 is defined by using such martingales. Before we introduce Dynkin martingales, we will quickly recall the definitions of a martingale and its quadratic variation. For a detailed introduction to martingales see Williams [158] and Rogers and Williams [132].

Filtrations and martingales

The idea of a martingale is that the expected value of the process in the future given all the information up to the current time is the current state of the process. To be more precise, we need to introduce filtrations. A filtration is a collection $\mathscr{F} = (\mathscr{F}_t, t \geq 0)$ of σ-algebras that is increasing in the sense that $\mathscr{F}_t$ contains $\mathscr{F}_s$ whenever $t \geq s$. $\mathscr{F}_t$ should be interpreted as the information up to time t. Indeed, given a stochastic process $X = (X_t, t \geq 0)$, a natural way to build a filtration is to let $\mathscr{F}_t$ be the σ-algebra generated by $\{X_s, 0 \leq s \leq t\}$. This is called the natural filtration generated by X.

Now a martingale is a real-valued stochastic process $M = (M_t, t \geq 0)$ such that,

i) M is integrable: for all $t \geq 0$, $\mathbb{E}[|M_t|] < \infty$.

ii) M is adapted: for all $t \geq 0$, M_t is measurable with respect to $\mathscr{F}_t$.

iii) M has the martingale property: for all $0 \leq s \leq t$, $\mathbb{E}[M_t|\mathscr{F}_s] = M_s$.

The characterising property is the martingale property. It implies that martingales are processes that have no drift. In fact, applying it with $s = 0$ shows that $\mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}[M_t|\mathscr{F}_0]] = \mathbb{E}[M_0]$, so the expected value of M is the same at each time point.

Example 2.9. As an example, we reconsider the Simple Symmetric Random Walk $X = (X_t, t \geq 0)$ on $\mathbb{Z}$ as described in Example 2.1 with the natural filtration $\mathscr{F}_t = \sigma(\{X_s, 0 \leq s \leq t\})$. This way X is adapted by construction. A direct computation shows that X is also integrable. For the martingale property we see

$$\mathbb{E}[X_t|\mathscr{F}_s] = \mathbb{E}[(X_t - X_s)|\mathscr{F}_s] + \mathbb{E}[X_s|\mathscr{F}_s] = \mathbb{E}[(X_t - X_s)] + X_s = X_s.$$

We used here that $(X_t - X_s)$ is independent of $\mathscr{F}_s$ and has mean 0.

Note that in general being Markov or being a martingale are unrelated properties of stochastic processes. Indeed, some processes are neither Markov nor a martingale, some are one but not the other and some are both. For instance standard Brownian motion is both Markov and a martingale, but as soon as we add a drift it is still Markov but no longer a martingale.

Quadratic variation

An important process that is associated to a martingale M (or generally to a stochastic process) is the quadratic variation process. It is defined as

$$\langle M, M\rangle_t = \lim \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2,$$

where the limit is in probability over all partitions $0 = t_0 \leq t_1 \leq .. \leq t_n = t$ of $[0, t]$ with mesh size tending to 0. The quadratic variation process is very useful in stochastic calculus. For martingales the following property is important. If M is a martingale with quadratic variation $\langle M, M\rangle$, then under some regularity conditions the process N defined by

$$N_t = M_t^2 - \langle M, M\rangle_t$$

is also a martingale. In particular $\mathbb{E}[M_t^2] = \mathbb{E}[\langle M, M\rangle_t]$. Because of this, one can show that a martingale (or, more precisely, a sequence of martingales depending on a parameter that goes to infinity) vanishes by showing that its quadratic variation process vanishes. This will be used in Chapter 4.

Dynkin martingale

Although we saw that the Markov property and the martingale property in principle have nothing to do with each other, the following result relates Markov processes to martingales. Roughly, it says that a function applied to a Markov process can be written as an additive functional of that process plus a martingale. To be more precise, let $(X_t, t \geq 0)$ be a Markov process and assume that it generates a strongly continuous semigroup $(S_t, t \geq 0)$ with generator $\mathscr{L}$. Suppose that f is a function such that $f, f^2 \in D(\mathscr{L})$. Then $M = (M_t, t \geq 0)$ defined by

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathscr{L} f(X_s) \mathrm{d}s, \tag{2.10}$$

is a martingale with respect to the natural filtration generated by X. This martingale M is called a Dynkin martingale. Moreover, M has the following quadratic variation:

$$\langle M, M\rangle_t = \int_0^t (\mathscr{L} f^2 - 2f\mathscr{L} f)(X_s) \mathrm{d}s. \tag{2.11}$$

We will now sketch the derivation of the martingale property of M_t. For the proof of (2.11) we refer to for instance Seppäläinen [136, Section 8.1].

Sketch proof of (2.10). Fix a function $f \in D(\mathscr{L})$ and let M_t be as defined in (2.10).

We want to show that for $0 \leq s \leq t, \mathbb{E}[M_t|\mathscr{F}_s] = M_s$. First we compute

$$\begin{aligned}\mathbb{E}[M_t|\mathscr{F}_s] &= \mathbb{E}\left[f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_r)\mathrm{d}r \middle| \mathscr{F}_s\right] \\ &= \mathbb{E}\left[f(X_t) - f(X_s) - \int_s^t \mathscr{L}f(X_r)\mathrm{d}r \middle| \mathscr{F}_s\right] \qquad (2.12) \\ &+ f(X_s) - f(X_0) - \int_0^s \mathscr{L}f(X_r)\mathrm{d}r, \qquad (2.13)\end{aligned}$$

where we wrote $f(X_t) = f(X_t) - f(X_s) + f(X_s)$ and used that functions of X_r for $r \leq s$ are $\mathscr{F}_s$-measurable. Now note that (2.13) equals M_s, so it remains to be shown that (2.12) equals 0. Using the definition of the semigroup and the Markov property, we see that (2.12) equals

$$\begin{aligned}&\mathbb{E}[f(X_t)|X_s] - \mathbb{E}[f(X_s)|X_s] - \mathbb{E}\left[\int_s^t \mathscr{L}f(X_r)\mathrm{d}r \middle| X_s\right] \\ = \; & S_{t-s}f(X_s) - f(X_s) - \int_s^t S_{r-s}\mathscr{L}f(X_s)\mathrm{d}r. \qquad (2.14)\end{aligned}$$

Now recall that for $f \in D(\mathscr{L})$, $S_t\mathscr{L}f = \frac{\mathrm{d}}{\mathrm{d}t}S_t f$. Therefore

$$\int_s^t S_{r-s}\mathscr{L}f\mathrm{d}r = \int_0^{t-s} S_r\mathscr{L}f\mathrm{d}r = \int_0^{t-s} \frac{\mathrm{d}}{\mathrm{d}r}S_r f\mathrm{d}r = S_{t-s}f - f. \qquad (2.15)$$

This implies that (2.14) equals 0 and therefore (2.12) equals 0, which is what we wanted to show. □

In some cases, the quadratic variation formula (2.11) can be computed explicitly. We give two examples, for a jump process and for a diffusion process.

Example 2.10. Let X be the random walk on the weighted graph V from Section 2.1.3 and fix $f : V \to \mathbb{R}$. Then we obtain that

$$f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)\mathrm{d}s$$

is a martingale with quadratic variation $\int_0^t \gamma^f(X_s)\mathrm{d}s$, where

$$\begin{aligned}\gamma^f(v) &:= \mathscr{L}f^2(v) - 2f(v)\mathscr{L}f(v) \\ &= \sum_w c_{vw}(f^2(w) - f^2(v)) - 2f(v)\sum_w c_{vw}(f(w) - f(v)) \\ &= \sum_w c_{vw}(f(w) - f(v))^2.\end{aligned}$$

Example 2.11. Now we consider a Brownian motion B_t in $\mathbb{R}$ and fix a smooth function f. We assume that f is in the domain of the generator, e.g. f and its first and second derivative vanish at infinity. Recall that the generator is given by $\mathscr{L}f = \frac{1}{2}f''$. This gives

$$\mathscr{L}f^2(x) - 2f(x)\mathscr{L}f(x) = f(x)f''(x) + f'(x)^2 - f(x)f''(x) = f'(x)^2. \tag{2.16}$$

Therefore we obtain that

$$M_t := f(B_t) - f(B_0) - \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s$$

is a martingale with quadratic variation

$$\langle M, M\rangle_t = \int_0^t f'(B_s)^2\mathrm{d}s.$$

The reader who is familiar with stochastic calculus may note that by Ito's formula we obtain the explicit expression $M_t = \int_0^t f'(B_s)\mathrm{d}B_s$. Then Ito's isometry

$$\mathbb{E}\left[\left(\int_0^t f'(B_s)\mathrm{d}B_s\right)^2\right] = \mathbb{E}\left[\int_0^t f'(B_s)^2\mathrm{d}s\right]$$

coincides with the identity $\mathbb{E}[M_t^2] = \mathbb{E}[\langle M, M\rangle_t]$ that we mentioned earlier.

2.2 Differential geometry

In this thesis we will consider a variety of models on Riemannian manifolds. In order to understand the coming chapters, some knowledge about differential geometry is required. In this section, we will introduce the main concepts. For details and proofs, we refer the reader to for instance the classic introduction in Spivak [140].

2.2.1 Basic geometric concepts

Smooth manifolds

As we noted in Chapter 1, an n-dimensional manifold (for some $n \in \mathbb{N}$) can loosely be described as a set that locally looks like $\mathbb{R}^n$. This is precisely what the definition intends to capture. Indeed, a n-dimensional manifold M is a metric space such that every point of M has a neighbourhood U that is homeomorphic to an open subset V of $\mathbb{R}^n$. Recall that homeomorphic means that there exists an invertible function $f : U \to V$ such that both f and f^{-1} are continuous.

A smooth manifold is a manifold M along with a set $\mathscr{A}$ that is called an atlas with elements called charts. These charts are coordinate systems on the manifold. More precisely, they are pairs (x, U) of an open subset U of M and a homeomorphism $x : U \to x(U) \subset \mathbb{R}^n$ such that two properties hold:

i) Every point $p \in M$ is contained in at least one chart.

ii) If two charts (x, U) and (y, V) have overlap, then the compositions $x \circ y^{-1}$ and $y \circ x^{-1}$ are smooth (infinitely differentiable).

These charts provide us with coordinates on the manifold. Now we can say that a function $f : M \to \mathbb{R}$ is smooth at a point p if for a chart (x, U) that contains p the composition $f \circ x^{-1}$ is smooth. Property ii) makes sure that this definition does not depend on which chart we use. We define the partial derivatives of f at p as

$$\frac{\partial f}{\partial x^i}(p) = D^i(f \circ x^{-1})(x(p)).$$

Note that this definition depends on the chart (x, U). However if p is also contained in some other chart (y, V) it is easy to transform the expression using the chain rule:

$$\frac{\partial f}{\partial y^j}(p) = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial y^j}. \tag{2.17}$$

Expressions like these appear a lot in differential geometry. Therefore the convention is to leave out the summation symbol whenever the same index appears both above and below in an expression. So if we leave out the summation sign in (2.17), the summation over i is implied by the two appearances of i.

Tangent spaces and vector fields

We can associate to each point $p \in M$ an n-dimensional vector space that we call the tangent space at p. This concept generalises the notion of a tangent line to a smooth curve in $\mathbb{R}^2$ or tangent plane to a smooth surface in $\mathbb{R}^3$. The tangent space T_pM can be viewed as the space of all possible tangent vectors to curves through M at p. The union of all tangent spaces is called the tangent bundle TM. TM inherits a smooth structure from the smooth structure of M. In fact, the tangent bundle is a manifold itself.

Given a tangent vector v at p (so an element of T_pM) and a smooth function $f : M \to \mathbb{R}$, one can compute the directional derivative $D_v f(p)$ of f at p in the direction of v. The directional derivative operator D_v can be identified with v itself, so the tangent space at p can be viewed as the space of all directional derivatives at p. If we fix a coordinate chart (x, U) around p, the partial derivatives $\partial_i := \frac{\partial}{\partial x^i}$ form a basis for T_pM. Note that ∂_i depends on p. Now we can write $v \in T_pM$ in local coordinates as the unique vector $(v^1, .., v^n)$ such that for all smooth f

$$D_v f(p) = \sum_i v^i \frac{\partial f}{\partial x^i}(p),$$

or, written more compactly, $D_v = v^i \partial_i$.

Now let $X : M \to TM$ be a mapping that assigns to each $p \in M$ a tangent vector $X_p \in T_pM$. If we fix a chart (x, U), we can write X restricted to U as

$$X_p = a^i(p)\partial_i,$$

where the $a^i(p)$ are the coordinates of X_p with respect to the basis of T_pM given by the partial derivatives with respect to x. Now X is smooth if and only if the functions a^i are smooth. Such smooth mapping $X : M \to TM$ is called a smooth vector field (or simply a vector field).

Riemannian metric

In this thesis we work with Riemannian manifolds. These are not special kinds of manifolds, but rather have an additional structure defined along with them. To be precise, a Riemannian manifold is a smooth manifold together with a Riemannian metric: an inner product $\langle \cdot, \cdot \rangle_p$ on each tangent space T_pM. This inner product varies smoothly in the sense that if X and Y are smooth vector fields, then the mapping $p \mapsto \langle X_p, Y_p \rangle_p$ is smooth. We will sometimes leave out the subscript p when the context is clear.

We can express the inner products in local coordinates (x, U) through the smooth functions

$$g_{ij}(p) = \langle \partial_i, \partial_j \rangle_p.$$

For each p, we interpret these numbers as elements of the matrix $G = (g_{ij})_{i,j=1}^n$, which we call the metric matrix. The elements of the inverse of G are denoted by g^{ij}.

Length of a curve

The inner products provide us with a norm $\|v\|_p = \langle v, v \rangle_p^{1/2}$ on T_pM for each p. Therefore we can measure the lengths of tangent vectors. This allows us to define the length of a smooth curve $\gamma : [0, 1] \to M$ as

$$\int_0^1 \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \right\| \mathrm{d}t.$$

Now we assume that the manifold is connected. Then we can define a distance d on M by setting $d(p, q)$ to be the infimum of the lengths of curves from p to q. The manifold M equipped with this distance is a metric space that is homeomorphic to M with its original metric.

Geodesics and the exponential map

Since we just defined the distance between two points p and q to be the infimum of the lengths of all curves from p to q, one might wonder whether this infimum is attained. In other words: is there a shortest path between p and q? And if so, is it unique? The answer to both questions is no, in general. However, under some conditions we do get a positive answer.

A curve γ that is the shortest curve between two points and additionally has constant speed is called a geodesic and satisfies the following geodesic equations (in local coordinates):

$$\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t} = 0.$$

Here Γ^k_{ij} are the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Geodesics are the manifold analogue of moving along a straight line. In fact, in $\mathbb{R}^n$ the geodesics are exactly the straight lines. On a sphere, the geodesics are the great circles.

Geodesics always exists locally, but they can sometimes be extended globally. If all geodesics can be extended globally, we call a manifold geodesically complete. It has been shown that a manifold is geodesically complete if and only if it is complete with respect to the metric induced by the Riemannian structure. For a compact Riemannian manifold this is always true. Therefore for every point p and every $v \in T_pM$ there exists a geodesic $\gamma : \mathbb{R} \to M$ through p that has tangent vector v at p. This allows us to define the exponential map

$$\exp_p : T_pM \to M$$

which maps a tangent vector $v \in T_pM$ to the point in M that is reached after following the geodesic through p with tangent vector v for time 1. As a consequence, the mapping $t \mapsto \exp_p(tv)$ describes the corresponding geodesic. It is also worth noting that $\exp_p$ maps the ball of radius ϵ around 0 in T_pM to the ball of radius ϵ around p in M. For ϵ small enough, this mapping is one-to-one. This will be useful in Chapter 3.

It follows from geodesic completeness that between any two points p and q there exists a geodesic. If p and q are close enough to each other, this geodesic is unique. Otherwise there can be many different geodesics. A clear example is when we take M to be the sphere and p and q the north pole and south pole, respectively. Every curve from p to q that goes straight to the south with constant speed is a geodesic. This implies that there are infinitely many shortest paths between p and q.

Normal coordinates

We can use the exponential map to define a very convenient coordinate chart. Fix a point p and fix an orthonormal basis $\phi_1, .., \phi_n$ of T_pM. Denote by H the function that maps $v \in T_pM$ to its coordinates $v^1, .., v^n$ with respect to $\phi_1, .., \phi_n$, i.e. the real numbers such that

$$v = v^i\phi_i.$$

Then there exists a neighbourhood U of p on which the following map is well-defined and homeomorphic to its image:

$$x : U \to \mathbb{R}^n, \qquad q \mapsto H \circ \exp_p^{-1}(q).$$

Note that U must in particular be small enough such that geodesics from p to points in U are unique. Now (x, U) defines a coordinate chart. We call these coordinates normal coordinates centered at p. Note that they depend on the choice of orthonormal basis.

The advantage of using normal coordinates is that some expressions are simplified. Since $g_{ij} = \langle \partial_i, \partial_j \rangle_p = \langle \phi_i, \phi_j \rangle_p$ equals 1 for $i = j$ and 0 otherwise, we see that by construction the metric matrix G at p equals the identity matrix when it is expressed in normal coordinates. Also, the first order partial derivatives of g_{ij} are 0 at p and therefore the Christoffel symbols vanish at p. Note that these properties only hold at p though, not at other points in U. Another nice property is that through these normal coordinates a small enough ball around p is in one-to-one correspondence with a ball in $\mathbb{R}^n$.

Volume measure

To integrate on a manifold, we need a measure that measures the volume of (measurable) subsets of the manifold. In a general smooth manifold there is no canonical way to do this, but the Riemannian structure induces a natural concept of volume. To describe it, we first specify the set $\Lambda(M)$ of measurable subsets of M. A subset $V \subset M$ is said to be measurable if for every chart (x, U) the set of coordinates $x(U \cap V)$ is Lebesgue measurable. In this way $\Lambda(M)$ in particular contains the Borel measurable subsets of M. Now we define the volume measure V in local coordinates (x, U) as

$$\mathrm{d}V = \sqrt{|G|}\mathrm{d}\lambda,$$

where $|G(s)|$ is the determinant of the Riemannian metric matrix at $x^{-1}(s)$ expressed in (x, U). This means that for a measurable function f with support in U

$$\int_M f \mathrm{d}V = \int_{x(U)} f(x^{-1}(s))\sqrt{|G(s)|}\lambda(\mathrm{d}s).$$

For the existence and uniqueness of the volume measure as given by this expression see Grigoryan [79]. To understand why the factor $\sqrt{|G|}$ shows up, one has to go through the theory of forms on a manifold, which we will not do here. We do mention that this factor makes sure that the expression $\sqrt{|G|}\mathrm{d}\lambda$ is invariant under coordinate transformations. Note that in $\mathbb{R}^n$ with the usual coordinates the matrix G is the identity matrix, so $\mathrm{d}V$ reduces to $\mathrm{d}\lambda$. In that sense the volume measure is a generalisation of the Lebesgue measure to manifolds. The total volume $V(M)$ of a manifold can be infinite, but if the manifold is compact the volume is finite. Therefore on a compact manifold we can normalise the volume measure by dividing by $V(M)$ to obtain a uniform probability measure, which we will denote by $\overline{V}$.

2.2.2 Brownian motion and related objects

So far we introduced general basic concepts on manifolds. Now we move to more specialised and sometimes probabilistic notions. Unless noted otherwise, we assume for the rest of this chapter that M is a complete, smooth, connected and compact n-dimensional Riemannian manifold.

Laplace-Beltrami operator

In order to treat Brownian motion, the Green's function and the heat equation on a manifold, an object of central importance is the Laplace-Beltrami operator, the manifold analogue of the Laplace operator. For more details about all of these topics and for proofs we refer the reader to Grigoryan [79] and Canzani [24]. We will only need to apply the Laplace-Beltrami operator to smooth functions, so we will define it for smooth functions. As in $\mathbb{R}^d$, the Laplace-Beltrami operator is the divergence of the gradient. Therefore we will first define those concepts.

First let f be a smooth function. Then the gradient of f is the unique smooth vector field ∇f such that for each point p and tangent vector $v \in T_pM$, the directional derivative at p in the direction of v satisfies

$$D_v f(p) = \langle v, \nabla f \rangle_p.$$

As in $\mathbb{R}^n$, the gradient of f at p is the vector that points in the direction of the largest growth of f and of which the norm corresponds to the size of this growth.

Now let X be a smooth vector field on M. Then the divergence of X is the unique smooth function $\mathrm{div}X$ such that for all smooth functions f

$$\int (\mathrm{div}X) f \mathrm{d}V = -\int \langle X, \nabla f \rangle \mathrm{d}V.$$

Note that this way the divergence is defined through the identity that is usually known as the divergence theorem. The usual term containing an integral over the boundary equals 0 because there is no boundary.

Now we can define the Laplace-Beltrami operator Δ_M as the divergence of the gradient, i.e. for a smooth function f

$$\Delta_M f = \mathrm{div}(\nabla f),$$

which is again a smooth function on M.

The advantage of the descriptions of ∇, div and Δ_M that we gave here, is that they are independent of coordinates, so we are sure that they do not depend on which chart we use to describe them. However, it is sometimes useful to express them in local coordinates. For the Laplace-Beltrami operator this yields the following two expressions:

$$\Delta_M = \frac{1}{\sqrt{|G|}} \partial_j (\sqrt{|G|} g^{ij} \partial_i) = g^{ij} \partial_i \partial_j - g^{ij} \Gamma_{ij}^k \partial_k.$$

In normal coordinates (x, U) centered at a point p the metric matrix is the identity and the Christoffel symbols vanish, so the expression reduces to

$$\Delta_M f(p) = \sum_i \partial_i^2 f(p),$$

which is what we are used to in $\mathbb{R}^n$. Note that this expression again only holds at the point p itself. An alternative description of Δ_M is obtained by fixing an orthonormal basis $v_1, .., v_n$ at p (for instance $\partial_1, .., \partial_n$ from the normal coordinates (x, U) we just used) and writing

$$\Delta_M f(p) = \sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\exp_p(tv_i))\Big|_{t=0}.$$

From what we saw above, we can interpret the Laplace-Beltrami operator as an operator $\Delta_M : C^\infty(M) \to C^\infty(M)$ on the smooth functions on the manifold. Δ_M has eigenvalues $0 = \lambda_1 \geq \lambda_2 \geq ...$, where the eigenvalue 0 comes from the fact that $\Delta_M \mathbf{1} = 0$. The corresponding eigenfunctions $\varphi_1 = \mathbf{1}, \varphi_2, ..$ are smooth functions that form an orthonormal basis for $L^2(M)$.

Heat semigroup, heat kernel and heat equation

The Laplace-Beltrami operator acting on the smooth functions can be closed both in $L^2(M)$ and in $C(M)$, the continuous functions on M. Now Δ_M generates a strongly continuous semigroup on $L^2(M)$ and on $C(M)$. We call both of them the heat semigroup and denote them by $S = (S_t, t \geq 0)$. This should not cause any confusion, since the semigroup on $L^2(M)$ extends the semigroup on $C(M)$, i.e. they agree on the continuous functions. Moreover, we will usually only apply the semigroup to smooth functions, where they agree by definition.

For $f \in L^2(M)$ and $t > 0$, $S_t f$ has a smooth version so we can take $S_t f$ to be smooth. There also exists a function $(t, p, q) \mapsto p_t(p, q)$ for $t > 0$ and $p, q \in M$ that is smooth jointly in all variables and symmetric in p and q such that for all $f \in L^2(M)$:

$$S_t f(p) = \int_M p_t(p, q) f(q) V(\mathrm{d}q).$$

$p_t(x, y)$ is called the heat kernel on M.

We define the heat equation on M as in $\mathbb{R}^n$:

$$\partial_t u(p, t) = \Delta_M u(p, t), \qquad u(p, 0) = u_0(p),$$

where u_0 is the initial density and Δ_M acts on the space variable of u. The function $(t, p) \mapsto S_t f(p)$ satisfies the heat equation with initial density f.

Green's function

With the tools developed so far, we can introduce the Green's function on M. For more details we refer to Aubin [4], Donaldson [45] and Grigoryan [79]. One would like to define the Green's function as the integral of the heat semigroup or as minus the inverse of the Laplacian. However, on a compact manifold (without boundary), the heat kernel is not integrable and the Laplace-Beltrami operator is not invertible. The reason is that the heat kernel converges to a constant as t goes to infinity and that $\Delta_M \mathbf{1} = 0$. However we can make adjustments to solve this.

The key to handling the Green's function on a compact manifold is to reduce to working with zero average functions. Then $S_t f$ still converges to a constant function, but this constant is 0. Also Δ_M is invertible on the smooth zero-average functions. In Chapter 6 we will define the Green operator G as

$$G = -\sum_{i=2}^{\infty} \frac{1}{\lambda_i} P_i,$$

where P_i is projection onto the eigenspace spanned by the eigenfunction φ_i of Δ_M.[2] This way G is the inverse of Δ_M on the space W of zero-average smooth functions and it is 0 on the constant functions (compare with (2.9)). G can be extended to a bounded, self-adjoint operator on $L^2(M)$. Finally it holds for f such that $\int f \mathrm{d}V = 0$ that in $L^2(M)$

$$Gf = \int_0^{\infty} S_t f \mathrm{d}t.$$

Alternatively, we could have defined the Green kernel as

$$\int_0^{\infty} \left(p_t(p,q) - \tfrac{1}{V(M)} \right) \mathrm{d}t \tag{2.18}$$

and then $Gf = \int g(p,q) f(q) V(\mathrm{d}q)$. The idea in (2.18) is that the limiting density is subtracted from the heat kernel to make sure that the integrand converges to 0 in the limit of t to infinity.

Brownian motion

Now we want to introduce Brownian motion on a manifold. We will not directly use it in this thesis (except in Appendix A), but Brownian motion on a manifold is an important object in its own right and is closely related to the heat equation, the Laplace-Beltrami operator and the random walk on a grid on a manifold that particles perform in Chapter 4. We refer to Hsu [87] and Émery [49] for more details and proofs.

Brownian motion on M can be defined as a Markov process $B = (B_t, t \geq 0)$ on M that has $p_{t/2}(p,q)$ as its transition density function. If we do that, we find that the

[2] Note the difference of sign compared to (6.4). This comes from the extra minus in the definition of the Laplace-Beltrami operator in Chapter 6.

semigroup of Brownian motion is $(S_{t/2}, t \geq 0)$ and that this semigroup is generated by $\frac{1}{2}\Delta_M$, like its counterpart in $\mathbb{R}^n$. Also like in $\mathbb{R}^n$, Brownian motion on M has a version with continuous paths. Under our assumption that the manifold M is compact, has no boundary and is complete we know that Brownian motion exists for all time. In $\mathbb{R}^n$ Brownian motion has independent Gaussian increments. On a manifold Brownian motion is still independent in non-overlapping time intervals, but since it is not possible to subtract points of the manifold we cannot speak of increments. However, the transition density function is the heat kernel (at time $\frac{t}{2}$), which is the density for the normal distribution in $\mathbb{R}^n$.

There are several other ways of constructing or characterising Brownian motion on a manifold. One method is called 'rolling without slipping' (Elworthy [48], Hsu [86]). It uses stochastic calculus and more specifically Stratonovich integrals. The idea is that a Brownian motion process in $\mathbb{R}^d$ is 'rolled out' over the manifold in an appropriate way to obtain Brownian motion on the manifold. Next, Brownian motion can be characterised as a martingale on the manifold with a certain quadratic variation (Émery [49]). Finally, Brownian motion can be described locally in a coordinate chart as a process that is generated by the Laplace-Beltrami operator expressed in local coordinates. However, this description is only valid until the process leaves the chart. For more general theory about (semi)martingales and stochastic calculus on manifolds, see Hsu [86] and Émery [49].

Geodesic random walk

As in the $\mathbb{R}^n$ case, we can interpret Brownian motion on a manifold as the scaling limit of a random walk. Of course one cannot interpret a random walk on a manifold as a sum of i.i.d. random increments. Also, its path cannot be rescaled in the same way as in $\mathbb{R}^n$. It turns out that the right random walk concept on a manifold is the geodesic random walk.

To define the geodesic random walk X, first specify a probability measure μ_p on each tangent space T_pM. Initialise the random walk by letting $X_0 = p_0$ be a possibly random point on the manifold. After an exponential time with rate 1, pick a random tangent vector v_0 from $T_{p_0}M$ according to μ_{p_0}. Then jump to $p_1 = \exp_{p_0}(v_0)$, i.e. the point that is reached by following the geodesic from p_0 in the direction of v_0 for time 1. Now wait again for an exponential time, pick v_1 from $T_{p_1}M$ according to μ_{p_1} and jump to $p_2 = \exp_{p_1}(v_1)$. Repeating this process yields the geodesic random walk.

As we said before, to rescale the random walk we cannot simply rescale its path. Instead, for fixed N, we set the parameter for the exponential waiting times to N^2. This scales time with a factor N^2. To rescale space with a factor N, we say that each geodesic should be followed for time $\frac{1}{N}$ instead of time 1, or, equivalently, we scale the tangent vectors that are picked at each point by $\frac{1}{N}$. Now under some conditions (that we will mention shortly) on the μ_p's, the rescaled geodesic random walk converges to Brownian motion on the manifold.

This invariance result was obtained in Jørgensen [93] and Blum [17]. A sufficient condition is for instance that each μ_p is the uniform measure on the unit ball of T_pM. In Appendix A we include a proof for more general jumping distributions that is tailor-made to be applied later in this thesis.

Part II

Particle systems and Gaussian Free Fields on manifolds

Chapter 3

Uniformly approximating grids

In this thesis we will use grids to define the Symmetric Exclusion Process and the Discrete Gaussian Free Field on a compact Riemannian manifold. In this chapter[1] these grids are introduced and we prove the necessary properties, the most important of which is the convergence of the graph Laplacians to the Laplace-Beltrami operator. We call these grids uniformly approximating grids.

3.1 Introduction

When considering an interacting particle system or a discrete field like the Discrete Gaussian Free Field on a continuous space, we usually define them on a discretisation of that space. We also discussed this in Chapter 1, in particular in Section 1.6.4. We refer to such discretisations as grids. In the well-studied case of $\mathbb{R}^d$ (or the flat torus), there is a very natural candidate grid, namely the lattice $\mathbb{Z}^d$. This grid has many nice properties. It is very homogeneous, the grid looks the same from the point of view of each grid point. In fact, it is symmetric around each point and translation invariant. Moreover, it can be naturally rescaled simply by multiplying it with a constant $\frac{1}{N}$, since $\frac{1}{N}\mathbb{Z}^d$ is again a subset of $\mathbb{R}^d$. And when N grows to infinity, $\frac{1}{N}\mathbb{Z}^d$ approximates $\mathbb{R}^d$ very evenly. The distance between neighbouring grid points is the same everywhere on the grid and it diminishes monotonically. Also the empirical measure of the grid points (i.e. $\frac{1}{N^d}$ times the sum of Dirac measures on all grid points) approximates the Lebesgue measure evenly (i.e. equally fast on compact boxes everywhere in $\mathbb{R}^d$).

[1] This chapter is based on research that was first started in van Ginkel [148] and then continued and finalised in van Ginkel and Redig [149].

When we move to a manifold it is not straightforward to define a good grid. And when we do, it will definitely not have all the nice lattice properties that we described above. In fact properties like translation invariance cannot be hoped for since translation itself has no fixed meaning on a manifold. However, we will also not need all of the properties of lattices. Some properties make proofs and notation easier but are not essential for treating interacting particle systems or discrete random fields. In this chapter we will assess which properties we need and show how to obtain grids with those properties. We will call these grids uniformly approximating grids.

For the rest of this chapter, let M be a complete, smooth, connected and compact d-dimensional Riemannian manifold. We will show two results.

i) If a sequence $(p_n)_{n=1}^{\infty}$ in M is such that the empirical measures $\frac{1}{N}\sum_{i=1}^{N}\delta_{p_i}$ converge to the uniform distribution in Kantorovich sense, then we can use the sequence to construct a sequence of grids. We will let the grid G^N be the first N points of the sequence and define weights on the edges between the grid points depending on the distance between the points. This is Theorem 3.4.

ii) If a sequence is sampled from the uniform measure on M, then with probability 1 the corresponding empirical measures converge to the uniform measure in Kantorovich sense, i.e. we can apply Theorem 3.4. This is Theorem 3.13.

The structure of this chapter is as follows. In Section 3.2 we describe what kind of properties the grids should satisfy. We mention other ways that have been used in the literature to approximate manifolds and explain why they do not suffice for our purposes. Then in Section 3.3 we introduce the grids that we will use and we state the main theorem, Theorem 3.4. The idea of this main theorem is that if we are given an appropriate sequence of points on the manifold, they can be used to construct weighted graphs that satisfy the properties that are required in Section 3.2. The proof of this theorem is given in Section 3.4 and Section 3.5. Then in Section 3.6 we state and prove Theorem 3.13, so we show that an appropriate sequence of points is obtained with probability 1 by sampling uniformly from the manifold. We conclude with some notes and references in Section 3.7.

For the result of Section 3.5, we will need a result that forms the core of proving the invariance principle for the geodesic random walk. For this we will refer to Appendix A.

3.2 Uniformly approximating grids

We will first explain which properties uniformly approximating grids should satisfy. Then we will mention some already existing discretisations of manifolds or grids on manifolds and we explain why they can not be used in our context.

3.2.1 Requirements for the grids

As we said earlier, the goal of this thesis is to consider interacting particle systems such as the Symmetric Exclusion Process (SEP) and the Discrete Gaussian Free Field (DGFF) on a manifold. To do this we need a sequence of grids that approximate the manifold. The fact that we want to construct SEP and the DGFF pose restrictions on the grids that we use. These restrictions naturally lead to the following three properties that the grids should satisfy.

Property 1: Symmetric conductances
First, both for SEP and for the GFF we need that the conductances between grid points are symmetric. Having symmetric conductances implies that we can can really speak of the conductance between points without specifying a direction. In the analysis of SEP, this means that we can view it as a process of flipping edges instead of letting particles jump (see Chapter 4). For the DGFF it also makes sense to have directionless conductances. The reason is that its density is a function of the weighted total squared difference of the DGFF values in neighbouring nodes. It is natural, both conceptually and for the calculations, that these edges weights only depend on the pair of nodes and not on the direction.

Property 2: Convergence of the graph Laplacians to Δ_M
This next property is probably conceptually the most important one. As we have seen in Chapter 1 and 2, the graph Laplacians corresponding to weighted graphs are crucial for the analysis of both SEP and the DGFF. In particular, we saw this in the sketch of the proof for the hydrodynamic limit of SEP in Section 1.3.3. Convergence of the graph Laplacians to the Laplace-Beltrami operator Δ_M is an essential ingredient to obtain the heat equation in the limit. Further, we saw in Section 1.5.4 that the convergence of the DGFF depends on convergence of the Green's functions. This in turn is directly related to convergence of the graph Laplacians.

To define the convergence of the graph Laplacians to the Laplace-Beltrami operator mathematically, let $(G^N)_{N=1}^{\infty}$ be a sequence of grids. For simplicity, we assume that G_N has size N and we write $G_N = (p_1, .., p_N)$[2]. As we argued above, on each G_N we will place symmetric conductances or edge weights W_{ij}^N. This defines the following graph Laplacians:

$$L^N f(p_i) = \sum_{j=1}^{N} W_{ij}^N (f(p_j) - f(p_i)). \tag{3.1}$$

Now we need that there exists a scaling function $a : \mathbb{N} \to [0, \infty)$ and some constant $C > 0$ such that

$$a(N)L^N \to C\Delta_M \qquad (N \to \infty)$$

[2]Note that we should write $p_1^N, .., p_N^N$, but since the meaning is clear in its context and we will later have graphs where p_i^N is the same for all N (and for fixed i), we leave out the superscript.

in the sense that for all smooth $\phi : M \to \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left| a(N) \sum_{j=1}^{N} W_{ij}^N (\phi(p_j) - \phi(p_i)) - C\Delta_M \phi(p_i) \right| = 0. \tag{3.2}$$

This is what we mean by convergence of the graph Laplacians to the Laplace-Beltrami operator.

Property 3: Convergence of the empirical measures to $\overline{V}$
The final requirement is implied by the previous two, but it is good to mention it separately. Consider for a moment the random walk $X^N = (X_t^N, t \geq 0)$ on G_N that jumps with rates given by W_N. The symmetry of the conductances implies that the uniform measure on the grid points is stationary and reversible for X. This means that when started from the uniform distribution, X^N spends roughly an equal amount of time in each grid point. Since the graph Laplacians converge to the Laplace-Beltrami operator, the random walk on the grids points should in some sense converge to Brownian motion (see Remark 3.5). Since Brownian motion is reversible with respect to the uniform measure $\overline{V} = \frac{V}{V(M)}$ on M, it spends an equal amount of time in regions of equal volume. Combining these arguments, we see that for large N, X^N spends roughly the same amount of time in each grid point and should spend time in a set proportional to the volume of the set. We conclude from this that the amount of grid points in a subset of M should in the limit be proportional to its volume. This suggests that we should have that

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{p_i} \to \overline{V} \tag{3.3}$$

where the limit is in the sense of weak convergence of measures.

Uniformly approximating grids
If a sequence of grids with corresponding weights $(G_N, W_N)_{N=1}^{\infty}$ satisfies all of these conditions we will call them uniformly approximating grids. In other words, uniformly approximating grids are grids with conductances $(G_N, W_N)_{N=1}^{\infty}$ such that the conductances are symmetric and (3.2) and (3.3) hold.

Remark 3.1 (Comparison with standard grids)**.** To give an idea of how known grids in Euclidean spaces can be incorporated in this framework, let S be the one-dimensional torus. Let S^N be the grid with grid points $p_k = \frac{k}{N}, k = 1, .., N$. We see directly that (3.3) holds. Now we can define a nearest neighbour random walk by putting $W_{ij}^N = \mathbf{1}_{|p_i - p_j| = 1/N}$. Note that these conductances are symmetric. Also set $a(N) = N^2$. Then we see for a point $p_i \in S^N$ that

$$\begin{aligned} a(N) \sum_{j=1}^{N} W_{ij}^N (\phi(p_j) - \phi(p_i)) &= N^2(\phi(p_i + 1/N) + \phi(p_i - 1/N) - 2\phi(p_i)) \\ &= \phi''(p_i) + O(N^{-1}). \end{aligned}$$

The compactness of the torus easily implies that this rest term can be bounded uniformly. This implies that (3.2) holds. Later in this chapter we encounter sequences of grids such that $G^N \subset G^{N+1}$. This property can be obtained here by considering the grids S^{2^m} with $m \in \mathbb{N}$.

3.2.2 Available grids

The problem of constructing weighted grids on a manifold such that the graph Laplacians in some sense approximate the Laplace-Beltrami operator has been studied within the analysis and statistics communities. We will get to the latter in the next section. Within the analysis community, the goal is usually to find approximations of the Laplace-Beltrami operator to obtain approximations of its eigenvalues. We mention some ways in which this was done and why they do not work for our purposes.

i) Triangulations. First there is the idea of approximating a manifold by triangulating it. Usually such triangulation is not intrinsic to the manifold but is performed in some ambient space, which is not what we want. There is also the idea of geodesic triangulations, which are intrinsic to the manifold, for instance in Aubry [5]. However, these triangulations are still technically complex and only spectral convergence is shown, not pointwise convergence of the graph Laplacians.

ii) ϵ-nets. One can also approximate a manifold by using ϵ-nets, grids such that each point of the manifold is within distance ϵ of a grid point and such that the grid points are at distance larger than ϵ from each other. Such approach is used in Burago et al. [22] and Fujiwara [66]. This ϵ-net property makes sure that points are distributed evenly when it comes to distance from each other. In a flat space, this can go together with being distributed according to volume. However, this is not true in curved space. The reason is that the volume of a ball of radius ϵ depends on the local curvature. Therefore an ϵ-net does not satisfy (3.3).

iii) Poisson-Voronoi tesselations. To make sure that (3.3) is satisfied, one can sample from the uniform distribution on the manifold (in fact, this is what we will do later in this chapter). Then the corresponding Voronoi tesselation can be constructed and points can be joined with an edge if the are in neighbouring cells. Properties of such tesselation have been studied in for instance Calka et al. [23]. However, in our case it is first not clear which weights should be put on these edges such that the graph Laplacians converge to the Laplace-Beltrami operator. Second, it is hard to prove pointwise convergence of these operators in this case, because of the lack of symmetry of the grid around a fixed grid point. See also Remark 3.2.

iv) Random geometric complexes. As in the previous point, one can sample uniformly from the manifold and join points with edges if they are a certain distance

from each other. This is in fact similar to what we will do, but not the same. For instance in Lerario and Mulas [105] such a grid is studied.

3.3 Model and motivation

In this section we introduce the grids that we will use in this thesis. To be more precise, we will show how a suitable sequence of grids points can be used to construct uniformly approximating grids. The requirement on this sequence of grid points is that the corresponding empirical measures converge in Kantorovich sense to the uniform distribution on the manifold. In Section 3.6, we will see how such sequence of points can be obtained.

Before we explain this, we will start with a motivation and some references to literature where similar models are used.

3.3.1 Motivation

In statistical data analysis the following setting is known and used in various contexts such as data clustering, dimension reduction, computer vision and statistical learning, see: Singer [138], Von Luxburg et al. [155], Giné and Koltchinskii [74], Belkin and Niyogi [11] and Belkin [8] and references therein for general background and various applications. Suppose we have a manifold M that is embedded in $\mathbb{R}^m$ for some m and we would like to recover the manifold from some observations of it, say an i.i.d. sample of uniform random elements of M. To do this we can describe the observations as a graph with as weight on the edge between two points a non-negative kernel with bandwidth ϵ applied to the Euclidean distance between those points. Then it can be shown that the graph Laplacian of the graph that is obtained in this way converges in a suitable sense to the Laplace-Beltrami operator on M as the number of observations goes to infinity and ϵ goes to 0.

We generalise this idea by taking a more general sequence of graphs, but our main example (in Section 3.6) will be this random graph. The main distinction between the statistical literature and our context is the following: for our purposes it is much more natural to view the manifold M on its own instead of embedded in a possibly high dimensional Euclidean space. This means that we have to use the distance that is induced by the Riemannian metric instead of the Euclidean distance. The latter is more suitable to purposes in statistics, because in that setting the Riemannian metric on M is not known beforehand. Also, a lot is known about the behaviour of the Euclidean distance in this type of situation and not so much about the distance on the manifold. We will have to make things work in M itself.

3.3.2 Model

We repeat our assumption that M is a complete, smooth, connected and compact d-dimensional Riemannian manifold.

Sequence approximating the uniform distribution
We call a function f on M Lipschitz with Lipschitz constant L_f if

$$\sup_{p,q\in M} \frac{|f(p)-f(q)|}{d(p,q)} = L_f < \infty.$$

Let $(p_n)_{n\geq 1}$ be a sequence in M such that $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{p_i}$ converges in the Kantorovich sense to $\overline{V}$ (the uniform distribution on M), i.e.

$$W_1(\mu^N, \overline{V}) = \sup_{f\in\mathcal{F}_1(M)} \left\{ \int_M f \mathrm{d}\mu^N - \int_M f \mathrm{d}\overline{V} \right\} \longrightarrow 0,$$

where $\mathcal{F}_1(M)$ denotes the set of Lipschitz functions f on M that have Lipschitz constant $L_f \leq 1$.

Weighted graphs
Define the N^{th} grid V_N as $V_N = \{p_1, .., p_N\}$. Set

$$\epsilon := \epsilon(N) := \left(\sup_{m\geq N} W_1(\mu^m, \overline{V}) \right)^{\frac{1}{4+d}}. \tag{3.4}$$

This ϵ rescales the distance over which particles will jump. Naturally, $\epsilon \downarrow 0$ as $N \to \infty$ (since $W_1(\mu^N, \overline{V}) \to 0$). Let $k : [0,\infty) \to [0,\infty)$ be Lipschitz and compactly supported (for instance $k(x) = (1-x)\mathbf{1}_{[0,1]}(x)$), we will call such k a kernel. Define

$$W_{ij}^\epsilon = k(d(p_i,p_j)/\epsilon)$$

as the jumping rate from p_i to p_j. Here d is the Riemannian metric on M. Note that the only dependence on N is through ϵ, hence the notation W_{ij}^ϵ instead of W_{ij}^N. If we regard two points p_i, p_j as having an edge between them if $W_{ij}^N > 0$, we want the resulting graph to be connected (to make sense of the random walk and later of the particle systems defined on it). If we assume that there is some α such that $k(x) > 0$ for $x \leq \alpha$, one can show that the resulting graph is connected for N large enough. The main reason is that the distance between points that are close to each other goes to zero faster than ϵ. The details of the proof are in the appendix of this chapter (see also Remark 3.2). These weights induce the following graph Laplacian

$$L^N f(p_i) = \sum_{j=1}^N W_{ij}^\epsilon (f(p_j) - f(p_i)). \tag{3.5}$$

Scaling
Finally we define the scaling factor

$$a(N) = \epsilon^{-2-d} N^{-1}.$$

To see why the scaling $a(N)$ is natural, we can rewrite

$$a(N)L^N f(p_i) = \frac{1}{\epsilon^2} \sum_{j=1}^{N} \frac{k\left(\frac{d(p_i,p_j)}{\epsilon}\right)}{N\epsilon^d}(f(p_j) - f(p_i)).$$

Since k is a kernel that is rescaled by ϵ inside, we need the $\frac{1}{\epsilon^d}$ to make sure the integral of the kernel stays of order 1 as ϵ goes to 0. Since the number of points that the process can jump to equals N, we also need the factor $\frac{1}{N}$ to make sure the jumping rate is of order 1 as N goes to infinity. Also note that the typical distance that a particle jumps with these rates is of order ϵ. This means that space is scaled by ϵ. Hence it is very natural for a diffusive scaling that time is rescaled by $\frac{1}{\epsilon^2}$.

Note that in the calculations N is the main parameter and ϵ an auxiliary parameter depending on N. However, conceptually, when the scaling is concerned, the most important parameter is ϵ. N is just the total number of positions and simply has to grow fast enough as ϵ goes to 0. To see why this is true, note in the rest of this chapter that any sequence $\epsilon(N)$ that goes to 0 more slowly than what we use here will also do. Hence ϵ should go to 0 slow enough with respect to N or, equivalently, N should go to infinity fast enough with respect to ϵ.

Remark 3.2. We have noted that N must grow to infinity fast enough as ϵ goes to 0. In fact, the number of points in a ball of radius ϵ goes to infinity (even though ϵ shrinks to 0). In particular, this means that the number of points that a particle can jump to, goes to infinity. This is very different from the $\mathbb{R}^d$ case, where the number of neighbours is constant. The reason why it should be different in the manifold case is the following. In $\mathbb{R}^d$, the natural grid $\frac{1}{N}\mathbb{Z}^d$ is very symmetric. Indeed, we can split the graph Laplacian into the contributions $N^2(f(x+\mathrm{e}_i/N)+f(x-\mathrm{e}_i/N)-2f(x))$ in each direction i, where e_i it the unit vector in direction i. Now when applying Taylor we see that the first order terms cancel perfectly, leaving us only with the second order terms, which we want for the Laplacian. In a manifold such perfect cancellation is not possible. Therefore the way to make the first order terms cancel is to sample more and more points around a grid point, such that the sum over the linear order terms becomes an integral that vanishes in the limit. This is exactly what happens in our proof. For this reason we need the number of grid points in a ball of size ϵ to go to infinity.

Remark 3.3. It is also possible to define W_{ij}^N as $p_\epsilon(p_i, p_j)$, the heat kernel after time ϵ, and rescale by ϵ^{-1} instead of ϵ^{-2-d}. Then the result of Section 3.4 can be proven in the same way by obtaining some good bounds on Lipschitz constants and suprema of the heat kernel and choosing $\epsilon = \epsilon(N)$ appropriately. We will follow this approach in Chapter 6. There the result of Section 3.5 is a direct consequence of the fact that the Laplace-Beltrami operator generates the heat semigroup. However, for purposes of application/simulation the weights that we have chosen here are much easier to calculate (since only the geodesic distances need to be known, not the heat kernel).

3.3.3 Main result

We now state the main result.

Theorem 3.4. *Let $V_N, W^\epsilon, a(N)$ be as defined above. Then $(V_N, W^\epsilon)_{N=1}^\infty$ is a sequence of uniformly approximating grids with scaling $a(N)$.*

By construction, the edge weights W_{ij}^ϵ are symmetric. Also by assumption on the underlying sequence, the empirical measures corresponding to the grids V_N converge in Kantorovich sense and therefore weakly to $\overline{V}$. Therefore to prove that the grids are uniformly approximating it remains to show (3.2), i.e. as the number of points N goes to infinity (and hence the bandwidth ϵ goes to 0)

$$\frac{1}{N}\sum_{i=1}^{N}\left|a(N)\sum_{j=1}^{N}W_{ij}^\epsilon(f(p_j)-f(p_i))-C\Delta_M f(p_i)\right|\longrightarrow 0 \quad (N\to\infty).$$

In fact, in Section 3.4 and 3.5 we will prove the following slightly stronger result:

$$\sup_{1\le i\le N}\left|a(N)\sum_{j=1}^{N}W_{ij}^\epsilon(f(p_j)-f(p_i))-C\Delta_M f(p_i)\right|\longrightarrow 0 \quad (N\to\infty). \tag{3.6}$$

We call (3.6) "convergence of the (rescaled) generators to Δ_M uniformly in the p_i's for $i \le N$" or just "convergence of the generators to Δ_M uniformly for $i \le N$". In fact, we will show that the rate of convergence does not depend on p_i, so we might as well call it "uniformly in the p_i's".

Remark 3.5. In fact, (3.6) implies more. Denote the semigroups corresponding to the generators $a(N)\sum_{j=1}^N W_{ij}^\epsilon(f(p_j)-f(p_i))$ by S_t^N and the semigroup corresponding to $C\Delta_M$ by S_t. Then (3.6) implies that uniformly on compact time intervals

$$\sup_{1\le i\le N}\left|S_t^N f|_{G^N}(p_i)-S_t f(p_i)\right|\longrightarrow 0 \quad (N\to\infty).$$

The proof is a straightforward application of Kurtz [102, Theorem 2.1] and a small argument that the extended limit of the generators above (as described in [102]) equals $C\Delta$ since they are equal on the smooth functions.

3.4 Replacing empirical measure by uniform measure

The first step to show (3.6), is to replace the summation over the grids points by integration with respect to the volume measure. In other words, we will replace the empirical measure corresponding to the grids by the uniform measure. We will use

the Kantorovich convergence of the empirical measures together with the Lipschitz property of the kernel k to show that the rest term vanishes.

To prove (3.6) we need to show that there is a C independent of i such that for all smooth f

$$\lim_{N\to\infty} \epsilon^{-2-d} N^{-1} \sum_{j=1}^{N} k(d(p_j, p_i)/\epsilon) \left[f(p_j) - f(p_i)\right] = C\Delta_M f(p_i)$$

uniformly in the p_i's. We can write

$$\epsilon^{-2-d} N^{-1} \sum_{j=1}^{N} k(d(p_j, p_i)/\epsilon) \left[f(p_j) - f(p_i)\right] = \epsilon^{-2-d} \int_M g^{\epsilon,i} \mathrm{d}\mu^N, \tag{3.7}$$

where

$$g^{\epsilon,i}(p) = k(d(p, p_i)/\epsilon) \left[f(p) - f(p_i)\right]$$

and

$$\mu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{p_i}$$

is the empirical measure corresponding to G^N. Now (3.7) equals

$$\epsilon^{-2-d} \int_M g^{\epsilon,i} \mathrm{d}\overline{V} + \epsilon^{-2-d} \int_M g^{\epsilon,i} \mathrm{d}(\mu^N - \overline{V}). \tag{3.8}$$

We will show later that the first term converges to $C\Delta_M f(p_i)$ (uniformly in the p_i's) as $N \to \infty$. Therefore it suffices for now to show that the second term converges to 0, uniformly in the p_i's.

3.4.1 Replacement

Note that k is Lipschitz so it has some Lipschitz constant $L_k < \infty$. This implies that

$$\left| k\left(\frac{d(q^1, p_i)}{\epsilon}\right) - k\left(\frac{d(q^2, p_i)}{\epsilon}\right) \right| \leq L_k \left| \frac{d(q^1, p_i)}{\epsilon} - \frac{d(q^2, p_i)}{\epsilon} \right| \leq \frac{L_k}{\epsilon} d(q^1, q^2),$$

by the reverse triangle inequality, so $k(d(\cdot, p_i)/\epsilon)$ has Lipschitz constant $\frac{L_k}{\epsilon}$. f is smooth, so it is Lipschitz too with Lipschitz constant L_f. Since $f(p_i)$ is just a constant, $f(\cdot) - f(p_i)$ is also Lipschitz with Lipschitz constant L_f. Since they are both bounded functions, we see for the Lipschitz constant of $g^{\epsilon,j}$:

$$\begin{aligned} L_{g^{\epsilon,j}} &\leq L_{k(d(\cdot,p_i)/\epsilon)} ||f(\cdot) - f(p_i)||_\infty + ||k(d(\cdot, p_i)/\epsilon)||_\infty L_{f(\cdot)-f(p_i)} \\ &\leq \frac{2L_k}{\epsilon} ||f||_\infty + ||k||_\infty L_f. \end{aligned}$$

Note that k is bounded since it is Lipschitz and compactly supported, so $||k||_\infty < \infty$. This shows that:

$$\begin{aligned}\left|\epsilon^{-2-d}\int_M g^{\epsilon,i}\mathrm{d}(\mu^N - \overline{V})\right| &\leq \epsilon^{-2-d}\left(\frac{2L_k}{\epsilon}||f||_\infty + ||k||_\infty L_f\right) W_1(\mu^N,\nu)\\ &= \epsilon(N)^{-3-d}\left(2L_k||f||_\infty + \epsilon(N)||k||_\infty L_f\right) W_1(\mu^N,\nu),\end{aligned}$$

where we denoted the dependence of ϵ on N explicitly. By (3.4), $W_1(\mu^N,\nu) \leq \epsilon(N)^{4+d}$, so we obtain

$$\left|\epsilon^{-2-d}\int_M g^{\epsilon,i}\mathrm{d}(\mu^N - \overline{V})\right| \leq \epsilon\left(2L_k||f||_\infty + \epsilon||k||_\infty L_f\right).$$

Note that this bound does not depend on p_i. Since $\epsilon \to 0$, it follows that the second term of (3.8) goes to 0 uniformly in the p_i's.

3.4.2 What remains

What we have seen above basically means that we can replace the empirical distribution μ^N by the uniform distribution $\overline{V}$. For convergence of the generators we still have to show that

$$\lim_{\epsilon\downarrow 0}\epsilon^{-2-d}\int_M k(d(p,p_i)/\epsilon)\left[f(p) - f(p_i)\right]\overline{V}(\mathrm{d}p) = C\Delta_M f(p_i)$$

uniformly in the p_i's. Note that we can replace $N \to \infty$ by $\epsilon \downarrow 0$, since the expression only depends on N via ϵ and $\epsilon(N) \downarrow 0$ as $N \to \infty$. Since the p_i's are all in M we can replace p_i by q and require that the convergence is uniform in $q \in M$.
Because of these considerations it remains to show that there exists $C > 0$ such that uniformly in $q \in M$:

$$\lim_{\epsilon\downarrow 0}\epsilon^{-2-d}\int_M k(d(p,q)/\epsilon)\left[f(p) - f(q)\right]\overline{V}(\mathrm{d}p) = C\Delta_M f(q). \tag{3.9}$$

Note that for every $\epsilon > 0$ this expression can be interpreted as the generator of a jump process on the manifold M. The process jumps from p to a (measurable) set $Q \subset M$ with rate $\int_Q \epsilon^{-2-d}k(d(p,q)/\epsilon)\mathrm{d}\overline{V}$.

Remark 3.6. Note that this is easy to show in $\mathbb{R}^d$. Indeed, using the transformation $u = (y-x)/\epsilon$ and Taylor, we see

$$\begin{aligned}&\epsilon^{-2-d}\int_{\mathbb{R}^d} k\left(\frac{\|y-x\|}{\epsilon}\right)(f(y) - f(x))\mathrm{d}y\\ =\ &\epsilon^{-2}\int_{\mathbb{R}^d} k(\|u\|)(f(x+\epsilon u) - f(x))\mathrm{d}u\\ =\ &\epsilon^{-1}\int_{\mathbb{R}^d} k(\|u\|)\nabla f(x)\cdot u\mathrm{d}u + \frac{1}{2}\int_{\mathbb{R}^d} k(\|u\|)u^T H(x)u\mathrm{d}u + O(\epsilon),\end{aligned}$$

where $H(x)$ is the Hessian of f in x. Now changing coordinates to integrate over each sphere B_r of radius r with respect to the appropriate surface measure S_r and then with respect to r, we obtain

$$\epsilon^{-1} \int_{\mathbb{R}} k(r) \int_{B_r} \nabla f(x) \cdot w S_r(\mathrm{d}w)\mathrm{d}r + \frac{1}{2} \int_{\mathbb{R}} k(r) \int_{B_r} w^T H(x) w S_r(\mathrm{d}w)\mathrm{d}r + O(\epsilon).$$

Now because of symmetry the integrals of w_i and of $w_i w_j$ over spheres vanish for each $i \neq j$. Moreover the integrals of w_i^2 do not depend on i, but only on r. Therefore the first term vanishes and we are left with

$$\frac{1}{2} \int_{\mathbb{R}} k(r) C(r) \Delta f(x) \mathrm{d}r + O(\epsilon) = C' \Delta f(x) + O(\epsilon).$$

This shows convergence (at least pointwise, for uniform convergence we have to be a little more careful about the $O(\epsilon)$).

3.5 Convergence result

As we argued above, it remains to show (3.9). Therefore let $p \in M$ be fixed (note that we reverse the notation of p and q with respect to (3.9)). We will pull the integral in (3.9) to the tangent space T_pM. There we analyse the measure with respect to which we integrate so that we can apply a result from Appendix A to obtain the convergence result.

3.5.1 Integral on the tangent space

First, let us introduce some notation. We will write

$$\begin{aligned} B(p,r) &= \{q \in M : d(p,q) < r\} \subset M \\ B_p(r) &= \{\eta \in T_pM : \|\eta\| < r\} \subset T_pM \\ B_d(r) &= \{v \in \mathbb{R}^d : \|v\| < r\} \subset \mathbb{R}^d. \end{aligned}$$

Since k is compactly supported, let $\alpha > 0$ be such that supp $k \subset [0, \alpha)$. Note that the integrand in (3.9) is

$$q \mapsto k(d(p,q)/\epsilon)(f(q) - f(p)),$$

which equals 0 outside of $B(p, \alpha\epsilon)$. Now fix normal coordinates (x, U) centered at p. For ϵ small enough $B(p, \alpha\epsilon) \subset U$. In that case we even know that

$$\exp_p : T_pM \supset B_p(\alpha\epsilon) \to B(p, \alpha\epsilon) \subset M$$

is a diffeomorphism. Using this we see that

$$\begin{aligned} & \int_M k\left(\tfrac{d(p,q)}{\epsilon}\right)(f(q)-f(p))\overline{V}(\mathrm{d}q) \\ = & \int_{B(p,\alpha\epsilon)} k\left(\tfrac{d(p,q)}{\epsilon}\right)(f(q)-f(p))\overline{V}(\mathrm{d}q) \\ = & \int_{B_p(\alpha\epsilon)} k\left(\tfrac{d(p,\exp_p(\eta))}{\epsilon}\right)(f(\exp_p(\eta))-f(p))\overline{V}\circ\exp(\mathrm{d}\eta). \end{aligned} \tag{3.10}$$

Now note that $d(p,\exp_p(\eta)) = \|\eta\|$, so (3.10) equals

$$\int_{B_p(\alpha\epsilon)} k\left(\tfrac{\|\eta\|}{\epsilon}\right)(f(\exp_p(\eta))-f(p))\overline{V}\circ\exp(\mathrm{d}\eta). \tag{3.11}$$

Now if we denote multiplication by ϵ in T_pM with λ_ϵ, we see that (3.11) equals

$$\int_{B_p(\alpha)} k(\|\eta\|)(f(\exp_p(\epsilon\eta))-f(p))\overline{V}\circ\exp_p\circ\lambda_\epsilon(\mathrm{d}\eta). \tag{3.12}$$

3.5.2 The measure $\overline{V}\circ\exp_p\circ\lambda_\epsilon$

Now we study the measure $\overline{V}\circ\exp_p\circ\lambda_\epsilon$ a bit more closely. Using the coordinate chart, we can write

$$\overline{V}\circ\exp_p\circ\lambda_\epsilon = (\overline{V}\circ x^{-1})\circ(x\circ\exp_p)\circ\lambda_\epsilon. \tag{3.13}$$

Now the measure $\overline{V}\circ x^{-1}$ equals $\sqrt{|G|}\mathrm{d}\lambda$ by definition, where λ is the Lebesgue measure on $\mathbb{R}^d$. We will analyse this expression later. Further, on $B_p(\alpha\epsilon)$, the mapping $x\circ\exp_p$ equals the isometry $\phi : T_pM \to \mathbb{R}^d$ that maps a vector in T_pM to its coordinates with respect to $(\partial_1,..,\partial_d)$, as we see in the following lemma.

Lemma 3.7. *$x\circ\exp = \phi$ on $B_p(\alpha\epsilon)_p$.*

Proof. Since (x,U) are normal coordinates centered at p, the geodesics through p are straight lines with respect to x. In other words, they are of the form $x(\gamma(t)) = ta+b$ with $a,b\in\mathbb{R}^d$. Let $\eta\in B_p(\alpha)$ and write $\eta = \eta^i\partial_i$. The geodesic starting at p with tangent vector η at p should satisfy

$$b = x(p) = 0, \qquad\qquad a_i = \eta^i = \phi^i(\eta)$$

for all i. Therefore we obtain the formula $x(\gamma(t)) = t\phi(\eta)$. Setting $t=1$, we see that $x(\exp_p(\eta)) = \phi(\eta)$. This implies that $x\circ\exp_p = \phi$ on $B_p(\alpha)$. □

From the properties of ϕ it now follows that

$$x\circ\exp : B_p(\alpha\epsilon) \to B_d(\alpha\epsilon)$$

is a homeomorphism. This situation is sketched in Figure 3.1. Therefore the inverse of $x \circ \exp$ is continuous, so in particular measurable. This implies that the pushforward measure of $\sqrt{|G|}\mathrm{d}\lambda$ through $(x \circ \exp)^{-1}$ is well-defined.

Lemma 3.7 implies that on $B_p(\alpha)$

$$(x \circ \exp_p) \circ \lambda_\epsilon = \phi \circ \lambda_\epsilon = \lambda_\epsilon \circ \phi. \tag{3.14}$$

The last equality holds since ϕ is an isometry and λ_ϵ is linear. We conclude from (3.13) and (3.14) that on $B_p(\alpha)$

$$\overline{V} \circ \exp_p \circ \lambda_\epsilon = \sqrt{|G|}\mathrm{d}\lambda \circ \lambda_\epsilon \circ \phi. \tag{3.15}$$

Finally, we find a useful expression for $\sqrt{|G|}\mathrm{d}\lambda \circ \lambda_\epsilon$.

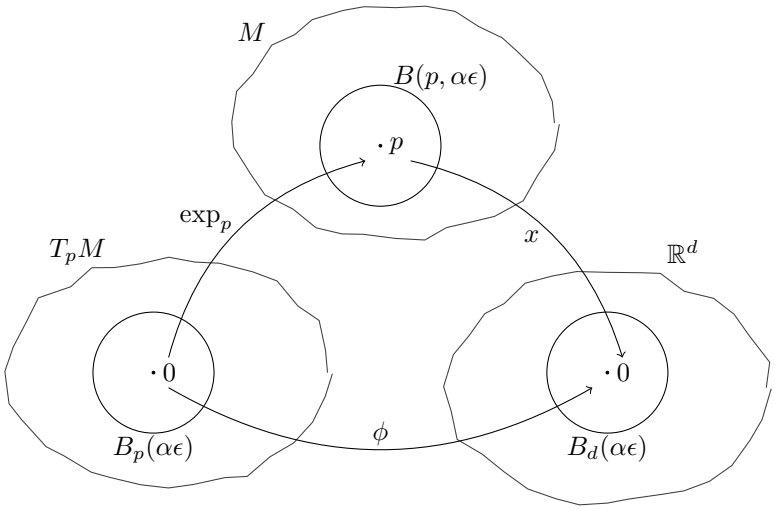


Figure 3.1: The situation in lemma 3.8. On $B_p(\alpha\epsilon)$: $x \circ \exp = \phi$. The uniform measure on $B(p, \alpha\epsilon)$ is moved via x to $B_d(\alpha\epsilon)$ using the formula $\sqrt{|G|}\mathrm{d}\lambda$. This measure can then be pulled back to $B(0, \alpha\epsilon)$ using ϕ. Since ϕ is an inner product space isomorphism, it will be easy to deal with orthogonal transformations later, in Lemma 3.11.

Lemma 3.8. *There exist a function $h : \mathbb{R}^d \to \mathbb{R}$ such that for t tending to 0, $h(t) = O(||t||^2)$ and*

$$\sqrt{|G|}\mathrm{d}\lambda \circ \lambda_\epsilon = \epsilon^d \frac{1 + h(\epsilon t)}{V(M)} \lambda(\mathrm{d}t).$$

Proof. According to [157, Cor 2.3], $\sqrt{|G|}$ can be expanded (in normal coordinates) as $1+h(x)$ where h is such that $h(x) = O(||x||^2)$. This implies that $\overline{V} \circ x^{-1} = \frac{1+h(t)}{V(M)}\lambda(\mathrm{d}t)$.

Now a simple transformation with respect to multiplication with ϵ yields

$$\left(\frac{1+h(t)}{V(M)}\lambda(\mathrm{d}t)\right)\circ\lambda_\epsilon = \epsilon^d\frac{(1+h(\epsilon t))}{V(M)}\lambda(\mathrm{d}t).$$

□

Remark 3.9. We used [157, Cor 2.3] in the proof above. In these notes the expansion of $\sqrt{|G(p,x)|}$ is calculated around a point p in normal coordinates x centered at p:

$$\sqrt{|G(p,x)|} = 1 - \frac{1}{6}\mathrm{Ric}(p)_{kl}x^k x^l + O\left(|x|^3\right). \tag{3.16}$$

As can be seen, there are no linear terms in the expansion. The coefficients for the quadratic terms are coefficients of the Ricci curvature of M in p. This implies that the way that the uniform distribution on a ball around p in M is pushed to the tangent space via the exponential map depends on the curvature of M in p. In particular, if there is no curvature, M is locally isomorphic to a neighbourhood in $\mathbb{R}^d$ so the same thing happens as in $\mathbb{R}^d$. This means that we get a constant time the uniform distribution on a ball around 0 in the tangent space.

Remark 3.10. We will need in Proposition 3.12 that the statement of Lemma 3.8 holds uniformly in all points of the manifold. This means that the difference between the uniform measure on a ball in the tangent space and the pulled back uniform measure on a geodesic ball in the manifold decays quadratically with ϵ uniformly in the manifold. Note that this uniform convergence is intuitively clear, since the difference between the two measures is caused by curvature and curvature is bounded in a compact manifold. As in the proof of Lemma 3.8, one needs to write

$$\sqrt{|G(p,x)|} = 1 + h_p(x)$$

for some function h_p that is $O(|x|^2)$ independent of p. Here for each p, (x,U) is a system of normal coordinates centered at p and $G(p,x)$ is the metric matrix expressed in local coordinates at the point with coordinate x. Note that this way for each p we use a different set of normal coordinates. Since $\sqrt{\ }$ and $|.|$ (the determinant function) are uniformly continuous in the right domains, it suffices to show that

$$G(p,x) = I + O(|x|^2), \tag{3.17}$$

where the $O(|x|^2)$ is independent of p. In other words we want that

$$||G(p,x) - I|| \leq C||x||^2, \tag{3.18}$$

where C does not depend on p. For all $p \in M$ (and for any system of normal coordinates centered at p) we have the following Taylor expansion (note that for fixed p, $G(p,\cdot)_{ij}$ is a map from a (subset of) $\mathbb{R}^d$ to $\mathbb{R}$):

$$G(p,x)_{ij} = \delta_{ij} + \frac{1}{3}R_{ijkl}x^k x^l + \sum_{|\beta|=3}\frac{3}{\beta!}\int_0^1 (1-t)^2 D^\beta G(p,\cdot)_{ij}(tx)\mathrm{d}t\cdot x^\beta. \tag{3.19}$$

From this we get (3.18) directly for fixed p, i.e. we have

$$||G(p,x) - I|| \leq C_p ||x||^2.$$

In order to obtain uniformity of C_p in p, we note that the functions of p and x appearing in the r.h.s. of (3.19) can be made smooth both in p and x. Smoothness in x is obvious (within the injectivity radius) and smoothness in p follows from a special choice of normal coordinates in such a way that they vary smoothly with p. A choice of normal coordinates is equivalent to a choice of an orthonormal basis, so one can construct smoothly varying normal coordinates by taking a smooth section of the orthonormal frame bundle (this can only be done locally, but it is enough to have the uniformity result locally, since then by compactness one has it globally). By compactness, the injectivity radius is bounded from below by some $\delta > 0$. Now for all $p \in M$ and $||x|| < \delta$, (3.19) holds and (locally) the quantities on the r.h.s. vary smoothly and therefore (again by compactness) one can show that $C := \sup_p C_p$ is finite.

3.5.3 Convergence to Δ_M

We define the measures μ and μ_R on T_pM by saying that

$$\mathrm{d}\mu = \left(\frac{1}{V(M)}\mathrm{d}\lambda\right) \circ \phi \qquad \text{and} \qquad \mathrm{d}\mu_R = \left(\frac{h(\epsilon\cdot)}{V(M)}\mathrm{d}\lambda\right) \circ \phi$$

on $B_p(0,\alpha)$ and 0 everywhere else. Then (3.15) and Lemma 3.8 imply that (3.12) equals

$$\epsilon^d \int_{B_p(0,\alpha)} k(||\eta||)(f(\exp_p(\epsilon\eta)) - f(p))(\mu + \mu_R)(\mathrm{d}\eta).$$

Now we define $\mathrm{d}\mu^k = k(||\cdot||)\mathrm{d}\mu$ (so the measure which has density $k(||\cdot||)$ with respect to μ) and analogously $\mathrm{d}\mu_R^k = k(||\cdot||)\mathrm{d}\mu_R$. Then we can write the integral above as

$$\epsilon^d \int_{T_pM} (f(\exp_p(\epsilon\eta)) - f(p))(\mu^k + \mu_R^k)(\mathrm{d}\eta).$$

In this way we transformed the integral to one that we work with in Section A.1 since we wrote it as the generator of a geodesic random walk (see L_N on page 191). [3] To use the theory that we obtained in Appendix A, we need the following lemma. It tells us that μ^k can be used as a stepping distribution for a geodesic random walk and it gives us the constant speed of the Brownian motion to which it converges (see section A.2).

[3] $p(\epsilon,\eta)$ in the Appendix is notation for $\exp_p(\epsilon\eta)$, so the point that is reached by following the geodesic from p in the direction of η for time ϵ. Also $\epsilon = \frac{1}{N}$ in Appendix A, but that is not the same N as in this chapter.

Lemma 3.11. *μ^k is canonical in the sense of Definition A.9. Moreover*

$$\int_{T_pM} ||\eta||^2 \mu^k(\mathrm{d}\eta) = \frac{2\pi^{d/2}}{V(M)\Gamma(d/2)} \int_0^\infty k(r) r^{d+1} \mathrm{d}r.$$

Proof. First of all recall that k is continuous and compactly supported, so the integral over k above makes sense and is finite. Define $\nu = \frac{1}{V(M)}\mathrm{d}\lambda$ on $B_n(\alpha)$ and 0 everywhere else. Then we can write $\mu = \nu \circ \phi$. Since ϕ preserves the norm, we see that $k(||\cdot||_{T_pM}) \circ \phi^{-1} = k(||\cdot||_{\mathbb{R}^d})$. This means that $\mu^k = \nu^k \circ \phi$, where $\nu^k := k(||\cdot||)\nu$. Since ϕ preserves the inner product, the measure μ^k behaves the same with respect to orthogonal transformations in T_pM as ν^k with respect to orthogonal transformations in $\mathbb{R}^d$. Since ν^k is clearly preserved under such transformations, so is μ^k. This shows that μ^k is canonical.
Now we calculate the corresponding constant.

$$\begin{aligned} & \int_{T_pM} ||\eta||^2_{T_pM} \mu^k(\mathrm{d}\eta) = \int_{\mathbb{R}^d} ||\phi^{-1}(v)||^2_{T_pM} \nu^k(\mathrm{d}v) \\ = & \int_{\mathbb{R}^d} ||v||^2_{\mathbb{R}^d} \nu^k(\mathrm{d}v) = \frac{1}{V(M)} \int_{B_d(\alpha)} ||v||^2_{\mathbb{R}^d} k(||v||_{\mathbb{R}^d}) \lambda(\mathrm{d}v). \end{aligned}$$

The first step holds because $\mu^k = \nu^k \circ \phi$ and the second step uses the fact that ϕ preserves the norm. Now we change coordinates in $\mathbb{R}^d$ and use the fact that $||v||$ is constant on spheres around the origin to obtain

$$\frac{1}{V(M)} \int_0^\alpha r^2 k(r) \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1} \mathrm{d}r = \frac{2\pi^{d/2}}{V(M)\Gamma(d/2)} \int_0^\infty k(r) r^{d+1} \mathrm{d}r.$$

Here $\frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}$ is the area of rS_{d-1}. In the last step we used that $\mathrm{supp}(k) \subset [0, \alpha]$. □

Now we use everything above to prove 3.9, which was the only remaining step to prove Theorem 3.4.

Proposition 3.12. *Set*

$$C = \frac{\pi^{d/2}}{V(M) d \Gamma(d/2)} \int_0^\infty k(r) r^{d+1} \mathrm{d}r.$$

Then as $\epsilon \to 0$ we have uniformly in $p \in M$:

$$\epsilon^{-2-d} \int_M k(d(p,q)/\epsilon) \left[f(q) - f(p)\right] \overline{V}(\mathrm{d}q) \longrightarrow C \Delta_M f(p).$$

Proof. Let $p \in M$. We can write

$$\begin{aligned} & \int_M k(d(p,q)/\epsilon)(f(q)-f(p))\overline{V}(\mathrm{d}q) \\ = \;& \epsilon^d \int_{T_pM} (f(\exp_p(\epsilon\eta))-f(p))(\mu^k+\mu_R^k)(\mathrm{d}\eta) \\ = \;& \epsilon^d \int_{T_pM} (f(\exp_p(\epsilon\eta))-f(p))\mu^k(\mathrm{d}\eta) + \epsilon^d \int_{T_pM} (f(\exp_p(\epsilon\eta))-f(p))\mu_R^k(\mathrm{d}\eta). \end{aligned}$$

From the results in Section A.1 and A.2 (Prop A.14) and Lemma 3.11, we see for the first term uniformly in p

$$\begin{aligned} & \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon^{2+d}} \epsilon^d \int_{T_pM} (f(\exp_p(\epsilon\eta))-f(p))\mu^k(\mathrm{d}\eta) \\ = \;& \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon^2} \int_{T_pM} (f(\exp_p(\epsilon\eta))-f(p))\mu^k(\mathrm{d}\eta) \\ = \;& \frac{1}{d} \frac{2\pi^{d/2}}{V(M)\Gamma(d/2)} \int_0^\infty k(r) r^{d+1} \mathrm{d}r \cdot \frac{1}{2}\Delta_M f(p) = C\Delta_M f(p). \end{aligned}$$

Now it suffices to show that the second term goes to zero at a rate independent of p. Let $\epsilon', K > 0$ such that $|h(s)| \leq K||s||^2$ for $s \in B_d(\epsilon')$. By Remark 3.10, K and ϵ' do not depend on p. Now note that for $\epsilon < \epsilon'$:

$$\begin{aligned} |\mu_R| &\leq \left(\sup_{t\in B_d(1)} |h(\epsilon t)| \right) \mu \leq \left(\sup_{t\in B_d(1)} K||\epsilon t||^2 \right) \mu \\ &= \left(\sup_{t\in B_d(1)} K\epsilon^2||t||^2 \right) \mu = K\epsilon^2\mu. \end{aligned}$$

Now we see:

$$\begin{aligned} & \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon^{2+d}} \epsilon^d \left| \int_{T_pM} f(\exp_p(\epsilon\eta)) - f(p)\mu_R^k(\mathrm{d}\eta) \right| \\ \leq \;& \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon^2} \int_{T_pM} \left| f(\exp_p(\epsilon\eta)) - f(p) \right| k(||\eta||)|\mu_R|(\mathrm{d}\eta) \\ \leq \;& \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon^2} \int_{T_pM} d(\exp_p(\epsilon\eta), p) L_f k(||\eta||) K\epsilon^2 \mu(\mathrm{d}\eta) \\ \leq \;& L_f K \lim_{\epsilon\downarrow 0} \int_{T_pM} \epsilon||\eta|| k(||\eta||)\mu(\mathrm{d}\eta) \\ = \;& L_f K \int_{T_pM} ||\eta|| k(||\eta||)\mu(\mathrm{d}\eta) \lim_{\epsilon\downarrow 0} \epsilon = 0, \end{aligned}$$

where we used that the integral is finite since k is bounded and has support in $[0, \alpha]$. Combining everything above gives what we wanted. □

3.6 Example grid

So far, we have seen that a sequence of grids is suitable for the hydrodynamic limit problem if the empirical distributions converge to the uniform distribution in the Kantorovich topology. We conclude by giving examples of such grids. To be more precise, we show that if one constructs a grid by adding uniformly sampled points from the manifold, this grid is suitable with probability 1.

Theorem 3.13. *Let $(P_n)_{n=1}^{\infty}$ be a sequence of i.i.d. uniformly random points of M. Recall that $\mu^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{P_i}$. Now it holds that $W_1(\mu^N, \overline{V}) \to 0$ as $N \to \infty$.*

To prove the theorem, we follow [153, Example 5.15]. In Section 3.6.1 we will show that the expectation of $W_1(\mu^N, \overline{V})$ goes to 0. Then we will show in Section 3.6.2 that it goes to 0 almost surely.

Remark 3.14 (Comparison with standard grids)**.** Recall the grids S^N on the one-dimensional torus S from Remark 3.1. We can show that the empirical measures corresponding to these grids along the subsequence $N = 2^m, m = 0, 1, 2, ..$ converge to the uniform measure on S with respect to the Kantorovich distance. To this end let $N = 2^m$ be fixed, call the corresponding empirical measure μ^N and call the uniform measure λ. Recall that the Kantorovich distance between these measures is alternatively given by

$$W_1(\mu^N, \lambda) = \inf_{\gamma \in \Gamma(\mu^N, \lambda)} \int_{S\times S} d(x, y)\gamma(dx, dy),$$

where $\Gamma(\mu^N, \lambda)$ is the set of all couplings of μ^N and λ. Now let Y be a uniform random variable on S and define

$$X = k/N \iff Y \in \left[\frac{k - 1/2}{N}, \frac{k + 1/2}{N}\right).$$

Denote the joint distribution of (X, Y) by ν. Then it is easy to see that $\nu \in \Gamma(\mu^N, \lambda)$. This implies that

$$W_1(\mu^N, \lambda) \leq \int_{S\times S} d(x, y)\nu(dx, dy) = \mathbb{E}_\nu(d(X, Y)) \leq \frac{1}{2N}.$$

This implies convergence with respect to the Kantorovich metric along the subsequence $N = 2^m, m = 0, 1, 2, ...$ Note, however, that the corresponding edge weights as described in this section are *not* the same as those in Remark 3.1.

3.6.1 Convergence of the expectation

Let $(P_n)_{n=1}^{\infty}$ be a sequence of i.i.d. uniformly random points of M and set $\mu^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{P_i}$. For now, let N be fixed. Let $\mathscr{F}_1$ be the set of Lipschitz function on M with Lipschitz constant ≤ 1. Then we define for $f \in \mathscr{F}_1$ the random variable

$X_f = \mu^N f - \overline{V} f$. Note that both μ^N and $\overline{V}$ are probability distributions, so $X_f(\omega)$ is Lipschitz in f for each ω:

$$|X_f - X_g| = |\mu^N f - \overline{V} f - (\mu^N g - \overline{V} g)| \leq |\mu^N (f-g)| + |\overline{V}(f-g)| \leq 2||f-g||_\infty.$$

Now note that since f has Lipschitz constant ≤ 1:

$$\sup_{p \in M} f(p) - \inf_{q \in M} f(q) = \sup_{p,q \in M} |f(p) - f(q)| \leq \sup_{p,q \in M} d(p,q) =: K.$$

M is compact, so $K < \infty$. Since adding constants to f does not change X_f, it suffices to consider $f \in \mathscr{F}_{1,K} = \{g \in \mathscr{F}_1 : 0 \leq g \leq K\}$. It follows that for each $f \in \mathscr{F}_{1,K}$ by writing

$$X_f = \sum_{i=1}^N \frac{f(X_i) - \overline{V} f}{N},$$

we see that it is a sum of i.i.d. random variables taking values in $[-\frac{K}{N}, \frac{K}{N}]$. By the Azuma-Hoeffding inequality, this implies that X_f is $\frac{K^2}{N}$-subgaussian for each $f \in \mathscr{F}_{1,K}$. Now [153, Lemma 5.7] shows that

$$\mathbb{E}[W_1(\mu^N, \overline{V})] \leq \inf_{\epsilon > 0} \left\{ 2\epsilon + \sqrt{\frac{2K^2}{N} \log N(W, ||\cdot||_\infty, \epsilon)} \right\},$$

where $N(\mathscr{F}_{1,K}, ||\cdot||_\infty, \epsilon)$ is the minimal number of points in some space containing $\mathscr{F}_{1,K}$ such that the balls of radius ϵ with respect to the uniform distance around those points cover $\mathscr{F}_{1,K}$.

We now need to estimate this covering number. To do this we need an upper bound of the covering number $N(M, d, \epsilon)$ of M (note that the d in $N(M, d, \epsilon)$ denotes the Riemannian metric on the manifold and should not be confused with the dimension of the manifold). Since M is compact there exist $a, \delta > 0$ such that for all $0 < \epsilon < \delta$: $N(M, d, \epsilon) \leq a\epsilon^{-d}$ (see for instance [112, Lemma 4.2]). Using this we can prove the following.

Lemma 3.15. *There is a $c > 0$ such that for all $0 < \epsilon < \delta$:*

$$N(\mathscr{F}_{1,K}, ||\cdot||_\infty, \epsilon) \leq \exp c/\epsilon^d.$$

Proof. Fix $\epsilon > 0$ and call $m = N(M, d, \epsilon/4)$. By definition of this number, we can find points $p_1, .., p_m \in M$ such that $\cup_{i=1}^m B(p_i, \epsilon/4) \supset M$. Now define $V_1 = B(p_1, \epsilon/4)$ and for $i \geq 2$: $V_i = B(p_i, \epsilon/4) \setminus \cup_{j=1}^{i-1} V_j$. Now for $f \in \mathscr{F}_{1,K}$, define $\pi^f : M \to \mathbb{R}$ by

$$\pi^f : V_i \ni p \mapsto \epsilon \left(\left\lfloor \frac{f(p_i)}{\epsilon} \right\rfloor + \frac{1}{2} \right).$$

Since each $p \in M$ is contained in exactly one V_i (by construction), this map is well-defined. Note that if $k\epsilon \leq f(p_i) < (k+1)\epsilon$, then $\pi^f = (k + 1/2)\epsilon$ on V_i. In particular

clearly $|f(p_i) - \pi^f(p_i)| \leq \epsilon/2$. Now denote $Y = \{\pi^f | f \in \mathscr{F}_{1,K}\}$.
Now fix $f \in \mathscr{F}_{1,K}$ and $p \in M$. Let i be such that $p \in V_i$. Then we see:

$$\begin{aligned} |\pi^f(p) - f(p)| &= |\pi^f(p_i) - f(p)| \leq |\pi^f(p_i) - f(p_i)| + |f(p_i) - f(p)| \\ &\leq \epsilon/2 + L_f d(p_i, p) \leq \epsilon/2 + \epsilon/4 < \epsilon. \end{aligned}$$

This shows that $||\pi^f - f||_\infty \leq \epsilon$, which implies that Y is an ϵ-net for $\mathscr{F}_{1,K}$. Hence $N(\mathscr{F}_{1,K}, ||\cdot||_\infty, \epsilon) \leq \#Y$.

All we have to do now is estimate $\#Y$.
First of all let $\pi^f \in Y$. Note that if $d(p_i, p_j) \leq \epsilon/2$, we see

$$\begin{aligned} |\pi^f(p_i) - \pi^f(p_j)| &\leq |\pi^f(p_i) - f(p_i)| + |f(p_i) - f(p_j)| + |f(p_j) - \pi^f(p_j)| \\ &\leq \epsilon/2 + L_f d(p_i, p_j) + \epsilon/2 = 3\epsilon/2. \end{aligned}$$

Since $|\pi^f(p_i) - \pi^f(p_j)| = k\epsilon$ for some $k \in \mathbb{Z}$, we conclude $|\pi^f(p_i) - \pi^f(p_j)| \in \{-\epsilon, 0, \epsilon\}$, so $\pi^f(p_i) \in \{\pi^f(p_j) - \epsilon, \pi^f(p_j), \pi^f(p_j) + \epsilon\}$.

Now define a graph G with vertices $p_1, .., p_m$ by putting an edge between p_i and p_j whenever $d(p_i, p_j) \leq \epsilon/2$. Any π^f is uniquely specified by its values on the nodes of G. Note further that whenever we know π^f for some point of the graph, there are only 3 possible values left for each of its neighbours (since neighbours are at distance at most $\epsilon/2$). Now $\#Y$ is dominated by the amount of ways in which we can assign values of the type $(k+1/2)\epsilon$ to nodes of G while keeping this restriction into account. Define, for $i \leq 0$, $S_i = \{p \in G : d_G(p_1, p) = i\}$, where $d_G(p, q)$ denotes the minimum amount of edges that need to be followed to walk from p to q in G.

Now we can start counting. For p_1, there are at most $\lceil K/\epsilon \rceil$ possible values (recall that any $f \in \mathscr{F}_{1,K}$ has $0 \leq f \leq K$). Each node in S_1 is a distance at most $\epsilon/2$ from p_1, so each node can take at most 3 values. This brings the possible amount of value assignments to (less than) $\lceil K/\epsilon \rceil 3^{\#S_1}$. Now each node in S_2 is at distance at most $\epsilon/2$ of a node in S_1, so each of these can take at most 3 different values. This brings the number of options so far to at most $\lceil K/\epsilon \rceil 3^{\#S_1} 3^{\#S_2}$. Continuing in this way, we obtain that the number of ways to assign values is at most

$$\left\lceil \frac{K}{\epsilon} \right\rceil \prod_{i=1}^{\infty} 3^{\#S_i} = \left\lceil \frac{K}{\epsilon} \right\rceil 3^{\sum_{i=1}^{\infty} \#S_i} = \left\lceil \frac{K}{\epsilon} \right\rceil 3^{m-1} = \left\lceil \frac{K}{\epsilon} \right\rceil 3^{N(M,d,\epsilon/4)-1}.$$

Recall that m is the total amount of balls as we defined at the beginning of the proof, which we chose equal to $N(M, d, \epsilon/4)$. Now we know that for $0 < \epsilon < \delta$

$$N(\mathscr{F}_{1,K}, ||\cdot||_\infty, \epsilon) \leq \left\lceil \frac{K}{\epsilon} \right\rceil 3^{a/(\epsilon/4)^d - 1} = \left\lceil \frac{K}{\epsilon} \right\rceil 3^{a4^d/\epsilon^d - 1}.$$

This implies that there exists $c > 0$ such that for all $0 < \epsilon < \delta$,

$$N(\mathscr{F}_{1,K}, ||\cdot||_\infty, \epsilon) \leq \mathrm{e}^{c/\epsilon^d}.$$

□

Now we see that for any $0 < \epsilon < \delta$:

$$\mathbb{E}[W_1(\mu^N, \overline{V})] \leq 2\epsilon + \sqrt{\frac{2K^2}{N} \log \exp c/\epsilon^d} = 2\epsilon + \sqrt{\frac{2cK^2}{N}} \epsilon^{-d/2}.$$

Elementary methods show that this value takes a minimum at $\epsilon = c_0 N^{\frac{-1}{d+2}}$ where c_0 is some constant (take N large enough such that $c_0 N^{\frac{-1}{d+2}} < \delta$). This shows that the optimal bound that we get is

$$2c_0 N^{\frac{-1}{d+2}} + \sqrt{\frac{2cK^2}{N}} \left(c_0 N^{\frac{-1}{d+2}} \right)^{-d/2} = 2c_0 N^{\frac{-1}{d+2}} + c_1 N^{\frac{-1}{d+2}}$$

where c_1 is the product of some constants that don't depend on N. This shows that

$$\mathbb{E}[W_1(\mu^N, \overline{V})] \leq (2c_0 + c_1) N^{\frac{-1}{d+2}} \to 0$$

as $n \to \infty$.

3.6.2 Almost sure convergence

It remains to show that $W_1(\mu^N, \overline{V})$ goes to zero almost surely. For a function $f : M^N \to \mathbb{R}$ define

$$D_i f(p_1, .., p_N) = \sup_{z \in M} f(p_1, .., p_{i-1}, z, p_{i+1}, .., p_N) - \inf_{z \in M} f(p_1, .., p_{i-1}, z, p_{i+1}, .., p_N).$$

Further, define the function $H : M^N \to \mathbb{R}$ by

$$(p_1, .., p_N) \mapsto \sup_{g \in \mathscr{F}_1} \left\{ \frac{1}{N} \sum_{i=1}^{N} g(p_i) - \int_M g \mathrm{d}\overline{V} \right\}.$$

Note that $H(p_1, .., p_N) = W_1(\mu^N, \overline{V})$.

Lemma 3.16. *Set (as before) $K = \sup_{p,q \in M} d(p, q)$. Then for each $1 \leq j \leq N$: $||D_j H||_\infty \leq K/N$.*

Proof. Let $1 \leq j \leq N$ and fix $p_1, .., p_N$. Denote for $p \in M$ and $g \in \mathscr{F}_1$

$$J^j(g, p) = \frac{1}{N} \left(\sum_{i=1, i \neq j}^{N} g(p_i) + g(p) \right) - \int_M g \mathrm{d}\overline{V}.$$

Now let $p, q \in M$. Then for any $g \in \mathscr{F}_1$:

$$|J^j(g, p) - J^j(g, q)| = \frac{1}{N} |g(p) - g(q)| \leq \frac{1}{N} d(p, q) \leq \frac{K}{N}.$$

This shows that $g \mapsto J^j(g,p)$ and $g \mapsto J^j(g,q)$ are always at most K/N apart from each other, which implies that

$$\left| \sup_{g \in \mathscr{F}_1} J^j(g,p) - \sup_{g \in \mathscr{F}_1} J^j(g,q) \right| \leq \frac{K}{N}.$$

Now $D_i H(p_1, .., p_N)$ equals

$$\begin{aligned} & \sup_{p \in M} H(p_1, .., p_{i-1}, p, p_{i+1}, .., p_N) - \inf_{q \in M} H(p_1, .., p_{i-1}, q, p_{i+1}, .., p_N) \\ = & \sup_{p,q \in M} |H(p_1, .., p_{i-1}, p, p_{i+1}, .., p_N) - H(p_1, .., p_{i-1}, q, p_{i+1}, .., p_N)| \\ = & \sup_{p,q \in M} \left| \sup_{g \in \mathscr{F}_1} J^j(g,p) - \sup_{g \in \mathscr{F}_1} J^j(g,q) \right| \leq \frac{K}{N}. \end{aligned}$$

Since $P_1, .., P_N$ were arbitrary, we conclude that $||D_j H||_\infty \leq \frac{K}{N}$. □

Now we are in position to prove the main result.

Proposition 3.17. *$W_1(\mu^N, \overline{V}) \to 0$ almost surely as $N \to \infty$.*

Proof. Since $P_1, .., P_N$ are independent, [153, Theorem 3.11] gives us that for any $t > 0$

$$\begin{aligned} \mathbb{P}(W_1(\mu^N, \overline{V}) - \mathbb{E}W_1(\mu^N, \overline{V}) > t) &= \mathbb{P}\left(H(P_1, .., P_N) - \mathbb{E}H(P_1, .., P_N) > t\right) \\ &\leq \exp\left(\frac{-2t^2}{\sum_{k=1}^N ||D_k H||_\infty^2} \right) \leq \exp\left(\frac{-2t^2 N}{K^2} \right), \end{aligned}$$

where the last inequality follows from lemma 3.16. For reasons of symmetry we obtain

$$\mathbb{P}\left(\left| W_1(\mu^N, \overline{V}) - \mathbb{E}W_1(\mu^N, \overline{V}) \right| > t \right) \leq 2 \exp\left(\frac{-2t^2 N}{K^2} \right).$$

By a standard application of the Borel-Cantelli lemma, this implies that $W_1(\mu^N, \overline{V}) - \mathbb{E}W_1(\mu^N, \overline{V}) \to 0$ a.s. Since we have already seen that $\mathbb{E}W_1(\mu^N, \overline{V}) \to 0$, we conclude that a.s. as $N \to \infty$

$$W_1(\mu^N, \overline{V}) \to 0.$$

□

We conclude that sampling uniformly from the manifold yields a suitable grid with probability 1.

3.7 Notes and perspectives

The goal of this chapter was to construct grids on which we can consider the Symmetric Exclusion Process and the Gaussian Free Field later in this thesis. As we will see in the coming chapters, the grids in this chapter will indeed suffice for that, so in that sense we have been successful. However, there are still directions in which we can improve.

First, as we mentioned in Remark 3.2, the amount of grid points within distance ϵ, the typical jumping distance of a particle that follows the rates W^ϵ, goes to infinity as N goes to infinity. This is very different from the $\mathbb{R}^d$ case, where particles only jump to the $2d$ nearest neighbours. As we explain in Remark 3.2, this is not a coincidence, but it is something that we need in the proof of convergence of the graph Laplacians to the Laplace-Beltrami operator. The reason is basically that the $2d$ neighbours of each grid point of $\mathbb{Z}^d$ are situated perfectly symmetrically with respect to each other, which makes first order terms disappear. To get rid of this, we will need to settle for something weaker than pointwise convergence of the graph Laplacians. This possibility is supported by other grids that we mentioned in Section 3.2.2. For instance the Poisson-Voronoi tesselations or constructions using ϵ-nets yield grids where the degrees of the grid points do not grow with N, the amount of grid points. In the cases of the ϵ-nets the spectra of the graph Laplacians converge to the spectrum of the Laplace-Beltrami operator. The most difficult part of this approach would be to find a proof for the hydrodynamic limit of SEP that depends in a less direct way on the pointwise convergence of the graph Laplacians. For convergence of the DGFF, this might be easier. In that case we only need convergence of the Green's operator applied to functions in an inner product. There pointwise convergence is less important, so spectral convergence of the graph Laplacians might be enough.

Another aspect that is different from the $\mathbb{R}^d$ case is the lack of an explicit description of a sequence of grids. Indeed, in $\mathbb{R}^d$ we can set the grids to be $\frac{1}{N}\mathbb{Z}^d$, which is very straightforward and explicit. We know that on a manifold we obtain a suitable sequence of grids with probability 1 by sampling uniformly from the manifold. However, it would be nice to have a way to deterministically construct such grids. A reason for this is that the the rate with which ϵ can go to 0 depends on the distance $W_1(\mu^N, \overline{V})$ of the empirical measures corresponding to the grids to the uniform measure. If the grids are not fixed, ϵ is not fixed (as a function of N). Also, one would like to use as few grid points as possible to have a good approximation of the hydrodynamic limit. For that it is important to find a sequence of grids for which $W_1(\mu^N, \overline{V})$ goes to 0 as fast as possible. It would be interesting to find an explicit way to construct grids like that. Alternatively, one could think of not sampling the points on the manifold uniformly and independently, but to let the distribution of point p_{N+1} depend on G_N by making the probability to sample from a region with few points higher.

So far we have considered a complete, smooth, connected and compact d-dimensional Riemannian manifold M. Of all of these properties probably the only one that would

make sense to let go is compactness. Indeed, instead of sampling uniformly from the manifold, we could use a Poisson Point Process on a non-compact manifold. Compare for instance with Faggionato et al. [58]. We could then try to prove the same results as in this chapter. It is likely that we will need some other condition like curvature bounds to ensure that the random walk on the grid does not 'run away' to infinity in finite time. We might not be able to obtain uniform convergence in the non-compact case, maybe again under curvature bounds. Otherwise we should consider the graph Laplacians as operators acting on smooth functions that are compactly supported.

Finally, we mention that the requirements that the grid points approximate the manifold uniformly, that the weights are symmetric and that the graph Laplacians approximate the Laplace-Beltrami operator are all in some sense due to the properties of SEP. We explained this in Section 3.2.1. When considering other particle systems on a manifold, one might be able to let some or all of these requirements go. However, treating other IPS than SEP brings along other issues. See also Section 4.5.

Appendix

Lemma. Let $(p_i)_{i=1}^{\infty}$ be a sequence for which the empirical measures converge to the volume measure in the Kantorovich sense. Define $\epsilon(N)$, W_{ij}^N and k as in Section 3.3. Additionally suppose that there exists some $\alpha > 0$ such that $k(x) > 0$ for all $x \leq \alpha$. Say that there is an edge between p_i and p_j whenever $W_{ij}^N > 0$. Then the corresponding graphs are eventually connected (in other words: there is some N_0 such that for all $N \geq N_0$, V_N with edges as just defined is connected).

Proof. Define

$$\begin{aligned} G_N(\beta) &:= \text{graph obtained from } V_N \text{ by connecting vertices at distance } \leq \beta \\ \beta_N &:= \inf\{\beta \geq 0 : G_N(\beta) \text{ is connected}\}. \end{aligned}$$

Since $G_N(0)$ is not connected (for $N > 1$), $G_N(\sup_{p,q\in M} d(p,q))$ is connected and $G_N(\beta_1)$ contains all edges of $G_N(\beta_2)$ for $\beta_1 \geq \beta_2$ it is clear that β_N is a finite number strictly larger than 0. Further note that $G_N(\beta_N)$ is connected (so the infimum is actually a minimum).

Now note that there must be two points $p', q' \in V_N$ such that p, q have an edge between them for $\beta = \beta_N$ and are not connected for $\beta < \beta_N$ (we call p and q connected if there is a path from p to q). Indeed if any pair $p, q \in V_N$ that has an edge between them for $\beta = \beta_N$ is still connected by some path for some $\beta_{pq} < \beta_N$, we see that for $\beta' = \sup_{p,q} \beta_{pq} < \beta_N$ the graph $G_N(\beta')$ is connected, which contradicts the definition of β_N (note that the supremum ranges over a finite amount of numbers, since V_N is finite). Fix such $p', q' \in V_N$.

Now let s_N be a point on M such that $d(p', s_N) = d(q', s_N) = \beta_N/2$. Then $B(s_N, \beta_N/4)$ does not contain any point of V_N (since by the triangle inequality such point would

have distance $\leq 3\beta_N/4$ to both p' and q' so p' and q' would be connected to each other via this point in $G_N(3\beta_N/4)$, which contradicts the choice of p' and q').

Now we define the following function $l_N : M \to \mathbb{R}$

$$l_N(p) = \begin{cases} d(p, s_N) - \frac{\beta_N}{4} & p \in B\left(s_N, \frac{\beta_N}{4}\right) \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $|l_N(p) - l_N(q)| \leq d(p, q)$, so l_N is Lipschitz with $L_{l_N} \leq 1$. This implies that

$$W_1(\mu^N, \overline{V}) \geq \int l_N \mathrm{d}\mu_N - \int l_N \mathrm{d}\overline{V}.$$

Since l_N is only non-zero on $B(s_N, \beta_N/4)$ and this set does not contain points of V_N, we see that

$$\int l_N \mathrm{d}\mu_N = 0.$$

Further, since l_N is non-positive and $l_N \leq -\beta_N/8$ on $B(s_N, \beta_N/8$, we see that

$$\int l_N \mathrm{d}\overline{V} \leq -\overline{V}\left(B\left(s_N, \frac{\beta_N}{8}\right)\right)\frac{\beta_N}{8},$$

so we conclude that $W_1(\mu^N, \overline{V}) \geq \overline{V}(B(s_N, \beta_N/8))\beta_N/8$. Since $W_1(\mu^N, \overline{V})$ goes to zero, it is easy to deduce from this inequality that $\beta_N \to 0$. Hence there are constants $C', C'' > 0$ (not depending on s_N), such that for N large enough

$$W_1(\mu^N, \overline{V}) \geq \overline{V}\left(B\left(s_N, \frac{\beta_N}{8}\right)\right)\frac{\beta_N}{8} \geq C''\left(\frac{\beta_N}{8}\right)^d \frac{\beta_N}{8} = C'\beta_N^{d+1}.$$

Now we see there is a $C > 0$ such that for N large enough

$$\epsilon_N = \left(\sup_{m \geq N} W_1(\mu^m, \overline{V})\right)^{\frac{1}{4+d}} \geq W_1(\mu^N, \overline{V})^{\frac{1}{4+d}} \geq C\beta_N^{\frac{d+1}{d+4}}.$$

This implies that there is some N_0 such that for all $N \geq N_0$, $\alpha\epsilon_N \geq \beta_N$. By our choice of k, all points at distance $\alpha\epsilon_N$ or less are joined by an edge, so this inequality combined with the definition of β_N shows that for all $N \geq N_0$, V_N with edges as defined in the lemma statement is connected. □

Chapter 4

Hydrodynamic limit of the Symmetric Exclusion Process

The main goal of this chapter[1] is to define the Symmetric Exclusion Process (SEP) on a manifold and prove that its hydrodynamic limit is the heat equation on the manifold. To do this, we will use uniformly approximating grids. In Chapter 3 we saw that such grids exist (and how they can be obtained by sampling points from the manifold and using the distance between points to define edge weights).

4.1 Introduction

Hydrodynamic limits of interacting particle systems is a well established subject. A large variety of parabolic equations (such as the non-linear heat equation) and hyperbolic conservation laws have been obtained from microscopic stochastic particle systems; see Kipnis and Landim [96], De Masi and Presutti [36], Seppäläinen [136] for overviews. Usually, the setting here is that in the underlying particle system the particles move on the lattice $\mathbb{Z}^d$, and after rescaling the limiting partial differential equation is defined on $\mathbb{R}^d$, or on a subdomain of $\mathbb{R}^d$ such as an interval, where then equations with boundary conditions on the ends of the interval are derived (e.g. Dirichlet boundary conditions for the case where at the right and left end the system is coupled to a reservoir fixing the density of particles, see Gonçalves [75]).

Motivated e.g. by the study of the motion of proteins in a cell-membrane, or more general motion of particles on curved interfaces, it is clear that there are many relevant physical systems of which the macroscopic motion takes place on a Riemannian

[1] This chapter is based on research that was first started in van Ginkel [148] and then continued and finalised in van Ginkel and Redig [149].

manifold rather than on Euclidean space. It is the aim of this chapter to provide first steps in this direction, by considering the simplest interacting particle system on a suitable discretisation of a Riemannian manifold and proving its hydrodynamic limit. The Symmetric Exclusion Process (SEP) is a well-known and well-studied interacting particle system for which in standard setting it is rather straightforward to obtain the hydrodynamic limit. As was explained in the introduction of this thesis (in Section 1.3.3), an important reason for this is the fact that the microscopic equation for the expectation of the density field is already a closed equation. The complexity in this chapter comes from the fact that the underlying space is curved. We consider SEP on a suitable discretisation (a notion defined more precisely below) of a compact Riemannian manifold and prove that its empirical density field, after appropriate rescaling, converges to the solution of the heat equation on the manifold.

We start in Section 4.2 by defining SEP on a sequence of uniformly approximating grids on a compact Riemannian manifold. Then, in Section 4.3, we define the hydrodynamic limit in this context and state the main result: Theorem 4.3. This result is proven in Section 4.4, where we generalise the strategy that is used in [136, Chapter 8] for the Euclidean case. We end the chapter with some notes and perspectives in Section 4.5.

4.2 Symmetric Exclusion Process on uniformly approximating grids

First we introduce the Symmetric Exclusion Process (SEP). SEP is an interacting particle system that was introduced in Spitzer [139] and studied in detail in Liggett [109, Chapter 8]. The idea is that there is some (possibly countably infinite) number of particles on a (possibly countably infinite) graph G. The particles are considered identical. Each particle jumps after independent exponential times with parameter 1 from x to y with probability $p(x, y)$, provided that the place that it wants to jump to is not already occupied. Otherwise, the jump is suppressed. We assume that $p(x, y) = p(y, x)$. Let $\eta_t \in \{0, 1\}^G$ denote the configuration of the particles at time t, i.e. $\eta_t(x) = 1$ if there is a particle at place $x \in G$ at time t and 0 else. We will sometimes write $\eta(p, t) = \eta_t(p)$. For any configuration η and points x, y define η^{xy} by

$$\eta^{xy}(z) = \begin{cases} \eta(x) & \text{if } z = y \\ \eta(y) & \text{if } z = x \\ \eta(z) & \text{if } z \neq x, y \end{cases}.$$

An equivalent description of this process is the following. The edges have independent exponential clocks with rate $p(x, y) = p(y, x)$ for $x, y \in G$. Whenever a clock rings, the particles that are at either side of the corresponding edge jump along the edge (and the clock is restarted). This means that if there are no particles, nothing happens. If there is one particle, it jumps. If there are two particles, they switch places. Since

we are not interested in individual particles, the configuration stays the same in the latter case. Note that in this way there can never be more than two particles at the same place. Using the notation introduced above, we see that the generator of this process is defined on the core of local functions as

$$Lf(\eta) = \frac{1}{2}\sum_{x,y} p(x,y)(f(\eta^{xy}) - f(\eta)).$$

The factor $\frac{1}{2}$ is there since we count every edge twice.

Before we can define SEP on a manifold, we need to introduce the grids. Let M be a complete, smooth, connected and compact n-dimensional Riemannian manifold. Let $(G_N, W_N)_{N=1}^{\infty}$ be a sequence of uniformly approximating grids with corresponding weights. In particular we assume the following. Let $(p_n)_{n=1}^{\infty}$ be a sequence in M such that the corresponding empirical measures converge weakly to the uniform measure $\overline{V}$ on the manifold. Set $G^N = \{p_1, .., p_N\}$. On each G^N, there is a random walk X^N which jumps from p_i to p_j with (symmetric) rate W_{ij}^N. We assume that there exists some function $a : \mathbb{N} \to [0, \infty)$ and some constant $C > 0$ such that for each smooth ϕ

$$a(N)\sum_{j=1}^{N} W_{ij}^N(\phi(p_j) - \phi(p_i)) \longrightarrow C\Delta_M\phi(p_i) \quad (N \to \infty)$$

where the convergence is in the sense that for all smooth ϕ

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \left| a(N)\sum_{j=1}^{N} W_{ij}^N(\phi(p_j) - \phi(p_i)) - C\Delta_M\phi(p_i) \right| = 0. \tag{4.1}$$

By dividing $a(N)$ by C if necessary, we can assume that $C = 1$.

Remark 4.1. Note that for the result of this section it is not necessary to construct grids from a sequence. Any sequence of finite grids such that (4.1) holds would do. However, since the grid that we constructed in Chapter 3 is of this form, we formulate our results in this chapter in the same way.

We now define the SEP $\eta^N = (\eta_t^N)_{t\geq 0}$ on G^N through the generator

$$L^N h(\eta) = \frac{a(N)}{2}\sum_{i,j=1}^{N} W_{ij}^N(h(\eta^{ij}) - h(\eta)), \quad h : \{0,1\}^{G^N} \to \mathbb{R}.$$

Here $\eta^{ij} := \eta^{p_i p_j}$. It follows from our considerations above that this process describes particles that perform independent random walks according to X^N with the restriction that jumps to occupied sites are suppressed. As initial configuration we set $\eta_0^N(p_i) = X_i$, where $(X_i)_{i=1}^{\infty}$ is some sequence of (possibly degenerate) random variables taking values in $\{0, 1\}$.

4.3 Hydrodynamic limit

We will use this section to give the basic definitions concerning the hydrodynamic limit. At a microscopic scale, the particles are just random walkers with some interaction, but at the macroscopic scale (where limits are taken in space and time), the behaviour is deterministic: it is described by a partial differential equation (in our case the heat equation). The goal is to prove this rigorously.

4.3.1 Trajectories of empirical measures

Write $R(M)$ for the space of Radon measures on M with the vague topology and let $D = D([0,\infty), R(M))$ denote the space of all paths $\gamma : [0,\infty) \to R(M)$ such that γ is right continuous and has left limits. On this space we can define the Skorokhod metric (see for instance Seppäläinen [136, Appendix A.2.2]). Since $R(M)$ is a Polish space, it can be shown that D with the Skorokhod metric is a Polish space too. Define

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{p_i} \eta_t^N(p_i),$$

where δ_p is the Dirac measure which places mass 1 at $p \in M$. It puts a point mass at each particle and rescales it by the amount of possible positions, which represents the particle configuration η_t^N at time t. In particular μ_t^N is a sub-probability measure and is in $R(M)$.

Instead of dealing with this problem pointwise for each t, we will look at trajectories. As the particles move according to the SEP, $\gamma^N : [0,\infty) \to R(M)$ defined by $t \mapsto \mu_t^N$ is a random trajectory and hence a random element of D. It represents the positions of the particles over time. The initial configuration $X_1, .., X_N$ and the dynamics of the SEP determine a distribution Q^N on D. In this way we obtain a sequence $(Q^N)_{N=0}^{\infty}$ of measures on D.

4.3.2 Initial configuration

We assume that there exists a measurable function $\rho_0 : M \to \mathbb{R}$ such that $0 \le \rho_0 \le 1$ and μ_0^N converges vaguely to $\rho_0 \mathrm{d}\overline{V}$ in probability, i.e. for any continuous ϕ as $N \to \infty$:

$$\int_M \phi \mathrm{d}\mu_0^N \to \int_M \rho_0 \phi \mathrm{d}\overline{V} \quad \text{in probability.} \tag{4.2}$$

If this is the case, we say that $\rho_0 \mathrm{d}V$ is the density profile corresponding to the configurations η_0^N. Note that using measures here to represent the particles provides a bridge between separate particles (discrete measures) and density profiles (measures that are absolutely continuous with respect to V). We would like to show that if this initial condition is given, then at any time t the configurations η_t^N have a corresponding density profile $\rho_t \mathrm{d}\overline{V}$. Moreover, we want to show that $t \mapsto \rho_t$ solves the heat equation with initial condition ρ_0.

Example 4.2. The weak convergence of the empirical measures corresponding to the grids G_N implies that for any continuous f: $\frac{1}{N}\sum_{i=1}^{N} f(p_i) \to \int_M f \mathrm{d}\overline{V}$. Define the random variables $(X_i)_{i=1}^{\infty}$ to be independent Bernoulli random variables with $\mathbb{E}X_i = \rho_0(p_i)$ for some continuous function $\rho_0 : M \to \mathbb{R}$ with $0 \leq \rho_0 \leq 1$. Then we see as $N \to \infty$:

$$\begin{aligned}\mathbb{E}\left[\int \phi \mathrm{d}\mu_0^N\right] &= \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\phi(p_i)\eta_0^N(p_i)\right] = \frac{1}{N}\sum_{i=1}^{N}\phi(p_i)\mathbb{E}\eta_0^N(p_i)\\ &= \frac{1}{N}\sum_{i=1}^{N}\phi(p_i)\rho_0(p_i) \to \int \phi\rho_0 \mathrm{d}\overline{V},\end{aligned}$$

since ϕ and ρ_0 are continuous. Further,

$$\begin{aligned}\mathrm{Var}\left[\int \phi \mathrm{d}\mu_0^N\right] &= \mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^{N}\phi(p_i)\eta_0^N(p_i)\right] = \frac{1}{N^2}\sum_{i=1}^{N}\phi(p_i)\mathrm{Var}(\eta_0^N(p_i))\\ &= \frac{1}{N^2}\sum_{i=1}^{N}\phi(p_i)\rho_0(p_i)(1-\rho_0(p_i)) \to 0.\end{aligned}$$

Together this implies that (4.2) holds here for any continuous ϕ.

4.3.3 Main result

After all these definitions, we can state the main result of this section.

Theorem 4.3. *Let M be complete, smooth, connected and compact n-dimensional Riemannian manifold and let $(G_N, W_N)_{N=1}^{\infty}$ be a sequence of uniformly approximating grids with corresponding weights. Let η_t^N be particle configurations that behave according to the SEP on (G_N, W_N) and let μ_t^N be its measure valued representation. Suppose that μ_0^N has density profile $\rho_0 \mathrm{d}V$ for some measurable function ρ_0. Then the trajectory $t \mapsto \mu_t^N$ converges in probability to the trajectory $t \mapsto \rho_t \mathrm{d}V$ in the Skorokhod topology, where $t \mapsto \rho_t$ satisfies the heat equation on M with initial condition ρ_0.*

4.4 Convergence result

The proof of the hydrodynamic limit follows the line of Seppäläinen [136, Chapter 8] which is a canonical method that is also discussed in Kipnis and Landim [96]. However, in our context, there are several new technical difficulties along the way which we have to tackle. The core idea is to rewrite integration of a test function with respect to the empirical measure process as an equation with a martingale. Then we can rewrite this equation and show that in the limit the martingale vanishes and the equation that is left is the heat equation. For the latter we need convergence of the graph Laplacians to the Laplace-Beltrami operator.

4.4.1 Dynkin martingale

First of all fix a smooth function ϕ on M. Define for $\eta \in \{0,1\}^{G^N}$: $f^N(\eta) = \frac{1}{N}\sum_{i=1}^N \eta(p_i)\phi(p_i) = \mu(\phi)$, where $\mu = \frac{1}{N}\sum_{i=1}^n \delta_i \eta(p_i)$. Note that since L^N is the generator of a random walk on a finite space of configurations, its domain consists of all functions on those configurations, so in particular f^N and $(f^N)^2$ are in it. Applying the Dynkin martingale theorem (as described in Chapter 2.1.4) in this situation shows that M^N defined by

$$M_t^N = f^N(\eta_t^N) - f^N(\eta_0^N) - \int_0^t L^N f(\eta_s^N)\mathrm{d}s \tag{4.3}$$

is a martingale with quadratic variation $\left\langle M^N, M^N \right\rangle_t = \int_0^t \gamma(s)\mathrm{d}s$, where $\gamma(s) = (L^N(f^N)^2 - 2f^N L^N f^N)(\eta_s)$. Some basic manipulations show that

$$f^N(\eta^{ij}) - f^N(\eta) = -\frac{1}{N}(\phi(p_j) - \phi(p_i))(\eta(p_j) - \eta(p_i). \tag{4.4}$$

Inserting definitions and leaving out some indexes (to keep everything clear) shows that the right hand side of (4.3) equals

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \phi(p_i)(\eta_t(p_i)) - \frac{1}{N}\sum_{i=1}^N \phi(p_i)(\eta_0(p_i)) \\
&- \left(- \int_0^t \frac{a(N)}{2N} \sum_{i,j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i))(\eta_s(p_j) - \eta_s(p_i))\mathrm{d}s \right) \\
= \;& \mu_t^N(\phi) - \mu_0^N(\phi) - \int_0^t \frac{a(N)}{N} \sum_{i,j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i))\eta_s(p_i)\mathrm{d}s \\
= \;& \mu_t^N(\phi) - \mu_0^N(\phi) \\
&\qquad - \int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s(p_i) \left(a(N) \sum_{j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i)) \right) \mathrm{d}s.
\end{aligned} \tag{4.5}$$

4.4.2 Using convergence of the graph Laplacians

By (4.1), we can write for any p_i:

$$a(N)\sum_{j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i)) = \Delta_M \phi(p_i) + E_{p_i}(N), \tag{4.6}$$

where

$$E(N) := \frac{1}{N}\sum_{i=1}^N |E_{p_i}(N)| \to 0 \qquad (N \to \infty). \tag{4.7}$$

This shows that

$$\begin{aligned}
&\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s(p_i)\left(a(N)\sum_{j=1}^N W_{ij}^N(\phi(p_j)-\phi(p_i))\right)\mathrm{d}s\\
=\;&\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s(p_i)\left(\Delta_M\phi(p_i)+E_{p_i}(N)\right)\mathrm{d}s\\
=\;&\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s(p_i)\Delta_M\phi(p_i)\mathrm{d}s+\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s(p_i)E_{p_i}(N)\mathrm{d}s\\
=\;&\int_0^t \mu_s(\Delta_M\phi)\mathrm{d}s+\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s(p_i)E_{p_i}(N)\mathrm{d}s.
\end{aligned}$$

Plugging this into (4.5) and (4.3), we obtain:

$$\mu_t^N(\phi)-\mu_0^N(\phi)-\int_0^t \mu_s^N(\Delta_M\phi)\mathrm{d}s = M_t^N+\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s^N(p_i)E_{p_i}(N)\mathrm{d}s, \tag{4.8}$$

so for any $T>0$:

$$\begin{aligned}
&\sup_{0\le t\le T}\left|\mu_t^N(\phi)-\mu_0^N(\phi)-\int_0^t \mu_s^N(\Delta_M\phi)\mathrm{d}s\right|\\
\le\;&\sup_{0\le t\le T}\left|M_t^N\right|+\sup_{0\le t\le T}\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s^N(p_i)E_{p_i}(N)\mathrm{d}s\right|.
\end{aligned}\tag{4.9}$$

We want to show that this expression converges to 0 in probability. We will deal with the terms on the right hand side separately.

First of all

$$\begin{aligned}
\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s^N(p_i)E_{p_i}(N)\mathrm{d}s\right| &\le \int_0^t \frac{1}{N}\sum_{i=1}^N |\eta_s^N(p_i)||E_{p_i}(N)|\mathrm{d}s\\
&\le \int_0^t E(N)\mathrm{d}s = tE(N),
\end{aligned}$$

so

$$\sup_{0\le t\le T}\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s(p_i)E_{p_i}(N)\mathrm{d}s\right| \le TE(N)\to 0 \quad \text{(by (4.7))}.$$

4.4.3 Vanishing of the martingale term

Now for the other term. Since the trajectory $t\mapsto\mu_t^N$ is càdlàg, so is M^N. Hence by Doob's inequality we see:

$$\mathbb{P}\left(\sup_{0\le t\le T}\left|M_t^N\right|>\delta\right)\le\frac{\mathbb{E}|M_T^N|}{\delta}. \tag{4.10}$$

To show that $\mathbb{E}|M_T^N|$ goes to 0, it suffices to show that $\mathbb{E}\left\langle M^N, M^N\right\rangle_T$ goes to 0 (since then $\mathbb{E}\left[(M_T^N)^2\right] = \mathbb{E}\left\langle M^N, M^N\right\rangle_T \to 0$ and hence $\mathbb{E}|M_T^N| \to 0$). This is what the following lemma tells us.

Lemma 4.4. *For any $T > 0$:*

$$\lim_{N\to\infty} \mathbb{E}\left\langle M^N, M^N\right\rangle_T = 0.$$

Proof. Recall that $\left\langle M^N, M^N\right\rangle_T = \int_0^T (L^N(f^N)^2 - 2f^N L^N f^N)(\eta_s)\mathrm{d}s$. By writing out, one simply obtains

$$(L^N(f^N)^2 - 2f^N L^N f^N)(\eta) = \sum_{i,j=1}^N \frac{a(N)}{2} W_{ij}^N (f(\eta^{ij}) - f(\eta))^2.$$

Using (4.4), we see

$$(f(\eta^{ij}) - f(\eta))^2 \le \left(\frac{1}{N}(\phi(p_j) - \phi(p_i))(\eta(p_j) - \eta(p_i))\right)^2 \le \frac{1}{N^2}(\phi(p_j) - \phi(p_i))^2,$$

since $\eta(p_i) \in \{0, 1\}$ for all i. This shows that

$$\begin{aligned} 0 &\le \left\langle M^N, M^N\right\rangle_T = \int_0^T (L^N(f^N)^2 - 2f^N L^N f^N)(\eta_s)\mathrm{d}s \\ &\le \int_0^T \frac{a(N)}{2N^2} \sum_{i,j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i))^2 \mathrm{d}s = T\frac{a(N)}{2N^2} \sum_{i,j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i))^2. \end{aligned}$$

This implies that also

$$0 \le \mathbb{E}\left\langle M^N, M^N\right\rangle_T \le T\frac{a(N)}{2N^2} \sum_{i,j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i))^2. \tag{4.11}$$

We can estimate this term by using (4.7). Some basic manipulations show that

$$\begin{aligned} &\frac{a(N)}{2} \sum_{i,j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i))^2 = -\sum_{i=1}^N \phi(p_i) a(N) \sum_{j=1}^N W_{ij}^N (\phi(p_j) - \phi(p_i)) \\ = \; &-\sum_{i=1}^N \phi(p_i) \left(\Delta_M \phi(p_i) + E_{p_i}(N)\right) = -\sum_{i=1}^N \phi(p_i) \Delta_M \phi(p_i) - \sum_{i=1}^N \phi(p_i) E_{p_i}(N), \end{aligned}$$

where the E_{p_i}'s are as before. This implies that

$$\begin{aligned}
&\limsup_{N\to\infty}\left|\frac{a(N)}{2N^2}\sum_{i,j=1}^{N}W_{ij}^N(\phi(p_j)-\phi(p_i))^2\right| \\
\leq\ &\limsup_{N\to\infty}\left\{\frac{1}{N^2}\sum_{i=1}^{N}|\phi(p_i)||\Delta_M\phi(p_i)|+\frac{1}{N^2}\sum_{i=1}^{N}|\phi(p_i)||E_{p_i}(N)|\right\} \\
\leq\ &\limsup_{N\to\infty}\frac{1}{N}||\phi||_\infty||\Delta_M\phi||_\infty+\limsup_{N\to\infty}\frac{1}{N}||\phi||_\infty E(N)=0,
\end{aligned}$$

where in the last step we used (4.7). So we obtain

$$\lim_{N\to\infty}\frac{a(N)}{2N^2}\sum_{i,j=1}^{N}W_{ij}^N(\phi(p_j)-\phi(p_i))^2=0.$$

Together with (4.11) this gives the result. □

We conclude from the lemma that the right hand side of (4.10) goes to zero as N goes to infinity and ϵ goes to zero, so

$$\lim_{\epsilon\downarrow 0}\lim_{N\to\infty}\sup_{0\leq t\leq T}\left|M_t^N\right|=0$$

in probability.

Combining everything above and using (4.9), we conclude that

$$\lim_{N\to\infty}\sup_{0\leq t\leq T}\left|\mu_t^N(\phi)-\mu_0^N(\phi)-\int_0^t\mu_s^N(\Delta_M\phi)\mathrm{d}s\right|=0$$

in probability. In particular, for any $\delta\geq 0$, define

$$H^\delta=\left\{\alpha\in D:\sup_{0\leq t<T}\left|\alpha_t(\phi)-\alpha_0(\phi)-\int_0^t\alpha_s(\Delta_M\phi)\mathrm{d}s\right|\leq\delta\right\}.$$

It can be shown, as in Seppäläinen [136, Chapter 8], that H^δ is closed for any $\delta>0$. Recall from page 88 that we write the distribution of $t\mapsto\mu_t^N$ as Q^N. Then the convergence result above implies that for any $\delta>0$:

$$\lim_{N\to\infty}Q^N(H^\delta)=1.$$

4.4.4 Tightness

We will need that the sequence of distributions $(Q^N)_{N=1}^\infty$ is tight. This can be shown in exactly the same way as Kipnis and Landim [96, p.55-56]. In fact all the most crucial calculations have already been performed above.

Lemma 4.5. *The sequence of distributions $(Q^N)_{N=1}^{\infty}$ is tight.*

Proof. It needs to be shown that the two conditions of [96, Chapter 4 Thm 1.3] are satisfied. Note that for any continuous f we can map a path $\nu \in D([0,T], R(M))$ to the path in $D([0,T],\mathbb{R})$ given by $t \mapsto \nu_t(f)$. This induces a sequence of distributions $Q^N f^{-1}$ on $D([0,T],\mathbb{R})$. By [96, Chapter 4 Prop 1.7] and the fact that the smooth functions are uniformly dense in the set of continuous functions on a manifold, it suffices to prove the conditions of [96, Chapter 4 Thm 1.3] for $\{Q^N f^{-1}, N \geq 0\}$ for all smooth f. Fix such f. Since each path stays in the set of sub-probability measures, the first condition is easily satisfied. For the second condition, it suffices to prove Aldous' tightness criterion, i.e. that

$$\lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{\tau\in\mathcal{I}_T, \theta\leq\gamma} Q^N \left[\left|\mu_\tau^N(f) - \mu_{\tau+\theta}^N(f)\right| > \epsilon\right] = 0, \tag{4.12}$$

where $\mathcal{I}_T$ denotes the set of all stopping times bounded by T. We know from equation (4.8) that there exists a martingale M (depending on f) such that

$$\mu_t^N(f) - \mu_0^N(f) - \underbrace{\int_0^t \mu_s^N(\Delta_M f)\mathrm{d}s}_{\text{(I)}} = \underbrace{M_t^N}_{\text{(II)}} + \underbrace{\int_0^t \frac{1}{N}\sum_{i=1}^N \eta_s^N(p_i) E_{p_i}(N)\mathrm{d}s}_{\text{(III)}}.$$

It therefore suffices to check the tightness criterion for the RHS of this equation and for the integral on the LHS (since the only other term is constant). Now we can make the following estimations.

(I). First of all, since μ_s^N is a sub-probability measure and $\Delta_M f$ is bounded:

$$\left|\int_0^{\tau+\theta} \mu_s^N(\Delta_M f)\mathrm{d}s - \int_0^\tau \mu_s^N(\Delta_M f)\mathrm{d}s\right| \leq \theta||\Delta_M f||_\infty.$$

This implies that

$$\begin{aligned}
&\sup_{\tau\in\mathcal{I}_T,\theta\leq\gamma} Q^N\left[\left|\int_0^{\tau+\theta} \mu_s^N(\Delta_M f)\mathrm{d}s - \int_0^\tau \mu_s^N(\Delta_M f)\mathrm{d}s\right| > \epsilon\right] \\
\leq\ & Q^N\left[\sup_{\tau\in\mathcal{I}_T,\theta\leq\gamma}\left|\int_0^{\tau+\theta} \mu_s^N(\Delta_M f)\mathrm{d}s - \int_0^\tau \mu_s^N(\Delta_M f)\mathrm{d}s\right| > \epsilon\right] \\
\leq\ & Q^N\left[\sup_{\tau\in\mathcal{I}_T,\theta\leq\gamma} \theta||\Delta_M f||_\infty > \epsilon\right] \\
\leq\ & Q^N\left[\gamma||\Delta_M f||_\infty > \epsilon\right] = \mathbf{1}_{\gamma||\Delta_M f||_\infty > \epsilon}.
\end{aligned}$$

This implies that the limit in (4.12) is smaller than

$$\lim_{\gamma\to 0}\limsup_{N\to\infty} \mathbf{1}_{\gamma||\Delta_M f||_\infty>\epsilon} = \lim_{\gamma\to 0} \mathbf{1}_{\gamma||\Delta_M f||_\infty>\epsilon} = 0,$$

so (I) satisfies the tightness criterion.

(II). For the second term, we first estimate $\mathbb{E}\left[(M^N_{\tau+\theta} - M^N_\tau)^2\right]$ (as is done in [96, p.56]). Naturally, the expectation is taken with respect to Q^N. Note that because of the martingale property:

$$\begin{aligned} 0 &\leq \mathbb{E}\left[(M^N_{\tau+\theta} - M^N_\tau)^2\right] = \mathbb{E}(M^N_{\tau+\theta})^2 - \mathbb{E}(M^N_\tau)^2 \\ &= \mathbb{E}\left\langle M^N, M^N\right\rangle_{\tau+\theta} - \mathbb{E}\left\langle M^N, M^N\right\rangle_\tau. \end{aligned}$$

We see from the calculations in the proof of Lemma 4.4 that

$$\mathbb{E}\left\langle M^N, M^N\right\rangle_{\tau+\theta} - \mathbb{E}\left\langle M^N, M^N\right\rangle_\tau \leq \theta \frac{a(N)}{2N^2} \sum_{i,j=1}^N W^N_{ij}(\phi(p_j) - \phi(p_i))^2.$$

Since the term after θ converges to 0, we see that it is bounded by some constant α. By Chebyshev's inequality we obtain:

$$Q^N\left(|M^N_{\tau+\theta} - M^N_\tau| > \epsilon\right) \leq \frac{\mathbb{E}\left[(M^N_{\tau+\theta} - M^N_\tau)^2\right]}{\epsilon^2} \leq \frac{\theta\alpha}{\epsilon^2}.$$

Since

$$\lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{\tau\in\mathcal{I}_T, \theta\leq\gamma} \frac{\theta\alpha}{\epsilon^2} = \lim_{\gamma\to 0} \limsup_{N\to\infty} \frac{\gamma\alpha}{\epsilon^2} = \lim_{\gamma\to 0} \frac{\gamma\alpha}{\epsilon^2} = 0,$$

this part satisfies (4.12) too.

(III). Now for the last term we see

$$\left| \int_0^{\tau+\theta} \frac{1}{N} \sum_{i=1}^N \eta^N_s(p_i) E_{p_i}(N) \mathrm{d}s - \int_0^\tau \frac{1}{N} \sum_{i=1}^N \eta^N_s(p_i) E_{p_i}(N) \mathrm{d}s \right| \leq \theta E(N) \leq \theta K.$$

Here K is some positive number which exists, because of (4.7). This part satisfies (4.12) in the same way as part (I). □

4.4.5 Limiting equation and continuity

We have just shown that $(Q^N)_{N=1}^\infty$ is a tight sequence of measures on D. This implies that every one of its subsequences is also tight and therefore has a weakly convergent subsequence. If these all have the same limit, then it follows from a basic result in metric spaces that the sequence itself converges weakly to that limit. It therefore suffices for weak convergence of $(Q^N)_{N=1}^\infty$ to show that every weakly convergent subsequence of $(Q^N)_{N=1}^\infty$ has the same limit. Let $(Q^{N_k})_{k=1}^\infty$ be any weakly convergent subsequence and denote its limit by Q. Since H^δ is closed, we know for any $\delta > 0$ that

$$Q(H^\delta) \geq \limsup_{k\to\infty} Q^{N_k}(H^\delta) = 1,$$

so $Q(H^\delta) = 1$. Since this holds for any $\delta > 0$, we see

$$Q(H^0) = Q\left(\bigcap_{m=1}^{\infty} H^{\frac{1}{m}}\right) = 1 - Q\left(\bigcup_{m=1}^{\infty} (H^{\frac{1}{m}})^C\right) \geq 1 - \sum_{m=1}^{\infty} Q\left(\left(H^{\frac{1}{m}}\right)^C\right) = 1.$$

This means that

$$Q\left(\alpha \in D : \sup_{0\leq t<T} \left|\alpha_t(\phi) - \alpha_0(\phi) - \int_0^t \alpha_s(\Delta_M \phi)\mathrm{d}s\right| = 0\right) = 1.$$

By doing this for a countable set of functions ϕ that is dense in C^∞ with respect to $||\cdot||_\infty + ||\Delta_M \cdot ||_\infty$ and arguing that this implies the same for any smooth function we see:

$$Q\left(\alpha \in D : \sup_{0\leq t<T} \left|\alpha_t(\phi) - \alpha_0(\phi) - \int_0^t \alpha_s(\Delta_M \phi)\mathrm{d}s\right| = 0 \quad \forall \phi \in C^\infty\right) = 1.$$

Since this holds for any $T > 0$, we see that $Q-$a.s. for every $t \geq 0$ and for all smooth ϕ:

$$\alpha_t(\phi) - \alpha_0(\phi) = \int_0^t \alpha_s(\Delta_M \phi)\mathrm{d}s. \tag{4.13}$$

Note that (4.13) is a weak, measure-valued formulation of the heat equation. We will argue and use shortly that this equation uniquely determines the trajectory $t \mapsto \alpha_t$ given the initial conditions.

Before we arrive at uniqueness, we also need to know that the trajectory is continuous. For the $\mathbb{R}^n$ case this is shown in Seppäläinen [136, Lemma 8.6]. The result can be shown in exactly the same way in our case, so we will not provide all the details. The topology on the space of measures is generated by the following metric:

$$d_M(\mu, \nu) = \sum_{j=1}^{\infty} 2^{-j} \left(1 \wedge |\mu(\phi_j) - \nu(\phi_j)|\right),$$

for some sequence $\phi_j \in C^\infty(M)$. It suffices to control

$$\sup_{t\geq 0} \mathrm{e}^{-t} d_M(\mu_t^N, \mu_{t-}^N).$$

Doing that can be reduced to showing that for any $T > 0$ and $\psi \in C^\infty(M)$:

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{E}\left[\sup_{0\leq s,t\leq T, |s-t|<\delta} \left|\mu_s^N(\phi) - \mu_t^N(\phi)\right|^2\right].$$

This can be done by using the Dynkin martingale representation (4.8) and bounding all the differences as in the proof of tightness. The only term that needs some attention

is $(M_t^N - M_s^N)^2$, but it can be controlled using Doob's maximal inequality:

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s,t\leq T,|s-t|<\delta}(M_t^N - M_s^N)^2\right] &\leq \mathbb{E}\left[\sup_{0\leq t\leq T} 4(M_t^N)^2\right] \\ &\leq 16\mathbb{E}(M_T^N)^2 = 16\mathbb{E}\left\langle M^N, M^N\right\rangle_T,\end{aligned}$$

which goes to zero according to Lemma 4.4.

4.4.6 Uniqueness

To obtain uniqueness of limits of subsequences of Q^N, we need to know that there is a unique continuous solution to (4.13) that has initial condition $\rho_0 \mathrm{d}\overline{V}$. We know that $t \mapsto \rho_t \mathrm{d}\overline{V}$ is a continuous solution to (4.13) with the right initial condition if $t \mapsto \rho_t$ satisfies the heat equation with initial condition ρ_0. Therefore it suffices to show that this solution is unique. This result is proven with a boundedness condition in [136, Thm A.28]. The main idea of the proof is that the measure valued path α_t is smoothed by taking its convolution with some smooth kernel with bandwidth $\epsilon > 0$. Then it is shown that this trajectory of functions satisfies the heat equation with initial condition ρ_0 in the strong sense (by interchanging integral and derivatives and using that these identities are known for sufficiently many ϕ), so it must equal $t \mapsto \rho_t$. Then by letting ϵ go to zero, it is shown that the original trajectory $t \mapsto \alpha_t$ must equal $t \mapsto \rho_t \mathrm{d}\lambda$, where λ is the Lebesgue measure.

To obtain the analogous result in our setting, we cannot use convolution, since this is not well-defined on a manifold. However, we can smooth the measures by integrating the heat kernel at time ϵ with respect to the measures. Using this smoothing, we can follow exactly the same approach, i.e. showing that the smoothed trajectory satisfies the heat equation in a strong sense and then letting ϵ go to 0. The boundedness condition is a bound on volumes, which is needed for some estimations in Seppäläinen [136] and for the uniqueness of the strong solution to the heat equation. Since we work in a compact setting and with probability measures, such a bound is not necessary. The uniqueness of the strong solution to the heat equation is a standard result in our case (so for a compact and connected Riemannian manifold). See for instance Grigoryan [79, Thm 8.18]. Results on the heat kernel on a manifold can also be found in [79].

4.4.7 Conclusion

Now let $t \mapsto \rho_t$ be the solution to the heat equation on M with initial condition ρ_0 and call $\beta := (t \mapsto \rho_t \mathrm{d}\overline{V})$. Recall that (4.13) holds $Q-$a.s. By the uniqueness result above, this implies that Q is a Dirac distribution with β as its support. Since this does not depend on Q^{N_k}, it must be the same for any convergent subsequence, so with arguments given above, we conclude that $Q^N \to Q$ weakly. Let γ^N denote

the random trajectory $t \mapsto \mu_t^N$. Since Q is degenerate, the weak convergence implies convergence in probability, so $\gamma^N \to \beta$ in probability. This is what we wanted to show.

4.5 Notes and perspectives

We have seen in this chapter that it is possible to generalise the Symmetric Exclusion Process to a process on a manifold. We have proved that the hydrodynamic limit is the heat equation on the manifold. The grids that we developed in Chapter 3 were suitable for this purpose. We will now mention some directions in which these results could be extended and possible related research questions.

The first few ideas that we mention are related to those in Section 3.7. We could extend the results to the non-compact case. In particular, this would include $\mathbb{R}^n$ itself again. As we mentioned before, this require some bounds on the curvature of the manifold. Further, we could try to use different grids than the one in this chapter to retrieve properties from $\mathbb{R}^n$ like nearest neighbour jumps. As we mentioned in 3.7, we need a different proof method for the hydrodynamic limit in this case, one that does not require pointwise convergence of the graph Laplacians to the Laplace-Beltrami operator.

Next, we could consider other IPS. The first idea would be to try to prove a hydrodynamic limit result for gradient models, since there are well-established methods in $\mathbb{R}^n$ in this case, as described in for instance Kipnis and Landim [96]. To follow these proofs, we need to prove the one block and two block estimates on our grids on a manifold. The idea of these estimates is that quantities are replaced by averages over blocks of grids points. Intuitively, there does not seem to be a reason for this not to be possible on a manifold. After all, in small neighbourhoods of a point on a manifold, the space is very close to $\mathbb{R}^n$ and in the limit we expect that (at least locally) the same averaging should work. However, the proofs in $\mathbb{R}^n$ heavily use the possibility of translating the grids and particle configurations. This translation is not defined on a manifold. Also in general a grid on a manifold looks different around each grid point. Therefore any possible 'blocks' on a manifold would look very different from each other. This complicates the proofs a lot.

Now that the hydrodynamic limit has been established, the logical next step is to study the fluctuations around the hydrodynamic limit. This is done in Chapter 5. It might be possible to also obtain a large deviations result. However, the proof probably requires again the existence of one block and two block estimates to follow for instance Kipnis et al. [98].

Further, it could be interesting to study more closely the behaviour of the particle configurations in the presence of curvature. We know for example about random fields (Adler and Taylor [2]) and diffusion semigroups (Wang [156]) that they are closely related to properties of the manifold. If such relations can be established for

interacting particle systems, one could for instance 'explore' a manifold by letting an interacting particle system run on it.

Finally, we mention the concept of a tagged particle. The idea is that one lets SEP run, but tags one particle and studies its trajectory. One might expect that the dynamics of such particle are subdiffusive, since its jumps are hindered by the presence of other particles. However, in $\mathbb{R}^n$ it turns out that in dimension ≥ 2 and in the non-symmetric case in dimension 1, the trajectory actually scales to Brownian motion. It would be interesting to generalise this to a manifold. Again, the main issue that one needs to address here is the lack of translation and translation invariance. The reason for this is that the main idea from the proof in the $\mathbb{R}^n$ case, as described in Kipnis and Varadhan [97], is to view the particle configuration from the perspective of the tagged particle. To do that, the whole configuration is shifted along with the particle whenever it jumps. Such shifts are not defined on a manifold. A first step could be to consider this problem in $\mathbb{R}^n$ or the flat torus where the underlying grid is a Poisson Point Process (compare with Faggionato et al. [58]). In that case the grid itself is not translation invariant, but its distribution is.

Chapter 5

Equilibrium fluctuations of the Symmetric Exclusion Process

In this chapter[1] we study again the Symmetric Exclusion Process on a compact Riemannian manifold, as introduced in Chapter 4. There it was shown that the hydrodynamic limit satisfies the heat equation. In this chapter we study the equilibrium fluctuations around this hydrodynamic limit. We define the fluctuation fields as functionals acting on smooth functions on the manifold and we show that they converge in distribution in the path space to a generalised Ornstein-Uhlenbeck process. This is done by proving tightness and by showing that the limiting fluctuations satisfy the corresponding martingale problem.

5.1 Introduction

In the study of interacting particle systems, one of the main targets is to understand the emergence of macroscopic phenomena from the underlying stochastic microscopic dynamics of the individual particles. The study of hydrodynamic limits concerns the derivation of the PDEs that govern macroscopic quantities from the (rescaled) dynamics of microscopic particle configurations. Hydrodynamic limits have been obtained for a large variety of interacting particle systems and by now there are multiple well-established methods to do this (see for instance De Masi and Presutti [36], Kipnis and Landim [96]). One can think of a hydrodynamic limit as a generalised law of large numbers. The natural question that one would like to answer after obtaining such result is how the limiting density field fluctuates around its deterministic hydrodynamic

[1] This chapter is based on van Ginkel and Redig [150].

limit. In other words: one would like to find a corresponding (infinite dimensional) central limit theorem. For these so-called fluctuations a lot of models have been studied and by now standard methods have been established (see for instance Kipnis and Landim [96, Chapter 11]).

Most of the results that were described above are set in (a subset of) some Euclidean space. However, some phenomena are naturally modelled in a space that is not Euclidean. One could for example think of the motion of proteins along cell membranes. Apart from motivation from potential physics applications, it is also an important mathematical challenge to understand the influence on hydrodynamic limits and their fluctuations of geometric properties of the underlying space such as curvature. Therefore, it is worthwhile to extend the study of interacting particle systems to non-Euclidean spaces. Here one could think of spaces with a fractal structure, such as the Sierpinski gasket. In this area results about the Symmetric Exclusion Process have been obtained in for instance Jara [90] and Chen and Gonçalves [26]. An advantage of fractals such as the Sierpinski gasket is that there is a natural discretisation available in the definition of the fractal.

In this chapter, we are interested in this kind of results on Riemannian manifolds. In van Ginkel and Redig [149] (Chapter 4 of this thesis) it was shown that one can set up the theory of hydrodynamic limits on a compact Riemannian manifold by defining suitable grid approximations of the manifold to define the microscopic particle systems. More precisely, we proved that the hydrodynamic limit of the Symmetric Exclusion Process on these grids is the heat equation on the manifold. Moreover, we showed that such grid approximations exist and can be obtained by sampling points uniformly from the manifold, and connecting them with edge weights depending on the Riemannian distance.

In this chapter, we continue the study of the exclusion process on compact Riemannian manifolds by looking at the trajectory of the fluctuation fields. We consider the Symmetric Exclusion Process started from equilibrium (so from the product of Bernoulli measures with fixed intensity ρ) and we show that the corresponding fluctuation fields converge in law in the space of distribution-valued trajectories to a generalised Ornstein-Uhlenbeck process. To do this, we follow the method that is described in Kipnis and Landim [96, Chapter 11], i.e. we show tightness and we prove that any limiting distribution satisfies the same martingale problem.
Working on a manifold instead of $\mathbb{R}^d$ or the torus poses several new challenges. The first challenge is how to discretise the manifold in a suitable way. As we mentioned earlier, this was dealt with in van Ginkel and Redig [149] (Chapter 3 of this thesis). The second challenge is to make sense of the fluctuation fields in the right space. The computations for tightness in a negatively indexed Sobolev space as performed in Kipnis and Landim [96] become intrinsically more involved on a manifold due to the absence of notions like translation and translation invariance, which implies that one cannot rely on standard Fourier analysis for discrete and continuous Laplacians. Therefore we resort to defining the fluctuation fields as elements of the dual of the

smooth functions $C^\infty(M)$ on the manifold. The advantage of this approach is that $C^\infty(M)$ is a nuclear space (see for instance Becnel and Sengupta [7] for more on nuclear spaces), which ensures that we only need to prove tightness of the distribution-valued trajectories applied to test functions. The drawback, however, is that the dual of $C^\infty(M)$ does not have a norm and that the space of càdlàg trajectories is not even metrisable. Therefore we must be careful when treating the convergence of the martingales in Section 5.5.
Finally, note that the result of this chapter is also a constructive proof for the existence of generalised Ornstein-Uhlenbeck processes on compact Riemannian manifolds. This type of processes have been studied (through their corresponding SPDEs) in for instance Christensen [30]. In our work, we show that the fluctuation fields converge to a well-defined limiting random field and that this field satisfies the martingale problem that is associated to a generalised Ornstein-Uhlenbeck process.

Overview of the chapter

In Section 5.2 we define the grids that approximate the manifold, the Symmetric Exclusion Process on the grids and the corresponding fluctuation fields as random elements of $D([0,T],(C^\infty)')$. We also state the main theorem and give a brief overview of the proof. Then in Section 5.3 we study the Dynkin martingale associated to the fluctuation fields. In Section 5.4 we prove tightness of the distributions on $D([0,T],(C^\infty)')$ of the fluctuation fields. Next, in Section 5.5, we show that all possible limiting measures of subsequences are the same, by showing that they satisfy the same martingale problem with the same initial conditions. We conclude the chapter with some notes and perspectives in Section 5.6.

5.2 Preliminaries

In all of this chapter we fix a complete, smooth, connected and compact d-dimensional Riemannian manifold M. In this section we will introduce approximating grids on M and the Symmetric Exclusion Process on these grids. Further we state the theorem and give an outline of its proof.

5.2.1 Definitions

In order to define interacting particle systems on a manifold, we need a suitable discretisation of the manifold. In $\mathbb{R}^d$ such discretisation is easily obtained by taking $\frac{1}{N}\mathbb{Z}^d$ (or something similar). A manifold, however, does not have these nice scaling properties, so another path must be taken (this is explained further in Chapter 3).

Let $(G^N, c^N)_{N=1}^\infty$ be a sequence of grids with $G^N = \{p_1^N, .., p_N^N\} \subset M$ (we usually write simply p_i instead of p_i^N) and edge weights $c^N = \{c_{ij}^N\}_{i,j\leq N}$ where c_{ij}^N is the weight of the edge between p_i^N and p_j^N and we assume that $c_{ij}^N = c_{ji}^N \geq 0$ for all

$i, j \leq N$. We denote by $\mathscr{L}^N$ the corresponding graph Laplacians

$$\mathscr{L}^N f(p_i) = \sum_{j=1}^{N} c_{ij}^N (f(p_j) - f(p_i)).$$

Note that $\mathscr{L}^N$ (acting on functions $G^N \to \mathbb{R}$) generates a random walk on G^N with jumping rates c^N. We assume that the graph Laplacians converge to the Laplace-Beltrami operator Δ_M in a uniform way, i.e. for all $f \in C^\infty$

$$\lim_{N\to\infty} \sup_{1\leq i\leq N} \left| \sum_{j=1}^{N} c_{ij}^N (f(p_j) - f(p_i)) - \Delta_M f(p_i) \right| = 0. \tag{5.1}$$

It will be convenient later to have the following notation for fixed smooth f

$$E_f(N) = \sup_{i\leq N} \left| \mathscr{L}^N f(p_i) - \Delta_M f(p_i) \right|.$$

Note that by assumption (5.1) for each smooth f, $E_f(N)$ goes to 0 as N goes to infinity.

Finally we assume that the empirical measures corresponding to the grids G^N converge weakly to the normalised volume measure $\overline{V}$ on M, i.e. for all continuous f

$$\frac{1}{N}\sum_{i=1}^{N} f(p_i) = \int f \mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{p_i}\right) \to \int f \mathrm{d}\overline{V} \qquad (N \to \infty).$$

Remark 5.1. Note that usually for results about hydrodynamic limits or fluctuations there is an explicit time scaling visible in the equations, i.e. in the diffusive case (like for the Symmetric Exclusion Process) one would typically consider $N^2\mathscr{L}^N$. However, in our case this rescaling is hidden in the conductances c_{ij}^N because of the assumption in (5.1). The reason for this approach is that it is less straightforward to define the space scale in a more general grid than a lattice. To see how the (diffusive) space and time scales do show up in a particular construction of such grid, see the paragraph on scaling in Section 3.3.2.

Remark 5.2. We formulate the results of this chapter in terms of these general uniformly approximating grids. Recall that these grids can always be obtained. It was shown in Chapter 3 that sampling a sequence of i.i.d. uniformly random elements from the manifold and setting G^N to be the first N of these elements yields a suitable sequence of grids with probability 1. Here the conductances c_{ij}^N between points p_i and p_j can be chosen as a function of the distance between p_i and p_j.

We can now define the Symmetric Exclusion Process (SEP) on G^N. The idea of this process is that it describes particles that perform independent random walks according to the jumping rates c^N with the restriction that jumps to occupied sites are

suppressed. Note that in this way every site contains at most one particle. Therefore the SEP $\eta^N = (\eta_t^N)_{t\geq 0}$ takes values in $\{0,1\}^{G^N}$, where 1 denotes the presence of a particle and 0 the absence. An equivalent way to describe the process is by saying that the edges have clocks that ring according to the rates c^N and that if particles are present at either ends of the edge, they jump to the other end. Therefore the dynamics are defined through the generator

$$L^N h(\eta) = \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}^N (h(\eta^{ij}) - h(\eta)), \quad h : \{0,1\}^{G^N} \to \mathbb{R}, \tag{5.2}$$

where $\eta^{ij} := \eta^{p_i p_j}$ denotes the configuration obtained from η by exchanging the values at p_i and p_j.

Since we want to consider equilibrium fluctuations, we want to initialise the process from a stationary measure. Therefore fix $\rho \in (0,1)$ and set as the initial configuration ν_ρ^N: the product of N Bernoulli distributions with parameter ρ. It is well known that SEP is reversible with respect to this measure, so in particular that this measure is invariant for SEP.

Now we define the fluctuation field Y^N as follows. For $f \in C^\infty$ we write

$$Y_t^N(f) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} f(p_i)(\eta_t^N(p_i) - \rho). \tag{5.3}$$

This way we can interpret Y_t^N as a distribution acting on smooth functions, i.e. as an element of $(C^\infty)'$. Now when t varies we obtain a càdlàg trajectory in $(C^\infty)'$, so $Y^N := (Y_t^N, 0 \leq t \leq T)$ is an element of $D([0,T],(C^\infty)')$, the space of càdlàg trajectories in $(C^\infty)'$. The topology of $D([0,T],(C^\infty)')$ is described in Mitoma [120]. We will also sometimes use the subspace of continuous trajectories $C([0,T],(C^\infty)')$, the topology of which is also described in Mitoma [120]. The law of the underlying process η induces a law $\mathcal{L}_N$ of the fluctuation field Y^N on $D([0,T],(C^\infty)')$.

Remark 5.3. To see that (5.3) is the right object with the right scaling, note the following. First of all, at fixed times η_t^N is distributed like a product of Bernoulli measures, so it is a very rough object and it makes sense to regard it as acting on functions (instead of considering its pointwise values). Second, the expectation of $\eta_t^N(p_i)$ equals ρ for every grid point, so the right quantity is subtracted. This makes sure that for any f, $\mathbb{E}Y_t^N(f) = 0$. Finally, $\mathrm{Var}Y_t^N(f)$ equals

$$\mathrm{Var}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N} f(p_i)(\eta_t^N(p_i) - \rho)\right) = \frac{1}{N}\sum_{i=1}^{N} f^2(p_i)\rho(1-\rho) \to \rho(1-\rho)\int f^2 \mathrm{d}\overline{V},$$

where we use that f is continuous and that the empirical measure of the grid points converges to the uniform measure on the manifold. This motivates that $\frac{1}{\sqrt{N}}$ provides the right scaling to get a meaningful, non-degenerate limit.

It is natural to expect that the fluctuation field converges to a generalised stationary Ornstein-Uhlenbeck process. Loosely, this process is the solution of the following (formal) SPDE

$$\mathrm{d}Y_t = \Delta Y_t \mathrm{d}t + \sqrt{2\rho(1-\rho)}\nabla \mathrm{d}W_t, \tag{5.4}$$

where Y takes values in $D([0,T],(C^\infty)')$ and W_t is space-time white noise. A process Y is then a mild solution of (5.4) if for any $f \in C^\infty$

$$Y_t(f) = Y_0(S_t f) + \sqrt{2\rho(1-\rho)} \int_0^t \nabla S_{t-s} f \mathrm{d}W_s.$$

Here $(S_t, t \geq 0)$ is the heat semigroup that is generated by the Laplace operator Δ. The solution should be a Gaussian process that is stationary with respect to white noise W_0 with covariance

$$\mathrm{Cov}(W_0(f), W_0(g)) = \rho(1-\rho)\langle f, g\rangle$$

and with stationary covariance

$$\mathrm{Cov}(Y_t(f), Y_s(g)) = \rho(1-\rho)\langle f, S_{|t-s|} g\rangle. \tag{5.5}$$

More precisely, we define this Ornstein-Uhlenbeck process via the following martingale problem. We call a random trajectory $Y \in C([0,T],(C^\infty)')$ a generalised stationary Ornstein-Uhlenbeck process if for each smooth test function f the following are martingales with respect to the natural filtration generated by $Y = (Y_t, 0 \leq t \leq T)$:

$$\begin{aligned} M_t^f(Y) &:= Y_t(f) - Y_0(f) - \int_0^t Y_s(\Delta_M f)\mathrm{d}s \\ N_t^f(Y) &:= (M_t^f)^2 - 2t\rho(1-\rho)\int (\nabla f)^2 \mathrm{d}\overline{V}. \end{aligned} \tag{5.6}$$

Here $(\nabla f)^2$ should be interpreted as $\langle \nabla f, \nabla f\rangle$, i.e. the function $p \mapsto \langle \nabla f(p), \nabla f(p)\rangle_p$, where $\langle \cdot, \cdot \rangle_p$ is the inner product on T_pM.

5.2.2 Main theorem and overview of the proof

The main theorem of this chapter is the following.

Theorem 5.4. *There exists a random element Y of $C([0,T],(C^\infty)') \subset D([0,T],(C^\infty)')$ with corresponding law $\mathcal{L}$ on $D([0,T],(C^\infty)')$ such that $\mathcal{L}_N \to \mathcal{L}$ as N goes to infinity. Moreover, this Y is a generalised stationary Ornstein-Uhlenbeck process solving the martingale problem* (5.6).

In other words, the theorem says that as N approaches infinity the trajectories of fluctuations converge to a generalised stationary Ornstein-Uhlenbeck process.

Remark 5.5. Note that Theorem 5.4 implies the existence of solutions to (5.6) and hence of the generalised stationary Ornstein-Uhlenbeck process. In Section 5.5 we will also need uniqueness of solutions (given the initial condition). This is explained in Proposition 5.14.

The proof consists of two parts. In Section 5.4 we will show tightness of $(\mathcal{L}_N, N \in \mathbb{N})$. By the first part of the proof of Proposition 5.1 from Mitoma [120], this implies that every subsequence of $(\mathcal{L}_N, N \in \mathbb{N})$ has a further subsequence that converges to some limit. Then in Section 5.5 we show that all limiting points are the same. This is done by showing that any limiting measure satisfies the same martingale problem with the same initial condition. This martingale problem also characterises the limiting process as a generalised stationary Ornstein-Uhlenbeck process like described above and we compute the limiting covariance to confirm this. Together these results imply Theorem 5.4. To do all this, we start by analysing martingales involving the fluctuation fields in Section 5.3.

5.3 Dynkin martingale

We know that η^N is a Markov process with generator given by (5.2) (we will usually leave out the superscript N). Now if we fix $f \in C^\infty$, we can define the function

$$\phi^{N,f} : \{0,1\}^{G^N} \longrightarrow \mathbb{R}, \quad \eta \mapsto \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(p_i)(\eta(p_i) - \rho).$$

Now we know that both the Dynkin martingale

$$M_t^{N,f} := \phi^{N,f}(\eta_t) - \phi^{N,f}(\eta_0) - \int_0^t L^N \phi^{N,f}(\eta_s) \mathrm{d}s$$

and

$$N_t^{N,f} := \left(M_t^{N,f}\right)^2 - \int_0^t \left(L^N(\phi^{N,f})^2(\eta_s) - 2\phi^{N,f}(\eta_s) L^N \phi^{N,f}(\eta_s)\right) \mathrm{d}s$$

are martingales with respect to the natural filtration generated by η (for this well-known result and approach see for instance Seppäläinen [136] or Kipnis and Landim [96]). Since these martingales will have an important role in the calculations later, we will calculate the components that are involved and study their limits.

5.3.1 The martingale $M^{N,f}$

First of all note that

$$\phi^{N,f}(\eta_t) = Y_t^N(f).$$

Second we want to calculate $L^N\phi^{N,f}(\eta)$ for $\eta \in \{0,1\}^{G^N}$. To do this, first we see

$$\begin{aligned}
&\phi^{N,f}(\eta^{ij}) - \phi^{N,f}(\eta) \\
= \;& \frac{1}{\sqrt{N}}\sum_{k=1}^{N} f(p_k)(\eta^{ij}(p_k) - \rho) - \frac{1}{\sqrt{N}}\sum_{k=1}^{N} f(p_k)(\eta(p_k) - \rho) \\
= \;& \frac{1}{\sqrt{N}}\sum_{k=1}^{N} f(p_k)(\eta^{ij}(p_k) - \eta(p_k)) \\
= \;& \frac{1}{\sqrt{N}}(f(p_i)(\eta(p_j) - \eta(p_i)) + f(p_j)(\eta(p_i) - \eta(p_j))) \\
= \;& \frac{1}{\sqrt{N}}(\eta(p_i)(f(p_j) - f(p_i)) + \eta(p_j)(f(p_i) - f(p_j))).
\end{aligned}$$

Now we obtain that

$$L^N\phi^{N,f}(\eta) = \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}^N \frac{1}{\sqrt{N}}(\eta(p_i)(f(p_j) - f(p_i)) + \eta(p_j)(f(p_i) - f(p_j))).$$

By symmetry of the weights, this equals

$$\begin{aligned}
\sum_{i,j=1}^{N} c_{ij}^N \frac{1}{\sqrt{N}}\eta(p_i)(f(p_j) - f(p_i)) &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\eta(p_i)\sum_{j=1}^{N} c_{ij}^N(f(p_j) - f(p_i)) \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\eta(p_i)\mathscr{L}^N f(p_i), \qquad (5.7)
\end{aligned}$$

where we recall that $\mathscr{L}^N$ is the generator of the random walk according to the weights c_{ij}^N on G^N. Now note that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\rho\mathscr{L}^N f(p_i) = \frac{\rho}{\sqrt{N}}\sum_{i,j=1}^{N} c_{ij}^N(f(p_j) - f(p_i)) = 0$$

to see that (5.7) equals

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathscr{L}^N f(p_i)(\eta(p_i) - \rho).$$

This implies that

$$M_t^{N,f} = Y_t^N(f) - Y_0^N(f) - \int_0^t Y_s^N(\mathscr{L}^N f)\mathrm{d}s. \qquad (5.8)$$

The next lemma shows that as N grows to infinity we can replace $\mathscr{L}^N$ by Δ_M.

Lemma 5.6. *For all $f \in C^\infty$,*

$$\lim_{N\to\infty} \mathbb{E}\left(\int_0^t Y_s^N(\mathscr{L}^N f)\mathrm{d}s - \int_0^t Y_s^N(\Delta_M f)\mathrm{d}s\right)^2 = 0.$$

Proof. First we see

$$\begin{aligned} & \mathbb{E}\left(\int_0^t Y_s^N(\mathscr{L}^N f)\mathrm{d}s - \int_0^t Y_s^N(\Delta_M f)\mathrm{d}s\right)^2 \\ = \;& \mathbb{E}\left(\int_0^t Y_s^N(\Delta_M f - \mathscr{L}^N f)\mathrm{d}s\right)^2 \\ \leq \;& t\int_0^t \mathbb{E}Y_s^N(\Delta_M f - \mathscr{L}^N f)^2\mathrm{d}s. \end{aligned} \tag{5.9}$$

Now, using that η_t is a vector of independent Bernoulli random variables, we compute

$$\begin{aligned} \mathbb{E}Y_s^N(\Delta_M f - \mathscr{L}^N f)^2 &= \frac{1}{N}\sum_{i=1}^N \left(\Delta_M f(p_i) - \mathscr{L}^N f(p_i)\right)^2 \rho(1-\rho) \\ &\leq \rho(1-\rho)E_f(N)^2, \end{aligned}$$

so (5.9) is bounded by $t^2\rho(1-\rho)E_f(N)^2$, which vanishes in the limit. □

5.3.2 The martingale $N^{N,f}$

Now we analyse the second martingale. First we calculate the integrand, which we will denote by $\Gamma^{N,f}(s)$, and its expectation and variance.

Lemma 5.7. *For all $f \in C^\infty$,*

$$\lim_{N\to\infty} \mathbb{E}\Gamma^{N,f}(s) = 2\rho(1-\rho)\int (\nabla f)^2 \mathrm{d}\overline{V}.$$

Moreover, $\mathbb{E}\Gamma^{N,f}(s)$ does not depend on s, so in particular the convergence is uniform in s.

Proof. Note that $\mathbb{E}\Gamma^{N,f}(s)$ only depends on s through the distribution of η_s, which is the same for each s, because it is stationary. Therefore $\mathbb{E}\Gamma^{N,f}(s)$ does not depend

on s. We calculate

$$\begin{aligned}
\Gamma^{N,f}(s) &= L^N(\phi^{N,f})^2(\eta_s) - 2\phi^{N,f}(\eta_s)L^N\phi^{N,f}(\eta_s) \qquad (5.10)\\
&= \frac{1}{2}\sum_{i,j=1}^N c_{ij}^N\left(\phi^{N,f}(\eta_s^{ij}) - \phi^{N,f}(\eta_s)\right)^2\\
&= \frac{1}{2}\sum_{i,j=1}^N c_{ij}^N\left(\frac{1}{\sqrt{N}}\Big[\eta_s(p_i)(f(p_j)-f(p_i)) + \eta_s(p_j)(f(p_i)-f(p_j))\Big]\right)^2\\
&= \frac{1}{2N}\sum_{i,j=1}^N c_{ij}^N(\eta_s(p_j)-\eta_s(p_i))^2(f(p_j)-f(p_i))^2 \qquad (5.11)\\
&= -\frac{1}{N}\sum_{i,j=1}^N c_{ij}^N(f(p_j)-f(p_i))f(p_i)(\eta_s(p_j)-\eta_s(p_i))^2.
\end{aligned}$$

Now we take the expectation and, using that $\mathbb{E}(\eta_s(p_j)-\eta_s(p_i))^2 = 2\rho(1-\rho)$, we obtain

$$\begin{aligned}
\mathbb{E}\Gamma^{N,f}(s) &= -\frac{1}{N}\sum_{i,j=1}^N c_{ij}^N(f(p_j)-f(p_i))f(p_i)\mathbb{E}(\eta_s(p_j)-\eta_s(p_i))^2\\
&= -\frac{2\rho(1-\rho)}{N}\sum_{i=1}^N f(p_i)\sum_{j=1}^N c_{ij}^N(f(p_j)-f(p_i))\\
&= -\frac{2\rho(1-\rho)}{N}\sum_{i=1}^N f(p_i)\mathscr{L}^N f(p_i). \qquad (5.12)
\end{aligned}$$

Note that

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{i=1}^N f(p_i)\mathscr{L}^N f(p_i) - \frac{1}{N}\sum_{i=1}^N f(p_i)\Delta_M f(p_i)\right|\\
\leq\ &\frac{1}{N}\sum_{i=1}^N |f(p_i)|\left|\mathscr{L}^N f(p_i) - \Delta_M f(p_i)\right|\\
\leq\ &\|f\|_\infty E_f(N) \longrightarrow 0 \qquad (N\to\infty).
\end{aligned}$$

This implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N f(p_i)\mathscr{L}^N f(p_i) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N f(p_i)\Delta_M f(p_i) = \int f\Delta_M f\mathrm{d}\overline{V}. \qquad (5.13)$$

Combining this with (5.12), we conclude that

$$\lim_{N\to\infty}\mathbb{E}\Gamma^{N,f}(s) = -2\rho(1-\rho)\int f\Delta_M f\mathrm{d}\overline{V} = 2\rho(1-\rho)\int(\nabla f)^2\mathrm{d}\overline{V}. \qquad (5.14)$$

□

Next, we want to prove that $\mathrm{Var}(\Gamma^{N,f}(s))$ vanishes in the limit, but we first need the following lemma and corollary.

Lemma 5.8. *For all $f \in C^\infty$,*

$$\lim_{N\to\infty} \sup_{1\le i\le N} \left| \sum_{j=1}^N c_{ij}^N (f(p_j)-f(p_i))^2 - \big(\Delta_M(f^2)(p_i) - 2f(p_i)\Delta_M f(p_i)\big)\right| = 0. \quad (5.15)$$

Proof. By writing $(f(p_j)-f(p_i))^2 = f(p_j)^2 - f(p_i)^2 - 2f(p_i)(f(p_j)-f(p_i))$ and the triangle inequality we see that (5.15) is bounded by

$$\begin{aligned} &\sup_{1\le i\le N} \left| \sum_{j=1}^N c_{ij}^N (f(p_j)^2 - f(p_i)^2) - \Delta_M(f^2)(p_i)\right| \\ + \quad & 2\|f\|_\infty \sup_{1\le i\le N} \left| \sum_{j=1}^N c_{ij}^N (f(p_j)-f(p_i)) - \Delta_M f(p_i)\right|, \end{aligned}$$

which goes to 0 by (5.1). □

Corollary 5.9. *For all $f \in C^\infty$, there exists a constant C (depending on f) such that for all $N \in \mathbb{N}$*

$$\sup_{1\le i\le N} \sum_{j=1}^N c_{ij}^N (f(p_j)-f(p_i))^2 \le C.$$

Proof. For all $i \le N$ by Lemma 5.8

$$\begin{aligned} &\sum_{j=1}^N c_{ij}^N (f(p_j)-f(p_i))^2 \le \sup_{1\le i\le N} \big|\Delta_M(f^2)(p_i) - 2f(p_i)\Delta_M f(p_i)\big| \\ + \quad & \sup_{1\le i\le N} \left| \sum_{j=1}^N c_{ij}^N (f(p_j)-f(p_i))^2 - \big(\Delta_M(f^2)(p_i) - 2f(p_i)\Delta_M f(p_i)\big)\right| \\ \le \quad & \|\Delta_M f^2\|_\infty + 2\|f\|_\infty \|\Delta_M f\|_\infty + h_f(N), \end{aligned}$$

where $h_f(N) = o(1)$. Since this bound does not depend on i and is bounded in N, the result follows. □

Now we can prove the following lemma.

Lemma 5.10. *For all $f \in C^\infty$,*

$$\lim_{N\to\infty} \mathrm{Var}\Gamma^{N,f}(s) = 0.$$

Moreover, $\mathrm{Var}\Gamma^{N,f}(s)$ does not depend on s, so in particular the convergence is uniform in s.

Proof. Like in Lemma 5.7, $\mathrm{Var}\Gamma^{N,f}(s)$ only depends on s through the distribution of η_s, which is the same for each s because it is stationary. Therefore $\mathrm{Var}\Gamma^{N,f}(s)$ does not depend on s. Using (5.11), we see that the variance of (5.10) equals

$$\frac{1}{4N^2}\sum_{i,j,k,l=1}^{N} c_{ij}^N c_{kl}^N (f(p_j)-f(p_i))^2(f(p_l)-f(p_k))^2 \\ \cdot\mathrm{Cov}((\eta_s(p_j)-\eta_s(p_i))^2,(\eta_s(p_l)-\eta_s(p_k))^2).$$

Now note that $|\mathrm{Cov}((\eta_s(p_j)-\eta_s(p_i))^2,(\eta_s(p_l)-\eta_s(p_k))^2)| \le 1$ since both random variables in the covariance take values in $\{0,1\}$. Moreover (denoting "not independent" by $\not\perp$),

$$\begin{aligned} & \mathrm{Cov}((\eta_s(p_j)-\eta_s(p_i))^2,(\eta_s(p_l)-\eta_s(p_k))^2) \neq 0 \\ \Longrightarrow\quad & (\eta_s(p_j)-\eta_s(p_i))^2 \not\perp (\eta_s(p_l)-\eta_s(p_k))^2 \\ \Longrightarrow\quad & (i=k \text{ or } i=l \text{ or } j=k \text{ or } j=l). \end{aligned}$$

Together, this implies that

$$|\mathrm{Cov}((\eta_s(p_j)-\eta_s(p_i))^2,(\eta_s(p_l)-\eta_s(p_k))^2)| \le \delta_{ik}+\delta_{il}+\delta_{jk}+\delta_{jl}.$$

Now by positivity of the summands and symmetry it suffices to show that

$$\frac{1}{4N^2}\sum_{i,j,k,l=1}^{N} c_{ij}^N c_{kl}^N (f(p_j)-f(p_i))^2(f(p_l)-f(p_k))^2\delta_{ik} \tag{5.16}$$

goes to 0. By rearranging we see that (5.16) equals

$$\frac{1}{4N^2}\sum_{i,j=1}^{N} c_{ij}^N (f(p_j)-f(p_i))^2 \sum_{l=1}^{N} c_{il}^N (f(p_l)-f(p_i))^2. \tag{5.17}$$

By Corollary 5.9, there exists $C=C(f)>0$ such that (5.17) is bounded by

$$\begin{aligned} & \frac{C}{4N^2}\sum_{i,j=1}^{N} c_{ij}^N (f(p_j)-f(p_i))^2 \\ = \quad & -\frac{C}{2N}\frac{1}{N}\sum_{i=1}^{N} f(p_i)\sum_{j=1}^{N} c_{ij}^N (f(p_j)-f(p_i)) \\ \longrightarrow \quad & 0\cdot\int f\Delta_M f \mathrm{d}\overline{V}=0, \end{aligned}$$

where in the last line we used (5.12) and (5.13). This implies that

$$\lim_{N\to\infty}\mathrm{Var}\left(\Gamma^{N,f}(s)\right)=0. \tag{5.18}$$

□

Putting this together, we can prove the following.

Lemma 5.11. *For all $f \in C^\infty$,*

$$\lim_{N\to\infty} \mathbb{E}\left(\int_0^t \Gamma^{N,f}(s)\mathrm{d}s - 2t\rho(1-\rho)\int (\nabla f)^2 \mathrm{d}\overline{V}\right)^2 = 0.$$

Proof. Combining Lemma 5.7 and 5.10 (and using the independence of s), we obtain

$$\begin{aligned} & \mathbb{E}\left(\int_0^t \Gamma^{N,f}(s)\mathrm{d}s - 2t\rho(1-\rho)\int (\nabla f)^2 \mathrm{d}\overline{V}\right)^2 \\ \leq\ & t\int_0^t \mathbb{E}\left(\Gamma^{N,f}(s) - 2\rho(1-\rho)\int (\nabla f)^2 \mathrm{d}\overline{V}\right)^2 \mathrm{d}s \\ =\ & t^2\left(\mathrm{Var}\Gamma^{N,f}(0) + \left(\mathbb{E}\Gamma^{N,f}(0) - 2\rho(1-\rho)\int (\nabla f)^2 \mathrm{d}\overline{V}\right)^2\right) \to 0. \end{aligned}$$

We used here that for a constant c, $\mathbb{E}(X-c)^2 = \mathrm{Var}(X) + (\mathbb{E}X - c)^2$. □

5.4 Tightness

In this section we show tightness. Note that for fixed $N \in \mathbb{N}$ and $f \in C^\infty$, $Y^N(f)$ is a trajectory in $D([0,T],\mathbb{R})$ (where $Y(f) := (Y_t(f), 0 \leq t \leq T)$). Since C^∞ is a nuclear space, by Mitoma [120, Thm 4.1] it suffices to prove for fixed $f \in C^\infty$ that $(Y^N(f), N \in \mathbb{N})$ is a tight collection of random elements of $D([0,T],\mathbb{R})$. By Kipnis and Landim [96, Section 4.1] it suffices to show that Aldous' criterion holds, i.e. that

i) for each $t \in [0,T]$ and $\epsilon > 0$, there is a compact $K(t,\epsilon) \subset \mathbb{R}$ such that

$$\sup_N \mathcal{L}^N(Y_t^N(f) \notin K(t,\epsilon)) \leq \epsilon$$

ii) for all $\epsilon > 0$

$$\lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{\tau\in\mathscr{T}_T, \theta\leq\gamma} \mathcal{L}^N\left(\left|Y_\tau^N(f) - Y_{\tau+\theta}^N(f)\right| > \epsilon\right) = 0,$$

where $\mathscr{T}_T$ is the set of all stopping times with respect to the natural filtration of Y^N that are bounded by T. Further, for ease of notation Y_t^N should be interpreted as Y_T^N whenever $t > T$.

Proof. We study more carefully the distribution of $Y_t^N(f)$. Since ν_ρ^N is invariant under the SEP dynamics, we know that the $\eta_t(p_i)$'s are i.i.d. Bernoulli with parameter ρ.

This means that $(\eta_t(p_i) - \rho)$ has mean 0 and variance $\rho(1-\rho)$. Since they are independent, we see that $Y_t^N(f)$ has mean 0 and variance

$$\mathrm{Var}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N f(p_i)(\eta_t(p_i)-\rho)\right) = \frac{1}{N}\sum_{i=1}^N f^2(p_i)\rho(1-\rho).$$

By the central limit theorem, the distribution of $Y_t^N(f)$ converges to the

$$N\left(0, \rho(1-\rho)\int f^2 \mathrm{d}\overline{V}\right)$$

distribution (where $N(\mu,\sigma^2)$ as usual denotes the normal distribution with mean μ and variance σ^2). This implies tightness of $(Y_t^N(f), N \in \mathbb{N})$, which is (i). For (ii) we use the Dynkin martingale representation (5.8) to write

$$Y_t^N(f) = M_t^{N,f} + Y_0^N(f) + \int_0^t Y_s^N(\mathscr{L}^N f)\mathrm{d}s. \tag{5.19}$$

It suffices to show (ii) for the integral term and the martingale term of (5.19). For the integral term we first calculate the following.

$$\begin{aligned}
&\mathbb{E}\left(\int_\tau^{\tau+\theta} Y_s^N(\mathscr{L}^N f)\mathrm{d}s\right)^2 \le \theta\mathbb{E}\int_\tau^{\tau+\theta}\left(Y_s^N(\mathscr{L}^N f)\right)^2\mathrm{d}s \\
\le\ & \theta\mathbb{E}\int_0^{T+\theta}\left(Y_s^N(\mathscr{L}^N f)\right)^2\mathrm{d}s = \theta\int_0^{T+\theta}\mathbb{E}\left(Y_s^N(\mathscr{L}^N f)\right)^2\mathrm{d}s \\
=\ & \theta\int_0^{T+\theta}\frac{1}{N}\sum_{i=1}^N \mathscr{L}^N f(p_i)^2\rho(1-\rho)\mathrm{d}s = \theta(T+\theta)\rho(1-\rho)\frac{1}{N}\sum_{i=1}^N \mathscr{L}^N f(p_i)^2.
\end{aligned} \tag{5.20}$$

Now we write $E_{p_i}(N) = \left|\mathscr{L}^N f(p_i) - \Delta_M f(p_i)\right|$ and note that

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^N \mathscr{L}^N f(p_i)^2 &\le \frac{1}{N}\sum_{i=1}^N (|\Delta_M f(p_i)| + E_{p_i}(N))^2 \\
&\le \frac{2}{N}\sum_{i=1}^N \Delta_M f(p_i)^2 + \frac{2}{N}\sum_{i=1}^N E_{p_i}(N)^2 \\
&\le \frac{2}{N}\sum_{i=1}^N \Delta_M f(p_i)^2 + 2E_f(N)^2 \to 2\int(\Delta_M f)^2\mathrm{d}\overline{V} + 0,
\end{aligned}$$

which implies that there exists some $C > 0$ independent of N such that (5.20) is bounded by $\theta(T+\theta)\rho(1-\rho)C$. Now we see that

$$\begin{aligned}
&\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\theta\le\gamma}\mathcal{L}^N\left(\left|Y^N_\tau(f)-Y^N_{\tau+\theta}(f)\right|>\epsilon\right)\\
\le\ &\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\theta\le\gamma}\frac{1}{\epsilon^2}\mathbb{E}\left(\int_\tau^{\tau+\theta}Y^N_s(\mathscr{L}^N f)\mathrm{d}s\right)^2\\
\le\ &\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\theta\le\gamma}\frac{\theta(T+\theta)\rho(1-\rho)C}{\epsilon^2}\\
=\ &\lim_{\gamma\to 0}\limsup_{N\to\infty}\frac{\gamma(T+\gamma)\rho(1-\rho)C}{\epsilon^2}=0.
\end{aligned}$$

Now for the martingale term, by the martingale property we see that

$$\begin{aligned}
\mathbb{E}\left(M^{N,f}_{\tau+\theta}-M^{N,f}_\tau\right)^2 &= \mathbb{E}\left(\langle M^{N,f},M^{N,f}\rangle_{\tau+\theta}-\langle M^{N,f},M^{N,f}\rangle_\tau\right)\\
&= \mathbb{E}\int_\tau^{\tau+\theta}L^N(\phi^{N,f})^2(\eta_s)-2\phi^{N,f}(\eta_s)L^N\phi^{N,f}(\eta_s)\mathrm{d}s.
\end{aligned}$$

By (5.11), the latter equals

$$\begin{aligned}
&\frac{1}{2N}\sum_{i,j=1}^N c^N_{ij}(f(p_j)-f(p_i))^2\,\mathbb{E}\int_\tau^{\tau+\theta}(\eta_s(p_i)-\eta_s(p_j))^2\mathrm{d}s\\
\le\ &\frac{1}{2N}\sum_{i,j=1}^N c^N_{ij}(f(p_j)-f(p_i))^2\theta=-\theta\frac{1}{N}\sum_{i,j=1}^N c^N_{ij}(f(p_j)-f(p_i))f(p_i)\\
=\ &-\theta\frac{1}{N}\sum_{i=1}^N f(p_i)\mathscr{L}^N f(p_i)\longrightarrow-\theta\int f\Delta_M f\mathrm{d}\overline{V}=\theta\int(\nabla f)^2\mathrm{d}\overline{V},
\end{aligned}$$

where in the last line we used (5.13). This implies that there exists some $C > 0$ independent of N such that

$$\mathcal{L}^N\left(\left|M^{N,f}_{\tau+\theta}-M^{N,f}_\tau\right|>\epsilon\right)\le\frac{1}{\epsilon^2}\mathbb{E}\left(M^{N,f}_{\tau+\theta}-M^{N,f}_\tau\right)^2\le\frac{\theta C}{\epsilon^2}.$$

As with the integral term, this implies (ii). □

5.5 Uniqueness of limits of subsequences

Now let $\mathcal{L}^*$ be the limit of a subsequence $\mathcal{L}_{N_k}$. We want to show that $\mathcal{L}^*$ satisfies certain initial conditions and a martingale problem, which will then uniquely determine it.

First of all the initial condition can be shown to be a Gaussian field in exactly the same way as Kipnis and Landim [96, Chapter 11 Lemma 2.1], i.e. $\mathcal{L}^*$ restricted to $\mathscr{F}_0$ (the natural filtration at time 0) is a Gaussian field with covariance

$$\mathbb{E}[Y_0(f)Y_0(g)] = \rho(1-\rho)\int fg\mathrm{d}\overline{V}. \tag{5.21}$$

We will need the following lemma.

Lemma 5.12. *Let Y have distribution $\mathcal{L}^*$. Then for each smooth f, $Y(f)$ is continuous almost surely.*

Proof. Fix $f \in C^\infty$. Recall that for a real-valued trajectory $X = (X_t, 0 \le t \le T)$ the continuous and càdlàg modulus of continuity w_δ and w'_δ are defined as

$$w_\delta(X) = \sup_{|s-t|<\delta} |X_t - X_s|, \qquad w'_\delta(X) = \inf_{\substack{0=t_0\le t_1\le..\le t_r=1 \\ t_i - t_{i-1}>\delta}} \max_{1\le i\le r} \sup_{t_{i-1}\le s\le t\le t_i} |X_t - X_s|.$$

By Aldous' tightness criterion (which we showed in Section 5.4), we know that

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathcal{L}^N(w'_\delta(Y^N(f)) \ge \epsilon) = 0.$$

Now note that

$$w_\delta(X) \le 2w'_\delta(X) + \sup_t |X_t - X_{t-}|.$$

Since in our case the last term can be a.s. bounded by $2N^{-1/2}\|f\|_\infty$, we get

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathcal{L}^N(w_\delta(Y^N(f)) \ge \epsilon) = 0.$$

This implies a.s. continuity of $Y(f)$. □

Now we can show that Y satisfies a martingale problem under $\mathcal{L}^*$.

Proposition 5.13. *Recall from* (5.6) *that we define for $Y \in D([0,T],(C^\infty)')$ and $f \in C^\infty$,*

$$\begin{aligned} M_t^f(Y) &:= Y_t(f) - Y_0(f) - \int_0^t Y_s(\Delta_M f)\mathrm{d}s \\ N_t^f(Y) &:= (M_t^f)^2 - 2t\rho(1-\rho)\int(\nabla f)^2\mathrm{d}\overline{V}. \end{aligned}$$

Denote by Y^* the random element of $D([0,T],(C^\infty)')$ with law $\mathcal{L}^*$. Then for each $f \in C^\infty$, $M_t^f(Y^*)$ and $N_t^f(Y^*)$ are martingales with respect to the natural filtration $\mathscr{F} = (\mathscr{F}_t, 0 \le t \le T\}$ generated by $Y^* = (Y_t^*, 0 \le t \le T)$.

Proof. The proof is analogous to the proof of Kipnis and Landim [96, Chapter 11 Prop 2.3]. Recall

$$\begin{aligned} M_t^{N,f} &= Y_t^N(f) - Y_0^N(f) - \int_0^t Y_s^N(\mathscr{L}^N f)\mathrm{d}s \\ N_t^{N,f} &= (M_t^{N,f})^2 - \int_0^t \Gamma^{N,f}(s)\mathrm{d}s. \end{aligned}$$

Fix $0 \le s \le t \le T$. We want to show that

$$\mathbb{E}[M_t^f(Y^*)|\mathscr{F}_s] = M_s^f(Y^*) \quad \text{and} \quad \mathbb{E}[N_t^f(Y^*)|\mathscr{F}_s] = N_s^f(Y^*).$$

Fix $n \in \mathbb{N}, s \ge 0, 0 \le s_1 \le .. \le s_n \le s, H_1, .., H_n \in C^\infty, \Psi \in C_b(\mathbb{R}^n)$ and define

$$\begin{aligned} I: &\quad D([0,T],(C^\infty)') \to \mathbb{R} \\ I: &\quad Y \mapsto \Psi(Y_{s_1}(H_1), .., Y_{s_n}(H_n)). \end{aligned}$$

Now it suffices to show that

$$\lim_{N\to\infty} \mathbb{E}M_t^{N,f} I(Y^N) = \mathbb{E}M_t^f(Y^*)I(Y^*), \quad \lim_{N\to\infty} \mathbb{E}N_t^{N,f} I(Y^N) = \mathbb{E}N_t^f(Y^*)I(Y^*), \tag{5.22}$$

since then by the martingale property of $M_t^{N,f}$

$$\mathbb{E}M_t^f(Y^*)I(Y^*) = \lim_{N\to\infty} \mathbb{E}M_t^{N,f} I(Y^N) = \lim_{N\to\infty} \mathbb{E}M_s^{N,f} I(Y^N) = \mathbb{E}M_s^f(Y^*)I(Y^*)$$

and analogous for the $N_t^f(Y^*)$ case, which implies that $M_t^f(Y^*)$ and $N_t^f(Y^*)$ are martingales with respect to $\mathscr{F}$.

We start with the first martingale. First we show that we can replace $M_t^{N,f}$ by $M_t^f(Y^N)$ in the first expectation in (5.22). Indeed, using Jensen we see that

$$\begin{aligned} &\left(\mathbb{E}M_t^{N,f} I(Y^N) - \mathbb{E}M_t^f(Y^N)I(Y^N)\right)^2 \le \|\Psi\|_\infty^2 \mathbb{E}(M_t^{N,f} - M_t^f(Y^N))^2 \\ = \;& \|\Psi\|_\infty^2 \mathbb{E}\left(\int_0^t Y_s^N(\Delta_M f)\mathrm{d}s - \int_0^t Y_s^N(\mathscr{L}^N f)\mathrm{d}s\right)^2, \end{aligned}$$

which goes to 0 by Lemma 5.6. Now it remains to show that

$$\lim_{N\to\infty} \mathbb{E}M_t^f(Y^N)I(Y^N) = \mathbb{E}M_t^f(Y^*)I(Y^*).$$

First of all note that

$$\begin{aligned} \mathbb{E}(M_t^f(Y^N))^2 &\le 4\left(\mathbb{E}Y_t^N(f)^2 + \mathbb{E}Y_0^N(f)^2 + \mathbb{E}\left(\int_0^t Y_s^N(\Delta_M f)\mathrm{d}s\right)^2\right) \\ &\le 2\rho(1-\rho)\int f^2 \mathrm{d}\overline{V} + t^2\rho(1-\rho)\int (\Delta_M f)^2 \mathrm{d}\overline{V} + o(1), \end{aligned}$$

which implies that there exists $C > 0$ such that

$$\sup_{N\in\mathbb{N}} \mathbb{E}(M_t^f(Y^N)I(Y^N))^2 \leq \|\Psi\|_\infty^2 \sup_{N\in\mathbb{N}} \mathbb{E}(M_t^f(Y^N))^2 \leq C < \infty.$$

This implies that the $M_t^f(Y^N)I(Y^N)$ are uniformly integrable, so it suffices to show that $M_t^f(Y^N)I(Y^N)$ converges to $M_t^f(Y^*)I(Y^*)$ in distribution.

We proceed in steps. The idea is that we would like to apply the Portmanteau theorem, i.e. show that $M_t^f I$ is a map of which the set of discontinuities has measure 0 (under $\mathcal{L}^*$) and conclude from the convergence in distribution of Y^N to Y^* that $M_t^f(Y^N)I(Y^N)$ converges in distribution to $M_t^f(Y^*)I(Y^*)$. However, the path space $D([0,T],(C^\infty)')$ is not metrisable, so the Portmanteu theorem does not apply. Therefore we construct two mappings P_1 and P_2 such that $P_2 \circ P_1 = M_t^f I$ and P_1 maps continuously to a metric space where we can apply the Portmanteau theorem to P_2. First of all, consider the mapping

$$\begin{aligned} P_1 : D([0,T],(C^\infty)') &\longrightarrow D([0,T],\mathbb{R})^{n+2} \\ Y &\longmapsto (Y(f), Y(\Delta_M f), Y(H_1), .., Y(H_n)). \end{aligned}$$

By Jakubowski [88, Thm 1.7] each of the components is continuous, hence P_1 is continuous. This implies that $P_1(Y^N)$ converges in distribution to $P_1(Y^*)$. Now consider the mapping

$$\begin{aligned} P_2 : D([0,T],\mathbb{R})^{n+2} &\longrightarrow \mathbb{R} \\ (X^1, X^2, .., X^{n+2}) &\longmapsto (X_t^1 - X_0^1 - \int_0^t X_s^2 \mathrm{d}s)\Psi(X_{s_1}^3, .., X_{s_n}^{n+2}). \end{aligned}$$

We want to show that the set of discontinuities of P_2 has measure 0 under the law of $P_1(Y^*)$. Suppose $(X^m)_{m\geq 1}$ is a sequence in $D([0,T],\mathbb{R})^{n+2}$ (denoting $X^m = (X^{m,1}, .., X^{m,n+2})$) that converges to $X \in D([0,T],\mathbb{R})^{n+2}$ such that X^i is a continuous path for each $i \leq n+2$. Note that this implies that for each $i \leq n$, $X^{m,i}$ converges uniformly to X^i. So in particular for fixed $i \leq n$ and $t \in [0,T]$, $X_t^{m,i}$ converges to X_t^i and, because of the uniform convergence, $\int_0^t X_s^{m,i}\mathrm{d}s$ converges to $\int_0^t X_s^i \mathrm{d}s$. Combining all of this with the knowledge that Ψ is continuous, we obtain that $P_2(X^m)$ converges to $P_2(X)$. By Lemma 5.12, for each f, $Y(f)$ is continuous with probability 1. Therefore we see that under the measure on $D([0,T],\mathbb{R})^{n+2}$ induced by $\mathcal{L}^*$ through P_1 (i.e. the law of $P_1(Y^*)$), almost every $X \in D([0,T],\mathbb{R})^{n+2}$ has the property that X^i is a continuous path for each $i \leq n+2$. Therefore the set of discontinuities of P_2 has measure 0 under this measure. Hence, by the Portmanteau theorem (and the fact that $P_1(Y^N) \to P_1(Y^*)$ in distribution), we conclude that

$$M_t^f(Y^N)I(Y^N) = P_2(P_1(Y^N)) \longrightarrow M_t^f(Y^*)I(Y^*)$$

in distribution, which is what we wanted. Note (again) that the Portmanteau theorem is only valid in the context of metric spaces, which was the reason for the reduction to the (metric!) space $D([0,T],\mathbb{R})^{n+2}$.

The proof for the second martingale is similar. To replace $N_t^{N,f} I(Y^N)$ with $N_t^f(Y^N)I(Y^N)$, we calculate

$$\begin{aligned}
&\mathbb{E}\left(N_t^{N,f} - N_t^f(Y^N)\right)^2 = \mathbb{E}\left(\left((M_t^{N,f})^2 - \int_0^t \Gamma^{N,f}(s)\mathrm{d}s\right)\right. \\
- \quad &\left.\left((M_t^f(Y^N))^2 - 2t\rho(1-\rho)\int(\nabla f)^2 \mathrm{d}\overline{V}\right)\right)^2 \\
\leq \quad & 2\mathbb{E}\left((M_t^{N,f})^2 - (M_t^f(Y^N))^2\right)^2 \\
+ \quad & 2\mathbb{E}\left(\int_0^t \Gamma^{N,f}(s)\mathrm{d}s - 2t\rho(1-\rho)\int(\nabla f)^2\mathrm{d}\overline{V}\right)^2.
\end{aligned}$$

The right term goes to 0 by Lemma 5.11. For the left term note that

$$\begin{aligned}
&\mathbb{E}\left((M_t^{N,f})^2 - (M_t^f(Y^N))^2\right)^2 \\
= \quad &\mathbb{E}(M_t^{N,f} - M_t^f(Y^N))^2(M_t^{N,f} + M_t^f(Y^N))^2 \\
\leq \quad &\left(\mathbb{E}(M_t^{N,f} - M_t^f(Y^N))^4 \mathbb{E}(M_t^{N,f} + M_t^f(Y^N))^4\right)^{1/2} \\
\leq \quad &\left(8\mathbb{E}(M_t^{N,f} - M_t^f(Y^N))^4\left(\mathbb{E}(M_t^{N,f})^4 + \mathbb{E}(M_t^f(Y^N))^4\right)\right)^{1/2}
\end{aligned}$$

(where we used that $(a+b)^4 \leq 8(a^4+b^4)$). Now we calculate

$$\begin{aligned}
&\mathbb{E}(M_t^{N,f} - M_t^f(Y^N))^4 \\
= \quad &\mathbb{E}\left(\int_0^t Y_s^N(\mathscr{L}^N f - \Delta_M f)\mathrm{d}s\right)^4 \leq t^3 \int_0^t \mathbb{E}\left(Y_s^N(\mathscr{L}^N f - \Delta_M f)\right)^4 \mathrm{d}s \qquad (5.23) \\
= \quad &t^4 \frac{1}{N^2} \sum_{i,j,k,l=1}^N (\mathscr{L}^N f - \Delta_M f)(p_i)..(\mathscr{L}^N f - \Delta_M f)(p_l) \\
&\qquad \cdot \mathbb{E}_N(\eta_0(p_i) - \rho)..(\eta_0(p_l) - \rho), \qquad (5.24)
\end{aligned}$$

where we used that the expectation in the first line does not depend on s, so we can just set $s = 0$. Now note that the expectation in the last line is only non-zero if every index is present either 2 or 4 times. This gives $O(N^2)$ non-zero terms. This means that there is some constant $C > 0$ such that

$$\mathbb{E}(M_t^{N,f} - M_t^f(Y^N))^4 \leq t^4 C\left(\sup_{i\leq N} |\mathscr{L}^N f - \Delta_M f)(p_i)|\right)^4 = t^4 C E_f(N)^4, \qquad (5.25)$$

which goes to zero as N goes to infinity.

Now we show that $\mathbb{E}(M_t^{N,f})^4$ is uniformly bounded, the term with $M_t^f(Y^N)$ can be treated analogously.

$$\mathbb{E}(M_t^{N,f})^4 \leq 64 \left(\mathbb{E}Y_t^N(f)^4 + \mathbb{E}Y_0^N(f)^4 + \mathbb{E}\left(\int_0^t Y_s^N(\mathscr{L}^N f)\mathrm{d}s \right)^4 \right).$$

All these terms can be bounded as in (5.23) to (5.25), i.e. by writing the terms as a sum over four indices with $O(N^2)$ non-zero terms and bounding the non-zero terms using the supremum norm of f and $\mathscr{L}^N f$, respectively. One should further use that

$$\sup_{i \leq N} |\mathscr{L}^N f(p_i)| \leq \|\Delta_M f\|_\infty + E_f(N) = O(1).$$

All of this implies (like for the $M_t^{N,f}$ case earlier in this proof) that $N_t^{N,f} I(Y^N)$ can be replaced with $N_t^f(Y^N)I(Y^N)$. With the same kind of computations, one can show that $N_t^f(Y^N)I(Y^N)$ has uniformly bounded second moments, so is uniformly integrable. Then using the same kind of continuity arguments as with $M_t^f(Y^N)I(Y^N)$, we can conclude that

$$N_t^f(Y^N)I(Y^N) \longrightarrow N_t^f(Y^*)I(Y^*)$$

in distribution. Therefore

$$\mathbb{E}N_t^f(Y^N)I(Y^N) \longrightarrow \mathbb{E}N_t^f(Y^*)I(Y^*),$$

which is what we wanted. □

Now we finally need to know that the martingale problem has unique solutions (given the initial conditions).

Proposition 5.14. *The martingale problem* (5.6) *together with the initial condition* (5.21) *uniquely determine* $\mathcal{L}^*$ *as a measure on* $D([0,T],(C^\infty)')$.

Proof. The proof of this theorem follows exactly like the proof of Kipnis and Landim [96, Chapter 11 Theorem 0.2], which is given in Paragraph 4 of the same chapter. The idea of the proof is that for fixed $f \in C^\infty$ one can use the martingales M^f and N^f from the martingale problem to calculate the transition probabilities for the corresponding process $Y = (Y_t, t \geq 0)$. Since this process is Markov, the transition probabilities combined with the initial condition uniquely determine it. □

We conclude that every convergent subsequence $\mathcal{L}_{N_k}$ converges to the same limit $\mathcal{L}^*$. This implies that $\mathcal{L}_N$ converges to $\mathcal{L}^*$. Moreover, since $\mathcal{L}^*$ satisfies the martingale problem (5.6), the limiting field is a generalised Ornstein-Uhlenbeck process.

To conclude this section, we directly calculate the covariance structure of the limiting field. This is indeed the covariance given in (5.5) that one would expect from a generalised Ornstein-Uhlenbeck process.

Proposition 5.15. *For all $f, g \in C^\infty, t, s \geq 0$*

$$\mathbb{E}[Y_{t+s}(f)Y_s(g)] = \rho(1-\rho)\int (S_t f) g \mathrm{d}\overline{V},$$

where $(S_t, t \geq 0)$ is the heat semigroup generated by Δ_M.

Proof. We start to calculate the following covariance. To do this we need to use duality of SEP with a random walk, as is shown in Liggett [108, Thm 4.74]. Denote by X^N the random walk on the grid G^N that is generated by $\mathscr{L}^N$. Then, using the duality argument, we see

$$\begin{aligned}
&\mathbb{E}\left[(\eta_t(p_i)-\rho)(\eta_0(p_j)-\rho)\right] = \int \mathbb{E}_{\eta'}\left[\eta_t(p_i)-\rho\right](\eta'(p_j)-\rho)\nu_\rho(\mathrm{d}\eta') \\
= &\int \mathbb{E}_{p_i}\left[(\eta'(X_t^N)-\rho)\right](\eta'(p_j)-\rho)\nu_\rho(\mathrm{d}\eta') \\
= &\mathbb{E}_{p_i}\left[\int (\eta'(X_t^N)-\rho)(\eta'(p_j)-\rho)\nu_\rho(\mathrm{d}\eta')\right] \\
= &\mathbb{E}_{p_i}\mathbf{1}_{X_t^N=p_j}\rho(1-\rho) = \rho(1-\rho)\mathbb{P}_{p_i}(X_t^N = p_j),
\end{aligned}$$

where $\mathbb{E}_{p_i}$ is the expectation with respect to the law of X^N starting from p_i. Now we calculate the covariance of the fluctuation fields.

$$\begin{aligned}
\mathbb{E}[Y_t^N(f)Y_0^N(g)] &= \frac{1}{N}\sum_{i,j=1}^N f(p_i)g(p_j)\mathbb{E}\left[(\eta_t(p_i)-\rho)(\eta_0(p_j)-\rho)\right] \\
&= \frac{\rho(1-\rho)}{N}\sum_{i=1}^N f(p_i)\sum_{j=1}^N \mathbb{P}_{p_i}(X_t^N = p_j)g(p_j) \\
&= \frac{\rho(1-\rho)}{N}\sum_{i=1}^N f(p_i)S_t^N g(p_i),
\end{aligned}$$

where $(S_t^N, t \geq 0)$ is the semigroup corresponding to the random walk X^N on the grid points. By Remark 3.5, for each $f \in C^\infty$,

$$\lim_{N\to\infty}\sup_{1\leq i\leq N}\left|S_t^N f|_{G_N}(p_i) - S_t f(p_i)\right| = 0.$$

Using this and stationarity of η, we obtain

$$\begin{aligned}
\mathbb{E}[Y_{t+s}(f)Y_s(g)] &= \lim_{N\to\infty}\mathbb{E}[Y_{t+s}^N(f)Y_s^N(g)] = \lim_{N\to\infty}\mathbb{E}[Y_t^N(f)Y_0^N(g)] \\
&= \lim_{N\to\infty}\frac{\rho(1-\rho)}{N}\sum_{i=1}^N f(p_i)S_t^N g(p_i) = \rho(1-\rho)\int f S_t g \mathrm{d}\overline{V}.
\end{aligned}$$

The first equality follows from convergence in distribution and uniform integrability, which can be derived with methods similar to the proof of Proposition 5.13. □

5.6 Notes and perspectives

In this chapter we continued our study of the Symmetric Exclusion Process on a compact manifold and proved that the equilibrium fluctuations around the hydrodynamic limit converge to the solution of a generalised Ornstein-Uhlenbeck process. We will now mention some ways in which these results can be improved and directions for further study.

First, we defined the fluctuations fields as acting on smooth functions. Because of this they are elements of a non-metrisable topological space. It would be interesting to extend these definitions to a Sobolev space of negative index, since this is a Hilbert space and hence has a much nicer structure. To prove the same results in this setting requires proving tightness of the fluctuations fields in the Sobolev space. In the tightness proof in a flat space, which can be found in Kipnis and Landim [96, Chapter 11], an important ingredient is translation invariance. Since such property is not available on a manifold, this poses technical difficulties. However, we do not see a more conceptual reason why it would not hold. Therefore it would be interesting to see if the technical challenges can be dealt with.

Next, we could try to extend the proof to a non-compact space. For this we cannot work with the whole set of smooth functions as in this chapter, since the smooth functions in general do not form a nuclear space. Probably it could still work if we require the smooth functions to vanish rapidly at infinity.

As we mentioned in Section 4.5, after establishing the equilibrium fluctuations a logical next step would be to prove a large deviations principle. However, as we explained there, we need results like the one block and two block estimates for that.

Further, it would be interesting to consider other interacting particle systems, such as gradient systems. To do this, we need a Boltzmann-Gibbs principle. However, to prove such principle one probably runs into problems similar to the problems for the one and two blocks estimates. In particular there is no translation and the grids look different around each grid point. Therefore it is for instance not clear how to partition the grids in a useful way, as is done in the proof of the Boltzmann-Gibbs principle in Kipnis and Landim [96, Chapter 11].

Chapter 6

Scaling limit of the Discrete Gaussian Free Field

In this chapter[1] we define the discrete Gaussian free field (DGFF) on a compact manifold. Since there is no canonical grid approximation of a manifold, we use grids like the grids from Chapter 3 to replace the square lattice $\mathbb{Z}^d$ in Euclidean space, and prove that the scaling limit of the DGFF is given by the manifold continuum Gaussian free field (GFF). Furthermore using Voronoi tessellations we can interpret the DGFF as element of a Sobolev space and show convergence to the GFF in law with respect to the strong Sobolev topology.

6.1 Introduction and main results

The discrete Gaussian free field has received a lot of attention over the last years thanks to its connections with several areas of mathematics. An on-the-fly definition of it can be given by means of a multivariate centered Gaussian variable on a finite graph, whose covariance matrix is the inverse of the graph laplacian. The DGFF is considered the discrete version of a random distribution, the Gaussian free field, and the interplay between the two has been highlighted in the mathematics literature starting with the work of Sheffield [137]. As far as the authors know, the DGFF has been considered mainly on lattices due to the reason that, outside of the Euclidean setting, it is difficult to choose a canonical grid that approximates space (see the question on Mathoverflow [116]). If one wants to construct the DGFF on a Riemannian manifold for example, one possible strategy to define it is to begin directly with the GFF on the manifold, then construct a triangulation of the space and project the GFF on test functions that are affine on triangles. This procedure is originally contained in Schramm and

[1] This chapter is based on Cipriani and van Ginkel [32].

Sheffield [135]. The drawback of this construction is that it does not link the DGFF to a metrised graph, in particular it does not give information on the edge weights that the underlying graph should have. We, on the other hand, start by setting the edge weights and from there constructing the DGFF. Another approximation of the GFF is obtained via a truncation of its Wiener series representation in terms of eigenfunctions of the Laplace–Beltrami operator, as done in Rivera [130]. This approach is analytical and does not yield a DGFF, which is the object we want to use to discretise the GFF.

Our contributions Indeed, the goal of our work is to approximate the GFF on a manifold by an appropriately defined DGFF. The main difficulty here is to create the right setting in which to make the necessary constructions. In particular this means finding a graph on which we can define the DGFF. However, grids on manifolds are in general far from regular, translation invariance and scaling properties are usually lacking. These properties are not directly necessary to define the DGFF, but studying the analogous proofs in $\mathbb{R}^d$ shows that they are key ingredients used in the scaling limit. In particular it is more complicated to assign suitable edge weights to obtain the GFF in the limit without these properties. Another object which plays a crucial role in the Euclidean case is the Green's function, for which one needs pointwise convergence to the continuum Green's function and upper bounds. These are however not available for our weighted graphs. Therefore we aimed at, and succeeded in, finding different conditions under which Green's functions converge in a weaker way, but still strong enough to ensure the scaling limit. These assumptions (as listed in Theorem 6.1 and as elaborated upon shortly) are natural, in the sense that a grid uniformly sampled from the manifold exhibits these properties with probability one (see Theorem 6.3). In contrast to the $\mathbb{R}^d$ case, showing this relies on a result on the spectral convergence of graph laplacians, which is found in the literature of spectral clustering.

Our construction starts by considering a sequence of weighted graphs for which the random walk semigroups converge in the sense described in Theorem 6.1. Another quite natural assumption we make is that the grid approximates the manifold in the sense of measures, that is, the empirical measure on the grid points converges to the uniform measure on the manifold. Given this, we have to add one final ingredient to the picture: a uniform bound from below on the spectral gap of the discrete laplacians. The reason behind this condition is that one wishes to stay in the region of the spectrum away from zero, where the graph laplacian is invertible.

We can now begin by giving the mathematical exposition of our results. Throughout we will be working with a connected and compact Riemannian manifold M of dimension $d \geq 1$ with normalised volume measure $\overline{V}$. We will use the space of smooth and zero-mean test functions, that is to say the set

$$W := \left\{ f \in C^\infty(M) : \int_M f \mathrm{d}\overline{V} = 0 \right\}.$$

For a graph $\mathcal{V}$ with positive symmetric edge weights c_{vw} we define the graph laplacian

acting on functions $f : \mathcal{V} \to \mathbb{R}$ as

$$Lf(v) := -\sum_{w \in \mathcal{V}} c_{vw}(f(w) - f(v)), \qquad v \in \mathcal{V}. \tag{6.1}$$

The laplacian L generates a simple random walk on $\mathcal{V}$ with associated semigroup $(S_t^{\mathcal{V}})_{t \geq 0}$. We define the zero-average discrete Gaussian free field $\phi_{\mathcal{V}}$ on $\mathcal{V}$ as the Gaussian field indexed by $\mathcal{V}$ whose covariance function is the inverse of L (for proper definitions see Subsection 6.2.2).

The first Theorem we present is concerned with the convergence of the zero-average DGFF to its continuum counterpart: the Gaussian free field on M, that is, the generalised Gaussian field ϕ with mean zero and covariance matrix G, the Green's function of the Laplace–Beltrami operator on M (these notions will be specified in Section 6.2). While the first two conditions in the Theorem specify how to choose a suitable graph laplacian approximating the Laplace–Beltrami operator, the third one regards the dispersion of the grid points.

Theorem 6.1. *Let a sequence of graphs $(V_N)_{N \in \mathbb{N}}$ be given such that $V_N = (p_i^N)_{i=1}^N$ and set the graph laplacians $L = L_N$ as in (6.1). Define ϕ_N to be the zero-average DGFF on V_N and recall that ϕ denotes the GFF on M. Assume that the following conditions hold.*

1. *Denoting by λ_2^N the spectral gap of L_N, we require $\inf_N \lambda_2^N > 0$.*
2. *For any $f : M \to \mathbb{R}$, set*

$$f_N := f|_{V_N} - \frac{1}{N}\sum_{i=1}^N f|_{V_N}(p_i^N).$$

and assume that for all $f \in W$ and $t \geq 0$

$$\lim_{N \to \infty} \frac{1}{N}(f_N, S_t^N f_N) = (f,\, S_t f), \tag{6.2}$$

where $(S_t)_{t \geq 0}$ is the heat semigroup of the Laplace-Beltrami operator.

3. *The following weak limit of measures holds:*

$$\lim_{N \to \infty} \frac{1}{N}\sum_{i=1}^N \delta_{p_i^N} = \overline{V},$$

where $\overline{V}$ is the uniform measure on M.

Then $\sqrt{N}\phi_N$ converges to ϕ in law in the space W' equipped with the weak topology.*

We will show (see Remark 6.11) that canonical grids in flat space satisfy the above mentioned assumptions, for example the equally spaced grid on the d-dimensional flat torus $\mathbb{T}^d$.

Remark 6.2. It will follow from our proofs that we do not necessarily have to work with the Laplace–Beltrami operator. In general, two properties are essential: first of all the operator needs to be symmetric and positive semi-definite. This ensures that we can use its (possibly generalised) inverse as covariance of a Gaussian field, as we are going to do in Section 6.2. Further the operator must generate a suitably regular semigroup for our approach to work. Then if we have a sequence of discrete approximations of this operator in the sense of Theorem 6.1 with the analogous properties, we get convergence of the corresponding Gaussian fields.

The second Theorem exhibits an example of a graph satisfying Assumptions (1)-(3). As it often happens in statistics and manifold learning (Singer [138], Belkin and Niyogi [9], Hein et al. [82], Giné and Koltchinskii [74] are only a few of the numerous works on the topic), the points (p_i^N) of the grid are obtained as uniform observations of the manifold, and edges between them are weighted by a semi-positive kernel with bandwidth t applied to the distance between those grid points. As the number of observations grows and the bandwidth goes to zero, one should be able to capture the convergence of the graph laplacian to the continuum one, and in turn the scaling limit of the random field. Concretely, we sample points uniformly from $\overline{V}$ and we define the vertex set of the N^{th} grid to be the first N points. We connect any two vertices with an edge and choose our kernel to be the heat kernel $p_t(\cdot, \cdot)$ on M divided by t (the more precise definitions are in Subsection 6.2.1). Given the sequence of grids we set a bandwidth t that satisfies

$$W_1(\mu^N, \overline{V}) = o\left(t^{\frac{d}{2}+2}\right), \tag{6.3}$$

where W_1 denotes the Kantorovich or 1-Wasserstein metric and μ^N is the empirical measure on $(p_i^N)_{i=1}^N$.[2] Finally, we modify the bandwidth so that it goes to 0 slowly enough to get convergence of the spectral gaps of the graph laplacians to the continuum one (see Subsection 6.3.2.3 for the details). We formulate the result in the following Theorem.

Theorem 6.3. *Let $V_N := (p_i^N)_{i=1}^N$ be a sequence of i.i.d. points sampled from the normalised volume measure on M. Let $p_t(\cdot, \cdot)$ be the heat kernel on M. Choose t_N such that* (6.3) *holds and the spectral gaps converge to the continuum one. Define the weights in* (6.1) *as*

$$c_{vw} := \frac{p_{t_N}(v, w)}{N t_N}, \quad v, w \in V_N.$$

Then Assumptions (1)-(3) are satisfied almost surely in the law of the sampled grid points.

Finally we extend the result to convergence in a stronger sense, namely in the Sobolev space $H^{-s}(M)$ for some $s > 0$. To do this we lift ϕ_N using Voronoi cells with centers

[2] Note the similarity of the rates of convergence in (6.3) and (3.4).

$(p_i)_{i=1}^N$ to a random distribution in $H^{-s}(M)$ by specifying the action

$$\langle \widetilde{\phi}_N,\, f\rangle := \frac{1}{N}\sum_{i=1}^{N} \phi_N(p_i)\frac{1}{v_i}\int_{C_i} f(p)\overline{V}(\mathrm{d}p)$$

with v_i the volume of the cell C_i. We also extend the definition of the CGFF ϕ to let it act on H^k functions. Then we get the following theorem.

Theorem 6.4. *Assume the conditions of Theorem 6.1. Then $\sqrt{N}\widetilde{\phi}_N$ converges to ϕ in law in the strong topology of H^{-s} for $s > d - 1/2$.*

Structure of the chapter In Section 6.2 we will give the precise definitions of the Gaussian fields we consider, as well as the necessary background on the geometry of the manifold and further insight on Assumptions (1)-(3). Section 6.3 is devoted to showing the first two main Theorems, respectively in Subsections 6.3.1 and 6.3.2. The result in $H^{-s}(M)$ is stated and proved in Section 6.4. We conclude the chapter with some notes and perspectives in Section 6.5.

Notation In the following we will use C, c, c', ... as absolute constants whose value may change from line to line even within the same equation. The norms with subscript N are those on the graphs V_N. We will also use square brackets to denote dual pairings and round brackets for inner products.

6.2 Preliminaries: definitions and assumptions

6.2.1 The manifold

We assume M to be a compact, connected and d-dimensional Riemannian manifold (for all of the following definitions see for instance Grigoryan [79]). The Riemannian structure induces the metric $d(\cdot,\cdot)$. We denote the volume measure on M by V and the uniform measure by $\overline{V} := V/V(M)$ (note that M is compact, so $V(M) < \infty$). On M we can define the heat semigroup[3] $(S_t,\, t \geq 0)$ generated by the Laplace–Beltrami operator Δ_M and the corresponding heat kernel $p_t(p,\, q)$ such that

$$S_t f(p) = \int_M p_t(p,\, q) f(q)\overline{V}(\mathrm{d}q), \quad f \in L^2.$$

Recall from the introduction that $W \subset C^\infty(M)$ consists of the zero-average smooth functions on M. It is equipped with the topology that is generated by the seminorms

$$\sup_K |\partial^\alpha u|,$$

[3]Note that one can construct the heat semigroup on either $C(M)$ or $L^2(M)$. We will need both representations in what follows. However since we will evaluate the semigroups on the set of smooth functions, where they agree, we do not need to specify which one we are using.

where K ranges over the compact sets that are contained in charts and ∂^α ranges over partial derivatives in charts containing K. For $f, g \in L^2(M)$ we denote

$$(f,g) := \int_M f(p)g(p)\mathrm{d}\overline{V}.$$

We recall some basic facts on the Green's function of $-\Delta_M$ (for more details we refer the reader to Aubin [4, Chapter 4], Donaldson [45], Grigoryan [79, Chapter 13]). One knows that on a compact manifold the spectrum of $-\Delta_M$ is discrete, and is given by $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$. The Green kernel on M is given by the following sum in $L^2(M)$:

$$G := \sum_{j \geq 2} \frac{1}{\lambda_j} P_j \tag{6.4}$$

with P_j the projection on the j-th eigenspace of $-\Delta_M$. We also recall that on a compact Riemannian manifold without boundary $f = G\rho$ solves $-\Delta_M f = \rho$ for the input datum $\rho \in W$ and the solution is normalised to have integral zero. Moreover in that case $f \in W$.

6.2.2 The zero-average discrete Gaussian free field

We will now recall some definitions concerning the discrete Gaussian free field. The idea behind the construction follows the use of fundamental matrices to define Gaussian processes (Aldous and Fill [3, Section 14.6.2]) and has been applied for example in studying the zero-average DGFF on the torus by Abächerli [1] .

Let $\mathcal{V}$ be a finite graph. For $v, w \in \mathcal{V}$, let $c_{vw} = c_{wv} \geq 0$ be the conductance between v and w. Assume that $\mathcal{V}$ is connected in the sense that for any $v, w \in \mathcal{V}$ there is a path from v to w such that each edge that is traversed has strictly positive conductance. We define the graph laplacian acting on functions $f : \mathcal{V} \to \mathbb{R}$ by

$$Lf(v) = -\sum_{w \in \mathcal{V}} c_{vw}(f(w) - f(v)), \qquad v \in \mathcal{V}.$$

Since the graph is symmetric all the eigenvalues are non-negative and the corresponding eigenspaces are orthogonal. Moreover, we can conclude from the connectedness that there is exactly one eigenvalue 0 (see for instance Chung and Graham [31, Chapter 1]) with eigenfunction the constant function $\mathbf{1}$. Because of this, the following definition makes sense.

Definition 6.5 (Discrete Green's function)**.** We define the Green's operator as the linear operator on functions $f : \mathcal{V} \to \mathbb{R}$ uniquely defined by the following action on two linear subspaces

$$G^{\mathcal{V}} f := \begin{cases} L^{-1} f & f \perp \mathbf{1} \\ 0 & f = c\mathbf{1} \end{cases}.$$

Here **1** is the function constantly equal to one. There is also an explicit characterisation of $G^{\mathcal{V}}$, which we are going to use in the following. Assume that $\mathcal{V}$ has n points. Denote the eigenvalues of L by $0 = \lambda_1^n < \lambda_2^n \leq \ldots \leq \lambda_n^n$, possibly with multiplicities. Since **1** is exactly the eigenspace corresponding to λ_1^n we can write

$$G^{\mathcal{V}} = \sum_{j=2}^{n} \frac{1}{\lambda_j^n} P_j^n. \tag{6.5}$$

Here P_j^n is the projection on the eigenspace corresponding to the j-th eigenvalue of L.

Now that we have introduced the Green's function, we can make the following definition.

Definition 6.6 (DGFF as a multivariate Gaussian)**.** The zero-average Gaussian free field $\phi_{\mathcal{V}}$ on $\mathcal{V}$ is the Gaussian vector indexed by $\mathcal{V}$ with mean 0 and covariance matrix $G^{\mathcal{V}}$.

Note that $G^{\mathcal{V}}$ is symmetric and positive definite on $\{f \perp \mathbf{1}\}$ (since L is) and 0 on the rest. Therefore $\phi_{\mathcal{V}}$ lives in an $(n-1)$-dimensional space and is degenerate in the direction of the constant vectors. Indeed, as the name indicates, $\phi_{\mathcal{V}}$ has average 0 almost surely. One can see this since

$$\mathrm{Var}\left(\sum_{v\in\mathcal{V}} \phi_{\mathcal{V}}(v)\right) = \mathbf{1}^T G \mathbf{1} = 0$$

so

$$\frac{1}{|\mathcal{V}|}\sum_{v\in\mathcal{V}} \phi_{\mathcal{V}}(v) = 0 \quad \text{a.s.}$$

One of the most important properties of the DGFF is the Markov property, i.e. that the DGFF restricted to a subset of the underlying graph only depends on the rest of the graph through the boundary of that subset (Sznitman [144, Proposition 2.3]). In a zero-average DGFF this is no longer true, since the total average should be zero. Moreover, the restriction of a zero-average DGFF to a subset is not even a zero-average DGFF. However, we can still study the restriction of the zero-average DGFF to a subset when we subtract the harmonic interpolation of its values on the boundary. This turns out to be a DGFF, as it is shown in Abächerli [1, Lemma 1.7] for the zero-average DGFF on the torus. The same proof works in our case, given a few generalisations of the definitions that are involved. We will now formulate the statement. To this end, let $X = (X_t, t \geq 0)$ denote the random walk on $\mathcal{V}$ generated by $-L$, denote by $\mathbb{E}_v$ and $\mathbb{P}_v$ the expectation and law of X started from $v \in \mathcal{V}$, respectively, and set $T_U = \inf\{t : X_t \notin U\}$.

Lemma 6.7. *Let $\mathcal{U} \subset \mathcal{V}$ be a proper subset and for $v \in \mathcal{V}$ define*

$$\phi^{\mathcal{U}}(v) := \phi^{\mathcal{V}}(v) - \mathbb{E}_v[\phi^{\mathcal{V}}(X_{T_{\mathcal{U}}})].$$

Then $\phi^{\mathcal{U}}$ is a centered Gaussian field with covariance matrix

$$G^{\mathcal{U}}(v,w) = \mathbb{E}_v\left[\int_0^{T_{\mathcal{U}}} \mathbf{1}_{\{X_t=w\}}\mathrm{d}t\right].$$

Proof. The proof of this Lemma is essentially the same of Abächerli [1], with two main remarks that we want to stress now. Firstly let us note that we have to use the continuous-time random walk (as opposed to the situation in [1]), since the rates of the exponential waiting times do not have to be equal. Secondly, to be able to mimic the proof given on the d-dimensional flat torus we need to prove that our Green's function is the same as Abächerli [1], i.e. that

$$G^{\mathcal{V}}(v,w) = \int_0^\infty (\mathbb{P}_v[X_t = w] - 1/n)\,\mathrm{d}t =: H^{\mathcal{V}}(v,w). \tag{6.6}$$

To show (6.6), note first of all that $H^{\mathcal{V}}\mathbf{1} = 0$. So it remains to show that $H^{\mathcal{V}} = L^{-1}$ on $W = \{f : f \perp \mathbf{1}\}$. First of all note that for $f \in W$

$$H^{\mathcal{V}}f(v) = \int_0^\infty \left[\sum_{w\in\mathcal{V}} \mathbb{P}_v(X_t = w)f(w) - \frac{1}{n}\sum_{w\in\mathcal{V}} f(w)\right]\mathrm{d}t = \int_0^\infty S_t f(v)\mathrm{d}t,$$

where $S_t = \exp(-tL)$ is the semigroup corresponding to X. In particular

$$\sum_{v\in\mathcal{V}} H^{\mathcal{V}}f(v) = \int_0^\infty \sum_{v\in\mathcal{V}} S_t f(v)\mathrm{d}t = 0$$

since by symmetry of the random walk

$$\begin{aligned}\sum_{v\in\mathcal{V}} S_t f(v) &= \sum_{v\in\mathcal{V}}\sum_{w\in\mathcal{V}} \mathbb{P}_v(X_t = w)f(w) = \sum_{w\in\mathcal{V}} f(w)\sum_{v\in\mathcal{V}} \mathbb{P}_w(X_t = v)\\ &= \sum_{w\in\mathcal{V}} f(w) = 0.\end{aligned}$$

This implies that $H^{\mathcal{V}}$ maps W into W. Moreover for $f \in W$

$$LH^{\mathcal{V}}f = L\int_0^\infty S_t f\mathrm{d}t = \int_0^\infty LS_t f\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}(-S_t f)\,\mathrm{d}t = -S_t f|_0^\infty = f.$$

Note that we used that $\lim_{t\to\infty} S_t f = 0$, since f is zero-average. This finishes the proof. □

Now suppose our graph $\mathcal{V}$ consists of points of a manifold (which we generally denote by p or q). To speak of convergence of the DGFF to the GFF, we need to define them as comparable objects. To this end, we interpret them as random linear functionals on W. For the DGFF $\phi_{\mathcal{V}}$ this means introducing the following definition.

Definition 6.8 (DGFF as random distribution)**.** For $f \in W$, we write

$$\langle \phi_{\mathcal{V}}, f \rangle := \frac{1}{|\mathcal{V}|} \sum_{p \in \mathcal{V}} f(p) \phi_{\mathcal{V}}(p).$$

This defines $\phi_{\mathcal{V}}$ as a random distribution on M, i.e. a random element of W' with the strong topology.

Proof. Note that, for each ω in the underlying probability space, $\phi_{\mathcal{V}}(\omega)$ is a well-defined linear functional on W, so an element of W'. To see that $\phi_{\mathcal{V}}$ is a well-defined random element of W' (on the Borel sets generated by the strong topology) it suffices to show that the mapping $\mathbb{R}^{\mathcal{V}} \to W'$ that maps the vector $\phi_{\mathcal{V}}(\omega)$ to the linear functional $\phi_{\mathcal{V}}(\omega)$ is continuous. By linearity, it suffices to show for fixed p that the mapping

$$\begin{aligned} H : \mathbb{R} &\longrightarrow W' \\ x &\longmapsto (f \longmapsto f(p)x) \end{aligned}$$

is continuous. Since $\mathbb{R}$ is a metric space, it suffices to show sequential continuity. Also, recall that the strong topology of W' is the topology of uniform convergence on bounded sets. Now let $x_N \to x$ in $\mathbb{R}$ and let $B \subset W$ be bounded. Since B is bounded in W, it is in particular bounded in the supremum norm, i.e.

$$\sup_{f \in B} \|f\|_\infty =: C < \infty.$$

Therefore we see that

$$\begin{aligned} \sup_{f \in B} |H(x_N)(f) - H(x)(f)| &= \sup_{f \in B} |f(p)x_N - f(p)x| \\ &= \sup_{f \in B} |f(p)||x_N - x| \leq C|x_N - x|, \end{aligned}$$

which goes to 0 as N goes to infinity. This concludes the proof. □

6.2.3 The continuum GFF

Recall $W = \{f \in C^\infty(M) : \int_M f \mathrm{d}\overline{V} = 0\}$. We now give the following definition.

Proposition 6.9 (GFF on M)**.** *There exists a unique centered Gaussian random distribution $\phi := \{\langle \phi, f \rangle : f \in W\}$ on W' with the weak* topology with covariance kernel G given in* (6.4)*, that is, for all $f, g \in W$,*

$$\mathbb{E}\left[\langle \phi, f \rangle \langle \phi, g \rangle\right] = (f, Gg).$$

We call this distribution the GFF on M.

Proof. Note that W is a nuclear space, being a subspace of the nuclear space $C^\infty(M)$ (see for instance Becnel and Sengupta [7] for a collection of properties of nuclear spaces). By the Bochner–Minlos theorem for nuclear spaces, Umemura [147, Theorem A][4], it suffices to show that the characteristic functional

$$\begin{aligned}\mathcal{L}_\phi : W &\to \mathbb{R} \\ f &\mapsto \exp\left(-\frac{1}{2}(f, Gf)\right)\end{aligned}$$

is continuous around 0, positive definite and satisfies $\mathcal{L}_\phi(0) = 1$. The latter is clear. To show positive definiteness one can use Lodhia et al. [111, Proposition 2.4], which says that L_ϕ is positive definite if

$$f \mapsto (f, f)_G := (f, Gf)$$

is an inner product on W. This follows from the fact that G is a self-adjoint positive definite operator on W (compare (6.4)). Finally, since $f = G\rho \in L^2(M)$ is the unique solution with integral zero to the Poisson equation with input datum $\rho \in L^2(M)$, also with integral zero, one can use the Poincaré inequality and

$$\|\nabla f\|_2^2 = (\Delta_M f, f) \leq \|f\|_2 \|\rho\|_2$$

to conclude that G is a bounded and hence continuous operator on the set of zero-average square integrable functions on M. Since convergence in W implies convergence in L^2, it is immediate to see with Cauchy-Schwarz that $f \mapsto (f, Gf)$ is continuous and hence that $\mathcal{L}_\phi$ is continuous. □

6.2.4 Comments on Assumptions (1)-(3)

Let $(V_N)_{N=1}^\infty$ be a sequence of finite subsets of the manifold M with corresponding conductances $c_{pq}^N = c_{qp}^N \geq 0$ for $p, q \in V_N$ such that each V_N is connected in the sense described in Subsection 6.2.2. Throughout this chapter we assume that V_N consists of N points, which we label $p_1^N, \ldots, , p_N^N \in M$.[5] Let $(L_N)_{N=1}^\infty$, $(G_N)_{N=1}^\infty$ and $(\phi_N)_{N=1}^\infty$ be the sequences of corresponding generators, Green's functions and zero-average discrete Gaussian free fields on V_N, respectively, and for each N let $\{S_t^N, t \geq 0\}$ denote the semigroup on V_N that is generated by L_N. Note that we can also interpret ϕ_N as a random function on W', as we described in Definition 6.8.
Let us comment more on the necessity of Assumptions (1)-(3) of Theorem 6.1. First of all, as we discussed above, all eigenvalues of L_N are non-negative and only one eigenvalue equals 0. We denote the second smallest eigenvalue (or the spectral gap)

[4]This is Problem A on page 16 of Umemura [147], which is solved for nuclear spaces in Theorem A on page 24.

[5]This is not an essential requirement, it just makes our notation less involved. For instance, for some natural sequences of grids the amount of points in V_N is N^d where d is the dimension of the ambient space. With some straightforward changes our results hold in those cases too.

by λ_2^N. Then we know that $\lambda_2^N > 0$, so each spectral gap is positive. Assumption (1) says that the spectral gaps are uniformly positive, i.e.

$$\inf_N \lambda_2^N > 0.$$

Without this condition what could happen is that the spectrum of the graph laplacian would eventually capture the 0-eigenvalue of $-\Delta_M$ (compare Von Luxburg et al. [155, Result 3] for a case in which spectral convergence fails). In this case, we would not be anymore in the domain of invertibility of the Green's function. Secondly, we define the zero-average discrete version of any function $f : M \to \mathbb{R}$ to be

$$\begin{aligned} f_N : V_N &\to \mathbb{R} \\ p_i^N &\mapsto f(p_i^N) - \frac{1}{N}\sum_{i=1}^N f(p_i^N). \end{aligned}$$

Moreover, we define an inner product on $\mathbb{R}^{V_N}$ by $(f, g) = \sum_{i=1}^N f(p_i^N)g(p_i^N)$. Now Assumption (2) states that for each $f \in W$

$$\lim_{N\to\infty} \frac{1}{N}(f_N, S_t^N f_N) = (f,\, S_t f).$$

Assumption (2) is probably the most natural one would expect in a convergence-to-GFF-type result: as we will see, it implies that the bilinear forms induced by the Green's functions converge pointwise (see Equation (6.7) for the precise statement). One can ensure this limit via a stronger result, namely the uniform convergence of the discrete laplacian to the continuum one. This will be our strategy in the proof of Theorem 6.3. Finally, the third Assumption makes sure that the empirical measures corresponding to the grids converge weakly to the uniform distribution on the manifold. Therefore summing over grid points approximates integrating over the manifold in the same way as discrete lattice sums in $\mathbb{Z}^d$ approximate integrals in $\mathbb{R}^d$.

6.3 Proofs

Here we present the proofs of our main results. In Subsection 6.3.1 we will show that Assumptions (1)-(3) entail the convergence of the rescaled DGFF to the continuum one. We will show, using a spectral decomposition, that the variance of the distribution ϕ_N tested against smooth functions converges to that of the continuum field under Assumption (2). Assumptions (1) and (3) will ensure enough regularity to get this convergence. Note that we will not use here the potential theory for the random walk to prove the scaling limit, in contrast to the $\mathbb{Z}^d$ case (a proof in $d = 2$ is for example carried out in Biskup [16, Section 1.4]).

Theorem 6.3 will be shown in Subsection 6.3.2. We will sample uniform points from the manifold, and choose as conductances the heat kernel as explained in the Introduction. The proof of the validity of Assumptions (1)-(3) is in three steps (each

step shows one assumption). First we will use the fact that the empirical measures corresponding to the grids almost surely converge in Kantorovich sense to the uniform measure $\overline{V}$, which implies weak convergence. Then we will show that the graph laplacians converge, uniformly over the grid points, to the Laplace–Beltrami operator. This will be done by choosing the bandwidth t_N appropriately, following the ideas of van Ginkel [148], van Ginkel and Redig [149] (Chapter 3 of this thesis), here we will need again the Kantorovich convergence of the empirical measures. Finally, to show the bound on the spectral gap, we will use techniques developed in Von Luxburg et al. [155], Belkin and Niyogi [10] by proving convergence to an "intermediate" operator whose eigenvalues approximate those of the Laplace–Beltrami. This will yield a second condition on the rate of growth of t_N, and by combining the two we will obtain the final result.

6.3.1 Proof of Theorem 6.1

We would like to prove that $\sqrt{N}\phi_N \to \phi$ in law in W'. Since W is a nuclear Fréchet space, by Meyer [119, Theorem 2] it suffices to prove pointwise convergence of the characteristic functional, i.e. that for any $f \in W$

$$\mathbb{E}\exp\left(\mathrm{i}\left\langle\sqrt{N}\phi_N, f\right\rangle\right) \to \mathbb{E}\exp(\mathrm{i}\,\langle\phi, f\rangle).$$

Remark 6.10. To be a bit more precise, note that we defined the ϕ_N's as random elements of W' with the strong topology (i.e. their laws are measures on the Borel sets that are generated by the strong topology). Therefore they are in particular also well-defined random elements when restricted to W' with the topology of uniform convergence on convex, compact subsets, which is used in Meyer [119]. Now Meyer [119, Theorem 2] yields a limiting random element ϕ^* of W' with the latter topology, which we can restrict to W' with the weak* topology. Since the restriction of ϕ^* then has the same characteristic functional as the GFF ϕ from Proposition 6.9, we conclude from the uniqueness of ϕ that ϕ^* is an extension of ϕ. Note, moreover, that then the restrictions of the ϕ_N's to W' with the weak* topology converge in law to ϕ.

Recall that we define $f_N : V_N \to \mathbb{R}$ by $f_N := f|_{V_N} - 1/N\sum_{i=1}^N f|_{V_N}(p_i^N)$. We could subtract any constant from $f|_{V_N}$ since ϕ_N has average 0, but we choose to subtract the discrete average since it ensures that f_N belongs to the discrete counterpart of W. We can abbreviate

$$G_N(i,\,j) := G_N(p_i^N,\,p_j^N)$$

and we see that

$$\begin{aligned}\mathbb{E}\exp\left(\mathrm{i}\left\langle\sqrt{N}\phi_N, f\right\rangle\right) &= \mathbb{E}\exp\left(\mathrm{i}\frac{1}{N}\sum_{i=1}^N f(p_i^N)\sqrt{N}\phi_N(p_i^N)\right)\\ &= \exp\left(-\frac{1}{2N}\sum_{i,j=1}^N f_N(p_i^N)f_N(p_j^N)G_N(i,\,j)\right) = \exp\left(-\frac{1}{2N}(f_N,\,G_N f_N)\right).\end{aligned}$$

Further $\mathbb{E}\exp(\mathrm{i}\,\langle\phi, f\rangle) = \exp(-1/2(f,\, Gf))$ (by definition of ϕ). Therefore it suffices to show that

$$\frac{1}{N}\sum_{i,j=1}^{N} f_N(p_i^N) f_N(p_j^N) G_N(i,\, j) \to (f,\, Gf) \tag{6.7}$$

for every $f \in W$.

We now want to make use of the spectral decomposition of the Green's function. Let $0 < \lambda_2^N \leq \lambda_3^N \leq \ldots \leq \lambda_N^N$ be the non-zero eigenvalues of L_N. Define the measure μ_N^f on $\sigma(L_N)$ by

$$\mu_N^f(A) := \sum_{j=2}^{N} \mathbf{1}_A(\lambda_i^N) \|P_{j,N} f_N\|_{2,N}^2$$

for $A \subset \sigma(L_N)$. The total mass of the measure μ_N^f is

$$\sum_{j=2}^{N} \|P_{j,N} f_N\|_{2,N}^2 = \|f_N\|_{2,N}^2.$$

Similarly define $\mu^f(A)$ on $\sigma(L)$ by

$$\mu^f(A) := \sum_{j=2}^{\infty} \mathbf{1}_A(\lambda_i) \|P_j f\|_2^2$$

for $A \subset \sigma(L)$ and $\lambda_2 \leq \lambda_3 \leq \ldots$ the positive eigenvalues of $-\Delta_M$. This is a measure with total mass

$$\sum_{j=2}^{\infty} \|P_j f\|_2^2 = \|f\|_2^2.$$

Note that since $P_{j,N}$ is a projection and since $P_{1,N} f_N = 0$ by construction of f_N, we see by (6.5) that

$$\begin{aligned}(f_N,\, G_N f_N) &= \sum_{j=2}^{N} \frac{1}{\lambda_j^N} (f_N, P_{j,N} f_N) = \sum_{j=2}^{N} \frac{1}{\lambda_j^N} (P_{j,N} f_N, P_{j,N} f_N) \\ &= \sum_{j=2}^{N} \frac{1}{\lambda_j^N} \|P_{j,N} f_N\|_{2,\,N}^2 = \int_{\sigma(L_N)} y^{-1} \mu_N^f(\mathrm{d}y). \end{aligned} \tag{6.8}$$

Analogously, by (6.4) one deduces

$$(f, Gf) = \int y^{-1} \mu^f(\mathrm{d}y).$$

Now note that by Tonelli's theorem

$$\begin{aligned}\frac{1}{N}(f_N,\, G_N f_N) &= \frac{1}{N} \int_{\sigma(L_N)} y^{-1} \mu_N^f(\mathrm{d}y) = \frac{1}{N} \int_{\sigma(L_N)} \int_0^{\infty} \mathrm{e}^{-ty} \mathrm{d}t\; \mu_N^f(\mathrm{d}y) \\ &= \int_0^{\infty} \frac{1}{N} \int_{\sigma(L_N)} \mathrm{e}^{-ty} \mu_N^f(\mathrm{d}y) \mathrm{d}t. \end{aligned} \tag{6.9}$$

Denote $\delta := \inf_N \lambda_2^N > 0$ by Assumption (1). Then we see that

$$0 \leq \frac{1}{N} \int_{\sigma(L_N)} \mathrm{e}^{-ty} \mu_N^f(\mathrm{d}y) \leq \frac{1}{N} \mu_N^f(\sigma(L_N)) \mathrm{e}^{-t\lambda_2^N} \leq \frac{1}{N} \|f_N\|_{2,N}^2 \mathrm{e}^{-t\delta}.$$

Now note that

$$\frac{1}{N} \|f_N\|_{2,N}^2 = \frac{1}{N} \sum_{i=1}^N \left(f(p_i^N) - \frac{1}{N} \sum_{j=1}^N f(p_i^N) \right)^2 \leq \frac{1}{N} \sum_{i=1}^N f(p_i^N)^2.$$

By the continuity of f and Assumption (3), the last term converges to $\|f\|_2^2$. Therefore there exists a $C > 0$ such that for all N

$$0 \leq \frac{1}{N} \int_{\sigma(L_N)} \mathrm{e}^{-ty} \mu_N^f(\mathrm{d}y) \leq C \mathrm{e}^{-\delta t}.$$

Since $\int_0^\infty C\mathrm{e}^{-\delta t} \mathrm{d}t < \infty$, by the dominated convergence theorem, this implies that

$$\begin{aligned} \lim_{N\to\infty} \int_0^\infty \frac{1}{N} \int_{\sigma(L_N)} \mathrm{e}^{-ty} \mu_N^f(\mathrm{d}y) \mathrm{d}t &= \int_0^\infty \lim_{N\to\infty} \frac{1}{N} \int_{\sigma(L_N)} \mathrm{e}^{-ty} \mu_N^f(\mathrm{d}y) \mathrm{d}t \\ &= \int_0^\infty \lim_{N\to\infty} \frac{1}{N} (f_N, S_t^N f_N) \mathrm{d}t. \end{aligned} \tag{6.10}$$

Now we conclude thanks to Assumption (2):

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} (f_N, G_N f_N) &\overset{(6.8)}{=} \lim_{N\to\infty} \frac{1}{N} \int_{\sigma(L_N)} y^{-1} \mu_N^f(\mathrm{d}y) \\ &\overset{(6.9),(6.10)}{=} \int_0^\infty \lim_{N\to\infty} \frac{1}{N} (f_N, S_t^N f_N) \mathrm{d}t = \int_0^\infty (f, S_t f) \mathrm{d}t = (f, Gf). \end{aligned}$$

Note that in the last equality we have used the fact that f has average zero on M. □

Remark 6.11 (Compatibility with known grids)**.** For any integer $N \in \mathbb{N}$ consider the quotient space $\mathbb{S}_N := \mathbb{Z}/N\mathbb{Z}$. A finite product of d copies of $\mathbb{S}_N$ defines a discrete torus $\mathbb{T}_N^d$ of side-length N. This object is naturally connected to the d-dimensional (flat) torus $\mathbb{T}^d$ given by a product of d copies of $\mathbb{S}^1$. The rescaled graph Laplacian L_N on $N^{-1}\mathbb{T}_N^d$ is the sum of the Laplacians $\mathcal{L}_N$ on each discrete $N^{-1}\mathbb{S}_N$ component. More precisely, $\mathcal{L}_N$ is defined for any $f : N^{-1}\mathbb{S}_N \to \mathbb{R}$ by the following difference operator:

$$\mathcal{L}_N f(k) := \frac{N^2}{4\pi^2} \left[(f(k) - f(k - 1/N)) + (f(k) - f(k + 1/N)) \right], \quad k \in N^{-1}\mathbb{S}_N.$$

The spectra of $\mathcal{L}_N$ and L_N are thus given by

$$\sigma(\mathcal{L}_N) = \left\{ \frac{N^2}{\pi^2} \sin^2\left(\frac{\pi k}{N}\right) : k \in \{0, 1, \ldots, N-1\} \right\},$$
$$\sigma(L_N) = \left\{ \frac{N^2}{\pi^2} \sum_{i=1}^{d} \sin^2\left(\frac{\pi k_i}{N}\right) : k_i \in \{0, 1, \ldots, N-1\},\, i \in \{1, \ldots, d\} \right\}.$$

One can show that, with the rescaling N^2, the eigenvalues of L_N converge to those of the Laplace–Beltrami operator on $\mathbb{T}^d$ as N grows. Since the spectral gap of the Laplace-Beltrami operator is strictly positive, this ensures Assumption (1). A Taylor expansion yields that

$$\mathcal{L}_N f(k) = \frac{f''(k)}{4\pi^2} + O\left(N^{-1}\right)$$

and the O-term can be bounded uniformly in k due to the compactness of the torus and the translation invariance of the situation. By summing over d coordinate directions, we obtain the approximation to the Laplace–Beltrami operator on $\mathbb{T}^d$ (which is simply the sum of the second derivatives). A theorem of Trotter and Kurtz gives convergence of the corresponding semigroups, after which Assumption (2) follows from a direct computation (see Corollary 6.17 and Proposition 6.18 for the details in the manifold case). Finally, Assumption (3) is a consequence of the approximation of integrals via Riemann sums in $\mathbb{R}^d$.

6.3.2 Proof of Theorem 6.3

Since the proof of Theorem 6.3 is divided into three steps, the next three paragraphs will be dedicated to showing the validity of each assumption separately.

Remark 6.12 (Quenched results)**.** Note that all the upcoming assertions and quantities like the bandwidths depend on the realisation of $(p_i^N)_{i=1}^N$. We will show a quenched result, meaning that we assume from now on that the grid points are fixed on M. Thus all the statements of this Subsection are meant in an almost-sure sense in the law of the grid points.

6.3.2.1 Assumption (3) holds

This Assumption, in the case of uniformly sampled grid points, is bypassed by the following stronger convergence result, which is proved in Section 3.6.

Lemma 6.13. *Let $(p_i^N)_{i=1}^N$ be a sequence of i.i.d. points sampled from the normalised volume measure on M and let μ^N be the corresponding empirical measure. Then*

$$\lim_{N\to\infty} W_1(\mu^N, \overline{V}) = 0.$$

6.3.2.2 Assumption (2) holds

This Subsection is based on proving one key Proposition:

Proposition 6.14. *Set the bandwidth parameter t'_N to satisfy* (6.3)*. Then the graph laplacian L_N on V_N is such that for all $f \in W$ the following holds:*

$$\lim_{N\to\infty} \left\| L_N f|_{V_N} - \big((-\Delta_M)f\big)|_{V_N} \right\|_{\infty,N} = 0.$$

In order to prove Proposition 6.14 we begin with a few remarks based on the approach of van Ginkel and Redig [149, Section 3.2] (Section 3.4 of this thesis), which we recall here for completeness. Choose $i \in \{1, \dots, N\}$. We see that

$$-L_N f(p_i^N) = \int_M g^{t'_N,\, i}(p)\mu^N(\mathrm{d}p)$$

where

$$g^{t'_N,\, i}(p) := \frac{p_{t'_N}(p,\, p_i^N)}{t'_N}(f(p) - f(p_i^N)), \quad p \in M.$$

To avoid cumbersome notation we will now drop the N sub/superscript in t'_N and p_i^N. It is clear that one can write

$$-L_N f(p) = \int_M g^{t',\, i}(p)\overline{V}(\mathrm{d}p) + \int_M g^{t',\, i}(p)(\mu^N - \overline{V})(\mathrm{d}p). \tag{6.11}$$

The strategy of the proof consists in showing that the first term converges to $(-\Delta_M)f$, and the second one becomes negligible in the limit $N \to \infty$. To this purpose, we need a bound on the supremum norm and the Lipschitz constant of the heat kernel. In the following we use L_f to denote the Lipschitz constant of a function f.

Lemma 6.15. *For t small enough one has*

$$\sup_{x,\, y\in M} |p_t(x,\, y)| \le Ct^{-\frac{d}{2}}$$

and

$$\sup_{x,\, y\in M} L_{p_t(x,\, y)} \le Ct^{-\frac{d}{2}-1}$$

where C depends only on the curvature of the manifold and on the dimension.

Proof. Let us first recall the classical Gaussian bound on the heat kernel (Li and Yau [107, Corollary 3.1]):

$$p_t(x,\, y) \le C\frac{\mathrm{e}^{-\frac{d^2(x,\, y)}{Ct} + CKt}}{\sqrt{V(x,\, \sqrt{t})V(y,\, \sqrt{t})}} \tag{6.12}$$

where $K \ge 0$ is such that $\mathrm{Ric}(M) \ge -K$ and where $V(x, r)$ denotes the volume of the ball around $x \in M$ with radius $r > 0$ in the geodesic distance. Note that such K exists in our situation, since M is compact. A simple argument (comparing with a space of constant curvature) shows that there is a $C > 0$ that does not depend

on x such that $\inf_{x\in M} V(x, \sqrt{t}) \geq Ct^{d/2} > 0$ for every x when t is small enough. This immediately entails the sup-norm bound for the function $p_t(\cdot, \cdot)$. As far as the gradient is concerned, we use the bound in Engoulatov [51, Theorem 1] to deduce that

$$\begin{aligned}\nabla p_t(x, y) &= \nabla \log p_t(x, y) \cdot p_t(x, y) \\ &\overset{(6.12)}{\leq} C(R, d)\left(\frac{D}{t} + \frac{1}{\sqrt{t}} + K\sqrt{t}\right)\left(\frac{\mathrm{e}^{-\frac{d^2(x, y)}{Ct} + CKt}}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}}\right) \qquad (6.13)\end{aligned}$$

and $D := \operatorname{diam}(M) < \infty$. Bounding the exponential term by an absolute constant and plugging this in (6.13) one obtains that

$$\nabla p_t(x, y) \leq C\left(\frac{D}{t} + \frac{1}{\sqrt{t}} + R\sqrt{t}\right) t^{-\frac{d}{2}}$$

which concludes the proof. □

This entails easily that the second summand on the right-hand side of (6.11) goes to zero as t' goes to zero, namely one can derive the following.

Corollary 6.16. *Uniformly over $i \in \{1, \ldots, N\}$ one has*

$$\lim_{t\to 0}\left|\int_M g^{t', i}(p)\mathrm{d}(\mu^N - \overline{V})(\mathrm{d}p)\right| = 0.$$

Proof. Observe that

$$L_{g^{t', i}} \leq \frac{1}{t'}\left(L_{p_{t'}(\cdot, p_i)}\|f(\cdot) - f(p_i)\|_\infty + \|p_{t'}(\cdot, p_i)\|_\infty L_{f(\cdot)-f(p_i)}\right).$$

Note that $L_f < \infty$ exists since f is smooth and that $L_{f(\cdot)-f(p_i)} = L_f$ since $f(p_i)$ is a constant. Therefore

$$\begin{aligned}&\left|\int_M g^{t', i}(p)(\mu^N - \overline{V})(\mathrm{d}p)\right| \\ &\leq \frac{1}{t'}\left(L_{p_{t'}(\cdot, p_i)}\|f(\cdot) - f(p_i)\|_\infty + \|p_{t'}(\cdot, p_i)\|_\infty L_f\right) W_1(\mu^N, \overline{V}) \\ &\leq \frac{C}{t'}\left((t')^{-\frac{d}{2}-1}C\|f\|_\infty + (t')^{-\frac{d}{2}}L_f\right) W_1(\mu^N, \overline{V})\end{aligned}$$

where in the last line we have used Lemma 6.15. The conclusion is a consequence of (6.3). Uniformity follows since the bounds do not depend on i. □

We can now begin with the proof of Proposition 6.14.

Proof of Proposition 6.14. Considering the break-up of the graph laplacian as in (6.11) and Corollary 6.16 (remember that $t' = t'_N$ is infinitesimal as N grows), all that is left to show is that

$$\lim_{N\to\infty} \sup_{1\le i\le N} \left| (-\Delta_M) f(p_i) - \int_M g^{t',\,i}(p)\overline{V}(\mathrm{d}p) \right| = 0.$$

Now observe that

$$\int_M g^{t',\,i}(p)\overline{V}(\mathrm{d}p) = -\left(\frac{\mathbf{1} - S_{t'}}{t'} f\right)(p_i).$$

Since Δ_M generates $(S_t, t \ge 0)$, we know for any smooth f that

$$\left(\frac{S_{t'} - \mathbf{1}}{t'} f\right)(p) \to \Delta_M f(p)$$

uniformly in $p \in M$ as t' goes to 0 (see for instance Grigoryan [79, Theorem 7.13]), so in particular uniformly in the p_i's. Since t' goes to 0 as N goes to infinity, this concludes the proof. □

As a consequence we obtain the following.

Corollary 6.17. *For all $t > 0$ and $f \in W$*

$$\lim_{N\to\infty} \|S_t^N f|_{V_N} - (S_t f)|_{V_N}\|_{\infty,N} = 0.$$

Proof. The proof is a direct application of Theorem 2.1 from Kurtz [102] and Proposition 6.14, combined with an argument that the extended limit of L_N (as defined in Kurtz's paper) equals the Laplace-Beltrami operator. The reason is that they are both generators and they agree on the set of smooth functions (by Proposition 6.14 they agree on W and it is easy to see that they are both 0 on constant functions), which forms a core for the Laplace-Beltrami operator. □

We are now ready to show Assumption (2).

Proposition 6.18. *For all $f \in W$, Assumption (2) holds.*

Proof. Denote $f|_{V_N}$ by $f|_N$ and $\frac{1}{N}\sum_{i=1}^N f(p_i)$ (both the constant and the constant function) by $\overline{f}^N$. Then $f_N = f|_N - \overline{f}^N$, which implies that

$$(f_N, S_t^N f_N) = (f_N, S_t^N f|_N) - (f|_N, S_t^N \overline{f}^N) + (\overline{f}^N, S_t^N \overline{f}^N). \tag{6.14}$$

Since $\overline{f}^N$ is constant, $S_t^N \overline{f}^N = \overline{f}^N$. Thus we see for the second summand above that

$$\begin{aligned} \frac{1}{N}(f|_N, S_t^N \overline{f}^N) &= \frac{1}{N}(f|_N, \overline{f}^N) = \frac{1}{N}\sum_{i=1}^N f(p_i) \frac{1}{N}\sum_{j=1}^N f(p_j) \\ &\longrightarrow \int_M f\mathrm{d}\overline{V} \int_M f\mathrm{d}\overline{V} = 0. \end{aligned}$$

For the same reason, we see for the third summand in (6.14)

$$\frac{1}{N}(\overline{f}^N, S_t^N \overline{f}^N) = \frac{1}{N}(\overline{f}^N, \overline{f}^N) \to \int_M f \mathrm{d}\overline{V} \int_M f \mathrm{d}\overline{V} = 0.$$

Now we deal with the first summand of the right-hand side of (6.14):

$$(f_N, S_t^N f|_N) = (f_N, (S_t f)|_N) + (f_N, S_t^N f|_N - (S_t f)|_N) \tag{6.15}$$

The first term gives

$$\begin{aligned} \frac{1}{N}(f_N, (S_t f)|_N) &= \frac{1}{N}(f|_N, (S_t f)|_N) - \frac{1}{N}(\overline{f}^N, (S_t f)|_N) \\ &= \frac{1}{N}\sum_{i=1}^N f(p_i) S_t f(p_i) - \frac{1}{N}\sum_{i=1}^N \overline{f}^N S_t f(p_i) \\ &\longrightarrow \int_M f S_t f \mathrm{d}\overline{V} - \int_M f \mathrm{d}V \int_M S_t f \mathrm{d}V = (f, S_t f) - 0. \end{aligned}$$

Now we need to show that the last term in the right-hand side of (6.15) goes 0. Note that

$$|(f_N, S_t^N f|_N - (S_t f)|_N)| \leq \sum_{i=1}^N |f_N(p_i)| ||S_t^N f|_N - (S_t f)|_N||_{N,\infty}. \tag{6.16}$$

Recall that $||S_t^N f|_N - (S_t f)|_N||_{N,\infty} \to 0$ by Corollary 6.17. Moreover,

$$\frac{1}{N}\sum_{i=1}^N |f_N(p_i)| \leq \frac{1}{N}\sum_{i=1}^N |f(p_i)| + \left|\overline{f}^N\right| \to \int_M |f| \mathrm{d}\overline{V} + |\int f \mathrm{d}\overline{V}| = \int |f| \mathrm{d}\overline{V} < \infty.$$

Combining these results with (6.16) yields

$$\limsup_{N\to\infty} \frac{1}{N} |(f_N, S_t^N f|_N - (S_t f)|_N)| \leq \int |f| \mathrm{d}\overline{V} \cdot 0 = 0.$$

We conclude that $(f,\, S_t f)$ is the only non-zero remaining term when taking the limit $N \to \infty$ in (6.14), which was to be shown. □

6.3.2.3 Assumption (1) holds

For this proof, we denote the graph laplacian as L_N^t, thus now highlighting the dependence on both N and t:

$$L_N^t f(v) = -\sum_{w \in V_N} \frac{p_t(v,w)}{Nt}(f(w) - f(v)).$$

The idea is that, by letting first N to infinity and then t to 0, we prove that the spectral gaps $\lambda_{N,2}^t$ of L_N^t converge to the spectral gap of the Laplace–Beltrami operator, i.e.

$$\lim_{t\to 0} \lim_{N\to\infty} \lambda_{N,2}^t = \lambda_2.$$

From this we will extract a sequence t_N such that the spectral gap of $L_N = L_N^{t_N}$ converges (i.e. $\lambda_{N,2} \to \lambda_2$). We will show that this sequence can be constructed in such a way that the convergence of Assumption (2) still holds. We will base our proof on the ideas employed by Belkin and Niyogi [10] to prove convergence of the graph laplacian eigenmaps to the continuum ones. In the article, the authors use the "intermediate" operator $L^t : L^2(M) \to L^2(M)$, $t > 0$, defined via

$$L^t f(p) := t^{-1} \int_M p_t(p, q)(f(p) - f(q))\overline{V}(\mathrm{d}q)$$

whose eigenvalues we denote by $\lambda_1^t \leq \lambda_2^t \leq \ldots$ In their case, the heat kernel edge weights were replaced by the Gaussian kernel in some Euclidean ambient space. Instead, with our choices note that

$$L^t = \frac{\mathbf{1} - S_t}{t}.$$

Therefore the i-th eigenvalue of L^t equals $t^{-1}(1 - \exp(-t\lambda_i))$, with λ_i the i-th eigenvalue of the Laplace–Beltrami, so in particular we see

$$\lim_{t \to 0} \lambda_2^t = \lambda_2. \tag{6.17}$$

Using Von Luxburg et al. [155, Theorem 21, Proposition 23] analogously to what is done by Belkin and Niyogi [10, Theorem 3.2], one also obtains that

$$\lim_{N \to \infty} \lambda_{N,2}^t = \lambda_2^t \quad \text{a.s.} \tag{6.18}$$

Note that this is an almost sure result in the law of the grid points. Since the intersection of two probability one sets still has probability one, we can safely assume that for the grid that was fixed in Remark 6.12 the limit above holds.

Now we want to construct a sequence $(t_N)_{N=1}^\infty$ such that we can reduce (6.17)- (6.18) to one limit:

$$\lim_{N \to \infty} \lambda_{N,2}^{t_N} = \lambda_2 \tag{6.19}$$

We constructed the sequence $(t'_N)_{N=1}^\infty$ in Subsubsection 6.3.2.2 to prove pointwise convergence of the Laplacians. It is direct from those calculations that any sequence that goes to 0 more slowly than $(t'_N)_{N=1}^\infty$ would also suffice. Therefore we first construct $(t_N)_{N=1}^\infty$ to ensure (6.19) and such that $t_N \geq t'_N$ for each N, after which we can simply replace t'_N in Subsubsection 6.3.2.2 by t_N.

Lemma 6.19. *There exists a sequence $(t_N)_{N=1}^\infty$ such that the following hold:*

- $t_N \downarrow 0$ *as* $N \to \infty$,
- $\lim_{N \to \infty} \lambda_{N,2}^{t_N} = \lambda_2$,
- $t_N \geq t'_N$ *for every* $N \in \mathbb{N}$.

Proof. For $j \in \mathbb{N}$ choose n_j such that:

(i) $n_j > n_{j-1}$ for $j \geq 2$,

(ii) $|\lambda_{n,2}^{1/j} - \lambda_2^{1/j}| \leq 1/j$ for all $n \geq n_j$,

(iii) $n_j \geq \min\{k \in \mathbb{N} : t'_k \leq 1/j\}$.

Such n_j exists because of (6.18) and because $t'_N \to 0$. Now for $N \in \mathbb{N}$ define $j(N) \in \mathbb{N}$ such that

$$n_{j(N)} \leq N < n_{j(N)+1}$$

and set

$$t_N := \frac{1}{j(N)}.$$

First of all $j(N)$ is well-defined for each N because of (i). Moreover, we directly see that $j(N) \uparrow \infty$, so $t_N \downarrow 0$. Note that it follows from (iii) and the fact that t'_N is decreasing that $t'_{n_j} \leq 1/j$. Using this and the monotonicity of t'_N, we see

$$t_N = \frac{1}{j(N)} \geq t'_{n_{j(N)}} \geq t'_N.$$

We also see

$$|\lambda_{N,2}^{t_N} - \lambda_2| \leq \underbrace{|\lambda_{N,2}^{t_N} - \lambda_2^{t_N}|}_{=:\text{(I)}} + \underbrace{|\lambda_2^{t_N} - \lambda_2|}_{=:\text{(II)}}.$$

(II) goes to 0 because of (6.17) and the fact that $t_N \downarrow 0$. Further we see

$$\text{(I)} = \left|\lambda_{N,2}^{1/j(N)} - \lambda_2^{1/j(N)}\right| \leq \frac{1}{j(N)},$$

because of (ii) and the assumption $N \geq n_{j(N)}$ by construction. Since $1/j(N) \to 0$, the result follows. □

6.4 Convergence of the Voronoi extension

In this Section we would like to state and prove Theorem 6.4. The proof consists of two main blocks: tightness in $H^{-s}(M)$ and finite-dimensional convergence.

We start with the necessary definitions.

6.4.1 Definitions

For $s \geq 0$ we define the space $H^s := H^s(M)$ as the closure of W with respect to the norm

$$\|f\|_s^2 := \sum_{j=2}^{\infty} \lambda_j^s \|P_j f\|_2^2 = \sum_{j=2}^{\infty} \lambda_j^s (f, e_j)^2,$$

and the corresponding inner product

$$(f,g)_s = \sum_{j=2}^{\infty} \lambda_j^s (f,e_j)(g,e_j),$$

where $(e_j, j \geq 0)$ is an $L^2(M)$-orthonormal basis of eigenfunctions of the Laplace-Beltrami operator. Note that all the e_j's are smooth. We denote by H^{-s} the Hilbert space dual of H^s.

We will need the following properties.

Lemma 6.20. *Our definition of H^s coincides with the usual definition of Sobolev space on M (as described in for instance Canzani [24, Section 6] for $s > 0$, for $s < 0$ they are just the dual of H^{-s}). Moreover the canonical norm on H^{-s} induced by H^s satisfies*

$$\|\psi\|_{-s}^2 = \sum_{j=2}^{\infty} \lambda_j^{-s} \langle \psi, e_j \rangle^2, \quad \psi \in H^{-s}.$$

Proof. The first statement follows from Canzani [24, Proposition 56].

For all $\psi \in H^{-s}$ by Riesz representation theorem, there exists $f_\psi \in H^s$ such that $\langle \psi, g \rangle = (f_\psi, g)_s$ for all $g \in H^s$. Also by isometry we have that $\|\psi\|_{-s} = \|f_\psi\|_s$. Now note that

$$\langle \psi, e_j \rangle = (f_\psi, e_j)_s = \sum_{k=2}^{\infty} \lambda_k^s (f_\psi, e_k)(e_j, e_k) = \lambda_j^s (f_\psi, e_j)$$

Hence we have

$$\|\psi\|_{-s}^2 = \|f_\psi\|_s^2 = \sum_{j=2}^{\infty} \lambda_j^s (f_\psi, e_j)^2 = \sum_{j=2}^{\infty} \lambda_j^{-s} \langle v, e_j \rangle^2.$$

□

Furthermore, we will need the following classical result to prove tightness (its proof is analogous to Roe [131, Theorem 5.8]).

Theorem 6.21 (Rellich's theorem)**.** *If $s < t$ then the inclusion operator $H^t \hookrightarrow H^s$ is compact.*

Now let $\{C_i^N, i = 1, \ldots, N\}$ be the Voronoi tessellation corresponding to the vertex set $V_N := (p_i)_{i=1}^N$, i.e.

$$C_i^N = \{p \in M : d(p,p_i) \leq d(p,p_j) \ \ \forall\, j \leq N\}, \quad i = 1, \ldots, N.$$

Also denote $v_i^N = \overline{V}(C_i^N)$. We will usually leave out the superscript N to ease notation.

Definition 6.22 (The DGFF in H^{-s})**.** Let ϕ_N be the zero-average DGFF on V_N as in Theorem 6.1. We define $\widetilde{\phi}_N \in H^{-s}$ by the following action on $f \in H^s$:

$$\left\langle \widetilde{\phi}_N, f \right\rangle := \frac{1}{N} \sum_{i=1}^{N} \phi_N(p_i) \frac{1}{v_i} \int_{C_i} f(p) \overline{V}(\mathrm{d}p).$$

Note that if we define

$$\begin{aligned} \widetilde{f}_N : V_N &\to \mathbb{R} \\ p_i &\mapsto \widetilde{f}_N(p_i) := \frac{1}{v_i} \int_{C_i} f(p) \overline{V}(\mathrm{d}p) \end{aligned}$$

then we can write

$$\langle \widetilde{\phi}_N, f \rangle = N^{-1} (\phi_N, \widetilde{f}_N) = \langle \phi_N, \widetilde{f}_N \rangle \tag{6.20}$$

with a slight abuse of notation (since ϕ_N acts on W, but in fact this action depends only on grid values). In order to prove Theorem 6.4 first of all we will show that the sequence $\{\widetilde{\phi}_N,\, N \in \mathbb{N}\}$ is tight in H^{-s} (Subsection 6.4.2). From this it follows that every sequence has a convergent subsequence. Then what remains is to show that the limit is unique. Since the limit is Gaussian, it is characterised by its finite-dimensional distributions. By the theory of abstract Wiener spaces, already described for example in Cipriani et al. [33, Section 3.2], it suffices to show that for all $f,\, g \in H^1$

$$\mathbb{E}\left(\langle \sqrt{N} \widetilde{\phi}_N, f \rangle \langle \sqrt{N} \widetilde{\phi}_N, g \rangle \right) \to (f,\, Gg)$$

as $N \to \infty$. This will be done in Subsection 6.4.3.

6.4.2 Tightness of $\widetilde{\phi}_N$

We prove the following Proposition.

Proposition 6.23. *The collection $\{\widetilde{\phi}_N, N \in \mathbb{N}\}$ is tight in H^{-s} for any $s > d - 1/2$.*

Proof. We will first prove that for $s > d - 1/2$ and for every $\epsilon > 0$, there exists $R = R(\epsilon) > 0$ such that for all N

$$\mathbb{P}(\|\sqrt{N} \widetilde{\phi}_N\|_{-s}^2 > R) \leq \epsilon. \tag{6.21}$$

First of all by Chebyshev's inequality

$$\mathbb{P}(\|\sqrt{N} \widetilde{\phi}_N\|_{-s}^2 > R) \leq \frac{1}{R} \mathbb{E}(\|\sqrt{N} \widetilde{\phi}_N\|_{-s}^2).$$

It suffices then to show that $\mathbb{E}(\|\sqrt{N} \widetilde{\phi}_N\|_{-s}^2)$ is bounded by some constant. We write

$$\mathbb{E}\left(\sum_{j=2}^{\infty} \lambda_j^{-s} \langle \sqrt{N} \widetilde{\phi}_N, e_j \rangle^2 \right) = \sum_{j=2}^{\infty} \lambda_j^{-s} \mathbb{E}\left(\langle \sqrt{N} \widetilde{\phi}_N, e_j \rangle^2 \right).$$

Now note that for any $h \in W$

$$\mathbb{E}\left(\langle\sqrt{N}\widetilde{\phi}_N, h\rangle^2\right) \overset{(6.20)}{=} \mathbb{E}\left(\langle\sqrt{N}\phi_N, \widetilde{h}_N\rangle^2\right) = \frac{1}{N}(\widetilde{h}_N, G_N\widetilde{h}_N) \leq \frac{1}{N}\|\widetilde{h}_N\|^2\|G_N\| \tag{6.22}$$

where $\|G_N\|$ is the operator norm of G_N from $\ell^2(V_N)$ to itself and $\|\widetilde{h}_N\|$ is the $\ell^2(V_N)$-norm. Since $\|G_N\| = (\lambda_2^N)^{-1}$ by Assumption (1) we can bound it by some constant independent of N. Moreover

$$\frac{1}{N}\|\widetilde{h}_N\|^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{v_i}\int_{C_i} h(p)\overline{V}(\mathrm{d}p)\right)^2 \leq \|h\|_\infty^2.$$

Now by Canzani [24, Theorem 82] $\|e_j\|_\infty \leq C\lambda_j^{(d-1)/4}$, so applying the previous argument to the bound (6.22) with $h := e_j$ we see that

$$\sum_{j=1}^{\infty}\lambda_j^{-s}\mathbb{E}\left(\langle\sqrt{N}\widetilde{\phi}_N, e_j\rangle^2\right) \leq \sum_{j=1}^{\infty}\lambda_j^{-s}\|G_N\|\|e_j\|_\infty^2 \leq C\sum_{j=1}^{\infty}\lambda_j^{(d-1)/2-s}.$$

Canzani [24, Theorem 72] states Weyl's lemma with the asymptotic $\lambda_j \sim Cj^{2/d}$ as $j \to \infty$, which shows that

$$C\sum_{j=1}^{\infty}\lambda_j^{(d-1)/2-s} \leq C\sum_{j=1}^{\infty}j^{2/d((d-1)/2-s)}.$$

This series is bounded as long as $2/d((d-1)/2-s) < -1$, so for $s > d - 1/2$. This means we have shown (6.21).

To conclude the argument, fix $s > d - 1/2$. Let s' be such that $s > s' > d - 1/2$ and let $\epsilon > 0$. We know there exists $R > 0$ such that (6.21) holds, i.e. for all N

$$\mathbb{P}(\widetilde{\phi}_N \notin \overline{B_{-s'}(0,R)}) \leq \epsilon,$$

where $\overline{B_{-s'}(0,R)}$ is the closed ball with radius R in $H^{-s'}$. Now by Theorem 6.21, we see that $\overline{B_{-s'}(0,R)}$ is compact in H^{-s} (since $s > s'$), so we have shown tightness in H^{-s}. □

6.4.3 Convergence of finite dimensional distributions

As mentioned before, we need to show that for all f, $g \in H^1$

$$\mathbb{E}\left(\langle\sqrt{N}\widetilde{\phi}_N, f\rangle\langle\sqrt{N}\widetilde{\phi}_N, g\rangle\right) \to (f, Gg).$$

Since W is dense in H^1 and by a polarisation argument, it suffices to show the following.

Proposition 6.24. *For all $f \in W$*

$$\mathbb{E}\left\langle \sqrt{N}\widetilde{\phi}_N, f\right\rangle^2 \to (f, Gf).$$

Before we move on to the proof, we prove the following technical lemma.

Lemma 6.25. *Define*

$$\epsilon_N := \sup_{1\leq i\leq N} \sup_{p\in C_i^N} d(p, p_i).$$

Then ϵ_N goes to 0 as $N \to \infty$.

Proof. To derive a contradiction, suppose that ϵ_N does not go to 0. This means that there is some $\delta > 0$ such that $\epsilon_N > 2\delta$ for infinitely many N. Consequently for each such N, there exists $1 \leq i \leq N$ and $p \in C_i$ such that $d(p, p_i) \geq \delta$. Since $p \in C_i$, p_i is the nearest grid point to it. This implies that $B(p, \delta)$ does not contain any grid points. We conclude from this that

(i) for infinitely many $N \in \mathbb{N}$ there must be a ball with radius δ that does not contain a grid point of V_N.

Now fix $p \in M$ and $r > 0$ and suppose that $B(p, r)$ does not contain grid points of V_N for infinitely many N. Now fix some positive non-zero continuous function f which has support contained in $B(p, r)$. Then $\int f \mathrm{d}\mu_N = 0$ for infinitely many N, but $\int f \mathrm{d}\overline{V} > 0$. However, by assumption (3),

$$\int f \mathrm{d}\mu_N \to \int f \mathrm{d}\overline{V} \quad (N \to \infty).$$

This is a contradiction. We conclude that

(ii) for every fixed ball B in M there exists an N_0 such that B contains grid points of V_N for every $N \geq N_0$.

To finish the argument let $B(q_1, \delta/2), B(q_2, \delta/2), \ldots, B(q_m, \delta/2)$ be a finite number of balls of radius $\delta/2$ that cover M. By (ii), each of these balls will eventually contain a grid point. This means that there exists an N_0 such that for all $N \geq N_0$ each of these balls contains a grid point of V_N. Now let $N \geq N_0$ and let p be any point of the manifold. Since p is at distance less than $\delta/2$ from some q_i and there is a grid point of V_N at distance less than $\delta/2$ from q_i, it follows that $B(p, \delta)$ contains at least one grid point of V_N. This implies that every ball of radius p contains at least one grid point of V_N, which contradicts (i). □

Proof of Proposition 6.24. First of all

$$\mathbb{E}\left\langle \sqrt{N}\widetilde{\phi}_N, f\right\rangle^2 \overset{(6.20)}{=} \mathbb{E}\langle \sqrt{N}\phi_N, \widetilde{f}_N\rangle^2 = \frac{1}{N}(\widetilde{f}_N, G_N\widetilde{f}_N).$$

Recall the notations $f|_N = f|_{V_N}$, $\overline{f}^N = 1/N \sum_{i=1}^N f(p_i)$ and $f_N = f|_N - \overline{f}^N$. We have shown in Section 6.3.1 that

$$\frac{1}{N}(f|_N, G_N f|_N) \to (f, Gf)$$

(actually we have shown this for f_N, but since G maps constant vectors to 0 this does not make a difference). Hence by the triangular inequality it suffices to show that

$$\left| \frac{1}{N}(\widetilde{f}_N, G_N \widetilde{f}_N) - \frac{1}{N}(f|_N, G_N f|_N) \right| \to 0.$$

By linearity and Cauchy–Schwarz, we see that

$$\begin{aligned} & \left| \frac{1}{N}(f|_N, G_N f|_N) - \frac{1}{N}(\widetilde{f}_N, G_N \widetilde{f}_N) \right| \\ \leq \; & \frac{1}{N} \left(|(f|_N - \widetilde{f}_N, G_N f|_N)| + |(\widetilde{f}_N, G_N (f|_N - \widetilde{f}_N))| \right) \\ \leq \; & \frac{1}{N} \|f|_N - \widetilde{f}_N\| \|G_N\| \|f|_N\| + \frac{1}{N} \|\widetilde{f}_N\| \|G_N\| \|f|_N - \widetilde{f}_N\|. \end{aligned} \tag{6.23}$$

Now we see that

$$\left(\frac{1}{\sqrt{N}} \|f|_N\| \right)^2 = \frac{1}{N} \sum_{i=1}^N f(p_i)^2 \leq \|f\|_{L^\infty}^2$$

and

$$\left(\frac{1}{\sqrt{N}} \|\widetilde{f}_N\| \right)^2 = \frac{1}{N} \sum_{i=1}^N \left(\frac{1}{v_i} \int_{C_i} f(p) \overline{V}(\mathrm{d}p) \right)^2 \leq \frac{1}{N} \sum_{i=1}^N \|f\|_{L^\infty}^2 = \|f\|_{L^\infty}^2.$$

Also $\|G_N\| = (\lambda_2^N)^{-1}$. Further, we see that for all $p \in C_i$,

$$f(p_i) - L_f \epsilon_N \leq f(p_i) - L_f d(p, p_i) \leq f(p) \leq f(p_i) + L_f d(p, p_i) \leq f(p_i) + L_f \epsilon_N,$$

which implies that

$$\left| f(p_i) - \frac{1}{v_i} \int_{C_i} f(p) \overline{V}(\mathrm{d}p) \right| \leq \frac{1}{v_i} \int_{C_i} |f(p_i) - f(p)| \overline{V}(\mathrm{d}p) \leq L_f \epsilon_N.$$

Now we see that

$$\begin{aligned} \left(\frac{1}{\sqrt{N}} \|f|_N - \widetilde{f}_N\| \right)^2 &= \frac{1}{N} \sum_{i=1}^N \left(f(p_i) - \frac{1}{v_i} \int_{C_i} f(p) \overline{V}(\mathrm{d}p) \right)^2 \\ &\leq \frac{1}{N} \sum_{i=1}^N L_f^2 \epsilon_N^2 = L_f^2 \epsilon_N^2, \end{aligned}$$

which goes to 0 as $N \to \infty$. Putting everything together, we deduce that (6.23) is bounded by

$$L_f \epsilon_N \frac{1}{\lambda_2^N} \|f\|_\infty + \|f\|_\infty \frac{1}{\lambda_2^N} L_f \epsilon_N = \frac{2\epsilon_N L_f \|f\|_{L^\infty}}{\lambda_2^N},$$

which goes to 0 as $N \to \infty$, since by lemma 6.25 $\epsilon_N \to 0$ and by Assumption (1) $\inf_N \lambda_2^N > 0$. □

6.5 Notes and perspectives

In this chapter we defined the zero-average DGFF on grids on a manifold and proved that it converges to the GFF on the manifold. We regarded it first as acting on smooth functions and later as random element of a Sobolev space of negative index. To define the DGFF we used the grids from Chapter 3. However, in order to find a lower bound on the spectral gaps, we needed to use different conductances. Instead of defining edge weights depending on the distance between points we used the heat kernel on the manifold. We will now mention some further aspects of these models to be studied and some possible directions for generalisation or improvement.

First, it would be interesting to study properties of the DGFF and the GFF on a manifold. We defined the DGFF as a normal distribution with the Green's function on the grid as its covariance function. Then we showed that it satisfies a version of the Markov property. However, it would be interesting to study its further properties and compare them to the DGFF on a flat space. An example would be to study the maxima and level sets of the DGFF on a manifold.

We know for instance from Adler and Taylor [2] that there is a close connection between properties of random fields and the manifolds on which they are defined. It would be interesting to study to which extent such relations can be found also for the DGFF and the GFF, for example with the curvature of the manifold.

Further, as we mentioned in previous chapters, we could try to generalise to non-compact manifolds. However, for the DGFF this leads to problem of definition, since we would need to define it as a distribution on countably many points. Whether this is possible probably depends on the dimension of the underlying manifold. An additional complication is that the smooth functions on a non-compact manifold are generally not a nuclear space. One probably needs conditions on how rapidly the functions vanish at infinity.

As we mentioned in Section 3.7, we can use more general grids if we relax the assumption of pointwise convergence of the graph Laplacians to the Laplace-Beltrami operator. An alternative could be convergence of the spectra of the graph Laplacians to the spectrum of the Laplace-Beltrami operator. Such convergence results have been shown for several alternative grids that we mention in Chapter 3. This might suffice to prove convergence of the DGFF to the GFF. The reason for this is that we only need to show convergence of the Green's function applied to test functions in

an inner product. This inner product can be expressed in terms of eigenvalues and projections on eigenspaces of the graph Laplacians, as we did in Section 6.3.1.

Next, the way in which we extended the DGFF convergence to Sobolev spaces was by letting the DGFF act on averages of test functions over Voronoi cells. This helped us formulate this action in terms of inner products that we used earlier in the chapter. Perhaps a more natural way to extend the DGFF on a Voronoi tesselation is by letting it be a piecewise constant random function of which the constant value on a Voronoi cell equals the value at the grid point in that cell. This leads to an action on test functions that is slightly different from our case. It would be interesting to see if also in this case it is possible to prove convergence to the GFF in Sobolev norm.

Another way to generalise is to consider a different field. For instance we could consider the membrane model on a manifold. This would amount to replacing the Laplacian by the bilaplacian, the Laplacian squared. However, the covariance matrix of the membrane model cannot be represented as the Green's function corresponding to a random walk. This means that there is no underlying semigroup, which makes the study of the membrane model different from the Gaussian Free Field. Therefore we need to find another method of proof. Perhaps an approach using spectral convergence as mentioned above would be worth trying for the membrane model.

Part III

Active particles

Chapter 7

Active particles and the role of reversibility

In this chapter[1] we study a model of active particles that perform a simple random walk and on top of that have a preferred direction determined by an internal state which is modelled by a stationary Markov process. First we calculate the limiting diffusion coefficient. Then we show that the 'active part' of the diffusion coefficient is in some sense maximal for reversible state processes. Further, we obtain a large deviations principle for the active particle in terms of the large deviations rate function of the empirical process corresponding to the state process. Again we show that the rate function and free energy function are (pointwise) optimal for reversible state processes. Finally, we show that in the case with two states, the Fourier-Laplace transform of the distribution, the moment generating function and the free energy function can be computed explicitly. Along the way we provide several examples.

7.1 Introduction

In this chapter we study run-and-tumble motion, which is often used as a model of active particles. The particle motion has two ingredients: first the particle performs a symmetric random walk, and second, independently it moves in a direction dictated by an internal state process. This internal state process is assumed to be a continuous-time stationary Markov process. In the sequel we will first describe how our results relate to various results on run-and-tumble particles in the literature. Next, we will briefly sketch how our model relates to the broader literature on active matter, stochastic slow-fast systems and directionally reinforced random walks.

[1] This chapter is based on van Ginkel et al. [152].

7.1.1 Model and contributions

The model that we study in this chapter is an instance of what is more generally called run-and-tumble motion. These are models of particles that follow a preferred direction which is reversed at random points in time. Recent articles include Großmann et al. [80], Demaerel and Maes [40], Malakar et al. [113], Le Doussal et al. [104], Dhar et al. [43] and Garcia-Millan and Pruessner [69].

We study an active particle of which the state process (that determines the preferred direction) is a stationary Markov process (under some technical assumptions), started from its unique ergodic measure. Then our main contribution is twofold. First we are able to calculate closed form formulas for the limiting diffusion coefficient of the active particle. This formula holds in great generality, including also the case where the state process is a diffusion (we will provide examples where an Ornstein-Uhlenbeck process or Brownian motion on a circle form the state process). In this formula we can interpret the different terms and observe where the activity is manifested. We also calculate the large deviations free energy function and rate function in the case where the state process has a finite state space.

Second, we study the role of reversibility of the state process in the diffusion coefficient and large deviations of the active particle (again for finite state spaces). In particular, we show that reversible processes in some sense optimise those quantities. To be more precise, we show that among all processes with the same symmetric part and the same stationary measure, the reversible process maximises the diffusion coefficient and the free energy function (pointwise) and minimises the large deviations rate function (also pointwise). The last two results are obtained by showing a pointwise inequality for the Donsker-Varadhan rate function of the empirical processes corresponding to the reversible and non-reversible state processes, respectively.

The calculations that we present are for an active particle in $\mathbb{R}$, but we explain for all of our results how they generalise to $\mathbb{R}^d$ and we also provide the explicit formulas in the $\mathbb{R}^d$ setting.

7.1.2 Context and related literature

First of all, the run-and-tumble motion is often used as a model of active matter. As we said before, our active particle performs a symmetric random walk and a random walk with preferred directions that are switched. The part of the motion that follows the internal state is called the active part of the motion, because for the switching between internal states some internal source of energy is needed. The passive part of the motion is the symmetric random walk part and comes from collisions with surrounding molecules.

Note that active particles should not be confused with activated random walk. In those models particles perform random walks, but fall asleep after a random time and are awakened (activated) when other particles jump to their position.

Second, the active particle motion studied in this chapter is an example of a stochastic slow-fast system. These are well-studied systems where coupled quantities evolve on different time scales. If one rescales the position of the active particle diffusively, the underlying state process behaves as a fast process and the (rescaled) particle position is a slow process. Asymptotically the fast state process averages out and has a deterministic influence on the slow process: the limiting diffusion coefficient will depend on the state process only through the stationary distribution and the covariance function. For an introduction to stochastic slow-fast systems see for instance Berglund and Gentz [13]. The large deviation results that we obtain are related to more general results for large deviations of slow-fast systems that were studied in for instance Freidlin and Wentzell [65] or, more recently, in Kraaij and Schlottke [99].

Third, the active particle motion studied in this chapter has strong similarities with a directionally reinforced random walk. This model was first studied by Coppersmith and Diaconis [34] and a multidimensional version in Mauldin et al. [117]. Then in Horváth and Shao [85] and (in a more general context) Ghosh et al. [73] it was shown for a process of this type that it converges to a multidimensional Brownian motion when rescaled diffusively.

Also we will compare the diffusion coefficients and large deviations rate functions for active particles with state processes that are either reversible or non-reversible with respect to the same invariant measure. In particular we will show that the Donsker-Varadhan rate function of reversible processes is dominated by the rate functions of non-reversible processes with the same symmetric part and the same invariant measure. A similar result (in a different context) was obtained in Pinsky [125].

7.1.3 Structure of this chapter

In Section 7.2, we introduce the active particle process as a stochastic integral. We split it into a random walk part, a martingale part and an active part.

In Section 7.3, we obtain the limiting diffusion coefficient of the active particle and show that it is the sum of the contributions of the random walk part, the martingale part and the active part. Then we generalise the formulas to the multidimensional case. The limiting diffusion coefficient (or matrix) is then calculated for several concrete examples, both with finite and with infinite state spaces. Finally, we sketch how one obtains a Central Limit Theorem for the active particle.

Next, in Section 7.4, we restrict ourselves to finite space spaces and study the active part of the diffusion coefficient, which is proportional to an inner product with the inverse of the generator of the state process. We show that among all stationary processes with respect to the same invariant measure and with equal symmetric part, the active part of the diffusion coefficient is maximal for the reversible process. We use the 1-dimensional case to show that this also holds for the active part of the diffusion matrix in higher dimensions.

Then in Section 7.5, we move to large deviations (still for finite state spaces). We compute the large deviations free energy function. Using Varadhan's lemma, we derive an expression for the free energy function of the active particle in terms of the Donsker-Varadhan rate function for the empirical process corresponding to the state process (which in turn gives us the large deviations rate function as the Legrende transform of the free energy). We show that the free energy function is maximal and the rate function is minimal in the reversible case (similar to the situation for the diffusion coefficient) by showing that Donsker-Varadhan rate function is maximal for reversible processes.

In Section 7.6 we give an analysis of the situation where the state space is $\{-1, 1\}$. In this two-state case we can explicitly calculate the Fourier-Laplace transform of the distribution of the active particle process, the moment generating function and the large deviations free energy function.

We conclude the chapter with notes and perspectives in Section 7.7.

7.2 Preliminaries

We consider the position $(X_t, t \geq 0)$ of a particle that moves in continuous time and space (see also Remark 7.1). For now we assume $X_t \in \mathbb{R}$, but we will generalise to $\mathbb{R}^d$ later. The particle has the following dynamics.

a) With rate 2κ the particle performs a simple symmetric random walk.

b) Independently, with rate λ the particle jumps in a preferred direction indicated by an inner state. If such jump occurs at time t, the particle jumps from X_t to $X_t + v_t^\gamma$.

c) This internal state evolves with 'rate' γ according to a stationary Markov process.

Because of the jump to a preferred direction based on the inner state, we call the particle an *active particle*.

To make this more precise we make the following definitions. We will assume that the processes in the coming definitions are jointly defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

i) Random walk part. Let $Y = (Y_t, t \geq 0)$ be a simple symmetric random walk, i.e. a random walk that starts from the origin ($Y_0 = 0$), jumps with rate 1 and jumps 1 to the left or to the right with equal probability. Fix a constant $\kappa > 0$. Then the random walk part of the process is $Y_{2\kappa t}$.

ii) Internal state process. Let $(M_t, t \geq 0)$ be a stationary Markov process (independent of the random walk) on a state space $\mathscr{S}$ with ergodic measure μ. We will call this process the *state process*. Since we will always start M from μ, we can assume without loss of generality that μ is the unique ergodic (and hence the unique invariant) measure of M. Denote by $(S_t, t \geq 0)$ and A the corresponding

semigroup and Markov generator on $L^2(\mu)$, respectively, and denote the inner product on $L^2(\mu)$ by $(\cdot,\cdot)$ and the corresponding norm by $\|\cdot\|$.

iii) Speed function. Let v be an element of $L^2(\mu)$. We will call v the *speed function*. For simplicity, we assume that $\int v\mathrm{d}\mu = 0$, meaning that the average of the speed with respect to the stationary measure on the internal state space is 0. This is not essential though, we will make some remarks on what happens without this assumption. The idea is that $v : \mathscr{S} \to \mathbb{R}$ is a mapping that indicates for each internal state the jump vector in case of an active jump when the particle has that internal state. In the simplest possible case, the state space $\mathscr{S}$ is the set $\{-1, 1\}$ and v is the identity function. In Section 7.3.2 we will see this example (Example 7.8) and several others, for instance where v maps three internal states to three numbers that sum to 0 (in Example 7.9) or where v is the sine function (in Example 7.11).

iv) Speed process. Fix a constant $\gamma > 0$. We define $v_t^\gamma = v(M_{\gamma t})$ and call $(v_t^\gamma, t \geq 0)$ the *speed process*. Note that this speed process does not need to be a Markov process. In the special case for $\gamma = 1$, we will simply write v_t. Note that $(v_t^\gamma, t \geq 0)$ is the process $(v_t, t \geq 0)$ speeded up by the factor γ. We make the following two technical assumptions on the speed process.

a) First we assume that

$$\lim_{t\to\infty} \int_0^t S_r v \mathrm{d}r \quad \text{exists in } L^2(\mu). \tag{7.1}$$

This implies that the limit $u := \int_0^\infty S_t v \mathrm{d}t$ satisfies $u \in D(A)$ and $-Au = v$, so we will write $\int_0^\infty S_t v \mathrm{d}t = -A^{-1}v$. We need this assumption to ensure that the limiting variance is finite. If it does not hold, there may not be a diffusive scaling limit. Sufficient conditions for Assumption (7.1) are for instance that the spectral gap of A is positive or that there exist $c, C > 0$ such that

$$\|S_t v\| \leq C\mathrm{e}^{-ct}.$$

The latter is a condition on the speed of relaxation, it ensures that the internal state process reaches equilibrium fast enough, which avoids large temporal covariances. In any case, Assumption (7.1) requires that $S_t v$ goes to 0 fast enough that it is integrable.

b) The second assumption is that for all $t > 0$

$$\lim_{\delta\downarrow 0} \sup_{\substack{0\leq s,s'\leq t \\ |s-s'|<\delta}} \mathbb{E}[(v_s - v_{s'})^2] = 0. \tag{7.2}$$

In other words: the speed process must be uniformly continuous in L^2. This assumption is purely technical, we will use it in Lemma 7.29 to show that the integral in (7.3) is well-defined.

Both of these assumptions are automatically satisfied in the case that the state space $\mathscr{S}$ of M is finite. Other internal state processes that satisfy these assumptions (with a suitable choice of v) include diffusion processes such as Brownian motion and the Ornstein-Uhlenbeck processes that we encounter in the examples in Section 7.3.2.

v) Active jumps. Finally, fix a constant $\lambda > 0$ and let $(N_t, t \geq 0)$ be a Poisson process with rate λ (independent of the random walk and the state process). This process marks the times at which the particle jumps in a preferred direction.

With these components we can define

$$X_t = Y_{2\kappa t} + \int_0^t v_s^\gamma \mathrm{d}N_s, \tag{7.3}$$

where the integral is defined as a limit in $L^2(\mathbb{P})$ (see in Lemma 7.29 how the well-definedness of the integral follows from Assumption (7.2)). This expression matches with our description above: $Y_{2\kappa t}$ is the random walk part and on top of that whenever the Poisson process N has a jump at time t, say, the number v_t^γ is added to X_t. Note that (7.3) implies that $X_0 = 0$. Also, we can write (7.3) as

$$X_t = Y_{2\kappa t} + \int_0^t v_s^\gamma \mathrm{d}\overline{N}_s + \lambda \int_0^t v_s^\gamma \mathrm{d}s, \tag{7.4}$$

where $\overline{N}_t = N_t - \lambda t$ is a compensated Poisson process. We call the first, second and third term of (7.4) the *random walk part*, the *martingale part* and the *active part*, respectively. This division will become more clearly visible in the diffusion coefficient.

Remark 7.1. Note that if v is integer-valued, X_t stays in the lattice $\mathbb{Z}$. In case v is not integer-valued, we can also directly consider a continuous process and define

$$X_t^c = B_{2\kappa t} + \lambda \int_0^t v_s^\gamma \mathrm{d}s, \tag{7.5}$$

where $(B_t, t \geq 0)$ is Brownian motion (independent of the state process) and where the speed process is followed continuously in time. As will become clear later, the change to Brownian motion is mostly aesthetic. However, the change from $\mathrm{d}N_t$ to $\lambda \mathrm{d}t$ leaves out the martingale part of X_t, which will have consequences for both the limiting diffusion coefficient and for the large deviations. We will makes remarks on this later, after the results concerned.

7.3 Diffusion coefficient

A first observation is that the expectation of X_t is 0. Indeed, using independence of the processes v_s^γ and N_s and the fact that $\mathbb{E}v_s^\gamma = 0$, we compute

$$\begin{aligned}\mathbb{E}X_t &= \mathbb{E}Y_{2\kappa t} + \mathbb{E}\int_0^t v_s^\gamma \mathrm{d}N_s = 0 + \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}[v_s^\gamma(N_{s_{i+1}} - N_{s_i})] \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}v_{s_i}^\gamma\lambda(s_{i+1}-s_i) = 0.\end{aligned}$$

In this section we determine the limiting diffusion coefficient of the active particle and extend this result to active particles in higher dimensions. Then we provide some examples. Finally, we discuss the invariance principle.

7.3.1 Calculating the diffusion coefficient

As a first result, we compute the limiting variance of the position of the active particle.

The 1-dimensional case
We start in dimension 1. Recall that $(\cdot,\cdot)$ denotes the inner product on $L^2(\mu)$.

Theorem 7.2. *The active particle has the following limiting diffusion coefficient*

$$\lim_{t\to\infty}\frac{\mathrm{Var}(X_t)}{t} = 2\kappa + \lambda\int v^2\mathrm{d}\mu + \frac{2\lambda^2}{\gamma}(v, -A^{-1}v). \tag{7.6}$$

Proof. First of all, note that the random walk part of X_t is independent of the rest. Second, note that using Lemma 7.29 and the independence of v^γ and $\overline{N}$,

$$\begin{aligned}&\mathrm{Cov}\left(\int_0^t v_s^\gamma \mathrm{d}\overline{N}_s, \lambda\int_0^t v_s^\gamma \mathrm{d}s\right) \\ =\ &\lim_{n\to\infty}\sum_{i,j=0}^{n-1}\mathrm{Cov}\left(v_{s_i}(\overline{N}_{s_{i+1}} - \overline{N}_{s_i}), \lambda v_{s_j}(s_{j+1}-s_j)\right) \\ =\ &\lim_{n\to\infty}\sum_{i,j=0}^{n-1}\lambda v_{s_j}(s_{j+1}-s_j)\mathrm{Cov}\left(v_{s_i}, v_{s_j}\right)\mathbb{E}\left[\overline{N}_{s_{i+1}} - \overline{N}_{s_i}\right] = 0.\end{aligned}$$

This implies that

$$\mathrm{Var}(X_t) = \mathrm{Var}(Y_{2\kappa t}) + \mathrm{Var}\left(\int_0^t v_s^\gamma \mathrm{d}\overline{N}_s\right) + \mathrm{Var}\left(\lambda\int_0^t v_s^\gamma \mathrm{d}s\right).$$

In other words, each of the parts of X_t in (7.4) has its own contribution to the variance of X_t and hence to the limiting diffusion coefficient. Similar to before, we will refer

to these as the random walk part, the martingale part and the active part of the diffusion coefficient. We will now calculate these contributions.

First, $Y_{2\kappa t}$ is the difference of two independent Poisson random variables with rate κt. Therefore

$$\lim_{t\to\infty} \frac{\mathrm{Var}(Y_{2\kappa t})}{t} = \lim_{t\to\infty} \frac{\kappa t + \kappa t}{t} = 2\kappa. \tag{7.7}$$

Second, using Lemma 7.29, the independence of v^γ and $\overline{N}$ and the fact that $\mathbb{E}v_s^\gamma = \mathbb{E}\left[\overline{N}_{s_{i+1}} - \overline{N}_{s_i}\right] = 0$, we see

$$\begin{aligned}
\mathrm{Var}\left(\int_0^t v_s^\gamma \mathrm{d}\overline{N}_s\right) &= \lim_{n\to\infty} \sum_{i,j=0}^{n-1} \mathrm{Cov}\left(v_{s_i}(\overline{N}_{s_{i+1}} - \overline{N}_{s_i}), v_{s_j}(\overline{N}_{s_{j+1}} - \overline{N}_{s_j})\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathrm{Var}\left(v_{s_i}(\overline{N}_{s_{i+1}} - \overline{N}_{s_i})\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathrm{Var}\left(v_{s_i}\right) \mathrm{Var}\left(\overline{N}_{s_{i+1}} - \overline{N}_{s_i}\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \int v^2 \mathrm{d}\mu \lambda(s_{i+1} - s_i) = \lambda t \int v^2 \mathrm{d}\mu.
\end{aligned}$$

Therefore

$$\lim_{t\to\infty} \frac{\mathrm{Var}\left(\int_0^t v_s^\gamma \mathrm{d}\overline{N}_s\right)}{t} = \lim_{t\to\infty} \frac{\lambda t \int v^2 \mathrm{d}\mu}{t} = \lambda \int v^2 \mathrm{d}\mu. \tag{7.8}$$

For the third part we calculate the limiting variance of an additive functional of a Markov process. This formula was already obtained for instance in Kipnis and Varadhan [97, Corollary 1.9] and De Masi et al. [38, Lemma 2.4] (for reversible Markov processes). In fact, it is known as Green-Kubo relations, which go back to Green [78] and Kubo [101]. For completeness, we provide the calculations for our specific context here. Using the stationarity of v^γ and the symmetry of covariance, we compute

$$\begin{aligned}
\mathrm{Var}\left(\int_0^t v_s^\gamma \mathrm{d}s\right) &= \int_0^t \int_0^t \mathrm{Cov}(v_s^\gamma, v_r^\gamma)\mathrm{d}r\mathrm{d}s = 2\int_0^t \int_0^s \mathrm{Cov}(v_s^\gamma, v_r^\gamma)\mathrm{d}r\mathrm{d}s \\
&= 2\int_0^t \int_0^s \mathrm{Cov}(v_{s-r}^\gamma, v_0^\gamma)\mathrm{d}r\mathrm{d}s = 2\int_0^t \int_0^s \mathrm{Cov}(v_r^\gamma, v_0^\gamma)\mathrm{d}r\mathrm{d}s \\
&= 2\int_0^t \int_r^t \mathrm{Cov}(v_r^\gamma, v_0^\gamma)\mathrm{d}s\mathrm{d}r = 2\int_0^t (t-r)\mathrm{Cov}(v_r^\gamma, v_0^\gamma)\mathrm{d}r \\
&= \frac{2}{\gamma}\int_0^{\gamma t} (t-r)\mathrm{Cov}(v(M_r), v(M_0))\mathrm{d}r = \frac{2}{\gamma}\int_0^{\gamma t} (t-r)(v, S_r v)\mathrm{d}r.
\end{aligned} \tag{7.9}$$

To compute this, first note that with Assumption (7.1) we see that

$$\lim_{t\to\infty}\int_0^t (v, S_r v)\mathrm{d}r = \left(v, \lim_{t\to\infty}\int_0^t S_r v \mathrm{d}r\right) = \left(v, -A^{-1}v\right). \tag{7.10}$$

Note that the convergence of $\int_0^t (v, S_r v)\mathrm{d}r$ also implies that

$$\lim_{t\to\infty}\int_0^t \frac{r}{t}(v, S_r v)\mathrm{d}r = 0. \tag{7.11}$$

Combining (7.9), (7.10) and (7.11), we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathrm{Var}\left(\lambda\int_0^t v_s^\gamma \mathrm{d}\overline{N}_s\right)}{t} &= \lim_{t\to\infty}\frac{2\lambda^2}{\gamma}\int_0^{\gamma t}(v, S_r v)\mathrm{d}r + \lim_{t\to\infty} 2\lambda^2\int_0^{\gamma t}\frac{r}{\gamma t}(v, S_r v)\mathrm{d}r \\ &= \frac{2\lambda^2}{\gamma}(v, -A^{-1}v). \end{aligned}\tag{7.12}$$

Now combining (7.7), (7.8) and (7.12), we obtain the result. □

Higher dimensions
So far we considered an active particle that only moves in one dimension. However, we can just as well treat a higher dimensional situation. To this end fix a dimension $d \in \mathbb{N}$. Let Y be a d-dimensional simple random walk, i.e. each component of Y is an independent copy of the Y that we had in the 1-dimensional situation. Let the speed function v be an element of $L^2((\Omega, \mu), \mathbb{R}^d)$ such that $\int v\mathrm{d}\mu = 0$ (in $\mathbb{R}^d$). We denote by Σ the covariance matrix of v under μ, i.e.

$$\Sigma_{ij} = \mathrm{Cov}(v(M_0)_i, v(M_0)_j).$$

Let again X_t denote the position of the active particle, now in $\mathbb{R}^d$, with random walk part Y and speed function v. The internal state process remains the same as the 1-dimensional case. To find the limiting diffusion matrix of the active particle, we can show that similar to the 1-dimensional case

$$\begin{aligned}\mathrm{Cov}((X_t)_i, (X_t)_j) &= \mathrm{Cov}((Y_{2\kappa t})_i, (Y_{2\kappa t})_j) + \mathrm{Cov}\left(\int_0^t (v_s^\gamma)_i \mathrm{d}\overline{N}_s, \int_0^t (v_s^\gamma)_j \mathrm{d}\overline{N}_s\right) \\ &+ \mathrm{Cov}\left(\int_0^t (v_s^\gamma)_i \mathrm{d}s, \int_0^t (v_s^\gamma)_j \mathrm{d}s\right).\end{aligned}$$

Now if we go through calculations that are very similar to the 1-dimensional case, we obtain the following.

Theorem 7.3. *Let X_t be the position in $\mathbb{R}^d$ of the active particle that we just defined. Then*

$$\lim_{t\to\infty}\frac{\mathrm{Cov}((X_t)_i, (X_t)_j)}{t} = 2\kappa\delta_{i,j} + \lambda\Sigma_{ij} + \frac{\lambda^2}{\gamma}[((v)_i, -A^{-1}(v)_j) + ((v)_j, -A^{-1}(v)_i)]. \tag{7.13}$$

Remark 7.4. The sum of inner products in (7.13) equals $2((v)_i, -\mathrm{sym}(A^{-1})(v)_j)$ (where for an operator B on $L^2(\mu)$, $\mathrm{sym}(B) = (B + B^*)/2$ is the symmetric part). Note that the 1-dimensional case can be retrieved from this by realising that for any operator B and function w, $(w, Bw) = (w, \mathrm{sym}(B)w)$.

Interpretation

We now briefly discuss the various terms appearing in the RHS of (7.6). First of all, as is clear directly from the definition of the process, the random walk part is independent of the rest and therefore produces the term 2κ.

Now, to understand the other two terms, let us first consider what happens in the limit of γ to infinity. In that case the state process is speeded up so much that it reaches equilibrium between subsequent jumps of the N-process. Therefore the jump sizes are just independent copies of $v(M_0)$ (so v under the stationary measure μ), so the process is simply a random walk with jump rate λ and jump size distribution $v(M_0)$. In this case the diffusion coefficient should be $\lambda \mathrm{Var}(v(M_0)) = \lambda \int v^2 \mathrm{d}\mu$, which is indeed what we find when we let γ go to infinity in (7.6).

Finally, the third term of (7.6) corresponds to the case where γ is finite. Therefore this term comes from the dependence between the active jumps due to the temporal dependence in the state process. Hence this term comes from the activity of the particle. These considerations justify the name 'active part' for the third part of (7.6). This is the only part that depends on the state process through more than just its stationary distribution. We will analyse this term more thoroughly in Section 7.4.

Remark 7.5. Note that for X^c (see Remark 7.1), the random walk part of X_t^c has variance $2\kappa t$, the martingale part is left out and the active part is the same as in X, so we obtain

$$\lim_{t\to\infty} \frac{\mathrm{Var}(X_t^c)}{t} = 2\kappa + \frac{2\lambda^2}{\gamma}(v, -A^{-1}v).$$

Remark 7.6. Note that instead of writing $(v, -A^{-1}v)$, we could also have kept the covariance in the expression in (7.9) to obtain in a similar way that the active part of the limiting diffusion coefficient equals

$$\frac{2\lambda^2}{\gamma} \int_0^\infty \mathrm{Cov}(v_0, v_r)\mathrm{d}r.$$

This might be easier to calculate for processes of which the covariance function is explicitly known.

Remark 7.7. The assumption that $\int v \mathrm{d}\mu = 0$ makes sure that $\mathbb{E}X_t = 0$. Considering a speed function that does not have average 0 is equivalent to setting the speed function to be $v + c$ where c is a constant and v still satisfies $\int v \mathrm{d}\mu = 0$. In this case the expectation equals $\mathbb{E}X_t = c\lambda t$. Of course the random walk part is not affected by this choice. Now it is easy to see following our calculations above that with the new speed function the expectation of the martingale part remains the same, but

the variance changes. Contrarily, the expectation of the active part changes, but the variance stays the same (since the change is deterministic). Overall, the limiting diffusion coefficient becomes:

$$\lim_{t\to\infty} \frac{\mathrm{Var}(X_t)}{t} = 2\kappa + \lambda\left(\int v^2 \mathrm{d}\mu + c^2\right) + \frac{2\lambda^2}{\gamma}(v, -A^{-1}v).$$

7.3.2 Examples

Now we give some examples. We start with two cases where the state process M is a Markov chain with 2 or 3 states. Then we take M to be an Ornstein-Uhlenbeck process and Brownian motion on a circle and finally we consider an Ornstein-Uhlenbeck process in $\mathbb{R}^2$.

First, in these examples we need to calculate $(v, -A^{-1}v)$ (cf. (7.6)). Now write $u = -A^{-1}v$ and recall that this means $u = \int_0^\infty S_t v \mathrm{d}t$, which implies $-Au = v$. In order to compute $(v, -A^{-1}v)$, we can procede as follows. First we find a function w such that $-Aw = v$. Then $(v, w) = (v, -A^{-1}v)$. Indeed, since μ is the unique ergodic measure, the only $h \in D(A)$ with $Ah = 0$ are constant functions, so if $-Au = -Aw$, u and w only differ by a constant. Therefore $(v, w) = (v, u + c\mathbf{1}) = (v, -A^{-1}v) + c\int v \mathrm{d}\mu = (v, -A^{-1}v)$.

Second, for all of the examples, we need to verify Assumptions (7.1) and (7.2). In Example 7.8 and 7.9, the state space is finite so both assumptions always hold. In Example 7.10, 7.11 and 7.12, Assumption (7.2) can be verified by a direct computation, since the correlation functions for Brownian motion and the Ornstein-Uhlenbeck process are explicitly known. As we noted before, for Assumption (7.1), it suffices to find constants $c, C > 0$ such that $\|S_t v\| \leq C\exp(-ct)$. This is implied by the Poincaré inequality (see van Handel [153, Thm 2.18]). The Poincaré inquality for the Ornstein-Uhlenbeck process is proved in van Handel [153, Lem 2.22, Thm 2.25] (and holds similarly in the higher dimensional case). By van Handel [153, Rem 2.19], the Poincaré inequality for Brownian motion with drift on the circle follows from the Poincaré inequality for driftless Brownian motion on the circle. The exponential ergodicity (and the corresponding Poincaré inequality) in this case is known and can be shown using Fourier analysis.

Example 7.8 (2 states)**.** We start with the case where M is a Markov chain on $\mathscr{S} = \{1, -1\}$ where the state switches with rate 1 and v is the identity function $[1, -1]^T$. Then $\mu = (\delta_{-1} + \delta_1)/2$,

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and indeed $\int v \mathrm{d}\mu = 0$. Now choose $w = [1, 0]$, then $-Aw = v$. So $(v, -A^{-1}v) = (v, w) = 1/2 * (1 * 1) + 1/2 * (-1 * 0) = 1/2$. Also we compute $\int v^2 \mathrm{d}\mu = \int 1 \mathrm{d}\mu = 1$.

Now applying Theorem 7.2 yields

$$\lim_{t\to\infty}\frac{\mathrm{Var}(X_t)}{t} = 2\kappa + \lambda\int v^2 \mathrm{d}\mu + \frac{2\lambda^2}{\gamma}(v,w) = 2\kappa + \lambda + \frac{\lambda^2}{\gamma}.$$

Note that the same diffusion coefficient is found in the calculation in Section 7.6.

Example 7.9 (3 states)**.** Now let M be a Markov chain on the triangle with nodes $\mathscr{S} = \{n_1, n_2, n_3\}$ where the state switches with rate 1 and jumps to the right with probability $1/2 + a$ and to the left otherwise (where $|a| \le 1/2$). Here $\mu = (\delta_{n_1} + \delta_{n_2} + \delta_{n_3})/3$, $v = [v_1, v_2, v_3]$ such that $v_1 + v_2 + v_3 = 0$ and A is

$$\begin{bmatrix} -1 & \frac{1}{2}+a & \frac{1}{2}-a \\ \frac{1}{2}-a & -1 & \frac{1}{2}+a \\ \frac{1}{2}+a & \frac{1}{2}-a & -1 \end{bmatrix}.$$

Now to find w we solve the linear system

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2}-a & -\frac{1}{2}+a & v_1 \\ -\frac{1}{2}+a & 1 & -\frac{1}{2}-a & v_2 \\ -\frac{1}{2}-a & -\frac{1}{2}+a & 1 & v_3 \end{array}\right],$$

Note that the last row is redundant. Therefore we can leave it away and set $w_3 = 0$, which leads us to solve

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-a & v_1 \\ -\frac{1}{2}+a & 1 & v_2 \end{array}\right],$$

which yields

$$w = \left[\frac{v_1 + (a+1/2)v_2}{3/4+a^2}, \frac{(1/2-a)v_1 + v_2}{3/4+a^2}, 0\right].$$

This gives

$$\begin{aligned}(v,w) &= \frac{v_1^2 + v_1v_2 + v_2^2}{3(3/4+a^2)} = \frac{v_1^2 + v_1v_2 + v_2^2 + v_3(v_1+v_2+v_3)}{3(3/4+a^2)} \\ &= \frac{(v_1+v_2+v_3)^2 - (v_1v_2 + v_2v_3 + v_1v_3)}{9/4+3a^2} = -\frac{v_1v_2 + v_2v_3 + v_1v_3}{9/4+3a^2}, \quad (7.14)\end{aligned}$$

where we used in the last step that $v_1 + v_2 + v_3 = 0$. Also we compute $\int v^2 \mathrm{d}\mu = (v_1^2 + v_2^2 + v_3^2)/3$. Now applying Theorem 7.2 yields

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathrm{Var}(X_t)}{t} &= 2\kappa + \lambda\int v^2\mathrm{d}\mu + \frac{2\lambda^2}{\gamma}(v,w) \\ &= 2\kappa + \frac{\lambda}{3}(v_1^2+v_2^2+v_3^2) + \frac{2\lambda^2}{\gamma}\frac{(-v_1v_2 - v_2v_3 - v_1v_3)}{9/4+3a^2}.\end{aligned}$$

Example 7.10 (Ornstein-Uhlenbeck process)**.** Now let us consider a different kind of example where M is a continuous process, namely an Orstein-Uhlenbeck process satisfying

$$\mathrm{d}M_t = -\theta M_t \mathrm{d}t + \sigma \mathrm{d}B_t,$$

where B_t is a Brownian motion independent of everything else (note that a similar process is studied in Szamel [143]). This process has stationary distribution $\mu \sim N(0, \sigma^2/(2\theta))$. We take $v(x) = x$ (indeed $\int x \mathrm{d}\mu = 0$). We know that the generator equals

$$A = -\theta x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\sigma^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

and has as domain $D(A)$ all functions in $L^2(\mu)$ of which the first and second (weak) derivative are also in $L^2(\mu)$. A quick inspection shows that if we set $w(x) = x/\theta$, then w in $D(A)$ and $-Aw = v$. Now we compute $(v, w) = \int x^2/\theta \mathrm{d}\mu = \sigma^2/(2\theta^2)$. Also $\int v^2 \mathrm{d}\mu = \int x^2 \mathrm{d}\mu = \sigma^2/(2\theta)$. Now Theorem 7.2 gives us

$$\lim_{t\to\infty} \frac{\mathrm{Var}(X_t)}{t} = 2\kappa + \lambda \int v^2 \mathrm{d}\mu + \frac{2\lambda^2}{\gamma}(v, w) = 2\kappa + \frac{\lambda\sigma^2}{2\theta} + \frac{\lambda^2}{\gamma}\frac{\sigma^2}{\theta^2}.$$

Note that the constant $(v, w) = \sigma^2/(2\theta^2)$ could also have been directly obtained by calculating

$$\mathrm{Var}\left(\lambda \int_0^t v_s^\gamma \mathrm{d}s\right) = \lambda^2 \int_0^t \int_0^t \mathrm{Cov}(v_s^\gamma, v_r^\gamma) \mathrm{d}s\mathrm{d}r \tag{7.15}$$

followed by rescaling and taking limits, since the covariance of the Ornstein-Uhlenbeck process is explicitly known. This yields the same result. Alternatively, one could have used the expression in Remark 7.6 to see

$$(v, -A^{-1}v) = \int_0^\infty \mathrm{Cov}(v_0, v_t)\mathrm{d}t = \int_0^\infty \frac{\sigma^2}{2\theta} \exp(-\theta t)\mathrm{d}t = \frac{\sigma^2}{2\theta^2}.$$

Example 7.11 (Sine of Brownian motion with drift)**.** In this example we want the speed process v_t to be $\sin(M_t)$ where $M_t = B_{2at} + bt$, $(B_t, t \geq 0)$ is Brownian motion and $a, b > 0$ are constants. However, X_t does not have a stationary (probability) distribution. Therefore we take M to be $B_{2at} + bt$ on a circle $\mathscr{S}$ with radius 1 and we set $v(\theta) = \sin(\theta)$. Now $\mu = \frac{1}{2\pi}\mathrm{d}\theta$, so indeed $\int v \mathrm{d}\mu = 0$. The generator is given by

$$A = a\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + b\frac{\mathrm{d}}{\mathrm{d}\theta}$$

with domain $D(A)$ containing all smooth functions on $\mathscr{S}$. Substituting $w(\theta) = c\sin(\theta) + d\cos(\theta)$ and solving for c, d shows that

$$w(\theta) = \frac{a}{a^2 + b^2}\sin(\theta) + \frac{b}{a^2 + b^2}\cos(\theta)$$

satisfies $-Aw = v$ with $w \in D(A)$. Now we calculate and see that $\int v^2 \mathrm{d}\mu = \frac{1}{2\pi}\int_0^{2\pi} \sin^2(\theta)\mathrm{d}\theta = 1/2$ and

$$(v,w) = \frac{1}{2\pi}\int_0^{2\pi} \sin(\theta)\left(\frac{a}{a^2+b^2}\sin(\theta) + \frac{b}{a^2+b^2}\cos(\theta)\right)\mathrm{d}\theta = \frac{a}{2(a^2+b^2)},$$

so applying Theorem 7.2, we see:

$$\lim_{t\to\infty}\frac{\mathrm{Var}(X_t)}{t} = 2\kappa + \lambda\int v^2\mathrm{d}\mu + \frac{2\lambda^2}{\gamma}(v,w) = 2\kappa + \frac{\lambda}{2} + \frac{2\lambda^2}{\gamma}\frac{a}{a^2+b^2}. \tag{7.16}$$

Note that first of all the last term vanishes when either a or b goes to infinity, similar to what happens when γ goes to infinity (see the considerations at the end of Section 7.3). However, note that this part also vanishes when a goes to 0, even when $b > 0$. Indeed, when $a = 0$, the speed process is $\sin(M_0 + bt)$, where M_0 is sampled from μ. Now it is easy to see that $\int_0^t v_s^\gamma \mathrm{d}s$ is bounded in t, so $\mathrm{Var}(\int_0^t v_s^\gamma \mathrm{d}s)/t$ goes to 0. In that sense the particle is not active in the limit.

Example 7.12. As example for the higher dimensional case, we take M to be the two-dimensional stationary Ornstein-Uhlenbeck process given by

$$\mathrm{d}M_t = -\Theta M_t \mathrm{d}t + \sigma \mathrm{d}W_t,$$

where W_t is a two-dimensional Brownian motion,

$$\Theta = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

and $\sigma, a > 0$ are constants. The invariant distribution is $N(0, \sigma^2/2I)$. We set v to be the identity function. The corresponding generator is

$$Af = -(\nabla f)^T \Theta x + \frac{\sigma^2}{2}\Delta f.$$

First we see that $\Sigma = \frac{\sigma^2}{2}I$. Now set

$$u_1(x) = \frac{1}{1+a^2}(x_1 - ax_2), \qquad u_2(x) = \frac{1}{1+a^2}(ax_1 + x_2),$$

then $-Au_1(x) = x_1 = (v)_1(x)$ and $-Au_2(x) = x_2 = (v)_2(x)$. Using these we obtain

$$\begin{aligned}((v)_1, -A^{-1}(v)_1) + ((v)_1, -A^{-1}(v)_1) &= 2\frac{1}{1+a^2}(x_1, x_1 - ax_2) \\ &= \frac{2}{1+a^2}(x_1,x_1) = \frac{\sigma^2}{1+a^2}.\end{aligned}$$

Also

$$\begin{aligned}((v)_1, -A^{-1}(v)_2) + ((v)_2, -A^{-1}(v)_1) &= (x_1, ax_1 + x_2) + (x_2, x_1 - ax_2) \\ &= a(x_1,x_1) - a(x_2,x_2) = 0.\end{aligned}$$

Here we used that under μ, $(x_1, x_1) = (x_2, x_2) = \sigma^2/2$ and $(x_1, x_2) = (x_2, x_1) = 0$.

Applying Theorem 7.3, we see that the limiting diffusion matrix equals

$$(2\kappa + \lambda\frac{\sigma^2}{2} + \frac{\lambda^2}{\gamma}\frac{\sigma^2}{1+a^2})I. \tag{7.17}$$

7.3.3 Invariance principle

So far we have calculated the limiting diffusion coefficient of the active particle. In a lot of cases one can in fact show a Central Limit Theorem (CLT) for (the trajectory of) the active particle. This type of problem has been dealt with in a lot of generality under several sets of assumptions before, so we will not provide all the details.

As we noted before the active particle process decomposes naturally into three parts. First of all, there is the random walk part, which is independent of the rest. The CLT for this case is well-known.

Then there is the martingale part

$$\int_0^t v_s^\gamma \mathrm{d}\overline{N}_s.$$

As the name suggests, this term is actually a martingale with respect to the filtration $\mathscr{F}_t = \sigma\{(M_{\gamma s}, N_s), 0 \le s \le t\}$ (see Remark 7.13). Moreover, the active part

$$\lambda \int_0^t v_s^\gamma \mathrm{d}s,$$

is an additive functional of a stationary Markov process and can (under some technical assumptions) be approximated by a martingale with respect to the filtration $\mathscr{F}'_t = \sigma\{M_{\gamma s}, 0 \le s \le t\}$ and hence (by independence of N and the active part) also with respect to $\mathscr{F}_t$. This type of result was obtained in Gordin and Lifšic [77], Kipnis and Varadhan [97], Tóth [146] and Maxwell and Woodroofe [118].

Therefore the sum of the martingale part and the active part

$$\int_0^t v_s^\gamma \mathrm{d}N_s$$

can be approximated by a martingale with respect to $\mathscr{F}_t$. Since the martingale part has a source of randomness (the Poisson process N) that is independent of the active part, the martingales cannot cancel each other out. Finally, as is done in the papers that were just cited, one can apply functional martingale central limit theorems such as in Durrett and Resnick [47] and Helland [83] to obtain the CLT for the active particle.

Remark 7.13. The fact that the martingale part is actually a martingale with respect to $\mathscr{F}_t$ can be shown from a direct computation. However, this martingale also naturally shows up as a Dynkin martingale. Because of the underlying state process, the position X_t itself is not a Markov process. However, the pair (X_t, M_t^γ) is (where M_t^γ is M speeded up by a factor γ). The corresponding generator L is given by

$$Lf(x,m) = \lambda(f(x+v(m),m) - f(x,m)) + \gamma(Af(x,\cdot))(m).$$

Setting $g(x,m) = x$, we see that the following is (formally) a martingale with respect to the natural filtration of (X_t, M_t^γ):

$$\begin{aligned}\mathscr{M}_t &:= g(X_t, M_t^\gamma) - g(X_0, M_0^\gamma) - \int_0^t Lg(X_s, M_s^\gamma)\mathrm{d}s \\ &= X_t - X_0 - \int_0^t \lambda v_s^\gamma \mathrm{d}s = \int_0^t v_s^\gamma \mathrm{d}\overline{N}_s.\end{aligned}$$

The quadratic variation of this martingale equals

$$\int_0^t (Lg^2 - 2gLg)(X_s, v_s^\gamma)\mathrm{d}s = \lambda \int_0^t (v_s^\gamma)^2 \mathrm{d}s.$$

Note that by ergodicity of M we have that almost surely

$$\lim_{t\to\infty} \frac{\lambda}{t} \int_0^t (v_s^\gamma)^2 \mathrm{d}s = \lambda \int v^2 \mathrm{d}\mu,$$

which confirms that the martingale part converges to a Brownian motion with diffusion coefficient $\lambda \int v^2 \mathrm{d}\mu$.

7.4 Diffusion coefficient: the role of reversibility

Now that we found an expression for the limiting diffusion coefficient of the active particle, we want to understand how it depends on the internal state process. In particular we want to understand the role of reversibility of the internal state process with respect to the stationary measure μ. Recall that we say that the state process M_t is reversible with respect to μ if the generator A is a self-adjoint operator on its domain in $L^2(\mu)$. We will fix the stationary measure μ and study processes with this stationary measure. We will also assume in the rest of this section that the internal state space $\mathscr{S}$ is finite, this is mainly to avoid technical complications.

When we inspect the different terms of the diffusion coefficient (7.6), we see the following.

a) The random walk part, 2κ, does not depend on the internal state process.

b) The martingale part, $\lambda \int v^2 \mathrm{d}\mu$, only depends on the internal state process through its stationary measure μ.

c) The active part, $\frac{2\lambda^2}{\gamma}(v, -A^{-1}v)$ depends on the whole internal state process, i.e. its stationary measure as well as its generator.

We conclude that given a stationary measure μ, only the active part might depend on the reversibility of the state process with respect to μ. Since also the factor $\frac{2\lambda^2}{\gamma}$ is fixed, we will dedicate the rest of this section to studying the behaviour of the term

$$\left(v, -A^{-1}v\right).$$

To further specify our results, note that the generator A can be decomposed into a symmetric part $\mathrm{sym}(A) = (A + A^*)/2$ and an antisymmetric part $\mathrm{asym}(A) = (A - A^*)/2$, where A^* denotes the adjoint of A as operators on $L^2(\mu)$. In particular the internal state process is reversible with respect to μ if $\mathrm{sym}(A) = A$ and accordingly $\mathrm{asym}(A) = 0$. We will show the following.

i) In Section 7.4.2 we will consider state processes with the same symmetric part. We will show that the active part of the diffusion coefficient is maximal for the process generated by the symmetric part itself, for any choice of the speed function v. In other words: the diffusion coefficient is maximal for the reversible process. Mathematically this means that we will prove that for all v that satisfy $\int v \mathrm{d}\mu = 0$,

$$\left(v, -A^{-1}v\right) \leq \left(v, -\mathrm{sym}(A)^{-1}v\right).$$

This is Proposition 7.17. We also generalise this to active particles in higher dimensions.

ii) In Section 7.4.3 we will consider reversible processes with the requirement that the total jumping rate from each point is the same. We will show that in this case there is no reversible process that maximises the diffusion coefficient for each choice of the speed function. In other words: within the class of reversible processes (with the same total jumping rates) there is no optimal reversible process.

Before this, we will start with some motivating examples in Section 7.4.1.

Remark 7.14. Note that the active part of the diffusion coefficient only depends on the "zero-average"-part of the speed function (see Remark 7.7). Therefore it remains the same when we replace the speed function v by $v + c$, where c is a constant. Similarly, the active part of the diffusion coefficient is the same for X^c (from Remark 7.1). Because of this, if we replace v by $v + c$ or if we consider the process X^c instead of X, the results of this section are still valid.

7.4.1 Motivation

As a motivating example, let us look back at Example 7.9. Note that for each $a \in [-1/2, 1/2]$, the state process has the same stationary distribution, namely the uniform distribution. However, only for $a = 0$ the process is reversible, whereas for $a = 1/2$

or $a = -1/2$ the process is completely asymmetric (it only jumps to the right or only to the left, respectively). Hence we can think of a as the parameter that tunes the non-reversibility of the state process. The expression that we found earlier (see (7.14)) is

$$(v, -A^{-1}v) = \frac{-(v_1v_2 + v_2v_3 + v_1v_3)}{9/4 + 3a^2}.$$

Since $-(v_1v_2 + v_2v_3 + v_1v_3) \geq 0$ for v with $\int v \mathrm{d}\mu = 1/3(v_1 + v_2 + v_3) = 0$, this expression is maximal for $a = 0$, the reversible case, and decreases like $1/(1 + a^2)$ for a away from 0. We conclude that out of this family of state processes, the reversible process maximises the diffusion coefficient.

Now for a more general result, we go back to the three states example and note that the symmetric part of the generator (as an operator in $L^2(\mu)$) was the same for each a and the antisymmetric part varied with a, indeed:

$$\frac{1}{3}\begin{bmatrix} -1 & \frac{1}{2}+a & \frac{1}{2}-a \\ \frac{1}{2}-a & -1 & \frac{1}{2}+a \\ \frac{1}{2}+a & \frac{1}{2}-a & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} + \frac{a}{3}\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

We want to show that this is true in general: out of all processes (with the same stationary measure μ) of which the symmetric part of the generator is the same, the purely reversible process (so the purely symmetric one) maximises $(v, -A^{-1}v)$.

Remark 7.15. Even though we restrict ourselves in this section to finite state spaces (mainly for technical reasons), notice that the same behaviour (the fact that the diffusion coefficient is maximal for reversible state processes) occurs in Example 7.11 and 7.12.

Indeed, in Example 7.11 the state process consists of a reversible part scaled with a constant a and an non-reversible part with constant b (so in particular the process is reversible if and only if $b = 0$). The active part of the diffusion coefficient in (7.16) equals

$$\frac{2\lambda^2}{\gamma}\frac{a}{a^2 + b^2}.$$

So when we keep a fixed, the active part is maximised in the reversible case.

In Example 7.12 the active part of the diffusion matrix in (7.17) equals

$$\frac{\lambda^2}{\gamma}\frac{\sigma^2}{1 + a^2}I.$$

This matrix is maximal for $a = 0$, which is the reversible case.

7.4.2 Comparing reversible and non-reversible processes

In order to prove the main result, Proposition 7.17 below, we first need the following lemma.

Lemma 7.16. *Let C be a skew-symmetric matrix. Then both $I + C$ and $I - C^2$ are invertible and for all w*

$$(w, (I + C)^{-1}w) = (w, (I - C^2)^{-1}w) \leq (w, w).$$

Proof. The invertibility of $I+C$ and $I-C^2$ is known, but we repeat it for completeness. Suppose that $I + C$ is not invertible. Then there exists $v \neq 0$ such that $(I + C)v = 0$, so $v = -Cv$. Then $(v, v) = -(v, Cv) = 0$, which is a contradiction. Similarly if $(I - C^2)v = 0$, then $v = C^2v$, so $(v, v) = (v, C^2v) = -(Cv, Cv) \leq 0$, which is a contradiction.

Now let w be arbitrary and set $g = (I - C^2)^{-1}w$ and $h = (I + C)^{-1}w$, which implies that $(I - C)g = h$. Then we see

$$(w, (I + C)^{-1}w) = ((I + C)h, h) = (h, h) + (Ch, h) = (h, h)$$

and

$$\begin{aligned}(w, (I - C^2)^{-1}w) &= ((I - C^2)g, g) = ((I + C)(I - C)g, g)\\ &= ((I + C)h, g) = (h, g) + (Ch, g)\\ &= (h, g) - (h, Cg) = (h, (I - C)g) = (h, h),\end{aligned}$$

which proves the equality.

To prove the inequality, first note that $-C^2$ is positive semidefinite. Therefore the eigenvalues of $I - C^2$ are greater than 1, so the eigenvalues of $(I - C^2)^{-1}$ are between 0 and 1, so $\|(I - C^2)^{-1}\| \leq 1$, which implies that $(w, (I - C^2)^{-1}w) \leq (w, w)$. □

Since we want to compare a Markov generator with its symmetric part (in $L^2(\mu)$), we recall some properties of this symmetric part. First of all, the symmetric part is again a Markov generator. Moreover, if the original generator has a unique ergodic measure, then the symmetric part generates a reversible process with the same unique ergodic measure. These properties are known, but for the reader's convenience we collect them with a proof in Lemma 7.30 in the appendix.

Now we can prove the following proposition.

Proposition 7.17. *Let A be the generator of a Markov process on a finite state space with unique ergodic measure μ. Then for all v such that $\int v \mathrm{d}\mu = 0$*

$$(v, -A^{-1}v) \leq (v, -\mathrm{sym}(A)^{-1}v),$$

where $\mathrm{sym}(A) = (A + A^*)/2$ is the symmetric part of A in $L^2(\mu)$. As a consequence, the diffusion coefficient (7.6) is maximised for reversible state processes.

Proof. Let $B = (-A + (-A)^*)/2$ be the symmetric part of $-A$ and $D = (-A - (-A)^*)/2$ the skew-symmetric part (in $L^2(\mu)$). Let v such that $\int v \mathrm{d}\mu = 0$. Note that B is (strictly) positive definite on the subspace of w such that $\int w \mathrm{d}\mu = 0$, so B^{-1} and $B^{-1/2}$ exist and are symmetric (in $L^2(\mu)$). Now we see

$$\begin{aligned}(v, -A^{-1}v) &= (v, (B+D)^{-1}v) = (v, (B^{1/2}(I + B^{-1/2}DB^{-1/2})B^{1/2})^{-1}v) \\ &= (v, B^{-1/2}(I + B^{-1/2}DB^{-1/2})^{-1}B^{-1/2}v) \\ &= (B^{-1/2}v, (I + B^{-1/2}DB^{-1/2})^{-1}B^{-1/2}v).\end{aligned}$$

Now write $w = B^{-1/2}v$ and $C = B^{-1/2}DB^{-1/2}$, so

$$(v, -A^{-1}v) = (w, (I+C)^{-1}w).$$

Note that for all u, u'

$$\begin{aligned}(u, Cu') &= (u, B^{-1/2}DB^{-1/2}u') = (B^{-1/2}u, DB^{-1/2}u') \\ &= -(DB^{-1/2}u, B^{-1/2}u') = -(B^{-1/2}DB^{-1/2}u, u') = -(Cu, u'),\end{aligned}$$

so C is skew-symmetric. Therefore applying Lemma 7.16 gives us that

$$\begin{aligned}(v, -A^{-1}v) &= (w, (I+C)^{-1}w) \leq (w, w) \\ &= (B^{-1/2}v, B^{-1/2}v) = (v, B^{-1}v) = (v, -\mathrm{sym}(A)^{-1}v).\end{aligned}$$

□

Remark 7.18. If we assume that $\|B^{-1/2}DB^{-1/2}\| < 1$, we use the Taylor expansion and obtain the more explicit formula:

$$(v, -A^{-1}v) = (v, -\mathrm{sym}(A)v) + (w, C^2(I - C^2)^{-1}w),$$

where w and C are as in the proof of Proposition 7.17. Indeed in that case

$$\begin{aligned}(w, (I+C)^{-1}w) &= \left(w, \sum_{n=0}^{\infty}(-1)^n C^n w\right) = \sum_{n=0}^{\infty}(-1)^n (w, C^n w) \\ = \sum_{n=0}^{\infty}(-1)^{2n}(w, C^{2n}w) &= \left(w, \sum_{n=0}^{\infty}(C^2)^n w\right) = (w,w) + \left(w, \sum_{n=1}^{\infty}(C^2)^n w\right) \\ = (w,w) + \left(w, C^2\sum_{n=0}^{\infty}(C^2)^n w\right) &= (v, -\mathrm{sym}(A)^{-1}v) + (w, C^2(I-C^2)^{-1}w).\end{aligned}$$

Note that in the third equality we used that C^n is skew-symmetric, so $(w, C^n w) = 0$ for n odd.

Now that we have Proposition 7.17 for active particles in $\mathbb{R}$, we can use it to generalise to d dimensions. Recall from Theorem 7.3 that the active part of the limiting diffusion matrix of an $\mathbb{R}^d$-valued random walk is $(2\lambda^2/\gamma)D^A$, where

$$D^A_{ij} := ((v)_i, -A^{-1}(v)_j) + ((v)_j, -A^{-1}(v)_i).$$

The next proposition tells us that in the same context as Proposition 7.17, this quantity is optimal for the reversible process.

Corollary 7.19. *Let A and μ be as in Proposition 7.17. Then for all $\mathbb{R}^d$-valued v such that $\int v\mathrm{d}\mu = 0$ (in $\mathbb{R}^d$), D^A is dominated by $D^{\mathrm{sym}(A)}$ in the sense that $D^{\mathrm{sym}(A)} - D^A$ is positive definite.*

Proof. It suffices to show that for all $\alpha \in \mathbb{R}^d$, $\alpha^T D^A \alpha \leq \alpha^T D^{\mathrm{sym}(A)}\alpha$. Let $\alpha \in \mathbb{R}^d$. Then $\alpha \cdot v$ is an $\mathbb{R}$-valued function such that $\int(\alpha \cdot v)\mathrm{d}\mu = \alpha \cdot (\int v\mathrm{d}\mu) = 0$. Therefore, using Proposition 7.17, we see

$$\begin{aligned}
\alpha^T D^A \alpha &= \sum_{i,j=1}^{d} \alpha_i\alpha_j(((v)_i, -A^{-1}(v)_j) + ((v)_j, -A^{-1}(v)_i)) \\
&= 2((\alpha \cdot v), -A^{-1}(\alpha \cdot v)) \\
&\leq 2((\alpha \cdot v), -\mathrm{sym}(A)^{-1}(\alpha \cdot v)) = \alpha^T D^{\mathrm{sym}(A)}\alpha.
\end{aligned}$$

□

7.4.3 Comparing reversible processes

Proposition 7.17 tells us that among all generators with the same symmetric part, the symmetric part itself maximises the diffusion coefficient of the active particle. Now one might wonder whether there are classes of reversible internal state processes that yield the same diffusion coefficient for each speed function v. The following lemma shows us that this is not the case.

Lemma 7.20. *Let A and B be Markov generators with reversible measure μ. Suppose that for every v with $\int v\mathrm{d}\mu = 0$, $(v, -A^{-1}v) = (v, -B^{-1}v)$. Then $A = B$.*

Proof. Define the following linear subspaces of $L^2(\mu)$: $V_\mu := \{v | \int v\mathrm{d}\mu = 0\}$ and $V_1 = \{c\mathbf{1} | c \in \mathbb{R}\}$. Note that V_μ and V_1 are orthogonal in $L^2(\mu)$ and in fact V_μ is the orthogonal complement of V_1 in $L^2(\mu)$, so the action on V_μ and V_1 together fully define A and B. Also note that A and B are 0 on V_1 and are invertible when restricting to $V_\mu \to V_\mu$. It suffices to show that A and B are equal on V_μ, so in turn it suffices to show that A^{-1} and B^{-1} are equal on V_μ. For this let $v, w \in V_\mu$. Then

$$\begin{aligned}
&(v, -A^{-1}w) = \frac{1}{2}((v+w), -A^{-1}(v+w)) - (v, -A^{-1}v) - (w, -A^{-1}w)) \\
= \quad &\frac{1}{2}((v+w), -B^{-1}(v+w)) - (v, -B^{-1}v) - (w, -B^{-1}w)) = (v, -B^{-1}w).
\end{aligned}$$

This shows that $A^{-1} = B^{-1}$ on V_μ, so we conclude that $A = B$. □

Now that we know that different reversible processes cannot yield the same diffusion coefficients, it could still be that certain reversible processes yield larger diffusion coefficients than others. To answer this question, we need to normalise in some way. Otherwise if we replace the generator A by cA for some constant $c > 1$, the diffusion coefficient is divided by that constant c, so A trivially yields larger diffusion coefficients than cA. We normalise here by comparing reversible processes that have the same total jumping rate from each point. The next lemma tells us that in that case no process strictly dominates all the others, it depends on the speed function v.

Lemma 7.21. *Let A and B be Markov generators on a finite state space that are reversible with respect to μ. Additionally assume that the total jump rate from each state is the same for A and B. Then either $A = B$ or there exist $v, w \in V_\mu$ such that*

$$(v, -A^{-1}v) > (v, -B^{-1}v) \quad \text{and} \quad (w, -A^{-1}w) < (w, -B^{-1}w).$$

Proof. Let A and B be as stated. Now assume that there are no $v, w \in V_\mu$ such that $(v, -A^{-1}v) > (v, -B^{-1}v)$ and $(w, -A^{-1}w) < (w, -B^{-1}w)$. Without loss of generality assume that for all $v \in V_\mu$, $(v, -A^{-1}v) \geq (v, -B^{-1}v)$. This implies that $-A^{-1} \geq -B^{-1}$ (in the sense that $-A^{-1} - (-B^{-1})$ is symmetric and positive definite on V_μ). With the fact that $-A, -B$ are positive definite, this in turn implies that $-B \geq -A$, so $A - B \geq 0$ on V_μ. Since also $Av = Bv = 0$ for $v \in V_1$, this implies that $A - B \geq 0$ on $L^2(\mu)$. Now if we define D to be the diagonal matrix with $D_{ii} = \mu_i$, then $D(A - B) \geq 0$ and $D(A - B)$ is symmetric with respect to the usual inner product in $\mathbb{R}^d$. Also, $A - B$ and (hence) $D(A - B)$ have zeroes on the diagonal (because of the equal jump rates), so the trace of $D(A - B)$ is 0. Therefore the eigenvalues of $D(A - B)$ are non-negative and sum to 0, so they are all 0. This implies that $D(A - B) = 0$, so $A = B$. □

7.5 Large deviations

In this section we derive a large deviation principle (LDP) for X_t/t.[2] The active particle that we are studying is what is called a slow-fast system in the literature and a lot of research has already been done about its large deviations. Because of this it is not our goal here to present this result in the highest possible generality. We would rather see which formulas are obtained and study their behaviour, in particular the relation between the rate function and the reversibility of M. Therefore we reduce (as in Section 7.4) to the case where the state space $\mathscr{S}$ of M is finite (and hence where $(v^\gamma, s \geq 0)$ is bounded).

Remark 7.22. Note that we don't need anywhere in this section that $\int v \mathrm{d}\mu = 0$.

[2] For the definition of the Large Deviation Principle and for Varadhan's lemma and the Gaertner-Ellis theorem, see for intance Dembo and Zeitouni [41] or Den Hollander [42].

Since we will express the rate function for X_t/t in terms of the rate function of the empirical process corresponding to the underlying state process, we quickly recall some results that we will use. We write

$$\chi_t = \frac{1}{t}\int_0^t \delta_{M_s}\mathrm{d}s$$

and denote by P_t the distribution of χ_t in the space of probability measures on $\mathscr{S}$. Then we know from Donsker and Varadhan [46] that $(P_t, t \geq 0)$ satisfies an LDP with good rate function I_e given by

$$I_e(\xi) = \sup_{u>0}\left(-\sum_{i=1}^n \xi_i \frac{(Au)_i}{u_i}\right). \tag{7.18}$$

In case A is symmetric, this reduces to

$$I_e(\xi) = (u, -Au), \tag{7.19}$$

where $u_i = \sqrt{\xi_i/\mu_i}$ (note that we assumed that μ has full support, so $\mu_i > 0$ for all i) and the inner product is (as usual) with respect to μ.

7.5.1 Large deviations rate function

To obtain the large deviations rate function of X_t/t, we start by calculating the logarithmic moment generating function (log-mgf) of X_t: $F_t(\alpha) = \log \mathbb{E}\left[\mathrm{e}^{\alpha X_t}\right]$ for $\alpha \in \mathbb{R}^d$. To calculate it we first observe that by independence of Y and the rest,

$$\begin{aligned} F_t(\alpha) &= \log \mathbb{E}\left[\mathrm{e}^{\alpha\left(\sqrt{2\kappa}Y_t + \int_0^t v_s^\gamma \mathrm{d}N_s\right)}\right] \\ &= \log \mathbb{E}\left[\exp(\alpha\sqrt{2\kappa}Y_t)\right] + \log \mathbb{E}\left[\exp\left(\alpha\int_0^t v_s^\gamma \mathrm{d}N_s\right)\right]. \end{aligned} \tag{7.20}$$

The first term is just the log-mgf of a simple random walk speeded up with a factor 2κ. Therefore at time t it equals the difference of two independent Poisson random variables with parameter κt, so we obtain that

$$\log \mathbb{E}\left[\exp(\alpha Y_{2\kappa t})\right] = \log(\exp(\kappa t(\mathrm{e}^\alpha - 1))\exp(\kappa t(\mathrm{e}^{-\alpha} - 1))) = 2\kappa t(\cosh(\alpha) - 1).$$

To calculate the second term, we first condition on $v^\gamma = (v^\gamma_s, 0 \le s \le t)$ and obtain

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\alpha\int_0^t v^\gamma_s \mathrm{d}N_s\right)\middle| v^\gamma\right] = \lim_{n\to\infty}\mathbb{E}\left[\exp\left(\alpha\sum_{i=0}^{n-1} v^\gamma_{s_i}(N_{s_{i+1}}-N_{s_i})\right)\middle| v^\gamma\right]\\
= &\lim_{n\to\infty}\prod_{i=0}^{n-1}\mathbb{E}\left[\exp\left(\alpha v^\gamma_{s_i}(N_{s_{i+1}}-N_{s_i})\right)|v^\gamma\right]\\
= &\lim_{n\to\infty}\prod_{i=0}^{n-1}\exp\left(\lambda\left(\mathrm{e}^{\alpha v^\gamma_{s_i}}-1\right)(s_{i+1}-s_i)\right)\\
= &\lim_{n\to\infty}\exp\left(\sum_{i=0}^{n-1}\lambda\left(\mathrm{e}^{\alpha v^\gamma_{s_i}}-1\right)(s_{i+1}-s_i)\right) = \exp\left(\lambda\int_0^t\left(\mathrm{e}^{\alpha v^\gamma_s}-1\right)\mathrm{d}s\right).
\end{aligned}$$

Therefore we see that the second term of (7.20) equals

$$\log\mathbb{E}\exp\left(\lambda\int_0^t\left(\mathrm{e}^{\alpha v^\gamma_s}-1\right)\mathrm{d}s\right).$$

We conclude that

$$F_t(\alpha) = 2\kappa t(\cosh(\alpha)-1) + \log\mathbb{E}\exp\left(\lambda\int_0^t\left(\mathrm{e}^{\alpha v^\gamma_s}-1\right)\mathrm{d}s\right). \tag{7.21}$$

Now we can compute the large deviation free energy function $F(\alpha)$ as the limit of $F_t(\alpha)/t$. We see for the first term that

$$\lim_{t\to\infty}\frac{2\kappa t(\cosh(\alpha)-1)}{t} = 2\kappa(\cosh(\alpha)-1). \tag{7.22}$$

Now for the second term define h_α as a function on measures on $\mathscr{S}$ given by

$$h^\gamma_\alpha(\xi) = \frac{\lambda}{\gamma}\int_{\mathscr{S}}\left(\mathrm{e}^{\alpha v(x)}-1\right)\xi(\mathrm{d}x).$$

This enables us to rewrite the second part of $F_t(\alpha)$ and use Varadhan's lemma to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left(\frac{\lambda}{\gamma}\int_0^{\gamma t}\left(\mathrm{e}^{\alpha v(M_s)}-1\right)\mathrm{d}s\right)\\
= &\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left(t\gamma\frac{\lambda}{\gamma}\int_{\mathscr{S}}\left(\mathrm{e}^{\alpha v(x)}-1\right)\left(\frac{1}{\gamma t}\int_0^{\gamma t}\delta_{M_s}\mathrm{d}s\right)(\mathrm{d}x)\right)\\
= &\gamma\lim_{t\to\infty}\frac{1}{\gamma t}\log\mathbb{E}\exp\left(\gamma t h^\gamma_\alpha(\chi_{\gamma t})\right) = \gamma\sup_\xi(h^\gamma_\alpha(\xi)-I_e(\xi)).
\end{aligned}$$

Note that the latter equals

$$\sup_\xi(\lambda(\varphi_\xi(\alpha)-1)-\gamma I_e(\xi)), \tag{7.23}$$

where $\varphi_\xi(\alpha) = \int_{\mathscr{S}} \exp(\alpha v(x))\xi(\mathrm{d}x)$ denotes the mgf of v under ξ evaluated at α. Taking together (7.22) and (7.23), we conclude that

$$F(\alpha) = \lim_{t\to\infty} \frac{F_t(\alpha)}{t} = 2\kappa(\cosh(\alpha) - 1) + \sup_\xi(\lambda(\varphi_\xi(\alpha) - 1) - \gamma I_e(\xi)). \tag{7.24}$$

Using the Gaertner-Ellis theorem, we now obtain the large deviation principle for X_t/t with rate function given by the Legendre transform of $F(\alpha)$:

$$I(x) = \sup_\alpha(\alpha x - F(\alpha)) = \sup_\alpha(\alpha x - 2\kappa(\cosh(\alpha) - 1) - \sup_\xi(\lambda(\varphi_\xi(\alpha) - 1) - \gamma I_e(\xi))).$$

Remark 7.23. A very similar computation shows that a similar expression holds in the multidimensional case. Indeed, if we set $F_t(\alpha) = \log \mathbb{E}\exp(\alpha \cdot X_t)$ for $\alpha \in \mathbb{R}^d$, we obtain

$$F(\alpha) = \lim_{t\to\infty} \frac{F_t(\alpha)}{t} = 2\kappa \sum_{i=1}^d (\cosh(\alpha_i) - 1) + \sup_\xi(\lambda(\varphi_\xi(\alpha) - 1) - \gamma I_e(\xi)),$$

where

$$\varphi_\xi(\alpha) = \int_{\mathscr{S}} \mathrm{e}^{\alpha\cdot v(x)}\xi(\mathrm{d}x).$$

Then again we can take the Legendre transform to find the rate function I.

Example 7.24. We return to Example 7.8 to obtain an explicit expression for the large deviations free energy function. Note that the state process is reversible with respect to the stationary measure $\mu = (1/2, 1/2)$. Using (7.19), fixing a probability measure ξ on $\{1, -1\}$ and setting $u_i = \sqrt{(\xi_i/(1/2)} = \sqrt{2\xi_i}$, we see

$$I_e(\xi) = (u, -Au) = \frac{1}{2}(\sqrt{2\xi_1} - \sqrt{2\xi_{-1}})^2 = (\sqrt{\xi_1} - \sqrt{\xi_{-1}})^2 = 1 - 2\sqrt{\xi_1\xi_{-1}}.$$

Parametrising $\xi = (r, 1 - r)$, we see

$$\begin{aligned} & \sup_\xi(\lambda(\varphi_\xi(\alpha) - 1) - \gamma I_e(\xi)) \\ = & \sup_{0\le r\le 1}(\lambda(r\mathrm{e}^\alpha + (1 - r)\mathrm{e}^{-\alpha} - 1) - \gamma(1 - 2\sqrt{r(1 - r)})) \\ = & \lambda(\mathrm{e}^{-\alpha} - 1) - \gamma + \sup_{0\le r\le 1}(2\lambda\sinh(\alpha)r + 2\gamma\sqrt{r(1 - r)}). \end{aligned}$$

A simple calculation shows that the latter equals

$$\begin{aligned} & \lambda(\mathrm{e}^{-\alpha} - 1) - \gamma + \sqrt{\gamma^2 + \lambda^2\sinh^2(\alpha)} + \lambda\sinh(\alpha) \\ = & \lambda(\cosh(\alpha) - 1) + \sqrt{\gamma^2 + \lambda^2\sinh^2(\alpha)} - \gamma, \end{aligned}$$

so with (7.24), we see

$$F(\alpha) = (2\kappa + \gamma)(\cosh(\alpha) - 1) + \sqrt{\gamma^2 + \lambda^2\sinh^2(\alpha)} - \gamma. \tag{7.25}$$

Remark 7.25. In the case of X^c (from Remark 7.1), the calculations become a bit easier. Instead of the symmetric random walk $Y_{2\kappa t}$ we directly work with the continuous limit $B_{2\kappa t}$. But more importantly, there is no additional randomness from the Poisson process N. Following the analogous computations for this part, we find the same results with $\varphi_\xi(\alpha)$ replaced by $\alpha \int v(x)\xi(\mathrm{d}x)$.

In this section we worked with a finite state space, so all the computations and quantities here are well-defined. However, for a more general state process, for the original process X one would need

$$\mathbb{E}\exp\left(\lambda \int_0^t \left(\mathrm{e}^{\alpha v_s^\gamma} - 1\right) \mathrm{d}s\right) < \infty$$

to get a finite free energy. Setting $t \ll 1$, this implies that we need something like

$$\mathbb{E}\mathrm{e}^{\mathrm{e}^{v_0}} < \infty,$$

which is a very strong assumption that for instance for the Ornstein-Uhlenbeck process is not satisfied.

Changing to X^c means getting rid of the Poisson jumps, which takes away one of the exponentials. So we expect that an LDP holds for a lot more state processes in the X^c case than for the original process X.

7.5.2 The role of reversibility

Our goal now is to show a result that is similar to Proposition 7.17. Indeed, we show that if an active particle has a state process generated by some generator A, then the rate function of this active particle is greater (pointwise) than the rate function of the active particle of which the state process is generated by the symmetric part of A. In other words: a reversible state process yields a lower rate function. Before we show this, we will prove the following lemma about a similar result for the rate functions of the empirical measures corresponding to the state processes.

Lemma 7.26. *Let A be a Markov generator with unique ergodic measure μ and let* $\mathrm{sym}(A)$ *be its symmetric part (in $L^2(\mu)$). Denote the rate functions of the corresponding empirical processes by I_e^A and $I_e^{\mathrm{sym}(A)}$, respectively. Then for all probability measures ξ, $I_e^{\mathrm{sym}(A)}(\xi) \leq I_e^A(\xi)$.*

Proof. Let ξ be a probability measure on $\mathscr{S}$. We set $u_i = \sqrt{\xi_i/\mu_i} \geq 0$. Also define for $m \in \mathbb{N}$, $u_i^m = u_i$ if $u_i > 0$ and $u_i^m = 1/m$ otherwise. Note that $u_i^m > 0$ for all i and that $u^m \to u$ in $L^2(\mu)$ (since it converges pointwise and $\mathscr{S}$ is finite). Finally

note that $\xi_i/u_i^m = \mu_i u_i$ for all i. Now, using (7.18), we see that for all m

$$\begin{aligned} I_e^A(\xi) &= \sup_{u'>0} -\sum_i^n \xi_i \frac{(Au')_i}{u_i'} \geq -\sum_i^n \xi_i \frac{(Au^m)_i}{u_i^m} \\ &= -\sum_{i=1}^n \mu_i u_i (Au^m)_i = (u, -Au^m). \end{aligned}$$

Therefore, using (7.19), we conclude

$$I_e^A(\xi) \geq \lim_{m\to\infty} (u, -Au^m) = (u, -Au) = (u, -\mathrm{sym}(A)u) = I_e^{\mathrm{sym}(A)}(\xi).$$

□

Now we use this to prove the following result.

Corollary 7.27. *Let A be a Markov generator with unique ergodic measure μ and let $\mathrm{sym}(A)$ be its symmetric part (in $L^2(\mu)$). Denote the rate functions of the corresponding active particle processes by I^A and $I^{\mathrm{sym}(A)}$ and the free energy functions of those processes by F^A and $F^{\mathrm{sym}(A)}$, respectively. Then for all $\alpha \in \mathbb{R} : F^A(\alpha) \leq F^{\mathrm{sym}(A)}(\alpha)$ and for all $x \in \mathbb{R} : I^{\mathrm{sym}(A)}(x) \leq I^A(x)$.*

Proof. Since for all ξ, $I_e^{\mathrm{sym}(A)}(\xi) \leq I_e^A(\xi)$, it follows that for all α,

$$\sup_\xi(\lambda(\varphi_\xi(\alpha) - 1) - \gamma I_e^{\mathrm{sym}(A)}(\xi)) \geq \sup_\xi(\lambda(\varphi_\xi(\alpha) - 1) - \gamma I_e^A(\xi)),$$

so $F^{\mathrm{sym}(A)}(\alpha) \geq F^A(\alpha)$. Since this holds for all α, similary it follows that for all x, $I^{\mathrm{sym}(A)}(x) \leq I^A(x)$. □

Note that the proof for the higher dimensional case is exactly the same.

Remark 7.28. In the case that $F(\alpha)$ is sufficiently smooth, the limiting diffusion coefficient (or matrix, in the higher dimensional case) is given by the second derivative, or, more generally, the Hessian of $F(\alpha)$ in 0. By Corollary 7.27, the free energy function is dominated by the free energy function of the active particle with state process generated by the symmetric part pointwise everywhere and they are equal for $\alpha = 0$. Therefore we see in that case that the Hessian at 0 (and therefore the limiting diffusion matrix) is dominated by the Hessian of the symmetric version. This is consistent with the results of Proposition 7.17 and Corollary 7.19.

7.6 The 2-state case: explicit formulas

In the case where there are just two states, we can compute a lot of things explicitly with different methods. Therefore this section is dedicated to the active particle

with two states. In this case the active particle has a position $x \in \mathbb{Z}$ and a velocity $v \in \{-1, 1\}$. The process $\{(X_t, v_t) : t \geq 0\}$ is described via the generator

$$\begin{aligned} Lf(x,v) &= \lambda(f(x+v,v) - f(x,v)) \\ &+ \kappa(f(x+1,v) + f(x-1,v) - 2f(x,v)) \\ &+ \gamma(f(x,-v) - f(x,v)). \end{aligned} \tag{7.26}$$

This is interpreted as follows: with rate λ the process makes a jump in the direction of the velocity, with rate κ it makes a random walk jump and with rate γ it flips velocity $v \to -v$. If we denote by $\mu(x,t,v)$ the probability to be at location $x \in \mathbb{Z}$ with velocity $v \in \{-1,1\}$ at time $t > 0$, the generator (7.26) corresponds to the master equation (or Kolmogorov forward equation)

$$\begin{aligned} \frac{d\mu(x,t,v)}{dt} &= \lambda\mu(x-v,t,v) + \kappa(\mu(x-1,t,v) + \mu(x+1,t,v)) \\ &+ \gamma\mu(x,t,-v) - (2\kappa + \lambda + \gamma)\mu(x,t,v). \end{aligned} \tag{7.27}$$

7.6.1 The Fourier Laplace transform of the distribution

The master equation (7.27) can be solved using a Fourier-Laplace transform. We define

$$\hat{\mu}(q,t,v) = \sum_x e^{iqx}\mu(x,t,v)$$

and view this quantity as a two-column, denoted $\overline{\mu}(q,t,\cdot)$ indexed by row index $v = 1, -1$. The master equation (7.27) then becomes, after a Fourier transform:

$$\frac{d}{dt}\overline{\mu}(q,t) = M(q)\overline{\mu}(q,t) \tag{7.28}$$

with $M(q)$ a symmetric two by two matrix of the form

$$M(q) = \begin{pmatrix} a & b \\ b & a^* \end{pmatrix}, \tag{7.29}$$

where $*$ denotes complex conjugate and where

$$\begin{aligned} a &= (2\kappa + \lambda)(\cos(q) - 1) - \gamma + i\lambda\sin(q) \\ b &= \gamma. \end{aligned} \tag{7.30}$$

For the analysis of the scaling behaviour of the position of the particle, it is convenient to further Laplace transform $\overline{\mu}(q,t)$, i.e. we define for $z > 0$ the column vector

$$\widehat{\mu}(q,z) = \int_0^\infty \overline{\mu}(q,t)e^{-zt}\, dt. \tag{7.31}$$

Then, from (7.28) we find

$$\widehat{\mu}(q,z) = (zI - M(q))^{-1}\overline{\mu}_0(q).$$

For the initial position and velocity we choose $X_0 = 0$, and $v = \pm 1$ with probability $1/2$. Then we have, $\overline{\mu}_0(q) = \frac{1}{2}(1,1)^T$ where T denotes transposition. We further define the Fourier Laplace transform of the distribution of the particle position:

$$S(q,z) = \int_0^\infty \mathbb{E}e^{iqX_t}e^{-zt}\,dt = \sum_v \widehat{\mu}(q,z,v) = (1,1)\widehat{\mu}(q,z).$$

Then we have, using (7.31)

$$S(q,z) = (\widehat{\mu}(q,z,1)) + (\widehat{\mu}(q,z,-1)) = \frac{1}{2}(1,1)(zI - M(q))^{-1}(1,1)^T.$$

Using the explicit formulas (7.29), (7.30), we obtain

$$S(q,z) = \frac{2\gamma + z - (\lambda + 2\kappa)(\cos(q) - 1)}{(\gamma + z - (\lambda + 2\kappa)(\cos(q) - 1))^2 - \gamma^2 + \lambda^2 \sin^2(q)}. \tag{7.32}$$

For a more general velocity distribution at time zero, i.e., $X_0 = 0$, and $v = 1$, resp. $v = -1$, with probability α, resp. $1 - \alpha$, we find

$$S(q,z) = \frac{i\lambda(2\alpha - 1)\sin(q) + 2\gamma + z - (\lambda + 2\kappa)(\cos(q) - 1)}{(\gamma + z - (\lambda + 2\kappa)(\cos(q) - 1))^2 - \gamma^2 + \lambda^2 \sin^2(q)}.$$

7.6.2 The limiting diffusion coefficient

We can now use the explicit formula (7.32) to obtain the limit distribution of $\epsilon X_{\epsilon^{-2}t}$ as $\epsilon \to 0$. This amounts to understanding the scaling behaviour of $\epsilon^2 S(\epsilon q, \epsilon^2 z)$. In particular $\epsilon X_{\epsilon^{-2}t} \to \mathcal{N}(0, \sigma^2 t)$ as $\epsilon \to 0$ (in distribution), where $\mathcal{N}(0, \sigma^2 t)$ denotes a normal with mean zero and variance $\sigma^2 t$, corresponds to the limiting scaling behaviour

$$\lim_{\epsilon \to 0} \epsilon^2 S(\epsilon q, \epsilon^2 z) = \frac{1}{z + \frac{q^2}{2}\sigma^2}.$$

If we obtain this scaling behaviour, we call σ^2 the (limiting) diffusion constant. Indeed, we compute from the exact formula (7.32)

$$\lim_{\epsilon \to 0} \epsilon^2 S(\epsilon q, \epsilon^2 z) = \frac{1}{z + \frac{q^2}{2}\sigma^2}$$

with the limiting diffusion constant

$$\sigma^2 = 2\kappa + \lambda + \frac{\lambda^2}{\gamma}. \tag{7.33}$$

This is consistent with the limiting diffusion coefficient that we obtained in Example 7.8.

7.6.3 Moment generating function and large deviations

We choose the starting point $X_0 = 0$ and with random initial velocity, i.e., $v = \pm 1$ with probability $1/2$. This allows us to compute the moment generating function via

$$\mathbb{E}(e^{\alpha X_t}) = \frac{1}{2}(1,1)e^{tM(-i\alpha)}(1,1)^T. \tag{7.34}$$

This amounts to computing the exponential of the matrix $M(q)$ from (7.29) which can be done using diagonalisation, and results in

$$e^{tM(q)} = \frac{e^{tA}}{2\gamma B}G(t,q),$$

where $G(t,q)$ is given by the symmetric two by two matrix

$$G(t,q) = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{11}^* \end{pmatrix},$$

where

$$\begin{aligned} A_{11} &= -2\gamma\lambda i \sin(k)\sinh(Bt) + 2\gamma B\cosh(tB) \\ A_{12} &= 2\gamma^2 \sinh(tB) \end{aligned}$$

and where

$$\begin{aligned} A &= (\cos(k)-1)(2\kappa+\lambda) - \gamma \\ B &= \sqrt{\gamma^2 - \lambda^2\sin^2(k)}. \end{aligned}$$

Moreover, we see from (7.34) that the free energy function

$$F(\alpha) = \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}\left(e^{\alpha X_t}\right)$$

is equal to the largest eigenvalue of the symmetric matrix $M(-i\alpha)$, which is explicitly given by

$$\begin{pmatrix} (2\kappa+\lambda)(\cosh(\alpha)-1) + \lambda\sinh(\alpha) - \gamma & \gamma \\ \gamma & (2\kappa+\lambda)(\cosh(\alpha)-1) - \lambda\sinh(\alpha) - \gamma \end{pmatrix}.$$

This gives

$$F(\alpha) = (2\kappa+\lambda)(\cosh(\alpha)-1) + \sqrt{\gamma^2 + \lambda^2\sinh^2(\alpha)} - \gamma, \tag{7.35}$$

which agrees with (7.25).

Let us look at three relevant limiting cases for the "free energy function" F from (7.35).

a) Expanding the free energy function F around $\alpha \approx 0$ gives

$$F(\alpha) = \frac{1}{2} D\alpha^2 + O(\alpha^4)$$

with $D = 2\kappa + \lambda + \frac{\lambda^2}{\gamma}$. This is consistent with the diffusion constant found in Example 7.8 and in (7.33). The function $F(\alpha)$ in (7.35) can be analytically extended in a neighbourhood of the origin in the complex plane, and as a consequence, we can reobtain the central limit theorem (which we found via the scaling behavior of the characteristic function) from the large deviation free energy, see Bryc [21].

b) In the limit $\gamma \to \infty$ the free energy function becomes

$$F(\alpha) = (\cosh(\alpha) - 1)(2\kappa + \lambda),$$

which corresponds to the large deviations of a symmetric random walk jumping with rates $\kappa + \lambda/2$ to the right or left. This is indeed the (slow-fast) scaling limit of the process as we saw before. For large values of γ we have

$$F(\alpha) = (\cosh(\alpha) - 1)(2\kappa + \lambda) + \frac{\lambda^2}{2\gamma} \sinh^2(\alpha) + o(1/\gamma).$$

Remark also that F in (7.35) is non-increasing as a function of γ.

c) In the continuum limit we rescale $\lambda \to \epsilon\lambda$, $\gamma \to \epsilon^2\gamma$, $X_t \to \epsilon X_{\epsilon^{-2}t}$, we find

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}^{\epsilon\lambda, \epsilon^2\gamma} \left(e^{\alpha \epsilon X_{\epsilon^{-2}t}} \right) = \kappa\alpha^2 + \sqrt{\gamma^2 + \lambda^2\alpha^2} - \gamma^2, \tag{7.36}$$

which corresponds to the large deviation free energy of the continuum model (see also Pietzonka et al. [124]), i.e., the limits $\epsilon \to 0$ and $t \to \infty$ in (7.36) commute.

7.7 Notes and perspectives

In this chapter we obtained the diffusion coefficient for an active particle process with a quite general underlying internal state process. In the case of a finite internal state space, we also obtained the large deviations rate function. Moreover, we found that both the diffusion coefficient and the large deviation free energy are maximal for reversible internal state processes. Here we will point to some possible generalisations and further questions to study.

First, as we just mentioned, to study the behaviour of the diffusion constant and to obtain the large deviations rate function we restricted ourselves to finite internal state spaces. The reasons for this are mostly computational. Moreover, the goal of this chapter was not to find the most general results for the diffusion coefficient or

the large deviations rate function, but rather to find expressions that we can work with and to analyse their behaviour. However, we do not see a reason why these results would not hold in more generality. In fact, we have seen that also in the case where the internal state process is a multidimensional Ornstein-Uhlenbeck process, the reversible process maximises the diffusion coefficient. Therefore it could be interesting to repeat the proof of Proposition 7.17 for non-finite state spaces, or, in other words, for general Markov generators. For the large deviations result some extra attention has to be paid in the non-finite case to see when the free energy is finite, see also Remark 7.25.

Further, the process that is considered in this chapter could be generalised by letting the dynamics of the internal state process depend on the position of the particle and possibly also on time. Of course in this case the rescaled process will not converge to Brownian motion, but will in the limit probably satisfy an SDE. However, it might be possible also in that case to analyse how this limiting SDE depends on the internal state process. In particular we could call the state process reversible if it is reversible for fixed values of the position of the particle. Then we can try to use the insights from this chapter to see how the limiting SDE depends on this reversibility.

Another interesting further step would be to let the state process be a metastable process. An easy example can be constructed by letting the state evolve with rate of order N between clusters of states and with rate N^2 within those clusters. If time is rescaled in such a way that the jumps between clusters happen on a macroscopic time scale and the jump within clusters on a microscopic time scale, the active particle should scale to a Brownian motion process that has a varying diffusion constant. The constant will depend on the cluster of the current state and will switch after exponential times, when the state switches to different clusters. After considering these somewhat artificial examples, one can probably show similar results for more general metastable state processes.

Appendix

Integral approximation in $L^2(\mathbb{P})$

First we show how it follows from Assumption (7.2) that the integral (7.3) is well-defined. An alternative more abstract way to establish the well-definedness of this integral is as follows. From Assumption (7.2) follows that v_s^γ admits a càdlàg version and hence the integral can be interpreted as an ordinary Riemann-Stieltjes integral of a càdlàg function against an integrator of bounded variation.

Lemma 7.29. *Let W_s be N_s, $\overline{N}_s$ or λs. Then Assumption* (7.2) *implies that*

$$\lim_{n\to\infty} \sum_{i=1}^{n} v_{s_i}^\gamma (W_{s_{i+1}} - W_{s_i}) =: \int_0^t v_s^\gamma \mathrm{d}W_s, \tag{7.37}$$

exists as a limit in $L^2(\mathbb{P})$ and the $s_i = s_i^n$ are partitions of $[0,t]$ of which the mesh sizes go to 0.

Proof. Without loss of generality, set $\gamma = 1$. By linearity it suffices to let W be either $\overline{N}$ or $W_s = s$. By the completeness of $L^2(\mathbb{P})$, it suffices to show that for each $\epsilon > 0$ there exists $\delta > 0$ such that if the meshes of two partitions are under δ, the L^2-distance between the corresponding Riemann sums is smaller than ϵ. Indeed, this implies both that for each sequence of partitions (with mesh going to 0), the Riemann sums form a Cauchy sequence in $L^2(\mathbb{P})$ (and hence has a limit) and that every sequence of partitions yields the same limit.

Let $\epsilon > 0$. Choose $\delta > 0$ such that for all $0 \le s, s' \le t$ with $|s - s'| < \delta$, $\mathbb{E}[(v_s - v_{s'})^2] \le \epsilon$.

Now let $s^1 = (s_i^1)_{i=0}^n$ and $s^2 = (s_i^2)_{i=0}^m$ denote partitions of $[0,t]$. Assume that s^1 is such that $\text{mesh}(s^1) \le \delta$ and that s^2 is a refinement of s^1. Denote by s_{i*}^1 the largest partition element of s^1 that is smaller than or equal to s_i^2 and note that for all i, $|s_i^2 - s_{i*}^1| \le \text{mesh}(s^1) \le \delta$. In particular for all i, j, using Cauchy-Schwarz,

$$|\mathbb{E}[(v_{s_i^2} - v_{s_{i*}^1})(v_{s_j^2} - v_{s_{j*}^1})]| = |\text{Cov}(v_{s_i^2} - v_{s_{i*}^1}, v_{s_j^2} - v_{s_{j*}^1})| \le \sqrt{\epsilon^2} = \epsilon.$$

Now

$$\sum_{i=0}^{m-1} v_{s_i^2}(W_{s_{i+1}^2} - W_{s_i^2}) - \sum_{i=1}^{n} v_{s_i^1}(W_{s_{i+1}^1} - W_{s_i^1}) = \sum_{i=0}^{m-1} (v_{s_i^2} - v_{s_{i*}^1})(W_{s_{i+1}^2} - W_{s_i^2}).$$

Therefore,

$$\begin{aligned} &\mathbb{E}\left(\sum_{i=0}^{m-1} v_{s_i^2}(W_{s_{i+1}^2} - W_{s_i^2}) - \sum_{i=1}^{n} v_{s_i^1}(W_{s_{i+1}^1} - W_{s_i^1}) \right)^2 \\ = &\sum_{i,j=0}^{m-1} \mathbb{E}\left[(v_{s_i^2} - v_{s_{i*}^1})(W_{s_{i+1}^2} - W_{s_i^2})(v_{s_j^2} - v_{s_{j*}^1})(W_{s_{j+1}^2} - W_{s_j^2}) \right]. \end{aligned} \tag{7.38}$$

In case $W = \overline{N}$, using the independence of v and N and the fact that $\overline{N}$ has increments with expectation 0, (7.38) equals

$$\sum_{i=0}^{m-1} \mathbb{E}(v_{s_i^2} - v_{s_{i*}^1})^2 \mathbb{E}(W_{s_{i+1}^2} - W_{s_i^2})^2 \le \sum_{i=0}^{m-1} \epsilon\lambda(s_{i+1}^2 - s_i^2) = \lambda t \epsilon.$$

In the case $W_s = s$, (7.38) equals

$$\begin{aligned} &\sum_{i,j=0}^{m-1} \mathbb{E}\left[(v_{s_i^2} - v_{s_{i*}^1})(v_{s_j^2} - v_{s_{j*}^1}) \right] (s_{i+1}^2 - s_i^2)(s_{j+1}^2 - s_j^2) \\ \le &\sum_{i,j=0}^{m-1} \epsilon(s_{i+1}^2 - s_i^2)(s_{j+1}^2 - s_j^2) = \lambda t^2 \epsilon. \end{aligned}$$

Set $C = \max(\lambda t, t^2)$. Then in both cases

$$\mathbb{E}\left(\sum_{i=0}^{m-1} v_{s_i^2}(W_{s_{i+1}^2} - W_{s_i^2}) - \sum_{i=1}^{n} v_{s_i^1}(W_{s_{i+1}^1} - W_{s_i^1})\right)^2 \leq C\epsilon,$$

so the L^2 distance between the Riemann sums is smaller than $\sqrt{C\epsilon}$. Now if s^1 and s^2 are any partitions of $[0,t]$ with mesh smaller than δ (so if it is not necessarily the case that one is a refinement of the other), let s^3 be a refinement of both. Then $\text{mesh}(s^3) \leq \delta$, so the Riemann sum corresponding to s^3 is close to the Riemann sums of both s^1 and s^2. Now the triangle inequality gives the desired result. □

Properties of the symmetric part of a Markov generator

Since we want to compare a Markov generator with its symmetric part, we will introduce this symmetric part and show some of its relevant properties in the next lemma. In particular, the symmetric part is again a Markov generator.

Lemma 7.30. *Let A be a Markov generator with unique ergodic measure μ on a finite state space and denote by A^* its adjoint in $L^2(\mu)$. Assume that μ has full support. Then* $\text{sym}(A) := (A + A^*)/2$ *is also a Markov generator with unique ergodic measure μ. Moreover* $\text{sym}(A)$ *is reversible with respect to μ.*

Proof. Denote by e^i the i^{th} unit vector. Then for every matrix B

$$(e^i, Be^j) = \sum_{k=1}^{n} e_k^i (Be^j)_k \mu_k = B_{ij}\mu_k.$$

Therefore

$$\text{sym}(A)_{ij} = \frac{1}{\mu_i}(e^i, \text{sym}(A)e^j) = \frac{1}{2\mu_i}((e^i, Ae^j) + (e^j, Ae^i)) = \frac{1}{2\mu_i}(\mu_i A_{ij} + \mu_j A_{ji}). \tag{7.39}$$

Since $A_{ij} \geq 0$ for all $i \neq j$, (7.39) implies that $\text{sym}(A)_{ij} \geq 0$ for $i \neq j$. Also, setting $i = j$, we obtain $\text{sym}(A)_{ii} = A_{ii} \leq 0$.

Since μ is a stationary distribution for A, we know that for all v,

$$(A^*\mathbf{1}, v) = (\mathbf{1}, Av) = \int Av \mathrm{d}\mu = 0,$$

so $A^*\mathbf{1} = 0$. Therefore

$$\text{sym}(A)\mathbf{1} = \frac{1}{2}(A\mathbf{1} + A^*\mathbf{1}) = 0.$$

We conclude (so far) that $\text{sym}(A)$ has negative diagonal elements, positive off-diagonal elements and 0 row sums, so $\text{sym}(A)$ is a Markov generator.

By construction, $\mathrm{sym}(A)$ is self-adjoint in $L^2(\mu)$, so $\mathrm{sym}(A)$ is reversible with respect to μ and in particular μ is stationary for $\mathrm{sym}(A)$.

Finally, since A has a unique ergodic measure with full support, (the Markov process generated by) it is irreducible. Since (by (7.39)) for all i, j, $A_{ij} \neq 0 \implies \mathrm{sym}(A)_{ij} \neq 0$, this implies that also (the Markov process generated by) $\mathrm{sym}(A)$ is irreducible. Therefore it can have at most one invariant measure, which implies that μ is the unique invariant measure and hence the unique ergodic measure. □

Conclusions

We concluded each chapter of this thesis with notes and perspectives on the results of the chapter. We mean to avoid repetition of those comments here. However, we do want to collect some of the overarching themes and make some concluding remarks as to the goal of the thesis as a whole.

As we wrote in the introduction (in Section 1.7), the goal of this thesis was to make the first steps in studying interacting particle systems and discrete random fields like the DGFF on manifolds. We believe that we have been successful in this goal. Indeed, we succeeded to define SEP and the DGFF on compact manifolds in such a way that we were able to replicate or generalise important results from the flat case. These results are new and form an important first step towards a more general theory of interacting particle systems and discrete random fields on manifolds.

The backbone of the results of the Chapters 4 through 6 is provided by the uniformly approximating grids that we constructed in Chapter 3. The most important property of these grids is that the graph Laplacians converge uniformly in the grid points to the Laplace-Beltrami operator. This is the key ingredient that we used to first derive the hydrodynamic limit and then also the equilibrium fluctuations for SEP. From the convergence of the graph Laplacians, we could additionally derive the convergence of the corresponding semigroups and eventually the Green's functions, which allowed us to prove convergence of the DGFF to the GFF.

This observation leads us to three main next steps that we recommend to further strengthen and expand our results. First, as we noted above, we rely on the convergence of the graph Laplacians in a somewhat strong sense: pointwise convergence, or even uniformly in the grid points. As we noted in Chapter 3 (Remark 3.2), this requires a certain symmetry of the grid around each grid point. The way that we obtain this symmetry is by letting the amount of points that a particle can jump to in a single jump go to infinity. It would be natural to use a more nearest-neighbour kind of grid. To get the same results with such grid will probably require methods that only rely on the convergence of the graph Laplacians in a weaker sense or maybe just on convergence of for instance the semigroups. As we pointed out before (in Section 6.5), the DGFF might be the most logical first candidate for such approach.

Second, in our analysis of SEP we used that it directly yields a closed hydrodynamic equation. For more general systems, one needs results such as one block and two block estimates where quantities are replaced by averages (as we mentioned in Section 4.5). For a more general theory of interacting particle systems on manifolds, it would therefore be good to investigate how such results can be obtained. The most important difficulty to overcome here is the lack of translation invariance.

This brings us to the third point. When studying these models on manifolds, one encounters many examples of nice properties of flat spaces in general and of lattice-like grids in particular that are not available on a manifold. Some of these are mostly technical, but one particularly conceptually challenging aspect of manifolds is the lack of translation invariance and even of translation itself. For this reason it would be interesting to generalise any result that usually relies on translation invariance to the manifold case. Here one can think of the one block and two block estimates that we just mentioned, but also of a result on a tagged particle (like we mentioned in Section 4.5).

Appendix A

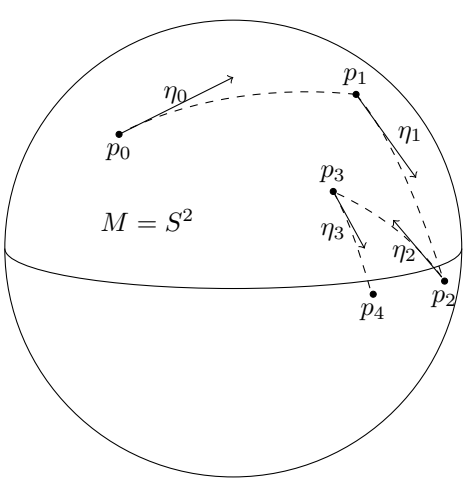
Invariance principle for geodesic random walks

Let M be an n-dimensional, compact and connected Riemannian manifold. Then we know that M is complete and hence geodesically complete. The main purpose of this section is to define the geodesic random walk and to show that it approximates Brownian motion when appropriately rescaled (in time and space). Such random walks and this so-called invariance principle have been studied in Jørgensen [93] and in a special case Blum [17]. The results in this Appendix were obtained in van Ginkel [148] and were published in the current form in van Ginkel and Redig [149]. The main reason for including these results in the Appendix of this thesis is for completeness, since they are tailor-made to apply them in Chapter 3. In particular, general assumptions are obtained on the jumping distributions of the geodesic random walk for it to converge to Brownian motion. In Section A.1, we define the geodesic random walk and show convergence of the generators to the generator of Brownian motion under certain assumptions on the jumping distributions. Section A.2 is devoted to finding out which distributions satisfy these assumptions.

A.1 Convergence of the generators

The process
Let $\{\mu_p, p \in M\}$ be a collection of positive, finite measures where each μ_p is a measure on T_pM. The measure μ_p represents the rate to jump in a particular direction of T_pM. More precisely, the Markov process $X^N = \{X_t^N, t \geq 0\}$ associated to $\{\mu_p, p \in M\}$ has generator

$$L_N f(p) = \int_{T_pM} f(p(1/N, \eta)) - f(p)\mu_p(\mathrm{d}\eta),$$



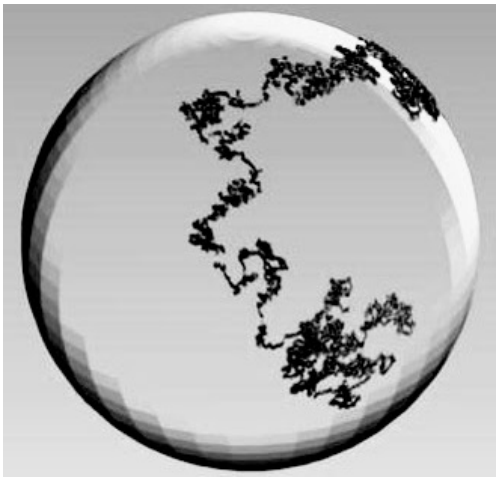


Figure A.1: Left: geodesic random walk on a sphere. Right: Brownian motion on a sphere (source: `https://en.wikipedia.org/wiki/Brownian_motion`, picture made by Christian Bayer and Thomas Steiner).

where for a vector $\xi \in T_pM$ we denote the geodesic through p with tangent vector ξ at p by $p(\cdot, \xi)$. We denote the corresponding semigroup by

$$S_t^N f(p) = \mathbb{E}_p f(X_t^N).$$

Both of these have the continuous functions on the manifold $C(M)$ as their domain.

We interpret this process as follows. When the process X^N is at a point p, it chooses a random direction η from T_pM with rates given by μ_p (i.e. it waits for an exponential time with rate $\mu_p(T_pM)$ and then independently picks a vector according to the probability distribution $\frac{\mu_p}{\mu_p(T_pM)}$). Then the process jumps to the position $p(1/N, \eta)$ that is reached by following the geodesic through p in the direction of η for time $\frac{1}{N}$. This situation is sketched in figure A.1. We assume that choosing random directions happens independently. In this Section we will specify restrictions that the measures μ_p should satisfy. Later (in Section A.2), we will show that we can take μ_p to be for instance the uniform distribution on the unit tangent vectors at p.

The $\mathbb{R}^n$ case

Before we go into the general case, we illustrate the above in $\mathbb{R}^n$. In $\mathbb{R}^n$ the exponential map is simply addition if we identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ itself. So in that case from a point p the process moves to $p(1/N, \eta) = p + \frac{1}{N}\eta$ where η is chosen from $T_p\mathbb{R}^n = \mathbb{R}^n$ randomly. This means that the discrete time jumping process when jumping as described above, can be denoted by $S_m^N = \sum_{i=1}^m \frac{1}{N}\eta_i = \frac{1}{N}\sum_{i=1}^m \eta_i$ where η_j is drawn

from $T_{S_{j-1}}\mathbb{R}^n = \mathbb{R}^n$ according to some distribution. Now let $\{N_t, t \geq 0\}$ be a Poisson process with rate one and define $X_t^N = S_{N_t}$. Then X makes the same jumps as S, but after independent exponential times. We see that $X^N = \{X_t^N, t \geq 0\}$ satisfies the description above. Now the invariance principle tells us that under some conditions on the jumping rates $X_{tN^2}^N \to B_t$ in distribution as N goes to infinity, where B is Brownian motion. We show the analogous result in the more general setting of a manifold.

Aim
We denote the Laplace-Beltrami operator on the manifold by Δ_M. The rest of this section will be devoted to the proof of the following result.

Proposition A.1. *Suppose that in the situation above we have:*

- $\sup_{p\in M} \sup_{\eta\in \mathrm{supp}\mu_p} ||\eta|| < \infty$
- $\sup_{p\in M} \mu_p(T_pM) < \infty$
- $\int \eta^i \mu_p(\mathrm{d}\eta) = 0$ *and* $\int \eta^i\eta^j \mu_p(\mathrm{d}\eta) = g^{ij}(p)$ *in each coordinate system around* p

Then for $f \in C^\infty$*:* $N^2 L_N f \to \frac{1}{2}\Delta_M f$ *uniformly on* M*.*

The first assumption requires that the supports of the measures and their total masses are bounded uniformly over all points of the manifold. We will loosely say that the measures are uniformly compactly supported and uniformly finite. Since $C^\infty(M)$ is a core for $\frac{1}{2}\Delta_M$ (Strichartz [142]), the Trotter-Kurtz theorem (see Kurtz [102]) implies the following corollary.

Corollary A.2. *In the situation of Proposition A.1 the geodesic random walk converges to Brownian motion in distribution in* $D([0,\infty), M)$ *(the space of càdlàg maps* $[0,\infty) \to M$*).*

Note that if we denote the random variable corresponding to μ_p by ζ_p, the second requirement of Proposition A.1 is that (in any coordinate system) $\mathbb{E}\zeta_p^i = 0$ and $\mathrm{Cov}(\zeta_p^i, \zeta_p^j) = g^{ij}(p)$. This shows that the mean vector m of ζ_p satisfies $m = 0$ and the covariance matrix Σ satisfies $\Sigma = (g^{ij})(p)$. In $\mathbb{R}^n$, this simplifies to $\mathbb{E}\zeta_p^i = 0$ and $\mathrm{Cov}(\zeta_p^i, \zeta_p^j) = \delta_j^i$. This is satisfied for instance when μ_p is the uniform distribution on the sphere with radius $\sqrt{N}$ in $\mathbb{R}^n$. Section A.2 deals with the question which measures satisfy the restrictions above. Some examples will be given at the end of that section as well.

Remark A.3. Although we study the jumping distributions later, something that can already be seen now, is that we do not require any relation between jumping measures at different points of the manifold (apart from the uniform bounds on the support and the total mass). This means that our result does not require the jumping measures to be identically distributed, so it really generalises Jørgensen [93].

Choosing suitable charts

Let f be a fixed smooth function from now on. Since we want the convergence $N^2 L_N f \to \frac{1}{2}\Delta_M f$ to be uniform on M, we cannot just consider this problem pointwise. To deal with this, we will choose specific coordinate charts.

Let ρ denote the original metric of the manifold and let d denote the metric that is induced by the Riemannian metric. Recall that these metrics induce the same topology. This means that we do not cause confusion when we speak about open and closed sets, continuous maps and compactness without explicitly mentioning the metric. For each $p \in M$, let (x_p, U_p) be a coordinate chart for M around p. U_p is open with respect to ρ and hence with respect to d. This means that there is some $\epsilon_p > 0$ such that $G_p := \overline{B_d(p, \epsilon_p)} \subset U_p$. Now define $O_p = B_d(p, \epsilon/2)$. Since M is compact, we can find $p_1, .., p_m$ such that $M \subset \cup_i O_{p_i}$. We have the following easy statement.

Lemma A.4. *Let $(g_k)_{k=1}^{\infty}$ and g be functions $M \to \mathbb{R}$. If $g_k \to g$ uniformly on each O_{p_i}, then $g_k \to g$ uniformly on M.*

Proof. Let $\epsilon > 0$. For each i there is an $N_i \in \mathbb{N}$ such that for all $k \geq N_i$: $\sup_{O_{p_i}} |g_k(q) - g(q)| < \epsilon$. Set $N = \max_{1 \leq i \leq m} N_i$ and let $q \in M$. Then there is a j such that $q \in O_{p_j}$. Now for all $k \geq N$, we see $k \geq N_j$, so $|g_k(q) - g(q)| \leq \sup_{O_{p_i}} |g_k(s) - g(s)| < \epsilon$. This shows that $\sup_M |g_k(q) - g(q)| \leq \epsilon$. Hence $g_k \to g$ uniformly on M. □

Now let $j \in \{1, .., m\}$ be fixed. Call $O := O_{p_j}$, $\epsilon := \epsilon_{p_j}$, $x := x_{p_j}$, $G := G_{p_j}$ and $U := U_{p_j}$ (this situation is shown in figure A.2). Because of the lemma, it suffices to show that $N^2 L_N f \to \frac{1}{2}\Delta_M f$ uniformly on O.

Technical considerations

To obtain good estimations later, we will need that $p(s, \eta)$ is still in our coordinate system (x, U) and even in the set G when $|s| \leq \frac{1}{N}$ for N large enough. Since the convergence must be uniform, how large N must be can not depend on the point p. The following lemma tells us how to choose such N.

Lemma A.5. *Call $K = \sup_{p \in M} \sup_{\eta \in \mathrm{supp}\mu_p} ||\eta|| < \infty$ (by assumption). Choose $N_\epsilon \in \mathbb{N}$ such that $\frac{1}{N_\epsilon} < \frac{\epsilon}{2K}$. Then for all $p \in O$ and $N \geq N_\epsilon$ we see*

$$\forall |s| \leq \frac{1}{N} : p(s, \eta) \in G.$$

Proof. Let $N \geq N_\epsilon$ and let $p \in O$. The situation of the proof is visually represented in figure A.2. Fix $s \in (-\frac{1}{N}, \frac{1}{N})$. Without loss of generality assume $s > 0$. Note that the speed of the geodesic $p(\cdot, \eta)$ equals $||\eta||$, so at time s, it has traveled a distance $s||\eta||$ from p. This means that there is a path of length $s||\eta||$ from $p(s, \eta)$ to p, so $d(p(s, \eta), p) \leq s||\eta|| \leq \frac{1}{N} K \leq \frac{1}{N_\epsilon} K < \epsilon/2$. Since $p \in O$, we know $d(p, p_j) < \epsilon/2$. Now

the triangle inequality shows that $d(p_j, p(s,\eta)) \leq d(p_j, p) + d(p, p(s,\eta)) < \epsilon/2 + \epsilon/2 = \epsilon$. This implies that $p(s,\eta) \in B_d(p_j, \epsilon) \subset G$. □

Fix N_ϵ as in the lemma and take N larger than N_ϵ.

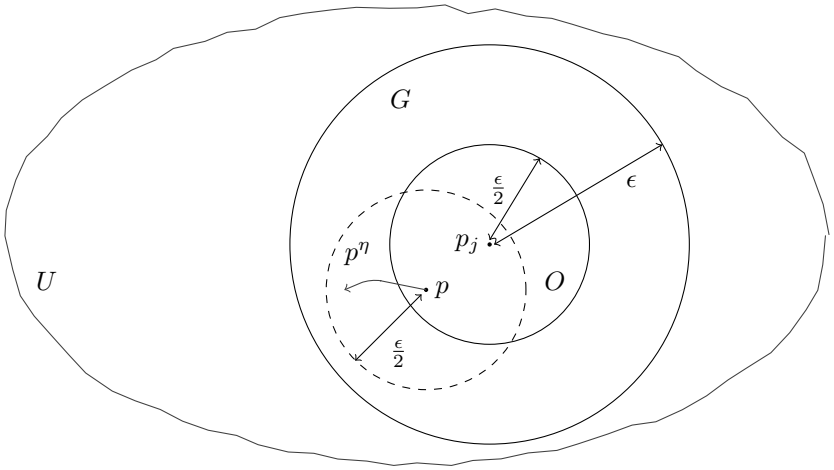


Figure A.2: The chart (x, U) with closed ball G and open ball O around p_j. As is shown in Lemma A.5, $p^\eta = p(t,\eta)$ does not leave the ball around p with radius $\epsilon/2$, as long as $|t| \leq 1/N$ for $N \geq N_\epsilon$. The importance for uniformity is that it does not matter where we choose p (in O).

Taylor expansion
Now fix $p \in O$ and $\eta \in T_pM$. Write p^η for the map $\mathbb{R} \to M$ that takes t to $p(t,\eta)$. We can locally write $f \circ p^\eta = (f \circ x^{-1}) \circ (x \circ p^\eta)$, which is a composition of smooth maps. This means that $f \circ p^\eta$ is just a smooth map $\mathbb{R} \to \mathbb{R}$, so we can use a Taylor expansion and obtain

$$f(p(1/N,\eta)) = f(p) + \frac{1}{N}\frac{\mathrm{d}(f\circ p^\eta)}{\mathrm{d}t}(0) + \frac{1}{2N^2}\frac{\mathrm{d}^2(f\circ p^\eta)}{\mathrm{d}^2 t}(0) + \frac{1}{6N^3}\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}^3 t}(t_{N,\eta,p}),$$

where $t_{N,\eta,p} \in (0, 1/N)$ is a number depending on N, η and p. This gives us

$$\begin{aligned} N^2 L_N f(p) &= N^2 \int_{M_p} f(p(1/N,\eta)) - f(p)\mu_p(\mathrm{d}\eta) \\ &= N^2 \int \frac{1}{N}\frac{\mathrm{d}(f\circ p^\eta)}{\mathrm{d}t}(0) + \frac{1}{2N^2}\frac{\mathrm{d}^2(f\circ p^\eta)}{\mathrm{d}^2 t}(0) \\ &+ \frac{1}{6N^3}\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}^3 t}(t_{N,\eta,p})\mu_p(\mathrm{d}\eta) \\ &= N\int \frac{\mathrm{d}(f\circ p^\eta)}{\mathrm{d}t}(0)\mu_p(\mathrm{d}\eta) + \frac{1}{2}\int \frac{\mathrm{d}^2(f\circ p^\eta)}{\mathrm{d}t^2}(0)\mu_p(\mathrm{d}\eta) \\ &+ \frac{1}{6N}\int \frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}(t_{N,\eta,p})\mu_p(\mathrm{d}\eta). \end{aligned} \tag{A.1}$$

We will examine these terms separately.

The first term

Recall that $p \in O$ and that O is contained in a coordinate chart (x, U). Since $N \geq N_\epsilon$, Lemma A.5 guarantees us that $p(s,\eta)$ stays in the coordinate chart for $|s| < \frac{1}{N}$. Writing $\eta = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i}|_p$, we see for $|s| < \frac{1}{N}$:

$$\begin{aligned} \frac{\mathrm{d}(f\circ p^\eta)}{\mathrm{d}t}(s) &= \frac{\mathrm{d}}{\mathrm{d}t}[(f\circ x^{-1})\circ(x\circ p^\eta)](s) \\ &= \sum_{i=1}^n D_i(f\circ x^{-1})(x(p^\eta(s))\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t}(s) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p^\eta(s))\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t}(s). \end{aligned}$$

Now setting $s = 0$, this becomes:

$$\sum_{i=1}^n \frac{\partial f}{\partial x^i}(p)\eta^i = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i}|_p f = \eta(f),$$

since $p^\eta(0) = p(0,\eta) = p$ and the tangent vector to the geodesic $p(\cdot,\eta)$ at 0 is η (so the i^{th} coordinate with respect x is just η^i). Now the first term of (A.1) becomes:

$$N\int \eta(f)\mu_p(\mathrm{d}\eta) = N\int \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i}|_p f \mu_p(\mathrm{d}\eta) = N\sum_{i=1}^n \frac{\partial}{\partial x^i}|_p f \int \eta^i \mu_p(\mathrm{d}\eta).$$

By assumption these integrals are 0. This shows that the first term of (A.1) vanishes.

The second term

Now we want to show that the remaining term equals $\frac{1}{2}\Delta_M f(p)$. Similarly to above

we see for $|s| < \frac{1}{N}$ (leaving out the arguments to keep things clear):

$$\begin{aligned}\frac{\mathrm{d}^2(f\circ p^\eta)}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t}\\ &= \sum_{i=1}^n\left\{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial x^i}\right)\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t} + \frac{\partial f}{\partial x^i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t}\right)\right\}\\ &= \sum_{i=1}^n\left\{\sum_{j=1}^n \frac{\partial^2 f}{\partial x^j\partial x^i}\frac{\mathrm{d}(x^j\circ p^\eta)}{\mathrm{d}t}\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}^2(x^i\circ p^\eta)}{\mathrm{d}t^2}\right\}.\end{aligned}$$

Since p^η is a geodesic, we know that it satisfies the geodesic equations. This shows that for each $i = 1, .., n$ we have

$$\frac{\mathrm{d}^2(x^i\circ p^\eta)}{\mathrm{d}t^2} + \sum_{k,l=1}^n \Gamma^i_{kl}\frac{\mathrm{d}(x^k\circ p^\eta)}{\mathrm{d}t}\frac{\mathrm{d}(x^l\circ p^\eta)}{\mathrm{d}t} = 0.$$

Using this yields the following expression for the second derivative:

$$\sum_{i=1}^n\left\{\sum_{j=1}^n \frac{\partial^2 f}{\partial x^j\partial x^i}\frac{\mathrm{d}(x^j\circ p^\eta)}{\mathrm{d}t}\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t} - \frac{\partial f}{\partial x^i}\sum_{k,l=1}^n \Gamma^i_{kl}\frac{\mathrm{d}(x^k\circ p^\eta)}{\mathrm{d}t}\frac{\mathrm{d}(x^l\circ p^\eta)}{\mathrm{d}t}\right\},$$

so

$$\frac{\mathrm{d}^2(f\circ p^\eta)}{\mathrm{d}t^2}(0) = \sum_{i=1}^n\left\{\sum_{j=1}^n \frac{\partial^2 f}{\partial x^j\partial x^i}(p)\eta^j\eta^i - \frac{\partial f}{\partial x^i}(p)\sum_{k,l=1}^n \Gamma^i_{kl}(p)\eta^k\eta^l\right\}.$$

Using linearity of the integral, we obtain the following expression for the second term of (A.1):

$$\frac{1}{2}\sum_{i=1}^n\left\{\sum_{j=1}^n \frac{\partial^2 f}{\partial x^i\partial x^j}(p)\int \eta^i\eta^j\mu_p(\mathrm{d}\eta) - \frac{\partial f}{\partial x^i}(p)\sum_{k,l=1}^n \Gamma^i_{kl}(p)\int \eta^k\eta^l\mu_p(\mathrm{d}\eta)\right\}.$$

Note that we also changed the order of the derivatives of f, this can be done since f is smooth. Now we want the term above to equal

$$\begin{aligned}&\frac{1}{2}\Delta_M f(p) = \frac{1}{2}\left\{g^{ij}\frac{\partial^2 f}{\partial x^i x^j} - g^{kl}\Gamma^i_{kl}\frac{\partial f}{\partial x^i}\right\}\\ =\ &\frac{1}{2}\sum_{i=1}^n\left\{\sum_{j=1}^n \frac{\partial^2 f}{\partial x^i\partial x^j}(p)g^{ij}(p) - \frac{\partial f}{\partial x^i}(p)\sum_{k,l=1}^n \Gamma^i_{kl}(p)g^{kl}(p)\right\}.\end{aligned}$$

This is true, since we required that for any coordinate chart around p and for all i, j: $\int_{M_p}\eta^i\eta^j\mu_p(\mathrm{d}\eta) = g^{ij}(p)$.

The rest term

If the last term goes to 0 uniformly on O, we have the result. Let N still be larger then N_ϵ.

$$\begin{aligned}\left|\frac{1}{6N}\int \frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}(t_{N,\eta,p})\mu_p(\mathrm{d}\eta)\right| &\leq \frac{1}{6N}\int\left|\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}(t_{N,\eta,p})\right|\mu_p(\mathrm{d}\eta)\\ &\leq \frac{K'}{6N}\sup_{\eta\in\mathrm{supp}\mu_p}\left|\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}(t_{N,\eta,p})\right|\end{aligned}$$

where $K' = \sup_{p\in M}\mu_p(T_pM) < \infty$ (by assumption). We know that $t_{N,\eta,p} \in [0, 1/N] \subset [0, 1/N_\epsilon]$. This means that the above is smaller than:

$$\frac{K'}{6N}\sup_{\eta\in\mathrm{supp}\mu_p}\sup_{t\in[0,1/N_\epsilon]}\left|\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}(t)\right| \leq \frac{K'}{6N}\sup_{\eta:||\eta||\leq K}\sup_{t\in[0,1/N_\epsilon]}\left|\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}(t)\right|.$$

Because of the $1/N$ in front of the equation, we only need to know that the rest is uniformly bounded to obtain uniform convergence. It thus suffices to show that $\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}(t)$ is bounded as a function of η with $||\eta|| < K$ and $t \in [0, 1/N_\epsilon]$. Lemma A.5 shows that $p(t, \eta)$ stays in G for all such η and t. We will use this fact multiple times.

We first express $\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}$ in local coordinates for $|t| \leq 1/N$.

$$\begin{aligned}\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}^2(f\circ p^\eta)}{\mathrm{d}t^2}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^n\left\{\sum_{j=1}^n\frac{\partial^2 f}{\partial x^j\partial x^i}\frac{\mathrm{d}(x^j\circ p^\eta)}{\mathrm{d}t}\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}^2(x^i\circ p^\eta)}{\mathrm{d}t^2}\right\}. \quad \text{(A.2)}\end{aligned}$$

To make notation more compact, we introduce the following notation (and f_i, f_{ijk} analogously):

$$f_{ij} := \frac{\partial^2 f}{\partial x^j\partial x^i}, \qquad p_k^i := \frac{\mathrm{d}^k(x^i\circ p^\eta)}{\mathrm{d}t^k}.$$

Combining this with Einstein summation, we can write (A.2) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(f_{ij}p_1^ip_1^j + f_ip_2^i) &= (f_{ijk}p_1^k)p_1^ip_1^j + f_{ij}(p_1^ip_2^j + p_2^ip_1^j) + (f_{ij}p_1^j)p_2^i + f_ip_3^i\\ &= f_{ijk}p_1^kp_1^ip_1^j + f_{ij}(p_1^ip_2^j + 2p_2^ip_1^j) + f_ip_3^i.\end{aligned}$$

Now, as before, we can deal with second derivatives of geodesics using the geodesic equations:

$$p_2^i = -\Gamma_{rs}^i p_1^r p_1^s.$$

We can also calculate the third derivative:

$$p_3^i = \frac{\mathrm{d}}{\mathrm{d}t}p_2^i = \frac{\mathrm{d}}{\mathrm{d}t}(-\Gamma_{rs}^i p_1^r p_1^s) = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{rs}^i\right)p_1^rp_1^s - \Gamma_{rs}^i(p_1^rp_2^s + p_2^rp_1^s).$$

This shows us that $\frac{\mathrm{d}^3(f\circ p^\eta)}{\mathrm{d}t^3}$ is a combination of products and sums of the following types of expressions: f_i, f_{ij}, f_{ijk}, p_1^i, Γ^i_{rs} and $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^i_{rs}$. If we can bound all of these on the right domains (independent of p and η), we are done.

Bounding f_i, f_{ij} and f_{ijk}
First of all, note that f is a smooth function on U. Further, ∂_i defines smooth vector field on U. Since $f_i = \frac{\partial f}{\partial x^i}$ is obtained by applying ∂_i on U to f, it is a smooth function on U. Continuing in this way, we see that f_{ij} and f_{ijk} are also smooth functions on U. In particular, they are smooth functions on G (since it is a subset of U). G is a closed subset of the compact M and is hence compact itself. This implies that f_i, f_{ij} and f_{ijk} are (for each choice of i,j,k) bounded on G. Since we evaluate these functions in the points $p(s,\eta)$ for $0 \leq s \leq 1/N$, $N \geq N_\epsilon$ and $||\mu|| \leq K$, our discussion above shows that we only evaluate them in points of G. This means that we have found bounds for f_i, f_{ij} and f_{ijk}.

Bounding p_1^i
We start with a technical lemma.

Lemma A.6. *Let $q \in M$ and let (y, V) be a coordinate chart around q. Let $v \in T_qM$ and write $v = v^i\partial_i$. Then $|v^i| \leq \sqrt{g^{ii}(q)}||v||$.*

Proof. Fix some $1 \leq i \leq n$. We see in the tangent space at q:

$$\left\langle v, g^{ij}\partial_j \right\rangle = \left\langle v^k\partial_k, g^{ij}\partial_j \right\rangle = v^k g^{ij} g_{kj} = v^k\delta^i_k = v^i.$$

Further,

$$||g^{ij}\partial_j||^2 = \left\langle g^{ij}\partial_j, g^{ik}\partial_k \right\rangle = g^{ij}g^{ik}g_{jk} = g^{ij}\delta^i_j = g^{ii}.$$

Using the relations above and the Cauchy-Schwarz inequality, we obtain:

$$|v^i| = |\left\langle v, g^{ij}\partial_j \right\rangle| \leq ||v|| \cdot ||g^{ij}\partial_j|| = \sqrt{g^{ii}}||v||.$$

□

Now we can use this to show the following.

Lemma A.7. $|p_1^i(t)| = \left|\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t}(t)\right| \leq \sqrt{g^{ii}(p(t,\eta))}||\eta||$.

Proof. The first equation is just a change of notation. Further we see

$$\frac{\mathrm{d}(x^i \circ p^\eta)}{\mathrm{d}t} = \left(p^\eta_* \frac{\mathrm{d}}{\mathrm{d}t}\right)(x^i) = \frac{\mathrm{d}p^\eta}{\mathrm{d}t}(x^i) = \left(\frac{\mathrm{d}p^\eta}{\mathrm{d}t}\right)^i.$$

This means that $\frac{\mathrm{d}(x^i\circ p^\eta)}{\mathrm{d}t}$ is just the i^{th} coordinate with respect to (x, U) of the tangent vector to p^η at time t so at the point $p(t,\eta) \in M$. Using Lemma A.6, we see

$$\left|\frac{\mathrm{d}(x^i \circ p^\eta)}{\mathrm{d}t}(t)\right| \leq \sqrt{g^{ii}(p(t,\eta))}\left|\left|\frac{\mathrm{d}p^\eta}{\mathrm{d}t}\right|\right|. \tag{A.3}$$

Since p^η is a geodesic, it has constant speed. Its speed at p is $||\eta||$, so this must be its speed anywhere else along the trajectory. Hence $||\frac{\mathrm{d}p^\eta}{\mathrm{d}t}|| = ||\eta||$. Inserting this in (A.3) yields the result. □

We can now easily obtain a bound for p_1^i. For $0 \leq t \leq 1/N$ and $||\eta|| \leq K$, we know $p(t,\eta)$ stays in G. g^{ii} is a smooth and hence continous function on U, so it is bounded on G (since G is compact). This means that $\sqrt{g^{ii}(p(t,\eta))}$ is bounded by some K^i for $||\eta|| \leq K$ and $0 \leq t \leq 1/N$. Now we see $|p_1^i| \leq \sqrt{g^{ii}(p(t,\eta))} \left|\left|\frac{\mathrm{d}p^\eta}{\mathrm{d}t}\right|\right| \leq K^i K$.

Bounding Γ^i_{rs} and $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^i_{rs}$

Each g_{ij} is a smooth function on U. This means that $\frac{\partial g_{ij}}{\partial x^k}$ is a smooth function on U. This implies that Γ^i_{rs} is just combination of products and sums of smooth functions, so it is smooth itself. Now, as before, Γ^i_{rs} is bounded on G. Since we only evaluate it in $p(t,\eta)$ with $0 \leq t \leq 1/N$ and $||\eta|| \leq K$, we only evaluate it in G, so we have bounded Γ^i_{rs}.
Now $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^i_{rs}$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^i_{rs} = \frac{\partial \Gamma^i_{rs}}{\partial x^j}\frac{\mathrm{d}(x^j \circ p^\eta)}{\mathrm{d}t} = (\Gamma^i_{rs})_j p_1^j,$$

with notation as above. Since Γ^i_{rs} is smooth function $U \to \mathbb{R}$, this expression can be bounded in exactly the same way as expressions like $f_j p_1^j$ above.

A.2 Stepping distribution

Constraints for a stepping distribution

The question now is which distributions μ_p on T_pM satisfy the assumptions of Proposition A.1. From here on we fix $p \in M$ and simply write μ for μ_p. Being compactly supported and finite are rather natural constraints, but the other assumptions are harder, especially since they involve local coordinates. In this section we address the question which distributions satisfy the other assumptions, i.e. for every coordinate system around p:

$$\begin{aligned} \int \eta^i \mu(\mathrm{d}\eta) &= 0 \qquad \forall i = 1,..,n \\ \int \eta^i \eta^j \mu(\mathrm{d}\eta) &= g^{ij} \qquad \forall i,j = 1,..,n. \end{aligned} \tag{A.4}$$

To generalise this a bit, suppose μ satisfies the following for some $c > 0$ for every coordinate system:

$$\begin{aligned} \int \eta^i \mu(\mathrm{d}\eta) &= 0 \qquad \forall i = 1,..,n \\ \int \eta^i \eta^j \mu(\mathrm{d}\eta) &= c g^{ij} \qquad \forall i,j = 1,..,n. \end{aligned} \tag{A.5}$$

Following the proof in the previous section, one sees directly that in this case the generators converge to the generator of Brownian motion that is speeded up by a factor c. We will look into this generalised situation and at the end we will see how to determine c.

Independence of (A.5) of coordinate systems
The following lemma shows that if (A.5) holds for a single coordinate system, it holds for any coordinate system.

Lemma A.8. *If (A.5) holds for some $c > 0$ and for some coordinate system (x, U) around p, then it holds for the same c for all coordinate systems around p.*

Proof. Let (x, U) be a coordinate system around p for which (A.5) holds with $c > 0$ and let (y, V) be any other coordinate system around p. It suffices to show that (A.5) holds with the same c for y. Denote the metric matrix with respect to x by g and the one with respect to y by $\hat{g}$. For any $\eta \in T_pM$ define $\eta^1, .., \eta^n$ as the coefficients of η with respect to x, so such that $\eta = \sum_i \eta^i \frac{\partial}{\partial x^i}$. Analogously let $\hat{\eta}^1, .., \hat{\eta}^n$ be such that $\eta = \sum_i \hat{\eta}^i \frac{\partial}{\partial y^i}$. Let $J = \frac{\partial(x^1,..,x^n)}{\partial(y^1,..,y^n)}$. If $\eta \in T_pM$, then

$$\hat{\eta}^j = \eta(y^i) = \sum_i \eta^i \frac{\partial}{\partial x^i} y^i = \sum_i \eta^i \frac{\partial y^j}{\partial x^i}.$$

This shows that for any j

$$\int \hat{\eta}^j \mu(\mathrm{d}\eta) = \int \sum_{i=1}^n \eta^i \frac{\partial y^j}{\partial x^i} \mu(\mathrm{d}\eta) = \sum_{i=1}^n \frac{\partial y^j}{\partial x^i} \int \eta^i \mu(\mathrm{d}\eta) = 0,$$

since for any i: $\int \eta^i \mu(\mathrm{d}\eta) = 0$. Moreover, for any i, j: $\int \eta^i \eta^j \mu(\mathrm{d}\eta) = c g^{ij}$, so for any i, j:

$$\begin{aligned} \int \hat{\eta}^i \hat{\eta}^j \mu(\mathrm{d}\eta) &= \int \sum_{k=1}^n \eta^k \frac{\partial y^i}{\partial x^k} \sum_{l=1}^n \eta^l \frac{\partial y^j}{\partial x^l} \mu(\mathrm{d}\eta) = \sum_{k,l=1}^n \frac{\partial y^i}{\partial x^k} \frac{\partial y^j}{\partial x^l} \int \eta^k \eta^l \mu(\mathrm{d}\eta) \\ &= \sum_{k,l=1}^n \frac{\partial y^i}{\partial x^k} \frac{\partial y^j}{\partial x^l} c g^{kl} = c(J^{-1} G^{-1} (J^{-1})^T)_{ij}. \end{aligned}$$

Since $J^{-1}G^{-1}(J^{-1})^T = J^{-1}G^{-1}(J^T)^{-1} = (J^T G J)^{-1} = \hat{G}^{-1}$, we see that $\int \hat{\eta}^i \hat{\eta}^j \mu(\mathrm{d}\eta) = c\hat{g}^{ij}$. We conclude that (A.5) holds for y with the same c. □

Orthogonal transformations and canonical measures
We now introduce a class of measures.

Definition A.9. Let V be an inner product space and let T be a linear map $V \to V$. We call T an *orthogonal transformation* if for any $u, v \in V$: $\langle Tu, Tv\rangle = \langle u, v\rangle$.
We call a measure μ on T_pM *canonical* if for any orthogonal transformation T on T_pM and for any coordinate system:

$$\int \eta^i \mu(\mathrm{d}\eta) = \int (T\eta)^i \mu(\mathrm{d}\eta) \text{ and } \int \eta^i\eta^j \mu(\mathrm{d}\eta) = \int (T\eta)^i (T\eta)^j \mu(\mathrm{d}\eta).$$

Remark A.10. In the same way as above, one can show that μ has the property above with respect to some coordinate system if and only if it has the property with respect to every coordinate system. Moreover, since $-I$ always satisfies $(-I)^T G(-I) = G$, we see that $\int \eta^i \mu(\mathrm{d}\eta) = \int (-\eta)^i \mu(\mathrm{d}\eta) = \int -\eta^i \mu(\mathrm{d}\eta) = -\int \eta^i \mu(\mathrm{d}\eta)$, so $\int \eta^i \mu(\mathrm{d}\eta)$ is 0 for any canonical μ.

In words, μ is canonical if orthogonal transformations do not change the mean vector and the covariance matrix of a random variable that has distribution μ. Remark A.10 shows that in fact the mean vector must be 0. Note that in particular measures that are invariant under orthogonal transformations are canonical, since then $\int (T\eta)^i \mu(\mathrm{d}\eta) = \int \eta^i (\mu \circ T^{-1})(\mathrm{d}\eta) = \int \eta^i \mu(\mathrm{d}\eta)$ and the other equation follows analogously. However a simple example shows that the converse is not true. Let $M = \mathbb{R}$ and let μ be any non-symmetric distribution on $T_pM = \mathbb{R}$ with mean 0. The only orthogonal transformation (apart from the identity) is $t \mapsto -t$. Under this transformation the mean (which is 0) and the second moment are obviously left invariant, but μ is not symmetric, so it is not invariant. We will give an example for $\mathbb{R}^n$ later.

If (x, U) is some coordinate system around p and $G = (g_{ij})$ is the matrix of the metric in p with respect to x, we can write a linear transformation $T : T_pM \to T_pM$ as a matrix (which we will also call T) with respect to the basis $\frac{\partial}{\partial x^1}, .., \frac{\partial}{\partial x^n}$. We see that

$$\begin{aligned}\langle T\eta, T\xi\rangle &= \sum_{i,j} g_{ij} (T\eta)^i (T\xi)^j = \sum_{i,j} g_{ij} \sum_k T_{ik}\eta^k \sum_l T_{jl}\xi^l \\ &= \sum_{k,l} \left(\sum_{i,j} g_{ij} T_{ik} T_{jl} \right) \eta^k \xi^l.\end{aligned}$$

If T is orthogonal, this must equal

$$\langle \eta, v\rangle = \sum_{k,l} g_{kl}\eta^k \xi^l,$$

so we see that $g_{kl} = \sum_{i,j} g_{ij} T_{ik} T_{jl} = (T^T G T)_{kl}$ and hence $G = T^T G T$.
Now for a measure μ on T_pM and a coordinate system (x, U), define the vector A_μ and the matrix B_μ by $A^i_\mu = \int \eta^i \mu(\mathrm{d}\eta)$ and $B^{ij}_\mu = \int \eta^i \eta^j \mu(\mathrm{d}\eta)$. Then we have the following.

Lemma A.11. *Let μ be a measure on T_pM. Then the following are equivalent.*

(i) μ is canonical.

(ii) For every linear transformation T and every coordinate system (x, U): if $G = T^TGT$, then $A_\mu = TA_\mu$ and $B_\mu = TB_\mu T^T$.

Proof. $(i) \Leftrightarrow (ii)$ because (ii) is just the definition of being canonical written in local coordinates. Indeed, we already saw that orthogonality or T translates in local coordinates to $G = T^TGT$, the other expressions follow in a similar way from the following equations:

$$A^i_\mu = \int (T\eta)^i \mu(\mathrm{d}\eta) = \int \sum_k T_{ik}\eta^k \mu(\mathrm{d}\eta) = \sum_k T_{ik} \int \eta^k \mu(\mathrm{d}\eta) = \sum_k T_{ik} A^k_\mu$$

and

$$\begin{aligned} B^{ij}_\mu &= \int (T\eta)^i (T\eta)^j \mu(\mathrm{d}\eta) = \int \sum_k T_{ik}\eta^k \sum_l T_{jl}\eta^l \mu(\mathrm{d}\eta) \\ &= \sum_{k,l} T_{ik}T_{jl} \int \eta^k\eta^l \mu(\mathrm{d}\eta) = \sum_{k,l} T_{ik}T_{jl}B^{kl}_\mu. \end{aligned}$$

□

Canonical measures are stepping distributions
Now we have the following result.

Proposition A.12. *Let μ be a probability measure on T_pM. Then μ is canonical if and only if it satisfies (A.5) for some $c > 0$.*

Proof. First assume that μ is canonical and let (x, U) be normal coordinates centered at p. Because of Lemma A.8 it suffices to verify (A.5) for x, so we need to show that $A_\mu = 0$ and $B_\mu = cG^{-1} = cI$ for some $c > 0$.
The fact that $A_\mu = 0$ is just Remark A.10. Now note that since B_μ is symmetric, it can be diagonalised as $TB_\mu T^{-1}$ where T is an orthogonal matrix (in the usual sense). This means that $T^T = T^{-1}$ and that $T^TGT = T^TIT = T^TT = I = G$, so Lemma A.11 tells us that the diagonalisation equals $TB_\mu T^T = B_\mu$. This implies that B_μ is a diagonal matrix. Now for $i \neq j$ let $\overline{I}^{ij}$ be the $n \times n$-identity matrix with the i^{th} and j^{th} column exchanged. It is easy to see that $(\overline{I}^{ij})^T\overline{I}^{ij} = I$, so we must also have $B_\mu = \overline{I}^{ij}B_\mu(\overline{I}^{ij})^T$. The latter is B_μ with the i^{th} and j^{th} diagonal element exchanged. This shows that these elements must be equal. Hence all diagonal elements are equal and $B_\mu = cI$ for some $c \in \mathbb{R}$. Since $c = B^{11}_\mu = \int \eta^1\eta^1\mu(\mathrm{d}\eta) \geq 0$, we know that $c \geq 0$. If $c = 0$, then $B_\mu = 0$, so $\mu = 0$, which is not possible. We conclude that $c > 0$.
Conversely let (x, U) be a coordinate system with corresponding metric matrix G and

assume that μ satisfies (A.5) for some $c > 0$. Let T be such that $G = T^T G T$. Then $A_\mu = 0 = T0 = TA_\mu$. We also see: $T^T G T = G \iff G = (T^T)^{-1} G T^{-1} \iff G^{-1} = TG^{-1}T^T \iff cG^{-1} = T(cG^{-1})T^T \implies B_\mu = TB_\mu T^T$ (since $B_\mu = cG^{-1}$), so by Lemma A.11 μ is canonical. □

Now we know that if the stepping distribution is canonical (and finite and compactly supported, uniformly on M), the generators converge to the generator of Brownian motion that is speeded up by some factor $c > 0$ (depending on μ). The question remains what this c is. The following lemma answers this question.

Lemma A.13. *Suppose μ satisfies (A.5) for some $c > 0$. Then $c = \frac{\int ||\eta||^2 \mu(\mathrm{d}\eta)}{n}$.*

Proof. We calculate the following (with respect to some coordinate system (x, U)):

$$\begin{aligned}
\int ||\eta||^2 \mu(\mathrm{d}\eta) &= \int \langle \eta, \eta \rangle \, \mu(\mathrm{d}\eta) = \int \left\langle \sum_i \eta^i \frac{\partial}{\partial x^i}, \sum_j \eta^j \frac{\partial}{\partial x^j} \right\rangle \mu(\mathrm{d}\eta) \\
&= \sum_{i,j} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \int \eta^i \eta^j \mu(\mathrm{d}\eta) = \sum_{i,j} g_{ij} c g^{ij} \\
&= c \sum_i \sum_j g_{ij} g^{ji} = c \sum_i 1 = cn.
\end{aligned}$$

Hence $c = \frac{\int ||\eta||^2 \mu(\mathrm{d}\eta)}{n}$. □

The nice part of this lemma is that the expression for c does not involve a coordinate system, only the norm (and hence inner product) of T_pM. In particular we see that $c = 1$ is equivalent to $\int ||\eta||^2 \mu(\mathrm{d}\eta) = n$. We summarise our findings in the following result.

Proposition A.14. *A probability measure μ on T_pM satisfies (A.5) for some $c > 0$ if and only if it is canonical and $c = \frac{\int ||\eta||^2 \mu(\mathrm{d}\eta)}{n}$. In particular, it satisfies (A.4) if and only if it is canonical and $\int ||\eta||^2 \mu(\mathrm{d}\eta) = n$.*

Remark A.15. Note that all we need of the jumping distributions is that their mean is 0, their covariance matrix is invariant under orthogonal transformations, they are (uniformly) compactly supported and they are (uniformly) finite. We don't need the measures to be similar in any other way, so we do not at all require the jumps to have identical distributions in the sense of Jørgensen [93].

Examples

We conclude with some examples.

Example A.16. To satisfy (A.4) for every coordinate system, by Lemma A.8 it suffices to choose a coordinate system and construct a distribution that satisfies (A.4)

for that coordinate system. Let (x, U) be any coordinate system around some point in M with corresponding metric matrix G in that point. Let X be any random variable in $\mathbb{R}^n$ that has mean vector 0 and covariance matrix G^{-1} (for instance let $X \sim N(0, G^{-1})$). Now let μ be the distribution of $\sum_i X^i \frac{\partial}{\partial x^i}$. Then by construction $\int \eta^i \mu(\mathrm{d}\eta) = \mathbb{E}X^i = 0$ and $\int \eta^i \eta^j \mu(\mathrm{d}\eta) = \mathbb{E}X^i X^j = \mathbb{E}X^i X^j - \mathbb{E}X^i \mathbb{E}X^j = g^{ij}$.

Example A.17. In the previous example (A.4) is immediate. Let us now consider an example that illustrates the use of Proposition A.14. Let μ_p be the uniform distribution on $\sqrt{n}S_pM$ (the vectors with norm $\sqrt{n}$). By definition of such a distribution, it is invariant under orthogonal transformations (rotations and reflections), so it is a canonical distribution. Since also $\int ||\eta||^2 \mu(\mathrm{d}\eta) = \int \sqrt{n}^2 \mu(\mathrm{d}\eta) = n$, we conclude that the uniform distribution on $\sqrt{n}S_pM$ satisfies (A.4). Moreover, $\sup_{p\in M} \sup_{\eta\in \mathrm{supp}\mu_p} ||\eta|| = \sqrt{n} < \infty$ and $\sup_{p\in M} \mu_p(T_pM) = 1 < \infty$. Together this shows that the μ_p's satisfy the assumption of Proposition A.1.

Example A.18. Let us conclude by showing for $\mathbb{R}^n$ that the class of canonical distributions is strictly larger than the class of distributions that are invariant under orthogonal transformations, even with the restriction that $\int ||\eta||^2 \mu(\mathrm{d}\eta) = n$. It suffices to find a distribution μ with mean 0 and covariance matrix I (since then μ satisfies (A.4) and Proposition A.14 then tells us that μ is canonical and has $\int ||\eta||^2 \mu(\mathrm{d}\eta) = n$) and an orthogonal T such that $\mu \neq \mu \circ T^{-1}$. Let ν be the distribution on $\mathbb{R}$ given by $\nu = \frac{1}{5}\delta_{-2} + \frac{4}{5}\delta_{1/2}$. Then, using the natural coordinate system, $\int t\nu(\mathrm{d}t) = \frac{1}{5}(-2) + \frac{4}{5}\frac{1}{2} = 0$ and $\int t^2 \mu(\mathrm{d}t) = \frac{1}{5}(-2)^2 + \frac{4}{5}(\frac{1}{2})^2 = 1$. Now let $\mu = \nu \times .. \times \nu$ (n times). Then we directly see that the mean vector is 0 and the covariance matrix is I. However $T = -I$ is an orthogonal transformation and $\mu \circ (-I)^{-1}$ equals the product of n times $\frac{1}{5}\delta_2 + \frac{4}{5}\delta_{-1/2}$, so obviously $\mu \neq \mu \circ (-I)^{-1}$.

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Summary

In this thesis we study the Symmetric Exclusion Process (SEP) and the Discrete Gaussian Free Field (DGFF) on compact Riemannian manifolds. In particular, we obtain the hydrodynamic limit and the equilibrium fluctuations of SEP and we show that the DGFF converges to its continuous counterpart. To define these discrete models, we construct grids with edge weights that approximate the underlying manifold in a suitable way. Additionally, we study a model of an active particle and the role of reversibility for its limiting diffusion coefficient and large deviations rate function.

Context and motivation

An important concept in statistical physics is that many natural objects and phenomena can be described on different scales. One of its goals is to derive the macroscopic properties (density, temperature, magnetisation) of matter and the equations that govern them from the microscopic behaviour, properties and/or interactions of the particles that they are made of. In mathematical statistical physics often, for simplicity, we use probabilistic models for the description on the microscopic level. Therefore also the tools and methods to relate them to macroscopic models come from probability theory.

The main probabilistic framework in which we analyse questions from statistical physics is interacting particle systems. The idea is that we want to model a space through which particles move randomly and where they influence each other. For this purpose, the continuum space is discretised into a grid and particles can jump between grid points (also called sites). The particle system that we study in this thesis is the Symmetric Exclusion Process (SEP). It consists of particles that perform independent random walks on a grid with the rule that there is a maximum of one particle per site, so jumps to occupied sites are suppressed. This induces a repulsion between the particles. This model is well-studied and in some sense has the most basic interaction rule. The difficulty in this thesis comes from the fact that we study it in a new setting, namely on a manifold. This poses a range of challenges, both of a conceptual and of a technical nature.

To pass from a microscopic description in terms of particle configurations to a macroscopic description in terms of densities and PDEs, the particle system needs to be rescaled. The grid on which it evolves should become finer and should approximate the continuum, while time is rescaled appropriately. Then we can relate sequences of particle configurations on grids that approximate the continuum to a density profile on the continuum space. We would then like to prove that if at time 0 the particle configurations have an associated density profile, then they will have an associated density profile at every later macroscopic time point and that, moreover, the density profiles satisfy a PDE. This PDE is then called the hydrodynamic limit of the particle system under consideration. When the hydrodynamic limit is obtained, we can rescale and study how the system fluctuates around the hydrodynamic limit. One aims to show that these so called equilibrium fluctuations satisfy an SPDE.

In addition to interacting particle systems, we study the Gaussian Free Field. The Discrete Gaussian Free Field (DGFF) is a random field on a graph that penalises the squared difference of the values of the DGFF at neighbouring vertices, scaled with the edge weight between those vertices. This induces a normal distribution with the Green's function on the graph as the covariance matrix. It can be viewed as a Gaussian fluctuation of a harmonic graph function. The DGFF is the discrete counter part of the continuum Gaussian Free Field (GFF). This continuum field is so rough that it takes values in the distributions, i.e. its realisations cannot be described by functions and it is not pointwise a random variable. In flat space it is known that the DGFF can be rescaled to obtain the GFF in the limit. We aim to show the same on a manifold.

The models that we discuss have been studied mostly on flat spaces, i.e. subsets of $\mathbb{R}^n$ or the flat torus. In this thesis we consider them on a manifold. The first motivation comes from the point of view of applications: since manifolds are ubiquitous in nature we would like to understand how particle systems and random fields behave on them. The second motivation is that there has been shown to be an interesting interplay between stochastics and geometry. Therefore understanding particle systems on manifolds could teach us more about geometry. Finally, studying models in a more general context like curved space helps us understand better what the essential ingredients are of the known results and their proofs.

Uniformly approximating grids

The main issue that one needs to address to define these models on a manifold is the absence of a natural discretisation or grid on a manifold. In Chapter 3, we study what we call uniformly approximating grids. We first argue that for our models the underlying grids should be such that their empirical measures converge weakly to the uniform measure on the manifold. Moreover, the edge weights or conductances should be symmetric and such that the corresponding graph Laplacians converge to the Laplace-Beltrami operator.

We then show that such grids can be constructed in the following way. We start with a sequence in the manifold for which the corresponding empirical measures converge to the uniform measure in Kantorovich sense. Then the N^{th} grid is defined to be the set of the first N points of the sequence. Next, we place weights on the edges between the grids points by applying a fixed function to the distance between the points, scaled by a parameter ϵ. ϵ represents the typical jumping distance of the random walk on the grid and goes to 0 as the number of grid points N goes to infinity.

The only thing that needs to be proven in this case is the convergence of the graph Laplacians. First we require that ϵ decreases slowly enough that the number of grid points within a ball of radius ϵ goes to infinity. Doing that allows us to replace the empirical measure corresponding to the grid points by the uniform measure on the manifold, so we are left with an integral on M. Then we fix a point p and push the integral over the space around p to the tangent space at p. We show that the resulting measure satisfies properties that allow us to use a result from the appendix. This result is related to the invariance principle of the geodesic random walk and lets us conclude that the graph Laplacians converge to the Laplace-Beltrami operator.

We started the construction with a sequence such that the corresponding empirical measures converge to the uniform measure. We conclude the chapter with a proof that such sequences are obtained almost surely by sampling i.i.d. points from the uniform distribution on the manifold.

Hydrodynamic limit of SEP

In Chapter 4 we use the uniformly approximating grids from Chapter 3 to define the Symmetric Exclusion Process (SEP) on a manifold. We then generalise the strategy of Kipnis and Landim [96, Chapter 4] to prove that the hydrodynamic limit of SEP is the heat equation on the manifold.

First we note that integration with respect to the empirical measures corresponding to the particle configurations is a function of the particle configurations. Therefore we can write it in terms of a Dynkin martingale. Then we show that this martingale vanishes in the scaling limit by studying its quadratic variation and that the remaining equation is the heat equation written in a weak form. Here we need convergence of the graph Laplacians to the Laplace-Beltrami operator. Together with tightness of the distributions of trajectories of particle configurations and uniqueness of solutions to the heat equation, this yields the desired result.

Equilibrium fluctuations of SEP

Having established the hydrodynamic limit of SEP, we proceed with the fluctuations in Chapter 5. We start SEP from its equilibrium measure: we place a particle at each site independently with probability ρ. Then we subtract the constant ρ from the particle configurations and rescale to study the fluctuations around the constant

density profile ρ. We generalise the strategy of Kipnis and Landim [96, Chapter 11] to prove that the limiting fields are a generalised Ornstein-Uhlenbeck process.

However, because of a lack of translation invariance it is harder to prove tightness when we interpret the fluctuation fields as elements of a Sobolev space of negative index. Therefore we consider them as distributions acting on smooth functions on the manifold. The advantage is that we only need to prove tightness of the trajectories obtained by applying the fluctuations field trajectories to a test function. The disadvantage is that the Skorokhod space of fluctuation field trajectories is a non-metrisable topological space, which is harder to treat.

In addition to tightness, we need to prove uniqueness of possible limit points. We rewrite the fluctuation fields acting on a smooth function as a Dynkin martingale as in Chapter 4. Because of the different scaling, the martingales do not vanish and we obtain a martingale problem in the limit. This martingale problem characterises the generalised Ornstein-Uhlenbeck process, which shows that any possible limit is unique.

The DGFF and convergence to the GFF

In Chapter 6 we use the uniformly approximating grids from Chapter 3 again, this time to construct the Discrete Gaussian Free Field (DGFF) on a manifold. Since there is no natural boundary on a manifold without boundary, we define a zero-average DGFF on the grids. We prove a property similar to the Markov property for this DGFF.

We want to take its limit and show that the DGFF converges to the continuum GFF. Initially, we interpret the DGFF and the GFF as random distributions acting on smooth functions on the manifold. The advantage of that is that the smooth functions on a compact manifold form a nuclear space. This allows us to reduce to proving pointwise convergence of the characteristic functional. For this we need to show convergence of the discrete Green's functions to their continuous counterpart. We prove that if the graph Laplacians have a lower bound on the spectral gap and the corresponding semigroups converge, then the DGFF converges to the GFF.

Next, we show that if we change the weights on the uniformly approximating grids from Chapter 3 to weights depending on the heat kernel on the manifold, we obtain grids that satisfy the assumptions above. In other words, on top of convergence of the graph Laplacians to the Laplace-Beltrami operator, the spectral gaps are bounded from below.

Finally, we extend the definition of the DGFF to a random field that takes values in a Sobolev space of negative index. We show that also in this case the DGFF converges to the continuum GFF. The first step here is showing tightness. Then we prove convergence of finite dimensional distributions. To do this we need to show that the Green's function applied to test functions in an inner product converges

to the continuum Green's function. We do this by proving that this inner product approximates the case with smooth functions that we discussed above.

Active particles and the role of reversibility

We conclude the thesis in Chapter 7 with a topic of a somewhat different nature. We study the motion of a particle that performs a random walk and on top of that has a preferred direction that is described by an internal state. This internal state process is assumed to be a stationary Markov process. This kind of model is known in the literature as run-and-tumble motion or as a directionally reinforced random walk.

We describe the position process of the active particle as a sum of three parts: a random walk part, a martingale part and an active part. Then we calculate the limiting diffusion coefficient and show that each of the parts that we just mentioned contributes a term of this coefficient. The active part of the diffusion coefficient is the only part that depends on the internal state process through more than just the stationary measure.

Next, we consider internal state processes with a finite state space and we analyse the effect that the reversibility of the internal state process has on the limiting diffusion coefficient. We show that out of all internal state processes with the same stationary measure and the same symmetric part of the generator, the reversible process produces the largest diffusion coefficient.

Finally, we derive a large deviations principle for the active particle, still with a finite internal state space. We show that the large deviations free energy is again maximal for reversible internal state processes, and as a consequence the rate function is minimal for those processes.

Samenvatting

In dit proefschrift bestuderen we het Symmetrisch Exclusieproces (SEP) en het Discrete Gaussische Vrije Veld (DGFF) op compacte Riemannse variëteiten. In het bijzonder vinden we de hydrodynamische limiet en evenwichtsfluctuaties van SEP en laten we zien dat de DGFF convergeert naar zijn continue tegenhanger. Om deze discrete modellen te definiëren, construeren we grids met gewichten die de onderliggende variëteit op een geschikte manier benaderen. Verder bestuderen we een model van een actief deeltje en de rol van reversibiliteit voor de diffusiecoëfficiënt in de limiet en voor de grote afwijkingen.

Context en motivatie

Een belangrijk concept in de statistische fysica is dat veel natuurlijke objecten en fenomenen op verschillende niveaus kunnen worden beschreven. Een van haar doelen is het afleiden van de macro-eigenschappen (dichtheid, temperatuur, magnetisatie) van materialen en de vergelijkingen die hen beschrijven vanuit het microscopische gedrag, de eigenschappen en/of de interacties van de deeltjes waaruit ze bestaan. In mathematische statistische fysica gebruiken we vaak omwille van de eenvoud kansmodellen voor de beschrijving op een microscopisch niveau. Om die reden zijn ook de methoden om de microscopische modellen te relateren aan macroscopische modellen afkomstig uit de kansrekening.

Het belangrijkste kansmodel waarin we vragen uit de statistische fysica analyseren is interacterende deeltjessystemen. Het idee is dat we een ruimte willen modelleren waarin deeltjes aan toeval onderhevig rondbewegen en elkaar beïnvloeden. Voor dit doel wordt de continue ruimte gediscretiseerd tot een grid en kunnen deeltjes springen tussen gridpunten. Het deeltjessysteem dat we in deze thesis bestuderen is het Symmetrisch Exclusieproces (SEP). SEP bestaat uit deeltjes die onafhankelijke toevalswandelingen uitvoeren op een grid met de regel dat er maximaal één deeltje per gridpunt aanwezig kan zijn, dus sprongen naar een bezet punt worden onderdrukt. Dit geeft een afstoting tussen de deeltjes. Dit model is veel bestudeerd en heeft in zekere zin de meest basale interactieregel. De moeilijkheid in dit proefschrift komt uit het feit dat we het bestuderen in een nieuwe omgeving, namelijk op een variëteit.

Dit levert allerlei nieuwe uitdagingen, zowel van conceptuele als van technische aard.

Om van een microscopische beschrijving in termen van deeltjesconfiguraties te gaan naar een macroscopische beschrijving in termen van dichtheden en partiële differentiaalvergelijkingen (PDVs), moet het deeltjessysteem worden herschaald. Het grid waarop het systeem evolueert moet steeds fijner worden en de continue ruimte benaderen, terwijl de tijd op een gepaste manier mee wordt geschaald. Dan kunnen we rijen van deeltjesconfiguraties op grids die de continue ruimte benaderen, relateren aan een dichtheidsprofiel op die continue ruimte. Vervolgens willen we bewijzen dat als op tijd 0 de deeltjesconfiguraties een aan hen geassocieerd dichtheidsprofiel hebben, ze op elk later macroscopische tijdpunt ook een geassocieerd dichtheidsprofiel hebben en, bovendien, dat de dichtheidsprofielen voldoen aan een PDV. Deze PDV noemen we dan de hydrodynamische limiet van het betreffende deeltjessysteem. Zodra de hydrodynamische limiet is afgeleid, kunnen we opnieuw herschalen en bestuderen hoe het systeem fluctueert rond de hydrodynamische limiet. Het doel is dan om aan te tonen dat deze zogenaamde evenwichtsfluctuaties voldoen aan een stochastische PDE (SPDE).

Naast interacterende deeltjessystemen bestuderen we het Gaussische Vrije Veld (GFF). Het Discrete Gaussische Vrije Veld (DGFF) is een toevalsveld op een graaf dat het kwadratische verschil van zijn waarden in naburige knopen, geschaald met het gewicht op de tak tussen de knopen, klein wil houden. Dit induceert een normale verdeling met de Greense functie op de graaf als covariantiematrix. Het kan beschouwd worden als een Gaussische fluctuatie van een harmonische functie op de graaf. De DGFF is de discrete tegenhanger van de continue GFF. Dit continue veld is zo ruw dat het waarden aanneemt in de distributies, m.a.w. de realisaties kunnen niet beschreven worden met functies en het veld is niet puntsgewijs een toevalsvariabele. Het is bekend dat in vlakke ruimten de DGFF zodanig herschaald kan worden dat het convergeert naar de GFF. Ons doel is om hetzelfde aan te tonen op een variëteit.

De modellen die we behandelen zijn voorheen voornamelijk bestudeerd in vlakke ruimten, d.w.z. deelverzamelingen van $\mathbb{R}^d$ of de vlakke torus. In dit proefschrift beschouwen we hen op een variëteit. De eerste motivatie hiervoor komt vanuit het oogpunt van toepassingen: aangezien variëteiten alomtegenwoordig zijn in de natuur willen we begrijpen hoe deeltjessystemen en toevalsvelden zich op variëteiten gedragen. De tweede motivatie is dat aangetoond is dat er een interessante wisselwerking is tussen stochastiek en geometrie. Daarom kan het begrijpen van deeltjessystemen op variëteiten ons meer leren over geometrie. Tenslotte helpt het bestuderen van modellen in een meer algemene omgeving zoals gekromde ruimte ons om beter te begrijpen wat de essentiële ingrediënten zijn van de al bekende resultaten en hun bewijzen.

Uniform benaderende grids

De voornaamste horde die genomen moet worden om deze modellen op een variëteit te definiëren is de afwezigheid van een natuurlijke discretisatie of grid op een variëteit.

In Hoofdstuk 3 bestuderen we grids die we uniform benaderende grids noemen. Eerst leggen we uit dat voor onze modellen de onderliggende grids de eigenschap moeten hebben dat de corresponderende empirische maten zwak convergeren naar de uniforme maat op de variëteit. Bovendien moeten de gewichten op de takken of geleidingen symmetrisch zijn en moeten de corresponderende graaflaplacianen convergeren naar de Laplace-Beltramioperator.

Vervolgens tonen we aan dat zulke grids alsvolgt geconstrueerd kunnen worden. We beginnen met een rij in de variëteit waarvan de corresponderende empirische maten in de zin van Kantorovich convergeren naar de uniforme maat. Dan definiëren we het N-de grid als de verzameling van de eerste N punten van de rij. Vervolgens plaatsen we gewichten op de takken tussen de gridpunten door een vast gekozen functie toe te passen op de afstand tussen de punten, herschaald met een parameter ϵ. ϵ representeert de typische sprongafstand van de toevalswandeling op het grid en gaat naar 0 wanneer het aantal gridpunten N naar oneindig gaat.

Het enige wat in dit geval nog moet worden bewezen is de convergentie van de graaflaplacianen. Eerst nemen we aan dat ϵ langzaam genoeg daalt zodat het aantal gridpunten binnen een bol van straal ϵ naar oneindig gaat. Daardoor kunnen we de empirische maat gecorrespondeerd aan de gridpunten vervangen door de uniforme maat op de variëteit, zodat we een integraal over M over hebben. Vervolgens kiezen we een vast punt p en duwen we de integraal over de omgeving van p naar de raakruimte bij p. We tonen aan dat de resulterende maat bepaalde eigenschappen heeft die ons in staat stellen om een resultaat uit de Appendix te gebruiken. Dit resultaat is gerelateerd aan het invariantieprincipe van de geodetische toevalswandeling en zorgt dat we kunnen concluderen dat de graaflaplacianen naar de Laplace-Belramioperator convergeren.

We begonnen de constructie met een rij waarvan de corresponderende empirische maten naar de uniforme maat convergeren. We sluiten het hoofdstuk af met een bewijs dat zulke rijen bijna zeker verkregen worden door onafhankelijk punten te trekken uit de uniforme verdeling op de variëteit.

Hydrodynamische limiet van SEP

In Hoofdstuk 4 gebruiken we de uniform benaderende grids uit Hoofdstuk 3 om het Symmetrisch Exlusieprocess (SEP) te definiëren op een variëteit. Vervolgens generaliseren we de strategie van Kipnis and Landim [96, Chapter 4] om te bewijzen dat de hydrodynamische limiet van SEP de warmtevergelijking op de variëteit is.

Eerst merken we op dat integratie ten opzichte van de aan de deeltjesconfiguraties corresponderende empirische maten een functie is van de deeltjesconfiguraties. Daardoor kunnen we het schrijven in termen van een Dynkinmartingaal. Dan tonen we aan dat deze martingaal verdwijnt in de schalingslimiet door zijn kwadratische variatie te bestuderen en dat de resterende vergelijking de warmtevergelijking is, geschreven in een zwakke vorm. Hier hebben we nodig dat de graaflaplacianen convergeren naar de

Laplace-Beltramioperator. Samen met de *tightness* van de verdelingen van de paden van de deeltjesconfiguraties en de uniciteit van oplossingen van de warmtevergelijking levert dit het gewenste resultaat.

Evenwichtsfluctuaties van SEP

Nadat we de hydrodynamische limiet van SEP vastgesteld hebben, gaan we in Hoofdstuk 5 verder met de fluctuaties. We starten SEP uit zijn evenwichtsverdeling: we plaatsen op ieder gridpunt een deeltje met kans ρ, onafhankelijk van elkaar. Vervolgens trekken we de constante ρ van de deeltjesconfiguraties af en herschalen we deze om de fluctuaties rond het constante dichtheidsprofiel ρ te bestuderen. We generaliseren de strategie van Kipnis and Landim [96, Chapter 11] om te bewijzen dat de velden in de limiet een gegeneraliseerd Ornstein-Uhlenbeckproces zijn.

Echter, door de afwezigheid van translatie-invariantie is het moeilijker om *tightness* aan te tonen als we de fluctuaties opvatten als elementen van een Sobolevruimte met negatieve index. Daarom beschouwen we ze als distributies die werken op gladde functies op de variëteit. Het voordeel hiervan is dat we alleen de *tightness* hoeven aan te tonen van de paden die verkregen worden door de fluctuatieveldpaden toe te passen op een testfunctie. Het nadeel is dat de Skorokhodruimte van fluctuatieveldpaden een niet-metriseerbare topologische ruimte is, deze ruimten zijn moeilijker om mee te werken.

Naast *tightness* moeten we de uniciteit van limietpunten aantonen. We schrijven de fluctuatievelden die op een gladde functie werken als Dynkinmartingaal zoals in Hoofdstuk 4. Door de veranderde schaling verdwijnen de martingalen niet in de limiet, maar verkrijgen we een martingaalprobleem. Dit martingaalprobleem karakteriseert het gegeneraliseerde Ornstein-Uhlenbeckproces, wat vervolgens impliceert dat een mogelijk limietpunt uniek is.

De DGFF en convergentie naar de GFF

In Hoofdstuk 6 gebruiken we de uniform benaderende grids uit Hoofdstuk 3 opnieuw, ditmaal om het Discrete Gaussische Vrije Veld (DGFF) te definiëren op een variëteit. Aangezien er geen natuurlijke rand is op een variëteit zonder rand, definiëren we een nul-gemiddeld DGFF op de grids. We bewijzen een eigenschap die vergelijkbaar is met de Markoveigenschap voor deze DGFF.

We willen de limiet nemen en bewijzen dat de DGFF convergeert naar de continue GFF. In eerste instantie interpreteren we de DGFF en de GFF als toevalsdistributies die werken op gladde functies op de variëteit. Het voordeel hiervan is dat de gladde functies op een compacte variëteit een nucleaire ruimte vormen. Dit stelt ons in staat om ons bewijs te reduceren tot het aantonen van puntsgewijze convergentie van de karakteristieke functionaal. Hiervoor moeten we bewijzen dat de discrete Greense functies convergeren naar hun continue tegenhanger. We bewijzen dat als de

spectrale *gaps* van de graaflaplacianen van onder begrensd zijn en de corresponderende halfgroepen convergeren, de DGFF convergeert naar de GFF.

Vervolgens tonen we aan dat we, als we de gewichten op de uniform benaderende grids uit Hoofdstuk 3 veranderen naar gewichten die afhangen van de warmtekern op de variëteit, grids verkrijgen die aan de aannames hierboven voldoen. In andere woorden, naast de convergentie van de graaflaplacianen naar de Laplace-Beltramioperator zijn de spectrale *gaps* van onder begrensd.

Tenslotte breiden we de definitie van de DGGF uit tot een toevalsveld dat waarden aanneemt in een Sobolevruimte met negatieve index. We tonen aan dat ook in dit geval de DGGF naar de GFF convergeert. De eerste stap hier is het aantonen van *tightness*. Vervolgens bewijzen we de convergentie van eindig-dimensionale verdelingen. Om dit te doen, moeten we aantonen dat de Greense functie toegepast op testfuncties in een inproduct naar de continue Greense functie convergeert. We doen dit door te bewijzen dat dit inproduct het geval met gladde functies dat hierboven werd beschreven benadert.

Actieve deeltjes en de rol van reversibiliteit

In Hoofdstuk 7 sluiten we het proefschrift af met een onderwerp van een wat andere aard. We bestuderen de beweging van een deeltje dat een toevalswandeling uitvoert en daarbij een voorkeursrichting heeft die wordt bepaald door een interne toestand. We nemen aan dat de interne toestand een stationair Markovproces is. Dit type model is bekend in de literatuur als ren-en-tuimelbeweging of als een directioneel versterkte toevalswandeling.

We beschrijven het positieproces van het actieve deeltje als de som van drie delen: een toevalswandelingdeel, een martingaaldeel en een actief deel. Vervolgens berekenen we de diffusiecoëfficiënt in de limiet en tonen we aan dat elk van de zojuist genoemde delen een term bijdraagt aan deze coëfficiënt. Het actieve deel van de diffusiecoëfficiënt is het enige deel dat via meer dan alleen de stationaire maat afhangt van het interne toestandsproces.

Vervolgens beschouwen we interne toestandsprocessen met een eindige toestandsruimte en analyseren we het effect van de reversibiliteit van het interne toestandsproces op de diffusiecoëfficiënt in de limiet. We tonen aan dat van alle interne toestandsprocessen met dezelfde stationaire maat en waarvan de generator hetzelfde symmetrische deel heeft, het reversibele proces de grootste diffusiecoëfficiënt geeft.

Tenslotte leiden we een grote afwijkingenprincipe af voor het actieve deeltje, nog altijd met een eindige interne toestandsruimte. We tonen aan dat de grote afwijkingen vrije energie opnieuw maximaal is voor reversibele interne toestandsprocessen en dat bijgevolg de *rate*-functie minimaal is voor die processen.

Acknowledgements

Writing these acknowledgements is something that I have looked forward to. I guess the reason is that they form a collection of memories of the many great people and experiences that I have encountered over the course of my PhD. They made it into a valuable time.

First, I would like to thank Frank. You were an important part of my stochastic trajectory at the TU, starting from my first probability course. My bachelor's project with you sparked my interest in mathematical research. After that project you suggested me to come to you again for a master's project that might be continued into a PhD project. And so it happened. Thank you for the countless number of hours that we spent in front of a whiteboard and for patiently explaining so many concepts, ideas and proofs (and for the occasional diversions into more philosophical topics). You are down to earth and always take my ideas and comments seriously, whether as a bachelor student or soon, hopefully, as a doctor.

Then I would like to thank Alessandra. We didn't know each other before signing our contracts, but I am happy to have you as my second advisor. I don't know how often I would hear a knock on my door and you were there to share a new idea or I would walk to your office with a quick computation or a new reference or proof. Working with you was a pleasant and dynamic process. Thanks for our many discussions about maths and the world of maths research, but also for the more social chats about travelling (it's a shame that the boat to Uruguay didn't go that day!) and culture (I hope I was a good representative of the Dutch people).

Also thanks to Jan, for arranging the funding for this project, for your participation in all the forms and meetings around it and for some of the joint seminars in the past years. Also thanks for the invitations to the yearly new year's drinks at your house.

I would also like to thank the other members of my committee for their willingness to study my thesis and for their comments: Joe Chen, Alexander Drewitz, Dion Gijswijt, Christian Maes and Henk Schuttelaars. I hope that you enjoyed reading my thesis. Joe, thanks also for our discussions during your visits in Delft.

Next, I would like to thank Bart van Gisbergen for his work with Frank that led to our joint paper on active particles that I used in my thesis.

I would also like to thank all the supporting staff of our group for their practical assistance: Carl, Xi-Wei, Stefanie and Cecilia.

Now I arrive at a group that has been very important to my everyday PhD life, namely my fellow PhDs (and some other "random people"). There is only room to mention a few memories (in no particular order), although of course there are many more.
Thank you Jasper (for all the mornings that we started our working day with a coffee), Martina (for the christmas parties), Sebastiano (for the Italian food and games), Lőrinc (it was always a pleasure to run into you on the days that you were at the TU), Marc (for always being enthusiastic to get a coffee or a beer), Larisa (for our walks in Delft), Elisa (for the office workouts), Federico (for all the usually unasked but welcome advice), Mikola (for many interesting conversations), Andrea B (for co-organising many seminars and meetings), Rik (for sharing the occasional frustrations about the inevitable complications in curved space), Francesca (for your kindness to take over some classes when we shared the office), Richard (for many joint coffee breaks). Mario, some of my best PhD memories were getting to play with mathematics by working on toy problems with you. Thank you for all those and other moments, both in the office and abroad. Simone, many memories come to my mind! Ranging from the westernmost point of Europe to real pizza at Old Scuola to your apartment in Paris, it was and is always a pleasure to catch up with you.
Also thanks to the others from the statistics and probability groups that were around during PhD seminars, coffee moments and lunches: Eni, Birbal, Kailun, Sven, Evelien, Inoni, Dan, Lixue, Vicente, Leandro, Michał, Kate, Bruno, Alan, Serena, Ardjen, Paul, Andrea F, Said.

Thanks to the PhD forum team, for all our (online) efforts to organise interesting meetings (and for dealing with the Dutch meetings style): Amey, Andrea, Gerrit, Marieke, Roel and Tom.

Thanks to my mentor group for the nice lunch meetings: Elli, Guillermo, Joshua, Marc and Willem. Hopefully it was also at times useful and I wish you the best of luck with your further paths at the TU and elsewhere.

I also want to thank some of my closest friends outside of academia. Thanks for hearing me out and for your advice along the way.

Thanks to my wonderful family: papa, mama, Willem, Maarten, Annelize and Frank. Whether I am happy or lacking motivation, you always provide a warm and welcoming place to go back to and I am very grateful for that.

Last but certainly not least I want to thank you Marjet for always being sweet and supportive. Thank you also for your patience and interest to listen to me talk about whatever topic I happen to find interesting at that moment.

Curriculum Vitae

Bart van Ginkel was born in Molenaarsgraaf, a small village close to Rotterdam. During his youth he moved several times, including a period of 3 years in Lubango, Angola. He finished his high school education *summa cum laude* at the "Driestar College" in Gouda.

After a gap year of reflection at the Evangelische Hogeschool in Amersfoort, he started studying Applied Mathematics at TU Delft in 2012. In 2013 he received the Young Talent Incentive Award for his results in the first year. He obtained his Bachelor of Science degree *cum laude* in 2015 with a thesis on "Duality in a Model of Wealth Distribution" under the supervision of Prof.dr. Frank Redig. His studies continued with the master Applied Mathematics, during which he performed a three-month internship at TNO in The Hague. In 2017 he graduated *cum laude* with a master's thesis on "Invariance Principle and Hydrodynamic Limits on Riemannian Manifolds", again under the supervision of Prof. Redig.

In October 2017, he started his PhD in the Applied Probability group of TU Delft under the joint supervision of Prof. Redig and Dr. Alessandra Cipriani with Prof.dr. Jan van Neerven as second promotor. This PhD position was funded by NWO, the national research council of the Netherlands.

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